A modal logic framework for reasoning about comparative distances and topology

Mikhail Sheremet[1], Frank Wolter[2] and Michael Zakharyaschev[1]

[1]School of Computer Science
Birkbeck College London
Malet Street, London WC1E 7HX , U.K.
{mikhail,michael}@dcs.bbk.ac.uk

[2]Department of Computer Science
University of Liverpool
Liverpool L69 3BX, U.K.
frank@csc.liv.ac.uk

Abstract

In 1944, McKinsey and Tarski proved that $\mathcal{S}4$ is the logic of the topological interior and closure operators of any separable dense-in-itself metric space. Thus, the logic of topological interior and closure over arbitrary metric spaces coincides with the logic of the real line, the real plane, and any separable dense-in-itself metric space; it is finitely axiomatisable and PSpace-complete. Because of this result $\mathcal{S}4$ has become a logic of prime importance in Qualitative Spatial Representation and Reasoning in Artificial Intelligence. And in Logic this result has triggered the investigation of a number of variants and extensions of $\mathcal{S}4$ designed for reasoning about qualitative aspects of metric spaces.

In parallel to this line of research (but without much interaction), Philosophical Logic and AI have suggested and investigated a variety of logics — such as conditional logics, certain non-monotonic logics, and logics of comparative similarity — which are naturally interpreted in metric (or more general distance) spaces and which contain a binary operator for comparing distances between points and sets in such spaces.

The contribution of this paper is as follows: We suggest a uniform framework covering large parts of these two lines of research, thus enabling a comparison of the logics involved and a systematic investigation of their expressive power and computational complexity. This framework is obtained by decomposing the underlying modal-like operators into first-order quantifier patterns. We then show that quite a powerful and natural fragment of the resulting first-order logic can be captured by one binary operator comparing distances between sets and one unary operator distinguishing between realised and limit distances (i.e., between minimum and infimum). Due to its greater expressive power, this logic turns out to behave quite differently from Tarski's $\mathcal{S}4$. We provide finite (Hilbert-style) axiomatisations and ExpTime-completeness proofs for the logics of various classes of distance spaces, in particular metric spaces. But we also show that the logic of the real line (and various other important metric spaces) is not recursively enumerable. This result is proved by an encoding of Diophantine equations.

1 Introduction

Tarski's work on the 'algebraisation' of mathematical theories has had numerous repercussions in logic and its applications. The one that is relevant to this paper is that propositional modal logics like $\mathcal{S}4$ (originally introduced as a logic of necessity and possibility by Lewis in [17] and a logic of 'provability' by Orlov [22] and Gödel [11]; see also [1] and references therein) can be used for reasoning about topological spaces.

In their seminal paper [20], McKinsey and Tarski showed in fact that $\mathcal{S}4$ is sound and complete with respect to the interpretation of its possibility and necessity operators as the topological closure and interior, respectively. More importantly, according to their main theorem, this interpretation can be taken over any Euclidean space where the topology is defined by the standard Euclidean metric. As $\mathcal{S}4$ (extended with the universal modality) can express many important topological relations over spatial regions (such as 'region X is a tangential proper part of region Y' or 'regions X, Y and Z are externally connected and share a common point'), on the one hand, and is of 'reasonable' computational complexity (compared with the corresponding first-order logics which can be even non-recursively enumerable; see [13, 4]), on the other hand, this logic and its various fragments and extensions provide basic formalisms for spatial representation and reasoning in AI; see, e.g., [8, 24, 2, 38, 26, 23, 34, 16] and references therein.

Topology abstracts away from the metric aspects of geometry, in particular the 'qualitative' notion of relative or comparative distance. For example, both Hilbert [15] and Tarski [28] used the 4-ary relation 'the distance between x and y is the same as the distance between u and v' in their axiomatisations of Euclidean geometry. In the context of comparative similarity, Lewis [18] considered the ternary predicate 'x is more similar to y than to z,' while Williamson [37] the 4-ary one 'x resembles y at least as much as u resembles v.' Quite recently Giritli [10] has axiomatised and investigated de Laguna's [6] 'can-connect' predicate: a solid X can connect two other solids Y and Z if X can be moved to a position where it contacts both Y and Z. Such relations of comparative distance have been exploited to provide a semantics for various modal formalisms: in conditional logic [18, 21, 27], the conditional implication $\varphi > \psi$ is regarded to be true in a world if ψ is true in every closest φ-world. Going further in this direction, the notion of relative similarity between worlds has been proposed as a semantic underpinning for belief revision and certain forms of non-monotonic reasoning [7, 27]. Finally, in modal logics for spatial reasoning, relative distances are used to interpret geometric modalities [33, 25].

In this paper we propose and investigate a uniform *modal logic framework* covering large parts of these two lines of research: reasoning about topology and relative distance in metric and more general distance spaces. In particular, we explore the interaction between these two notions.

Our starting point is to analyse the explicit quantifier patterns that are used to define the truth conditions for the corresponding 'modal' operators. Let (Δ, d) be a metric space and $\mathbb{R}^{>0}$ the set of positive real numbers. Then

- the interior of a set $X \subseteq \Delta$ is the set

$$\Box X \;=\; \big\{u \in \Delta \mid \exists x \in \mathbb{R}^{>0}\, \forall v \in \Delta\ (d(u,v) < x \to v \in X)\big\},$$

- the universal modality $\forall$ is defined by

$$\forall X \;=\; \big\{u \in \Delta \mid \forall x \in \mathbb{R}^{>0}\, \forall v \in \Delta\ (d(u,v) < x \to v \in X)\big\}$$

  (that is, $\forall X = \Delta$ if $X = \Delta$ and $\forall X = \emptyset$ otherwise),

- the derived set of X is

$$\partial X \;=\; \big\{u \in \Delta \mid \forall x \in \mathbb{R}^{>0}\, \exists v \in \Delta\ (v \in X \wedge 0 < d(w,v) < x)\big\}.$$

To make the quantifier patterns above more explicit, we can introduce modal-like *parameterised operators* of the form $\exists^{<x}$, $\exists^{\leq x}$, $\exists^{=x}$, $\exists^{\leq x}_{>0}$ (and their duals $\forall^{<x}$, $\forall^{>x}$, etc.), where the variable x ranges over $\mathbb{R}^{>0}$ and can be bound by the quantifiers $\forall x$ and $\exists x$. Intuitively, if x is assigned a value $a \in \mathbb{R}^{>0}$, then $\exists^{<x} X$ is the set of all points that are located at distance $< a$ from at least one point in X. In this language, the intended meaning of the operators considered above can be represented in a clear and concise manner:

$$\begin{aligned}\Box X &= \exists x \forall^{<x} X,\\ \forall X &= \forall x \forall^{<x} X,\\ \partial X &= \forall x \exists^{\leq x}_{>0} X.\end{aligned}$$

Observe that in all our examples so far the quantifiers $\exists x$ and $\forall x$ over the real numbers have been followed by exactly one parameterised operator ranging over the metric space. Restricted to the parameterised operators of the form $\exists^{<x}$, $\exists^{\leq x}$, $\forall^{<x}$ and $\forall^{\leq x}$, the resulting logic over metric spaces is equivalent to $\mathcal{S}4_u$, that is, $\mathcal{S}4$ with the universal modality [29].

A number of other well-known 'modal' operators can be obtained if we allow formulas in which the quantifiers $\exists x$ and $\forall x$ are applied to *Boolean combinations* of formulas starting with a parameterised operator. For example, the binary 'closer operator' $X \leftleftarrows Y$ returning all the points that are closer to X than to Y (first introduced in [30]) requires a Boolean combination of parameterised operators over the metric space:

$$X \leftleftarrows Y \;=\; \exists x\, (\exists^{<x} X \wedge \neg \exists^{<x} Y).$$

Another example of a modal operator of this sort is the conditional implication $X > Y$ [18]. If our metric (or distance) space (Δ, d) satisfies the so-called *limit assumption*

$$d(X,Y) \;=\; \inf\{d(u,v) \mid u \in X, v \in Y\} \;=\; \min\{d(u,v) \mid u \in X, v \in Y\}, \tag{1}$$

for all $X, Y \subseteq \Delta$, then a natural system for conditional implication [18, 7, 27] is obtained by setting[1]

$$X > Y \;=\; \exists x \exists^{<x} X \to \exists x \left(\exists^{<x} X \wedge \neg \exists^{<x}(X \wedge \neg Y)\right).$$

The interpretation of $X > Y$ over spaces without the limit assumption has been proposed and investigated by Veltman [35, 21]:

$$X > Y \;=\; \exists x \exists^{<x} X \to \exists x \big(\exists^{\leq x} X \wedge \neg(\exists^{\leq x}(X \wedge \neg Y) \vee (X \wedge \neg Y))\big).$$

In this paper, we consider the language (called $\mathcal{QML}$) obtained by considering all formulas in which the quantifiers $\exists x$ and $\forall x$ over the reals are applied to Boolean combinations of formulas starting with the parameterised operators $\exists^{<x}$, $\exists^{\leq x}$, $\forall^{<x}$ or $\forall^{\leq x}$. As we have seen above, this language covers the most important modal languages introduced in the literature so far for reasoning about topology and comparative distance/similarity in metric

[1]It should be noted that instead of distance spaces, in conditional logic one mostly considers models consisting of worlds w which come with additional strict partial orders $\prec_w$ over the set of worlds to represent the relative distance to w. This semantics is more flexible than using the order $\prec^d_w$ induced by a distance space defined by $\prec^d_w (x, y)$ iff $d(w, x) < d(w, y)$. However, according to the classification in [9], the system obtained using distance spaces corresponds to the conditional logic of frames satisfying the normality, reflexivity, strict centering, uniformity and connectedness conditions.

and distance spaces. Note that the syntactic condition imposed on $\mathcal{QML}$-formulas resembles the definition of the computational tree logic $\mathcal{CTL}^+$ in which path quantifiers are applied to Boolean combinations of formulas starting with a temporal operator; see [36] and references therein. The fragment $\mathcal{S}4_u$ of $\mathcal{QML}$ corresponds then to the standard computational tree logic $\mathcal{CTL}$ where path quantifiers can only be directly applied to formulas starting with a temporal operator.

For $\mathcal{QML}$, we analyse the following problems:

- Is it possible to capture the language $\mathcal{QML}$ by means of a (natural) modal language without first-order quantifiers and parameterised operators? Once such a modal language has been found, do languages previously introduced in the literature correspond to some of its natural fragments?
- Is the resulting logic axiomatisable over interesting classes of distance spaces? Are modular axiomatisations possible? In contrast to $\mathcal{S}4_u$, does the resulting modal language have enough expressive power to distinguish between important spaces?
- What is the computational complexity of deciding validity over important classes of distances spaces?

Our main results are as follows: We show that there is indeed a modal language (called $\mathcal{CSL}$) with one binary and one unary operators (for comparing distances and distinguishing between inf and min; see (1)) which is expressively complete for $\mathcal{QML}$. We provide modular axiomatisations of the logics in the language $\mathcal{CSL}$ interpreted over arbitrary and symmetric distance space, distance spaces with the triangle inequality, as well as standard metric spaces. The validity problem is proved to be ExpTime-complete for all those classes of spaces. In contrast, the logic of the real line (and other Euclidean and discrete spaces) is shown to be non-recursively enumerable. This is proved by a reduction of the solvability problem for Diophantine equations.

2 The logics

We begin by defining the syntax and semantics of the *qualitative metric logic* $\mathcal{QML}$ outlined in the introduction. Starting with a countably infinite set $\{p_1, p_2, \dots\}$ of *atomic terms* (unary predicates or spatial variables), we define *closed $\mathcal{QML}$-terms* τ and *$\mathcal{QML}$-terms* σ by the following inductive rules:

$$\begin{array}{lcl} \tau & ::= & p_i \mid \neg\tau \mid \tau_1 \sqcap \tau_2 \mid \exists x\sigma, \\ \sigma & ::= & \tau \mid \neg\sigma \mid \sigma_1 \sqcap \sigma_2 \mid \exists^{<x}\tau \mid \exists^{\leq x}\tau. \end{array}$$

Other Boolean operators will be used as abbreviations: $\tau_1 \sqcup \tau_2$ is a shorthand for $\neg(\neg\tau_1 \sqcap \neg\tau_2)$, $\tau_1 \to \tau_2$ for $\neg(\tau_1 \sqcap \neg\tau_2)$, $\tau_1 \leftrightarrow \tau_2$ for $(\tau_1 \to \tau_2) \sqcap (\tau_2 \to \tau_1)$, $\top$ for $p_1 \to p_1$, and $\bot$ for $\neg\top$. As usual, we introduce the universal quantifiers as the duals of the existential ones: $\forall x\sigma$ is an abbreviation for $\neg\exists x\neg\sigma$, $\forall^{<x}\tau$ for $\neg\exists^{<x}\neg\tau$, and similarly for $\forall^{\leq x}\tau$. It should be emphasised that expressions like $\exists^{<x}(p_1 \sqcap \exists^{<x}p_2)$ are not well-formed terms of $\mathcal{QML}$: each pair of nested occurrences of operators of the form $\exists^{<x}$ and $\exists^{\leq x}$ must be interleaved with a quantifier $\exists x$.

$\mathcal{QML}$-terms are interpreted over *distance spaces*, that is pairs (Δ, d) where $\Delta \neq \emptyset$ and $d : \Delta \times \Delta \to \mathbb{R}^+$ with $d(u, u) = 0$, for all $u \in \Delta$ (here $\mathbb{R}^+$ is the set of non-negative real numbers). If the *distance function* d on Δ is symmetric and satisfies the triangle inequality, that is, if the conditions

$$d(u,v) \;=\; d(v,u), \tag{sym}$$
$$d(u,w) \;\leq\; d(u,v) + d(v,w) \tag{tri}$$

hold for all points $u, v, w \in \Delta$, then (Δ, d) is clearly a standard *metric space*. We remind the reader that, for a point $u \in \Delta$ and a set $A \subseteq \Delta$, the *distance* $d(u, A)$ *from* u *to* A is defined by taking

$$d(u, A) \;=\; \inf_{v \in A} d(u, v). \tag{2}$$

If $A = \emptyset$ then, by definition, $d(u, A) = +\infty$. We say that the distance $d(u, A)$ is *realised* (*by a point* $v \in A$) if $d(u, A) = d(u, v)$; in this case $d(u, A) = \min_{v \in A} d(u, v)$.

A *distance model* is a structure of the form

$$\mathfrak{I} \;=\; (\Delta^{\mathfrak{I}}, d^{\mathfrak{I}}, p_1^{\mathfrak{I}}, p_2^{\mathfrak{I}}, \dots), \tag{3}$$

where $(\Delta^{\mathfrak{I}}, d^{\mathfrak{I}})$ is a distance space and the $p_i^{\mathfrak{I}}$ are subsets of $\Delta^{\mathfrak{I}}$. Let σ be a $\mathcal{QML}$-term and $a \in \mathbb{R}^+$. We define the *extension* $\sigma^{\mathfrak{I}}[a]$ *of* σ *in* $\mathfrak{I}$ *on* a inductively by taking

$$\begin{aligned}
p_i^{\mathfrak{I}}[a] &= p_i^{\mathfrak{I}},\\
(\neg\sigma)^{\mathfrak{I}}[a] &= \Delta^{\mathfrak{I}} \setminus \sigma^{\mathfrak{I}}[a],\\
(\sigma_1 \sqcap \sigma_2)^{\mathfrak{I}}[a] &= \sigma_1^{\mathfrak{I}}[a] \cap \sigma_2^{\mathfrak{I}}[a],\\
(\exists^{<x}\tau)^{\mathfrak{I}}[a] &= \{u \in \Delta^{\mathfrak{I}} \mid \exists v\, (d(u,v) < a \;\wedge\; v \in \tau^{\mathfrak{I}}[a])\},\\
(\exists^{\leq x}\tau)^{\mathfrak{I}}[a] &= \{u \in \Delta^{\mathfrak{I}} \mid \exists v\, (d(u,v) \leq a \;\wedge\; v \in \tau^{\mathfrak{I}}[a])\},\\
(\exists x \sigma)^{\mathfrak{I}}[a] &= \bigcup_{b \in \mathbb{R}^+} \sigma^{\mathfrak{I}}[b].
\end{aligned}$$

Note that the extension $\tau^{\mathfrak{I}}[a]$ of a closed term τ does not depend on a; in such a case we simply write $\tau^{\mathfrak{I}}$. Note also that in the definition above we allow quantification $\exists x$ over *non-negative real numbers*. To restrict quantification to the *positive* reals (as in most examples in the introduction), one can use terms of the form $\exists x(\exists^{<x}\top \sqcap \sigma)$. Indeed, in this case we have

$$(\exists x(\exists^{<x}\top \sqcap \sigma))^{\mathfrak{I}} \;=\; \bigcup_{b \in \mathbb{R}^{>0}} \sigma^{\mathfrak{I}}[b]$$

because $(\exists^{<x}\top)^{\mathfrak{I}}[0] = \emptyset$ and $(\exists^{<x}\top)^{\mathfrak{I}}[a] = \Delta^{\mathfrak{I}}$, for $a > 0$.

We say that a closed $\mathcal{QML}$-term τ is *satisfiable* (in a class $\mathcal{C}$ of models) if there is a distance model $\mathfrak{I}$ (in $\mathcal{C}$) such that $\tau^{\mathfrak{I}} \neq \emptyset$. And we say that τ is *valid* (in $\mathcal{C}$) if $\tau^{\mathfrak{I}} = \Delta^{\mathfrak{I}}$, for all models $\mathfrak{I}$ (in $\mathcal{C}$). Terms τ_1 and τ_2 are called *equivalent* ($\tau_1 \equiv \tau_2$, in symbols) if $\tau_1^{\mathfrak{I}} = \tau_2^{\mathfrak{I}}$, for every distance model $\mathfrak{I}$.

Our first result in this paper is that the logic $\mathcal{QML}$ turns out to be as expressive as its fragment which only deals with *comparing* distances. Given two terms τ_1 and τ_2, how can we define the property 'τ_1 is closer than τ_2'? The language of $\mathcal{QML}$ suggests four possibilities:

$$\begin{aligned}
&\exists x(\exists^{<x}\tau_1 \sqcap \neg\exists^{<x}\tau_2), \qquad && \exists x(\exists^{<x}\tau_1 \sqcap \neg\exists^{\leq x}\tau_2),\\
&\exists x(\exists^{\leq x}\tau_1 \sqcap \neg\exists^{\leq x}\tau_2), \qquad && \exists x(\exists^{\leq x}\tau_1 \sqcap \neg\exists^{<x}\tau_2).
\end{aligned}$$

In fact, the difference between them is quite subtle. Let us see first their semantical meaning:

$$\big(\exists x(\exists^{<x}\tau_1 \sqcap \neg\exists^{<x}\tau_2)\big)^{\mathfrak{I}} \;=\; \big(\exists x(\exists^{<x}\tau_1 \sqcap \neg\exists^{\leq x}\tau_2)\big)^{\mathfrak{I}} \;=\; \big\{u \in \Delta^{\mathfrak{I}} \;\big|\; d^{\mathfrak{I}}(u,\tau_1^{\mathfrak{I}}) < d^{\mathfrak{I}}(u,\tau_2^{\mathfrak{I}})\big\}, \tag{4}$$

$$\big(\exists x(\exists^{\leq x}\tau_1 \sqcap \neg\exists^{\leq x}\tau_2)\big)^{\mathfrak{I}} \;=\; \big\{u \in \Delta^{\mathfrak{I}} \;\big|\; \exists v \in \tau_1^{\mathfrak{I}}\, \forall w \in \tau_2^{\mathfrak{I}}\; d^{\mathfrak{I}}(u,v) < d^{\mathfrak{I}}(u,w)\big\}, \tag{5}$$

$$\big(\exists x(\exists^{\leq x}\tau_1 \sqcap \neg\exists^{<x}\tau_2)\big)^{\mathfrak{I}} \;=\; \big\{u \in \Delta^{\mathfrak{I}} \;\big|\; \exists v \in \tau_1^{\mathfrak{I}}\, \forall w \in \tau_2^{\mathfrak{I}}\; d^{\mathfrak{I}}(u,v) \leq d^{\mathfrak{I}}(u,w)\big\}. \tag{6}$$

Note that we always have (4) $\subseteq$ (5) $\subseteq$ (6), and the difference between these sets can only contain points u with $d^{\mathfrak{I}}(u,\tau_1^{\mathfrak{I}}) = d^{\mathfrak{I}}(u,\tau_2^{\mathfrak{I}})$. More precisely, it is not hard to compute that $u \in (6) \setminus (5)$ iff $d^{\mathfrak{I}}(u,\tau_1^{\mathfrak{I}}) = d^{\mathfrak{I}}(u,\tau_2^{\mathfrak{I}})$ and $d^{\mathfrak{I}}(u,\tau_1^{\mathfrak{I}}) = d^{\mathfrak{I}}(u,v_1)$ for some $v_1 \in \tau_1^{\mathfrak{I}}$, i.e., the distance $d^{\mathfrak{I}}(u,\tau_1^{\mathfrak{I}})$ is *realised* (by the point v_1). And $u \in (5) \setminus (4)$ iff $d^{\mathfrak{I}}(u,\tau_1^{\mathfrak{I}}) = d^{\mathfrak{I}}(u,\tau_2^{\mathfrak{I}})$, $d^{\mathfrak{I}}(u,\tau_1^{\mathfrak{I}})$ is realised, and $d^{\mathfrak{I}}(u,\tau_2^{\mathfrak{I}})$ is not realised. Thus, (5) and (6) can be expressed using (4) and the following 'diagonal' of (6):

$$\big(\exists x(\exists^{\leq x}\tau \sqcap \neg\exists^{<x}\tau)\big)^{\mathfrak{I}} \;=\; \big\{u \in \Delta^{\mathfrak{I}} \;\big|\; d^{\mathfrak{I}}(u,\tau^{\mathfrak{I}}) \text{ is realised}\big\}. \tag{7}$$

Denote the $\mathcal{QML}$-terms from the left-hand sides of (4), (5), (6) and (7) by $\tau_1 \leftleftarrows \tau_2$, $\tau_1 \mathrel{\underline{\leftarrow}} \tau_2$, $\tau_1 \mathrel{\underline{\leftleftarrows}} \tau_2$ and $\textcircled{r}\tau$, respectively. Then we have $\textcircled{r}\tau \equiv (\tau \mathrel{\underline{\leftleftarrows}} \tau)$ and

$$\begin{aligned} \tau_1 \mathrel{\underline{\leftarrow}} \tau_2 \;&\equiv\; (\tau_1 \leftleftarrows \tau_2) \sqcup (\neg(\tau_2 \leftleftarrows \tau_1) \sqcap \textcircled{r}\tau_1 \sqcap \neg\textcircled{r}\tau_2), \\ \tau_1 \mathrel{\underline{\leftleftarrows}} \tau_2 \;&\equiv\; (\tau_1 \leftleftarrows \tau_2) \sqcup (\neg(\tau_2 \leftleftarrows \tau_1) \sqcap \textcircled{r}\tau_1). \end{aligned} \tag{8}$$

Note also that, over the class of models satisfying the limit assumption, the following equivalences hold: $\textcircled{r}\tau \;\leftrightarrow\; \tau \leftleftarrows \bot$ and $\varphi \leftleftarrows \psi \;\leftrightarrow\; \varphi \mathrel{\underline{\leftarrow}} \psi \;\leftrightarrow\; \neg(\psi \mathrel{\underline{\leftleftarrows}} \varphi)$.

Consider now the sublanguage $\mathcal{CSL}$ (for *comparative similarity logic*) of $\mathcal{QML}$ with *terms* τ defined by the rule

$$\tau \quad ::= \quad p_i \;\mid\; \neg\tau \;\mid\; \tau_1 \sqcap \tau_2 \;\mid\; \textcircled{r}\tau \;\mid\; \tau_1 \leftleftarrows \tau_2.$$

Given a $\mathcal{QML}$- or $\mathcal{CSL}$-term τ, denote by $\mathsf{at}\,\tau$ the set of atomic terms occurring in τ, and by $\mathsf{com}\,\tau$ the set of all subterms φ of τ such that

- τ contains a subterm of the form $\textcircled{r}\varphi$, $\varphi \leftleftarrows \psi$, or $\psi \leftleftarrows \varphi$, if τ is a $\mathcal{CSL}$-term;
- τ contains a subterm of the form $\exists^{<x}\varphi$ or $\exists^{\leq x}\varphi$, if τ is a $\mathcal{QML}$-term.

Theorem 1. *For every closed $\mathcal{QML}$-term τ, there exists an equivalent $\mathcal{CSL}$-term $\bar{\tau}$ such that* $\mathsf{at}\,\bar{\tau} = \mathsf{at}\,\tau$ *and* $\mathsf{com}\,\bar{\tau} = \{\bar{\varphi} \mid \varphi \in \mathsf{com}\,\tau\}$.

Proof. We proceed by induction on the construction of τ. The basis of induction and the case of the Booleans are trivial. Suppose now that τ starts with $\exists x$. Using the equivalence $\exists x(\varphi \sqcup \psi) \equiv \exists x\varphi \sqcup \exists x\psi$ and the classical transformation to disjunctive normal form, we obtain

$$\tau \;\equiv\; \exists x\Big(\prod_{i \in I_0} \exists^{<x}\varphi_i \;\sqcap\; \prod_{i \in I_1} \exists^{\leq x}\varphi_i \;\sqcap\; \prod_{j \in J_0} \neg\exists^{\leq x}\psi_j \;\sqcap\; \prod_{j \in J_1} \neg\exists^{<x}\psi_j\Big) \;\sqcap\; \tau', \tag{9}$$

where τ' is a closed $\mathcal{QML}$-term (here we also use the obvious $\exists x(\chi \sqcap \tau') \equiv (\exists x\chi \sqcap \tau')$). Thus, without loss of generality we may assume that τ is just the first conjunct of (9). We can also assume that I_0, I_1, J_0, J_1 are all nonempty as $\exists^{<x}\top \equiv \neg\exists^{<x}\bot \equiv \exists^{\leq x}\top \equiv \neg\exists^{\leq x}\bot \equiv \top$.

Let $I = I_0 \cup I_1$, $J = J_0 \cup J_1$. We show now that τ is equivalent to the $\mathcal{CSL}$-term

$$\bar{\tau} \;=\; \prod_{i \in I_0, j \in J} (\bar{\varphi}_i \leftleftarrows \bar{\psi}_j) \;\sqcap\; \prod_{i \in I_1, j \in J_0} (\bar{\varphi}_i \mathrel{\underline{\leftharpoonup}} \bar{\psi}_j) \;\sqcap\; \prod_{i \in I_1, j \in J_1} (\bar{\varphi}_i \mathrel{\underline{\underline{\leftharpoonup}}} \bar{\psi}_j), \tag{10}$$

where $\mathrel{\underline{\leftharpoonup}}$ and $\mathrel{\underline{\underline{\leftharpoonup}}}$ are regarded as abbreviations defined by (8).

It follows from definitions and the induction hypothesis that $\tau^{\mathfrak{I}} \subseteq \bar{\tau}^{\mathfrak{I}}$, for every distance model $\mathfrak{I}$. Clearly, we also have $\mathsf{at}\,\bar{\tau} = \mathsf{at}\,\tau$ and $\mathsf{com}\,\bar{\tau} = \{\bar{\varphi} \mid \varphi \in \mathsf{com}\,\tau\}$.

Conversely, suppose that $w \in \bar{\tau}^{\mathfrak{I}}$, for some distance model $\mathfrak{I}$. Then, by the induction hypothesis, there exist $a_{ij} \in \mathbb{R}^+$, for $i \in I, j \in J$, such that

$$\begin{array}{ll} w \in (\exists^{<x}\varphi_i \sqcap \neg\exists^{\leq x}\psi_j)^{\mathfrak{I}}[a_{ij}],\ i \in I_0, j \in J_0; & w \in (\exists^{<x}\varphi_i \sqcap \neg\exists^{<x}\psi_j)^{\mathfrak{I}}[a_{ij}],\ i \in I_0, j \in J_1; \\ w \in (\exists^{\leq x}\varphi_i \sqcap \neg\exists^{\leq x}\psi_j)^{\mathfrak{I}}[a_{ij}],\ i \in I_1, j \in J_0; & w \in (\exists^{\leq x}\varphi_i \sqcap \neg\exists^{<x}\psi_j)^{\mathfrak{I}}[a_{ij}],\ i \in I_1, j \in J_1. \end{array}$$

We need to find an $a \in \mathbb{R}^+$ such that w belongs to each of the sets

$$\begin{array}{ll} (\exists^{<x}\varphi_i)^{\mathfrak{I}}[a],\ i \in I_0, & (\neg\exists^{\leq x}\psi_j)^{\mathfrak{I}}[a],\ j \in J_0, \\ (\exists^{\leq x}\varphi_i)^{\mathfrak{I}}[a],\ i \in I_1, & (\neg\exists^{<x}\psi_j)^{\mathfrak{I}}[a],\ j \in J_1. \end{array} \tag{11}$$

Let $m_i = \min\{a_{ij} \mid j \in J\}$, for $i \in I$. Then w belongs the sets in the first column of (11), for any $a \geq \max\{m_i \mid i \in I\}$. Similarly, let $m^j = \max\{a_{ij} \mid i \in I\}$. Then w belongs to the sets in the second column of (11), for any $a \leq \min\{m^j \mid j \in J\}$. Thus, w belongs to all of the sets in (11) whenever $m_i \leq a \leq m^j$, for all $i \in I, j \in J$. By definition we have $m_i \leq a_{ij} \leq m^j$ for all $i \in I$ and $j \in J$, and so the required a must exist. ❑

Note that the translation $\tau \mapsto \bar{\tau}$ defined in the proof above involves two exponential blowups: the reduction to the disjunctive normal form (9) and the multiple occurrences of the $\bar{\varphi}_i$ and $\bar{\psi}_j$ in (10). We conjecture that $\mathcal{QML}$ is exponentially more succinct than $\mathcal{CSL}$ — similarly to $\mathcal{CTL}^+$ being exponentially more succinct than $\mathcal{CTL}$ [36]. However, according to Theorems 8, 17 and 28 below, $\mathcal{CSL}$ and $\mathcal{QML}$ turn out to have the same computational complexity as far as the satisfiability problem is concerned.

We have already mentioned in the introduction that the modal logic $\mathcal{S}4_u$ is equivalent to a proper fragment of $\mathcal{CSL}$. Indeed, let us introduce the following abbreviations:

$$\Box\tau \;=\; (\top \leftleftarrows \neg\tau), \quad \Diamond\tau \;=\; \neg(\top \leftleftarrows \tau), \quad \forall\tau \;=\; \neg(\neg\tau \leftleftarrows \bot), \quad \exists\tau \;=\; (\tau \leftleftarrows \bot). \tag{12}$$

Then, clearly, $\Box\tau$ is dual to $\Diamond\tau$, $\forall\tau$ is dual to $\exists\tau$, and these operators represent, respectively, the interior operator, the closure operator, the universal modality, and the existential modality. Thus, $\mathcal{S}4_u$ can be defined as the logic

$$\tau \quad ::= \quad p_i \;\mid\; \neg\tau \;\mid\; \tau_1 \sqcap \tau_2 \;\mid\; \Box\tau \;\mid\; \forall\tau$$

interpreted over distance models based on metric spaces.

Recall from the introduction that over spaces satisfying the limit assumption the conditional implication $>$ can be defined as

$$\varphi > \psi \;\equiv\; (\varphi \mathrel{\underline{\leftharpoonup}} \bot) \to (\varphi \mathrel{\underline{\leftharpoonup}} (\varphi \sqcap \neg\psi)).$$

Conversely, over such spaces we also have

$$\varphi \mathrel{\underline{\leftharpoonup}} \psi \;\equiv\; \neg(\varphi > \bot) \sqcap ((\varphi \sqcup \psi) > \neg\psi).$$

Thus, conditional logic interpreted over distance spaces with the limit assumption corresponds to the $\leftleftharpoons$-fragment of $\mathcal{CSL}$. In this paper, we do not consider models with the limit assumption; for a comparison between the logics interpreted in models with and without the limit assumption the reader is referred to the discussion at the end of the paper.

Let us now turn to the decidability, complexity and axiomatisation problems for $\mathcal{QML}$ and $\mathcal{CSL}$. The most transparent case, which actually demonstrates some basic ideas and constructions required for the more complex ones, is the class of all and all symmetric distance models.

3 $\mathcal{CSL}$ over arbitrary and symmetric distance models

Our plan is as follows. First we show that the satisfiability problem for $\mathcal{CSL}$-terms over the classes of all and only symmetric distance models is decidable. As a consequence of the proof we obtain the ExpTime upper bound for this problem, and we establish the matching lower one by interpreting in $\mathcal{CSL}$ the global consequence relation for the modal logic $\mathcal{K}$ which is known to be ExpTime-complete. Finally, we use the decidability proof to find a transparent Hilbert-style axiomatisation of $\mathcal{CSL}$ (and so of $\mathcal{QML}$ as well).

3.1 Decidability and complexity

The general scheme of our decidability proof is similar to many other decidability proofs for modal (temporal, dynamic, etc.) logics. Given a term τ, we take an appropriate 'closure' $\mathsf{cl}\,\tau$ of the set $\mathsf{sub}\,\tau$ of subterms of τ, introduce a syntactical notion of a 'type' approximating those subsets of $\mathsf{cl}\,\tau$ that can be realised in distance models, and then try to construct a model realising a given type t with $\tau \in t$ by providing first a 'witness type' $t_\varphi \ni \varphi$, for each $\varphi \leftleftharpoons \bot \in t$, then witness types for comparisons in all these t_φ, and so on. So far the scheme is pretty standard. The most essential difference of our construction is that some of the witness types t_φ represent sets of isolated points, while others — namely, those t_φ for which $\neg Ⓡ\varphi \in t$ — represent infinite converging sequences of points. The main difficulty of the proof is to define a distance function over all these points which respects the comparisons $\psi \leftleftharpoons \chi$ in their types.

Recall that, for a $\mathcal{CSL}$-term τ, the set $\mathsf{com}\,\tau$ of 'comparisons' in τ was defined as

$$\mathsf{com}\,\tau \;=\; \{\varphi, \psi \mid \varphi \leftleftharpoons \psi \in \mathsf{sub}\,\tau\} \cup \{\varphi \mid Ⓡ\varphi \in \mathsf{sub}\,\tau\} \cup \{\bot, \top\}.$$

Define $\mathsf{cl}\,\tau$ to be the closure under (single) negation of the set

$$\mathsf{sub}\,\tau \cup \{\varphi \leftleftharpoons \psi \mid \varphi, \psi \in \mathsf{com}\,\tau\} \cup \{Ⓡ\varphi \mid \varphi \in \mathsf{com}\,\tau\}.$$

To understand what a type for τ could be, consider first a distance model $\mathfrak{I}$ of the form (3) and a point $u \in \Delta^{\mathfrak{I}}$. The τ-*type* of u in $\mathfrak{I}$ is the set

$$t^{\mathfrak{I}}(u) \;=\; \{\varphi \in \mathsf{cl}\,\tau \mid u \in \varphi^{\mathfrak{I}}\}.$$

Clearly, this set is *Boolean closed* in the sense that

- $\neg\varphi \in t^{\mathfrak{I}}(u)$ iff $\varphi \notin t^{\mathfrak{I}}(u)$, for $\neg\varphi \in \mathsf{cl}\,\tau$, and
- $\varphi \sqcap \psi \in t^{\mathfrak{I}}(u)$ iff $\varphi, \psi \in t^{\mathfrak{I}}(u)$, for $\varphi \sqcap \psi \in \mathsf{cl}\,\tau$.

Besides, $t^{\mathfrak{I}}(u)$ provides us with some information as to which of the sets $\varphi^{\mathfrak{I}}, \psi^{\mathfrak{I}} \in \mathsf{com}\,\tau$ is closer to u and whether the distance from u to $\varphi^{\mathfrak{I}}$ is realised. Indeed, this information is given by the linear quasi-order $\leq_{t^{\mathfrak{I}}(u)}$ on $\mathsf{com}\,\tau$ and the subset $\varrho_{t^{\mathfrak{I}}(u)}$ of $\mathsf{com}\,\tau$ defined by taking, for all $\varphi, \psi \in \mathsf{com}\,\tau$,

$$\begin{aligned} &\varphi \leq_{t^{\mathfrak{I}}(u)} \psi \ \text{ iff } \ d^{\mathfrak{I}}(u, \varphi^{\mathfrak{I}}) \leq d^{\mathfrak{I}}(u, \psi^{\mathfrak{I}}) && \text{iff } \ \neg(\psi \leftleftarrows \varphi) \in t^{\mathfrak{I}}(u), \\ &\varphi \in \varrho_{t^{\mathfrak{I}}(u)} \ \text{ iff } \ \exists v \in \varphi^{\mathfrak{I}} \ d^{\mathfrak{I}}(u, v) = d^{\mathfrak{I}}(u, \varphi^{\mathfrak{I}}) && \text{iff } \ \textcircled{r}\varphi \in t^{\mathfrak{I}}(u). \end{aligned} \tag{13}$$

We will need the following obvious facts:

- $\varphi \in \mathsf{com}\,\tau$ is a $\leq_{t^{\mathfrak{I}}(u)}$-minimal element iff $d^{\mathfrak{I}}(u, \varphi^{\mathfrak{I}}) = 0$, which means that u is in the closure of the set $\varphi^{\mathfrak{I}}$ (in this case $\varphi \in \varrho_{t^{\mathfrak{I}}(u)}$ iff $u \in \varphi^{\mathfrak{I}}$);
- $\varphi \in \mathsf{com}\,\tau$ is a $\leq_{t^{\mathfrak{I}}(u)}$-maximal element iff $d^{\mathfrak{I}}(u, \varphi^{\mathfrak{I}}) = +\infty$, which means that $\varphi^{\mathfrak{I}} = \emptyset$ (in this case $\varphi \notin \varrho_{t^{\mathfrak{I}}(u)}$).

Denote by $<_{t^{\mathfrak{I}}}$ the strict (partial) order induced by $\leq_{t^{\mathfrak{I}}}$:

$$\varphi <_{t^{\mathfrak{I}}(u)} \psi \ \text{ iff } \ d^{\mathfrak{I}}(u, \varphi^{\mathfrak{I}}) < d^{\mathfrak{I}}(u, \psi^{\mathfrak{I}}) \ \text{ iff } \ \varphi \leftleftarrows \psi \in t^{\mathfrak{I}}(u). \tag{14}$$

These considerations suggest the following syntactical approximation of the 'real' τ-types. For a Boolean closed subset t of $\mathsf{cl}\,\tau$, we define, by analogy with (13)–(14), binary relations $<_t$, $\leq_t$ and a set ϱ_t as follows: for $\varphi, \psi \in \mathsf{com}\,\tau$,

$$\varphi <_t \psi \ \text{ iff } \ \varphi \leftleftarrows \psi \in t, \qquad \varphi \leq_t \psi \ \text{ iff } \ \neg(\psi \leftleftarrows \varphi) \in t, \qquad \varphi \in \varrho_t \ \text{ iff } \ \textcircled{r}\varphi \in t.$$

Now we say that a Boolean closed subset t of $\mathsf{cl}\,\tau$ is a *τ-type* if it satisfies the following conditions:

- $\leq_t$ is a linear quasi-order on $\mathsf{com}\,\tau$,
- $t \cap \mathsf{com}\,\tau$ is the set of those $\leq_t$-minimal elements that belong to ϱ_t,
- $\bot$ is a $\leq_t$-maximal element, and no $\leq_t$-maximal term belongs to ϱ_t.

Let $\min t$ and $\max t$ denote the sets of $\leq_t$-minimal and $\leq_t$-maximal elements, respectively. We also write $\varphi \simeq_t \psi$ if $\varphi \leq_t \psi$ and $\varphi \leq_t \psi$.

Lemma 2. *Every τ-type t is determined by the sets $t \cap \mathsf{at}\,\tau$, ϱ_t, and the order $<_t$. In particular, the number of distinct τ-types does not exceed $2^{|\mathsf{at}\,\tau|} \cdot 2^{|\mathsf{com}\,\tau|} \cdot 2^{|\mathsf{com}\,\tau|^2}$.*

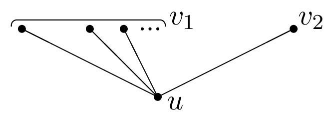
To motivate our next definition of a link, which will be used to provide witnesses for comparisons $\varphi \leftleftarrows \bot$ (i.e., $\exists\varphi$), consider a simple example.

Example 3. Suppose that we want to construct a model $\mathfrak{I}$ satisfying the term

$$\tau \ = \ (\top \leftleftarrows p_1) \sqcap (p_1 \leftleftarrows p_2) \sqcap \textcircled{r}p_2 \sqcap \neg\textcircled{r}p_1 \tag{15}$$

at some point u. As $u \in (\textcircled{r}p_2)^{\mathfrak{I}}$, we should also have $u \in (p_2 \leftleftarrows \bot)^{\mathfrak{I}}$ (the distance to the empty set cannot be realised), and so $u \in (p_1 \leftleftarrows \bot)^{\mathfrak{I}}$. Therefore, we need witnesses v_1 and v_2 (which can be a single point or a sequence of points) such that $d^{\mathfrak{I}}(u, v_i) = d^{\mathfrak{I}}(u, p_i^{\mathfrak{I}})$ with $0 < d^{\mathfrak{I}}(u, v_1) < d^{\mathfrak{I}}(u, v_2)$. As $u \in (\textcircled{r}p_2)^{\mathfrak{I}}$ and $u \in (\neg\textcircled{r}p_1)^{\mathfrak{I}}$, v_2 should be a single point, while v_1 should be an infinite set, say, $\{v_1^n \mid n \in \mathbb{N}\}$, such that $\lim_{n\to\infty} d^{\mathfrak{I}}(u, v_1^n) = d^{\mathfrak{I}}(u, p_1^{\mathfrak{I}})$ with

$d^{\mathfrak{I}}(u, v_1^n) > d^{\mathfrak{I}}(u, p_1^{\mathfrak{I}})$ for all n. Thus, we arrive to a model $\mathfrak{I}$ such as the one in the figure below:



The following obvious lemma ensures that even if we need an infinite set as a witness, all of its points can be chosen to have the same type. For a model $\mathfrak{I}$ and a type t we write $t^{\mathfrak{I}}$ for $\{v \in \Delta^{\mathfrak{I}} \mid t^{\mathfrak{I}}(v) = t\}$.

Lemma 4. *Let $\mathfrak{I}$ be a distance model.*

(i) *For all $u, v \in \Delta^{\mathfrak{I}}$ and $\psi \in \mathsf{com}\,\tau$, we have $\psi <_{t^{\mathfrak{I}}(u)} \bot$ iff $\psi <_{t^{\mathfrak{I}}(v)} \bot$ iff $\psi^{\mathfrak{I}} \neq \emptyset$.*

(ii) *Suppose that $u \in \Delta^{\mathfrak{I}}$ and $\varphi^{\mathfrak{I}} \neq \emptyset$ for some $\varphi \in \mathsf{com}\,\tau$. Then there is a τ-type t such that $\varphi \in t$ and $d(u, \varphi^{\mathfrak{I}}) = d(u, t^{\mathfrak{I}})$, with $d(u, \varphi^{\mathfrak{I}})$ and $d(u, t^{\mathfrak{I}})$ being realised or not realised simultaneously. Moreover, for all $\psi \in \mathsf{com}\,\tau$ we have:*

$$\begin{aligned} &\psi \in t \quad \textit{implies} \quad \psi \leq_s \varphi, \\ &\varphi \in \varrho_s, \;\; \varphi \simeq_s \psi, \;\; \textit{and}\; \psi \in t \quad \textit{imply} \quad \psi \in \varrho_s. \end{aligned} \tag{16}$$

This observation suggests the following definition. Let s, t be τ-types, $\varphi \in \mathsf{com}\,\tau$, and $\varphi \notin s$. The pair (s, t) is called a *φ-link* if, for every $\psi \in \mathsf{com}\,\tau$, the conditions (16) are satisfied and we have $\psi <_s \bot$ iff $\psi <_t \bot$. Thus, a φ-link (s, t) provides s with a φ-witness t. A complete set of such links will be called a τ-diagram. More precisely, a set D of τ-types is a *τ-diagram* if the following conditions are satisfied:

$$\begin{aligned} &\text{there exists } t_* \in D \text{ with } \tau \in t_*, \\ &\text{for all } s, t \in D \text{ and } \psi \in \mathsf{com}\,\tau, \text{ we have } \psi <_s \bot \text{ iff } \psi <_t \bot. \\ &\text{for every } s \in D \text{ and every } \varphi \notin s \text{ with } \varphi <_s \bot, \text{ there exists } t \in D \\ &\text{such that } (s, t) \text{ is a } \varphi\text{-link}, \end{aligned} \tag{17}$$

Theorem 5. *The following conditions are equivalent, for every $\mathcal{CSL}$-term τ:*

(i) *τ is satisfied in a distance model;*

(ii) *there exists a τ-diagram;*

(iii) *τ is satisfied in a symmetric distance model.*

Proof. (i) $\Rightarrow$ (ii) If $\mathfrak{I}$ is a distance model with $\tau^{\mathfrak{I}} \neq \emptyset$ then it is not hard to check that the set $D = \{t^{\mathfrak{I}}(x) \mid x \in \Delta^{\mathfrak{I}}\}$ is a τ-diagram.

(ii) $\Rightarrow$ (iii) Suppose now that D is a τ-diagram with $\tau \in t_*$. Let

$$\{\varphi_0, \dots, \varphi_{k-1}\} \;=\; \{\varphi \in \mathsf{com}\,\tau \mid \varphi <_t \bot, \text{ for all } t \in D\},$$

with all the φ_i being distinct.

Consider the tree Γ whose nodes are the words over the alphabet $\{(i, j) \mid i < k,\; j \in \mathbb{N}\}$, the root is the empty word λ, and the immediate successors (children) of a node $\alpha \in \Gamma$ are the words of the form $\alpha(i, j)$. We are going to 'unravel' D into a subtree Δ of Γ endowed with a labelling function $tp : \Delta \to D$. Then we will define a symmetric distance function d

on Δ that respects the comparisons from the types given by tp. Thus, the triple (Δ, d, tp) will provide us with the components for the required model satisfying τ.

The tree Δ and labelling tp are defined by the following inductive procedure. First we set $\lambda \in \Delta$ and $tp(\lambda) = t_*$. Then, at every next step, we choose some shortest word $\alpha \in \Delta$ that does not have children in Δ yet. As D is a diagram, for each $i < k$ with $\varphi_i \notin tp(\alpha)$ there exists $t_i \in D$ such that $(tp(\alpha), t_i)$ is a φ_i-link. We extend Δ according to the following rules:

- if $\varphi_i \in tp(\alpha)$ then $\alpha(i,j) \notin \Delta$, for all $j \in \mathbb{N}$ — $tp(\alpha)$ does not require φ_i-witnesses,
- if $\varphi_i \in \varrho_{tp(\alpha)} \setminus tp(\alpha)$, then $\alpha(i,0) \in \Delta$, $tp(\alpha(i,0)) = t_i$, and $\alpha(i,j) \notin \Delta$, for all $j > 0$ — $tp(\alpha)$ requires a single φ_i-witness,
- if $\varphi_i \notin \varrho_{tp(\alpha)}$, then $\alpha(i,j) \in \Delta$ and $tp(\alpha(i,j)) = t_i$, for all $j \in \mathbb{N}$ — $tp(\alpha)$ requires infinitely many φ_i-witnesses.

For $\alpha \in \Delta$ and $i < k$, we set

$$\alpha + i = \begin{cases} \{\alpha\} & \text{if } \varphi_i \in tp(\alpha), \\ \{\alpha(i,0)\} & \text{if } \varphi_i \in \varrho_{tp(\alpha)} \setminus tp(\alpha), \\ \{\alpha(i,j) \mid j \in \mathbb{N}\} & \text{if } \varphi_i \notin \varrho_{tp(\alpha)}. \end{cases}$$

Thus, the set $\alpha+ = \bigcup_{i<k}(\alpha+i)$ consists of α and its children in Δ (we always have $\top \in tp(\alpha)$).

Let us now define the distance function d. To simplify notation, we write d^α for $d(\alpha', \alpha)$, where α' is the parent of α. The values d^α, for $\alpha \in \Delta$, are defined inductively as follows.

For convenience we set $d^\lambda = 1$. Now suppose that d^α is already defined, for some $\alpha \in \Delta$. As $tp(\alpha)$ is a type, we can choose numbers $d^{\alpha+i} \in [0, d^\alpha)$, for all $i < k$, satisfying the following conditions, for all $i, l < k$:

$$d^{\alpha+i} \leq d^{\alpha+l} \text{ iff } \varphi_i \leq_{tp(\alpha)} \varphi_l, \qquad d^{\alpha+i} = 0 \text{ iff } \varphi_i \in \min tp(\alpha). \tag{18}$$

Then we set, for all $i < k$ and $j \in \mathbb{N}$:

$$\begin{aligned} d^{\alpha(i,0)} &= d^{\alpha+i}, & &\text{if } \varphi_i \in \varrho_{tp(\alpha)} \setminus tp(\alpha), \\ d^{\alpha(i,j)} &= d^{\alpha+i} + (d^\alpha - d^{\alpha+i})/(2+j), & &\text{if } \varphi_i \notin \varrho_{tp(\alpha)}. \end{aligned} \tag{19}$$

Thus, we obtain $d(\alpha, \alpha+i) = d^{\alpha+i}$. Note that we always have $d(\alpha, \alpha(i,j)) < d^\alpha \leq 1$. Finally we define distances between arbitrary nodes in Δ by taking

$$d(\alpha, \beta) \;=\; \begin{cases} 0 & \text{if } \alpha = \beta, \\ d^\alpha & \text{if } \alpha \text{ is a child of } \beta, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, d is a symmetric distance function on Δ.

For $\varphi \in \mathsf{cl}\,\tau$, let $\varphi^\Delta = \{\alpha \in \Delta \mid \varphi \in tp(\alpha)\}$. Then $\varphi^\Delta = \emptyset$ if $\varphi \in \mathsf{com}\,\tau \setminus \{\varphi_0, \dots, \varphi_{k-1}\}$, and $\alpha + i \subseteq \varphi_i^\Delta$, for all $i < k$ and $\alpha \in \Delta$.

Lemma 6. *Let $\alpha \in \Delta$ and $i < k$. Then $d(\alpha, \varphi_i^\Delta) = d^{\alpha+i}$. More precisely,*

- *if $\varphi_i \in \varrho_{tp(\alpha)} \setminus tp(\alpha)$ then $d(\alpha, \varphi_i^\Delta) = d^{\alpha(i,0)}$,*
- *if $\varphi_i \notin \varrho_{tp(\alpha)}$ then $d(\alpha, \varphi_i^\Delta) = \lim_{j\to\infty} d^{\alpha(i,j)}$ and $d(\alpha, \varphi_i^\Delta)$ is not realised.*

Proof. According to our choice of the distances, it suffices to prove the following property:

$$\forall \beta \in \varphi_i^{\Delta} \ \ \exists \beta_1 \in \alpha + i \ \ d(\alpha, \beta_1) \leq d(\alpha, \beta). \tag{20}$$

To this end we first note that $d(\alpha, \varphi_i^{\Delta} \cap \alpha+) < 1 \leq d(\alpha, \varphi_i^{\Delta} \setminus \alpha+)$. Therefore we can assume that $\beta \in \alpha+$ in (20).

Suppose that $\varphi_i \in \varrho_{tp(\alpha)} \setminus tp(\alpha)$. Then $\alpha(i,0) \in \varphi_i^{\Delta}$ and $(tp(\alpha), tp(\alpha(i,0)))$ is a φ_i-link by the construction. Consider an arbitrary $\beta \in \varphi_i^{\Delta} \cap \alpha+$. Then $\varphi_i \in tp(\beta)$ and $(tp(\alpha), tp(\beta))$ is a φ_l-link, for some $l < k$. Hence $\varphi_i \leq_{tp(\alpha)} \varphi_l$ by the definition of a link, and then $d^{\alpha(i,0)} = d^{\alpha+i} \leq d^{\alpha+l} \leq d^{\beta}$ by (18).

Suppose now that $\varphi_i \notin \varrho_{tp(\alpha)}$. Then $\alpha(i,j) \in \varphi_i^{\Delta}$ and $(tp(\alpha), tp(\alpha(i,j)))$ is a φ_i-link, for all $j \in \mathbb{N}$. Consider an arbitrary $\beta \in \varphi_i^{\Delta} \cap \alpha+$. Then $\varphi_i \in tp(\beta)$ and $(tp(\alpha), tp(\beta))$ is a φ_l-link, for some $l < k$. Since $\varphi_i \notin \varrho_{tp(\alpha)}$ and by the definition of a link, we have either $\varphi_i <_{tp(\alpha)} \varphi_l$, or $\varphi_l \simeq_{tp(\alpha)} \varphi_i$ and $\varphi_l \notin \varrho_{tp(\alpha)}$. Hence we obtain, respectively, that either $d^{\alpha+i} < d^{\alpha+l} = d^{\beta}$ or $d^{\alpha+i} = d^{\alpha+l} < d^{\beta}$. ❑

Define now a model $\mathfrak{I}$ by setting $\Delta^{\mathfrak{I}} = \Delta$, $d^{\mathfrak{I}} = d$, and $p^{\mathfrak{I}} = p^{\Delta}$, for all atomic terms p.

Lemma 7. *For each $\varphi \in \mathsf{cl}\,\tau$, we have $\varphi^{\mathfrak{I}} = \varphi^{\Delta}$.*

Proof. We proceed by induction on the construction of φ. The basis of induction and the case of Boolean operators are trivial. So two cases remain.

Case 1: $\varphi = (\psi_0 \leftleftarrows \psi_1)$. Suppose that $\alpha \in (\psi_0 \leftleftarrows \psi_1)^{\mathfrak{I}}$. Then $\psi_0^{\mathfrak{I}} \neq \emptyset$. By the induction hypothesis, this means that $\psi^{\Delta} \neq \emptyset$, i.e., $\psi_0 = \varphi_i$ for some $i < k$. If $\psi_1 \notin \{\varphi_0, \dots, \varphi_{k-1}\}$ then $\psi_1 \in \max tp(\alpha)$, and so $\psi_0 \leftleftarrows \psi_1 \in tp(\alpha)$. Let $\psi_1 = \varphi_l$ for some $l < k$. In view of the induction hypothesis and Lemma 6, we obtain $d^{\alpha+i} < d^{\alpha+l}$, and so $\varphi_i \leftleftarrows \varphi_l \in tp(\alpha)$ by the choice of d^{α}.

Conversely, suppose that $\alpha \in (\psi_0 \leftleftarrows \psi_1)^{\Delta}$. Then $\psi_0 <_{tp(\alpha)} \psi_1$; hence $\psi_0 \notin \max tp(\alpha)$ and $\psi_0 = \varphi_i$, for some $i < k$. By the induction hypothesis, $\psi_0^{\mathfrak{I}} = \varphi_i^{\Delta}$, which is nonempty. If $\psi_1 \notin \{\varphi_0, \dots, \varphi_{k-1}\}$ then $\psi^{\mathfrak{I}} = \psi^{\Delta} = \emptyset$, and so $\alpha \in \Delta = \varphi^{\mathfrak{I}}$. Let $\psi_1 = \varphi_l$ for some $l < k$. Then $d^{\alpha+i} < d^{\alpha+l}$ by definition. By Lemma 6 and the induction hypothesis we obtain $d(\alpha, \psi_0^{\mathfrak{I}}) < d(\alpha, \psi_1^{\mathfrak{I}})$, i.e., $\alpha \in \varphi^{\mathfrak{I}}$.

Case 2: $\varphi = ⓡ\psi$. Suppose that $\alpha \in (ⓡ\psi)^{\mathfrak{I}}$, i.e., $d(\alpha, \psi^{\mathfrak{I}})$ is realised. We have $\psi^{\mathfrak{I}} = \psi^{\Delta}$ by the induction hypothesis, and hence $\psi \in \varrho_{tp(\alpha)}$, i.e., $\alpha \in (ⓡ\psi)^{\Delta}$, by Lemma 6.

Conversely, suppose $\alpha \in (ⓡ\psi)^{\Delta}$, that is $\psi \in \varrho_{tp(\alpha)}$. Then $d(\alpha, \psi^{\Delta})$ is realised by Lemma 6. Thus, $\alpha \in (ⓡ\psi)^{\mathfrak{I}}$. ❑

Recall now that $\tau \in tp(\lambda)$ by the construction. It follows that $\mathfrak{I}$ is a symmetric distance model with $\tau^{\mathfrak{I}} \neq \emptyset$.

The implication (iii) $\Rightarrow$ (i) is trivial. ❑

It follows from Theorem 5 that $\mathcal{CSL}$ 'does not feel' the difference between symmetric and non-symmetric models that in general do not satisfy the triangle inequality. Moreover, we have the following:

Theorem 8. *The satisfiability problem for $\mathcal{CSL}$-terms and closed $\mathcal{QML}$-terms in both the class of all distance models and the class of symmetric distance models is* ExpTime*-complete.*

Proof. We begin by establishing the upper bound. Consider first the case of $\mathcal{CSL}$. Let τ be a $\mathcal{CSL}$-term and B the set of all τ-types. By Lemma 2, we have $|B| \leq 2^{|\mathsf{at}\,\tau|\cdot|\mathsf{com}\,\tau|^3}$.

As the third property of a diagram (see (17)) is preserved under set unions, B contains a unique maximal subset D satisfying this property. It is not hard to see that D can be constructed using the following elimination procedure (cf. [14]).

Step 0. Set $D_0 = B$.

Step $n+1$. For each $s \in D_n$, we check whether

$$\text{for every } \varphi <_s \bot \text{ with } \varphi \notin s, \text{ there exists } t \in D_n \text{ such that } (s,t) \text{ is a } \varphi\text{-link.} \tag{21}$$

Once we find $s \in D_n$ which does not satisfy (21), we set $D_{n+1} = D_n \setminus \{s\}$ and go to step $n+2$. Otherwise (in particular, if $D_n = \emptyset$), we set $D = D_n$ and halt.

This procedure will halt in at most $|B|$ steps, each of which takes at most $|B \times \mathsf{com}\,\tau|$ checks. Therefore the construction of D requires at most $2^{O(|\mathsf{at}\,\tau|\cdot|\mathsf{com}\,\tau|^3)}$ operations.

Suppose now that τ does not belong to any type in D. Then obviously no τ-diagram exists, and so τ is not satisfiable by Theorem 5. Now let $\tau \in t_* \in D$. Then the set

$$\{t_n \mid (t_*, t_0), \ldots, (t_{n-1}, t_n) \text{ are links, for some } t_0, \ldots, t_n \in D\}.$$

is clearly a τ-diagram, and so τ is satisfiable by Theorem 5.

It follows that satisfiability of τ can be checked in time $\leq 2^{O(|\mathsf{at}\,\tau|\cdot|\mathsf{com}\,\tau|^3)} \leq 2^{O(|\tau|^4)}$.

If τ is a closed $\mathcal{QML}$-term then, by Theorem 1 and Lemma 2, there exists an equivalent $\mathcal{CSL}$-term $\bar{\tau}$ such that $|\mathsf{at}\,\tau| = |\mathsf{at}\,\bar{\tau}|$ and $|\mathsf{com}\,\tau| = |\mathsf{com}\,\bar{\tau}|$. It follows that satisfiability of τ can be checked in time $\leq 2^{O(|\tau|^4)}$ as well.

The proof of the matching lower bound is by reduction of the global $\mathcal{K}$-consequence relation that is known to be ExpTime-hard [31]. As $\mathcal{CSL}$ is a fragment of $\mathcal{QML}$, it suffices to consider the case of $\mathcal{CSL}$ only.

We remind the reader that the language $\mathcal{L}_\mathcal{K}$ of the basic modal logic $\mathcal{K}$ extends the language of classical propositional logic (with propositional variables $p_1, p_2, \ldots$) by means of one unary operator $\Diamond$. $\mathcal{L}_\mathcal{K}$ is interpreted in models of the form

$$\mathfrak{N} \;=\; (W, R, p_1^{\mathfrak{N}}, p_2^{\mathfrak{N}}, \ldots), \tag{22}$$

where W is a nonempty set, $S \subseteq W \times W$ and $p_i^{\mathfrak{N}} \subseteq W$. The *value* $\varphi^{\mathfrak{N}} \subseteq W$ of an $\mathcal{L}_\mathcal{K}$-formula φ in $\mathfrak{N}$ is defined inductively as follows:

- $(\varphi \wedge \psi)^{\mathfrak{N}} = \varphi^{\mathfrak{N}} \cap \psi^{\mathfrak{N}}$;
- $(\neg\varphi)^{\mathfrak{N}} = W \setminus \varphi^{\mathfrak{N}}$;
- $(\Diamond\varphi)^{\mathfrak{N}} = \{v \in W \mid \exists w\; (vRw \wedge w \in \varphi^{\mathfrak{N}})\}$.

We say that φ_1 *follows globally from* φ_2 and write $\varphi_2 \vdash \varphi_1$ if, for every model $\mathfrak{N}$, $\varphi_1^{\mathfrak{N}} = W$ whenever $\varphi_2^{\mathfrak{N}} = W$.

Now we define inductively a translation $^\#$ from $\mathcal{L}_\mathcal{K}$ into the set of $\mathcal{CSL}$-terms. Let $\kappa_0 = q_0$, $\kappa_1 = \neg q_0 \sqcap q_1$, $\kappa_2 = \neg q_0 \sqcap \neg q_1$, for some fresh variables q_0, q_1. Then we set $p_i^\# = p_i$, $(\neg\varphi)^\# = \neg\varphi^\#$, $(\varphi_1 \wedge \varphi_2)^\# = \varphi_1^\# \sqcap \varphi_2^\#$, and

$$(\Diamond\varphi)^\# \;=\; \prod_{i<3} \big(\kappa_i \to ⓡ(\kappa_{i\oplus 1} \sqcap \varphi^\#) \sqcap \neg(\kappa_{i\oplus 1} \leftleftarrows (\kappa_{i\oplus 1} \sqcap \varphi^\#))\big),$$

where $\oplus$ is addition modulo 3. We show now that, for any $\varphi, \psi \in \mathcal{L}_{\mathcal{K}}$, ψ follows globally from φ iff $(\forall\varphi^{\#} \to \psi^{\#})$ is valid in all distance models iff $(\forall\varphi^{\#} \to \psi^{\#})$ is valid in all metric models (the universal modality $\forall$ was defined by (12) on page 7).

Suppose first that $\varphi \not\models \psi$. This means that there is a $\mathcal{K}$-model $\mathfrak{N}$ of the form (22) such that $\varphi^{\mathfrak{N}} = W$ and $r \notin \psi^{\mathfrak{N}}$ for some $r \in W$. As is well-known from modal logic (see, e.g., [3]), without loss of generality we may assume that (W, R) is an irreflexive intransitive tree with root r. Let d be the standard tree metric on (W, R), i.e., $d(u, v) = d(v, u)$ is the length of the shortest *undirected* path from u to v in (W, R). Consider the tree metric model

$$\mathfrak{I} \;=\; (W, d, p_1^{\mathfrak{N}}, p_2^{\mathfrak{N}}, \dots, q_0^{\mathfrak{I}}, q_1^{\mathfrak{I}}),$$

where $q_i^{\mathfrak{I}}$ $(i = 0, 1)$ consists of all points $u \in W$ such that $d(u, r) = 3n + i$ for some $n \in \mathbb{N}$. Then it is easily checked by induction that $\chi^{\mathfrak{N}} = (\chi^{\#})^{\mathfrak{I}}$, for every $\chi \in \mathcal{L}_{\mathcal{K}}$. Hence $(\varphi^{\#})^{\mathfrak{I}} = W$ and $r \notin (\psi^{\#})^{\mathfrak{I}}$, and so the term $(\forall\varphi^{\#} \to \psi^{\#})$ is not valid in the class of distance models.

Conversely, suppose that $\forall\varphi^{\#} \sqcap \neg\psi^{\#}$ is satisfied in some distance model

$$\mathfrak{I} \;=\; (\Delta^{\mathfrak{I}}, d^{\mathfrak{I}}, p_1^{\mathfrak{I}}, p_2^{\mathfrak{I}}, \dots, q_0^{\mathfrak{I}}, q_1^{\mathfrak{I}}).$$

Consider the $\mathcal{K}$-model

$$\mathfrak{N} \;=\; (\Delta^{\mathfrak{I}}, R^{\mathfrak{I}}, p_1^{\mathfrak{I}}, p_2^{\mathfrak{I}}, \dots)$$

where $uR^{\mathfrak{I}}v$ for $u, v \in \Delta^{\mathfrak{I}}$ iff, for some $i < 3$, $u \in \kappa_i^{\mathfrak{I}}$, $v \in \kappa_{i\oplus 1}^{\mathfrak{I}}$, and $d^{\mathfrak{I}}(u, v) = d^{\mathfrak{I}}(u, \kappa_{i\oplus 1}^{\mathfrak{I}})$. Again, it is easily checked by induction that $\chi^{\mathfrak{N}} = (\chi^{\#})^{\mathfrak{I}}$, for every formula $\chi \in \mathcal{L}_{\mathcal{K}}$. It follows that $\varphi \not\models \psi$. ❑

3.2 Axiomatisation

Now we present a Hilbert-style axiomatisation of the set of valid $\mathcal{CSL}$-terms. Our *axiom schemas* are all tautologies of classical propositional logic as well as the following ones:

$$\begin{aligned} ((\varphi \leftleftarrows \psi) \sqcap (\psi \leftleftarrows \chi)) \;&\to\; (\varphi \leftleftarrows \chi), \\ (\neg(\varphi \leftleftarrows \psi) \sqcap \neg(\psi \leftleftarrows \chi)) \;&\to\; \neg(\varphi \leftleftarrows \chi), \end{aligned} \tag{23}$$

$$\neg((\varphi \sqcup \psi) \leftleftarrows \varphi) \;\sqcup\; \neg((\varphi \sqcup \psi) \leftleftarrows \psi), \tag{24}$$

$$\forall(\varphi \to \psi) \;\to\; \neg(\varphi \leftleftarrows \psi), \tag{25}$$

$$ⓡ(\varphi \sqcup \psi) \;\to\; (ⓡ\varphi \sqcup ⓡ\psi), \tag{26}$$

$$(ⓡ(\varphi \sqcup \psi) \sqcap (\varphi \leftleftarrows \psi)) \;\to\; ⓡ\varphi \tag{27}$$

$$ⓡ\varphi \sqcap \neg(\psi \leftleftarrows \varphi) \;\to\; ⓡ(\varphi \sqcup \psi) \tag{28}$$

$$\forall(\varphi \leftrightarrow \psi) \;\to\; (ⓡ\varphi \leftrightarrow ⓡ\psi), \tag{29}$$

$$\varphi \;\leftrightarrow\; (ⓡ\varphi \sqcap \neg(\top \leftleftarrows \varphi)), \tag{30}$$

$$\top \leftleftarrows \bot, \tag{31}$$

$$\neg ⓡ\bot, \tag{32}$$

$$(\neg(\varphi \leftleftarrows \bot) \leftleftarrows \bot) \;\to\; \neg((\varphi \leftleftarrows \bot) \leftleftarrows \bot). \tag{33}$$

Informally, the meaning of these schemas is as follows:

- (23) expresses transitivity of the relations 'closer' and 'not closer,'
- (24) says that the union of two sets cannot be closer (to any given point) than either of these sets,
- (25) that a smaller set cannot be closer than a larger one (and, in particular, that Boolean equivalence preserves the relation 'closer'),
- (26) is, in a sense, an $\textcircled{r}$-counterpart of (24): if the distance to the union of two sets is realised, then the distance to at least one of these sets must be realised as well,
- (27) specialises (26): if the distance to the union of two sets is realised and one of them is closer than the other, then the distance to the former set is realised,
- (28) is a partial inverse of (26): if the distance to one set is realised, and another set is not closer than the former one, then the distance to the union of these sets is realised,
- the meaning of (29) is clear,
- (30) says that the distance to some set is realised and equal to zero if, and only if, we are actually in that set (recall that $\top \leftleftarrows \varphi$ gives those points whose distance from φ is positive),
- (31) says that the whole space is closer than the empty set, and
- (32) that the distance to the empty set cannot be realised,
- finally, (33)—by definition (12)—is just the classical implication $\exists\neg\exists\varphi \to \neg\exists\exists\varphi$; it will be used to prove various properties like $\exists\varphi \to \forall\exists\varphi$, $\forall\varphi \to \forall\forall\varphi$, etc.

It is worth noting that (24)–(29) can actually be replaced with just two axiom schemas using the operator $\leftleftarrows$. Namely, the conjunction of (24), (26) and (27) is equivalent to

$$\neg(\varphi \sqcup \psi) \leftleftarrows \varphi) \;\sqcup\; \neg((\varphi \sqcup \psi) \leftleftarrows \psi),$$

while the conjunction of (25), (28), and (29) is equivalent to

$$\forall(\varphi \to \psi) \;\to\; \neg(\varphi \leftleftarrows \psi).$$

The *inference rules* of our axiomatic system are standard:

$$\text{Modus ponens:} \qquad \frac{\varphi,\ \varphi \to \psi}{\psi}, \tag{MP}$$

$$\text{Generalisation:} \qquad \frac{\varphi}{\forall\varphi}. \tag{Gen}$$

As usual, the fact that a $\mathcal{CSL}$-term τ is *deducible* in the axiomatic system above is denoted by $\vdash \tau$, and we write $\varphi_0, \dots, \varphi_{n-1} \vdash \varphi_n$ if there exists a derivation of φ_n from the premises $\varphi_0, \dots, \varphi_{n-1}$ in which (Gen) is not applied to terms that depend on $\varphi_0, \dots, \varphi_{n-1}$.

It easy to see that all of our axioms are valid in the class of distance models, and that the rules (MP) and (Gen) preserve validity. Therefore, the axiomatic system is *consistent*. Clearly, we also have the standard *deduction theorem*:

Lemma 9. *If $\boldsymbol{\varphi}, \varphi \vdash \psi$ then $\boldsymbol{\varphi} \vdash \varphi \to \psi$, for any set $\boldsymbol{\varphi} \cup \{\varphi, \psi\}$ of terms.*

Another standard property, the *replacement theorem*, is a consequence of the following:

Lemma 10. *If ψ_1 results from ψ_0 by replacing some occurrences of φ_0 with φ_1, then*

$$\vdash \forall(\varphi_0 \leftrightarrow \varphi_1) \;\to\; (\psi_0 \leftrightarrow \psi_1).$$

Proof. As $\vdash \varphi \to \top$, $\vdash \varphi \to \varphi$ and $\vdash \bot \to \varphi$, we obtain $\vdash \forall(\varphi \to \top)$, $\vdash \forall(\varphi \to \varphi)$ and $\vdash \forall(\bot \to \varphi)$ by (Gen), and then, by (25) and (MP),

$$\vdash \neg(\varphi \leftleftarrows \top), \quad \vdash \neg(\varphi \leftleftarrows \varphi), \quad \vdash \neg(\bot \leftleftarrows \varphi). \tag{34}$$

Thus, in particular, 'not closer' is reflexive which, together with transitivity (23) of 'closer' and its negation, gives linearity of the latter:

$$\vdash \neg(\varphi \leftleftarrows \psi) \;\sqcup\; \neg(\psi \leftleftarrows \varphi). \tag{35}$$

It follows that the relation $\{(\varphi,\psi) \mid \Phi \vdash \neg(\psi \leftleftarrows \varphi)\}$, for any set Φ of terms, is a linear quasi-order on the set of all terms. In particular,

$$\vdash ((\varphi \leftleftarrows \psi) \sqcap \neg(\chi \leftleftarrows \psi)) \;\to\; (\varphi \leftleftarrows \chi), \quad \vdash (\neg(\psi \leftleftarrows \varphi) \sqcap (\psi \leftleftarrows \chi)) \;\to\; (\varphi \leftleftarrows \chi). \tag{36}$$

By (30), (31), and the substitution instance $(\neg(\top \leftleftarrows \varphi) \sqcap (\top \leftleftarrows \bot)) \to (\varphi \leftleftarrows \bot)$ of (36) we obtain

$$\vdash \varphi \to \exists\varphi, \quad \vdash \forall\varphi \to \varphi \tag{37}$$

(the latter implication is obtained from the former in view of the definition $\forall\varphi = \neg\exists\neg\varphi$).

By (25) and (36),

$$\begin{aligned} &\vdash \forall(\varphi_0 \to \varphi_1) \;\to\; ((\varphi_0 \leftleftarrows \psi) \to (\varphi_1 \leftleftarrows \psi)),\\ &\vdash \forall(\psi_1 \to \psi_0) \;\to\; ((\varphi \leftleftarrows \psi_0) \to (\varphi \leftleftarrows \psi_1)), \end{aligned} \tag{38}$$

from which we obtain

$$\vdash \forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi). \tag{39}$$

And using (33), (37), and (38) we prove

$$\vdash \exists\varphi \to \forall\exists\varphi, \quad \vdash \neg\exists\varphi \to \forall\neg\exists\varphi, \quad \vdash \forall\varphi \to \forall\forall\varphi. \tag{40}$$

It follows then from (39) and (40) that

$$\vdash \forall\varphi \to \psi \quad \text{implies} \quad \vdash \forall\varphi \to \forall\psi. \tag{41}$$

By (39) we have $\vdash \forall(\varphi \sqcap \psi) \to (\forall\varphi \sqcap \forall\psi)$, and so $\vdash \forall(\varphi_0 \leftrightarrow \varphi_1) \to \forall(\varphi_0 \to \varphi_1) \sqcap \forall(\varphi_1 \to \varphi_0)$. By combining this with (38) and (41) we obtain

$$\begin{aligned} &\vdash \forall(\varphi_0 \leftrightarrow \varphi_1) \;\to\; \forall((\varphi_0 \leftleftarrows \psi) \leftrightarrow (\varphi_1 \leftleftarrows \psi)),\\ &\vdash \forall(\varphi_0 \leftrightarrow \varphi_1) \;\to\; \forall((\psi \leftleftarrows \varphi_0) \leftrightarrow (\psi \leftleftarrows \varphi_1)). \end{aligned} \tag{42}$$

Finally, (39) also yields

$$\begin{aligned} &\vdash \forall(\varphi_0 \leftrightarrow \varphi_1) \;\to\; \forall(\neg\varphi_0 \leftrightarrow \neg\varphi_1),\\ &\vdash \forall(\varphi_0 \leftrightarrow \varphi_1) \;\to\; \forall((\varphi_0 \sqcap \psi) \leftrightarrow (\varphi_1 \sqcap \psi)). \end{aligned} \tag{43}$$

Now, using (43), (42), and (29), we complete the proof of our lemma by an easy induction on the construction of ψ_0. ❑

We are now in a position to prove completeness of our axiomatic system with respect to the class of (symmetric) distance models. Say that a term φ is *consistent* if $\not\vdash \neg\varphi$. A finite set Φ of terms is *consistent* if $\bigsqcap \Phi$ is consistent. (Note that Φ is consistent iff $\Phi \leftleftarrows \bot$ is consistent.) Our aim is to prove that if a term τ is consistent then there exists a τ-diagram.

Lemma 11. *Every maximal consistent subset t of $\mathsf{cl}\,\tau$ is a τ-type.*

Proof. Clearly, t is Boolean closed. As was observed above, the relation $\leq_t$ is a linear quasi-order on $\mathsf{com}\,\tau$ with $\top \in \min t$ and $\bot \in \max t$. We also have $t \cap \mathsf{com}\,\tau = \varrho_t \cap \min t$ by (30). To show that $\varrho_t \cap \max t = \emptyset$, we use the replacement theorem to obtain $\vdash \neg\exists\varphi \;\leftrightarrow\; \forall(\varphi \leftrightarrow \bot)$, which yields, by (29) and (32), $\vdash \text{ⓡ}\varphi \;\rightarrow\; \exists\varphi$. ❑

Our next lemma will show how to construct links between consistent types. To prove it we require some more derivable terms. Let us show first that we have

$$\begin{array}{ll} \vdash ((\varphi \leftleftarrows \psi) \sqcap \exists\chi) \rightarrow ((\varphi \sqcap \exists\chi) \leftleftarrows \psi), & \vdash (\neg(\varphi \leftleftarrows \psi) \sqcap \exists\chi) \rightarrow \neg(\varphi \leftleftarrows (\psi \sqcap \exists\chi)), \\ \vdash ((\varphi \leftleftarrows \psi) \sqcap \neg\exists\chi) \rightarrow ((\varphi \sqcap \neg\exists\chi) \leftleftarrows \psi), & \vdash (\neg(\varphi \leftleftarrows \psi) \sqcap \neg\exists\chi) \rightarrow \neg(\varphi \leftleftarrows (\psi \sqcap \neg\exists\chi)). \end{array} \tag{44}$$

Indeed, we obtain $\vdash \forall(\xi \rightarrow (\varphi \rightarrow \varphi \sqcap \xi))$ by (Gen) from a tautology, and then continue:

$$\begin{array}{rl} \vdash \forall\xi \rightarrow \forall(\varphi \rightarrow (\varphi \sqcap \xi)) & \text{by (39)}, \\ \vdash \forall(\varphi \rightarrow (\varphi \sqcap \xi)) \rightarrow \neg(\varphi \leftleftarrows (\varphi \sqcap \xi)) & \text{by (25)}, \\ \vdash \big(\neg(\varphi \leftleftarrows (\varphi \sqcap \xi)) \sqcap (\varphi \leftleftarrows \psi)\big) \rightarrow ((\varphi \sqcap \xi) \leftleftarrows \psi) & \text{by (36)}, \\ \vdash (\forall\xi \sqcap (\varphi \leftleftarrows \psi)) \rightarrow ((\varphi \sqcap \xi) \leftleftarrows \psi) & \text{by propositional logic.} \end{array}$$

A similar argument shows that $\vdash (\forall\xi \sqcap \neg(\varphi \leftleftarrows \psi)) \rightarrow \neg(\varphi \leftleftarrows (\psi \sqcap \xi))$. It remains to take $\xi = \exists\chi$ or $\xi = \neg\exists\chi$ and make use of (40).

We shall also need the following theorem:

$$\vdash \bigsqcup_{i=0,1} \big(\neg((\varphi_0 \sqcup \varphi_1) \leftleftarrows \varphi_i) \sqcap (\text{ⓡ}(\varphi_0 \sqcup \varphi_1) \rightarrow \text{ⓡ}\varphi_i)\big). \tag{45}$$

To prove it suppose that it does not hold. Then the term

$$\varphi \;=\; \bigsqcap_{i=0,1} \big(((\varphi_0 \sqcup \varphi_1) \leftleftarrows \varphi_i) \sqcup (\text{ⓡ}(\varphi_0 \sqcup \varphi_1) \sqcap \neg\text{ⓡ}\varphi_i)\big).$$

must be consistent. We show that in this case we would have $\vdash \varphi \rightarrow \bot$, which is a contradiction. Clearly, $\vdash \varphi \leftrightarrow (\psi_0 \sqcap \psi_1) \sqcup (\chi_0 \sqcap \chi_1) \sqcup (\psi_0 \sqcap \chi_1) \sqcup (\chi_0 \sqcap \psi_1)$, where

$$\psi_i \;=\; ((\varphi_0 \sqcup \varphi_1) \leftleftarrows \varphi_i) \quad \text{and} \quad \chi_i \;=\; (\text{ⓡ}(\varphi_0 \sqcup \varphi_1) \sqcap \neg\text{ⓡ}\varphi_i), \quad i = 0, 1.$$

In view of (24), we have $\vdash \psi_0 \sqcap \psi_1 \rightarrow \bot$ and, in view of (26), $\vdash \chi_0 \sqcap \chi_1 \rightarrow \bot$. Consider now the case of $\psi_0 \sqcap \chi_1$ ($\psi_1 \sqcap \chi_0$ is treated analogously). By (24) we obtain $\psi_0 \vdash \neg((\varphi_0 \sqcup \varphi_1) \leftleftarrows \varphi_1)$, and so $\psi_0 \vdash \varphi_1 \leftleftarrows \varphi_0$ by (36). But this implies $\psi_0 \sqcap \text{ⓡ}(\varphi_0 \sqcup \varphi_1) \vdash \text{ⓡ}\varphi_1$ by (27), which together with the conjunct $\neg\text{ⓡ}\varphi_1$ of χ_1 gives $\psi_0 \sqcap \chi_1 \vdash \bot$. It follows that $\vdash \varphi \rightarrow \bot$ and therefore (45) does hold.

For finite sets s, t of terms and a term φ, let

$$s \circ_\varphi t \;=\; \bigsqcap s \sqcap \neg(\varphi \leftleftarrows \bigsqcap t) \sqcap (\text{ⓡ}\varphi \rightarrow \text{ⓡ}\bigsqcap t).$$

Then, for every $\neg\psi \in \mathsf{cl}\,\tau$, we have

$$s \circ_\varphi t \;\vdash\; (s \circ_\varphi (t \cup \{\psi\})) \sqcup (s \circ_\varphi (t \cup \{\neg\psi\})) \tag{46}$$

Indeed, let $\psi_0, \psi_1 \in \mathsf{cl}\,\tau$, where one of these terms is the negation of the other. Then we obtain:

$$\vdash \forall(((\textstyle\bigsqcap t \sqcap \psi_0) \sqcup (\bigsqcap t \sqcap \psi_1)) \leftrightarrow \bigsqcap t) \quad \text{by (Gen) from a tautology,}$$

$$\vdash \bigsqcup_{i=0,1} (\neg(\textstyle\bigsqcap t \leftleftarrows (\bigsqcap t \sqcap \psi_i)) \sqcap (\text{ⓡ}\bigsqcap t \to \text{ⓡ}(\bigsqcap t \sqcap \psi_i))) \quad \text{by (45) and replacement,}$$

$$s \circ_\varphi t \vdash \bigsqcup_{i=0,1} (s \circ_\varphi (t \cup \{\psi_i\})) \quad \text{by (23) and the definition of } \circ_\varphi.$$

Lemma 12. *Suppose that $\varphi <_s \bot$ and $\varphi \notin s$, for some consistent τ-type s. Then there exists a consistent τ-type t such that (s, t) is a φ-link.*

Proof. It is easy to see that the term $s \circ_\varphi \{\varphi\}$ is consistent. Let $t \subseteq \mathsf{cl}\,\tau$ be maximal with the properties: $\varphi \in t$ and $s \circ_\varphi t$ is consistent. By assumption, $s \circ_\varphi t \vdash (\varphi \leftleftarrows \bot) \sqcap \neg(\varphi \leftleftarrows \bigsqcap t)$, and so $s \circ_\varphi t \vdash \bigsqcap t \leftleftarrows \bot$ by (36). Therefore, t is consistent. By the maximality of t, it follows from (46) that either $\psi \in t$ or $\neg\psi \in t$, for every $\neg\psi \in \mathsf{cl}\,\tau$. Therefore t is a maximal consistent subset of $\mathsf{cl}\,\tau$, and so, by Lemma 11, it is a τ-type. We now prove that (s, t) is a τ-link.

Suppose that $\psi <_s \bot$, i.e., $(\psi \leftleftarrows \bot) \in s$. As we have already observed, $s \circ_\varphi t \vdash \bigsqcap t \leftleftarrows \bot$. Hence $s \circ_\varphi t \vdash (\bigsqcap t \sqcap (\psi \leftleftarrows \bot)) \leftleftarrows \bot$ by the first theorem in (44), and $(\psi \leftleftarrows \bot) \in t$ by the maximality of t. Similarly, $\psi \simeq_s \bot$ implies $\psi \simeq_t \bot$ by the third theorem in (44).

Suppose now that $\psi \in t$, $\psi \simeq_s \varphi$, and $\varphi \in \varrho_s$. We need to show that $\psi \in \varrho_s$, i.e., ⓡ$\psi \in s$. By assumption, ⓡ$\varphi, \neg(\psi \leftleftarrows \varphi) \in s$. Hence $s \circ_\varphi t \vdash \neg(\psi \leftleftarrows \bigsqcap t) \sqcap$ ⓡ$\bigsqcap t$ in view of (23), $s \circ_\varphi t \vdash$ ⓡ$(\bigsqcap t \sqcup \psi)$ by (28), and $s \circ_\varphi t \vdash$ ⓡψ by (29) [because $\vdash \forall((\bigsqcap t \sqcup \psi) \leftrightarrow \psi)$]. Therefore, $\{$ⓡ$\psi\} \cup s$ is consistent, whence ⓡ$\psi \in s$, as s is a τ-type and ⓡ$\psi \in \mathsf{cl}\,\tau$. ❑

Theorem 13. *A $\mathcal{CSL}$-term τ is valid in the class of all (symmetric) distance models iff $\vdash \tau$.*

Proof. As $\vdash \varphi \leftrightarrow \neg\neg\varphi$, it suffices to show that the following conditions are equivalent:

$$\begin{gathered}\tau \text{ is satisfied in a distance model}, \quad \tau \text{ is satisfied in a symmetric model}, \\ \tau \text{ is consistent.}\end{gathered} \tag{47}$$

The equivalence of the first two conditions in (47) follows from Theorem 5. And we have already observed that every deducible term is valid, and so every satisfiable term is consistent.

Suppose now that τ is a consistent term. Then τ is contained in some consistent τ-type t_*. Take the set D of all consistent τ-types t such that $\psi <_t \bot$ iff $\psi <_{t_*} \bot$, for all $\psi \in \mathsf{cl}\,\tau$. By Lemma 12 and the definition of a link, D is a τ-diagram, and so τ is satisfiable in a symmetric model by Theorem 5. ❑

4 $\mathcal{CSL}$ over distance models with the triangle inequality

Now we extend the results and techniques from Section 3 to the class of distance models satisfying the triangle inequality (tri).

4.1 Decidability and complexity

To understand the main problem we face in this case, consider the following example.

Recall from (12) that the term $\Diamond p = \neg(\top \leftleftarrows p)$ represents the (topological) closure of the set defined by the atom p. Now take the term $\tau = \Diamond p \leftleftarrows p$, which says that the closure of p is (strictly) closer than p. It is easy to see that no model $\mathfrak{I}$ with (tri) can satisfy τ. Indeed, consider points $u \in \Delta^{\mathfrak{I}}$, $v \in (\Diamond p)^{\mathfrak{I}}$ and $w \in p^{\mathfrak{I}}$. By (tri), we must have

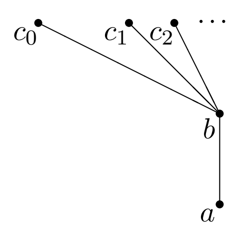
$$d^{\mathfrak{I}}(u, w) \leq d^{\mathfrak{I}}(u, v) + d^{\mathfrak{I}}(v, w).$$

By taking first the infimum over $w \in (\Diamond p)^{\mathfrak{I}}$ and then the infimum over $v \in p^{\mathfrak{I}}$, we obtain

$$d^{\mathfrak{I}}(u, p^{\mathfrak{I}}) \leq d^{\mathfrak{I}}(u, v) + d^{\mathfrak{I}}(v, p^{\mathfrak{I}}) \leq d^{\mathfrak{I}}(u, (\Diamond p)^{\mathfrak{I}}) + d^{\mathfrak{I}}((\Diamond p)^{\mathfrak{I}}, p^{\mathfrak{I}}) = d^{\mathfrak{I}}(u, (\Diamond p)^{\mathfrak{I}})$$

because $d^{\mathfrak{I}}((\Diamond p)^{\mathfrak{I}}, p^{\mathfrak{I}}) = 0$. Therefore, $\tau^{\mathfrak{I}} = \emptyset$. However, $\tau^{\mathfrak{S}} \neq \emptyset$ in the symmetric model $\mathfrak{S}$, where

$$\begin{aligned}
&\Delta^{\mathfrak{S}} = \{a, b, c_i \mid i \in \mathbb{N}\},\\
&p^{\mathfrak{S}} = \{c_i \mid i \in \mathbb{N}\},\\
&d^{\mathfrak{S}}(a, c_i) = 2, \quad i \in \mathbb{N},\\
&d^{\mathfrak{S}}(a, b) = 1, \quad d^{\mathfrak{S}}(b, c_i) = 1/2^i, \quad i \in \mathbb{N},
\end{aligned}$$



and all other distances are defined by symmetry. Clearly, $(\Diamond p)^{\mathfrak{S}} = \{b, c_i \mid i \in \mathbb{N}\}$. Therefore $d^{\mathfrak{S}}(a, (\Diamond p)^{\mathfrak{S}}) < d^{\mathfrak{S}}(a, p^{\mathfrak{S}})$, i.e., $a \in \tau^{\mathfrak{S}}$. Obviously, the only reason for this 'strange' behaviour is that the distance from a to c_i is 2, whereas by (tri) it should be $\leq 1 + 1/2^i < 2$.

This example suggests that we should slightly modify the second condition in the definition of a φ-link from Section 3.1. Let s, t be τ-types, $\varphi \in \mathsf{com}\,\tau$ and $\varphi \notin s$. The pair (s, t) will be called now a *φ-link* if, for every $\psi \in \mathsf{com}\,\tau$, we have:

$$\begin{aligned}
&\psi <_s \bot \ \text{ iff } \ \psi <_t \bot,\\
&\psi \in \min t \ \text{ implies } \ \psi \leq_s \varphi,\\
&\varphi \in \varrho_s, \ \varphi \simeq_s \psi, \text{ and } \psi \in t \ \text{ imply } \ \psi \in \varrho_s
\end{aligned} \tag{48}$$

Thus, now we require that $\psi \leq_s \varphi$ holds for all $\leq_t$-minimal terms ψ in $\mathsf{cl}\,\tau$, not only for those in t; in other words, we take into account those terms ψ that are 'infinitely close' to t.

We illustrate the new definition by the example considered above. Let $s = t^{\mathfrak{S}}(a)$ and $t = t^{\mathfrak{S}}(b)$. Then (s, t) is a $\Diamond p$-link for the case of models without (tri), because $b \in \Diamond p^{\mathfrak{S}}$ and $d^{\mathfrak{S}}(a, b) = d^{\mathfrak{S}}(a, \Diamond p^{\mathfrak{S}})$. On the other hand, $\Diamond p \in t$ means that $p \in \min t$, and so in the case of models with (tri) we should have $p \leq_s \Diamond p$, contrary to $\Diamond p \leftleftarrows p \in s$.

In fact, this turns out to be the only change we need to prove the following:

Theorem 14. *A $\mathcal{CSL}$-term τ is satisfied in a distance model with the triangle inequality iff there exists a τ-diagram.*

Proof. ($\Rightarrow$) Let $\mathfrak{I}$ be a distance model satisfying the triangle inequality and such that $\tau^{\mathfrak{I}} \neq \emptyset$. Then one can readily check that the set D of τ-types of elements in $\mathfrak{I}$ is a τ-diagram.

($\Leftarrow$) Conversely, suppose that D is a τ-diagram and $\tau \in t_* \in D$. Let $\varphi_0, \ldots, \varphi_{k-1}$ be all the distinct terms in $\mathsf{com}\,\tau$ such that $\varphi_i <_t \bot$ for some (and so all) $t \in D$. We construct the tree $\Delta \subseteq (\{0,\dots,k-1\} \times \mathbb{N})^*$ and the labelling $tp : \Delta \to D$ in exactly the same way as in the proof of Theorem 5. However, the definition of the distance function d on Δ is quite different now. The main reason is that, unlike the previous definition which did not comply with (tri), by providing a witness for some node α we can 'spoil' witnesses for the ancestors of α, and this cannot be repaired now by simply assigning a sufficiently large value to the distances between nodes which are not immediate successors or predecessors of each other.

To cope with this problem, for each $\alpha \in \Delta$ we introduce a new numerical parameter $e(\alpha)$, the main purpose of which is to ensure the following condition, for all $\alpha, \alpha(i,j) \in \Delta$:

$$\text{either} \quad \lim_{j\to\infty} d(\alpha, \alpha(i,j)) = 0 \quad \text{or} \quad 1 - 2e(\alpha) \leq d(\alpha, \alpha(i,j)) < 1 - e(\alpha). \tag{49}$$

The distances d^α, for $\alpha \in \Delta$ (here we use the notation from the proof of Theorem 5; in particular, d^α stands for $d(\alpha', \alpha)$, where α' is the parent of α), are defined inductively as follows. First we set $d^\lambda = 1$ and $e(\lambda) = 1/4$. Suppose now that d^α is already defined. Since D is a diagram, we can find a type $t_i \in D$ such that $(tp(\alpha), t_i)$ is a φ_i-link, for all $i < k$. And since $tp(\alpha)$ is a type, we can choose the values $d^{\alpha+i} \in \{0\} \cup [\,1 - 2\,e(\alpha), 1 - e(\alpha))$ such that, for all $i, l < k$,

$$d^{\alpha+i} \leq d^{\alpha+l} \ \text{ iff } \ \varphi_i \leq_{tp(\alpha)} \varphi_l, \qquad d^{\alpha+i} = 0 \ \text{ iff } \ \varphi_i \in \min tp(\alpha). \tag{50}$$

Then we set, for all $\alpha(i,j) \in \Delta$:

$$\begin{aligned}
d^{\alpha(i,0)} &= d^{\alpha+i} \ \text{ if } \ \varphi_i \in \varrho_{tp(\alpha)},\\
d^{\alpha(i,j)} &= d^{\alpha+i} + (1 - e(\alpha) - d^{\alpha+i})/(2+j) \ \text{ if } \ \varphi_i \notin \varrho_{tp(\alpha)},\\
e(\alpha(i,j)) &= \begin{cases} e(\alpha) & \text{if } \ \varphi_i \notin \min tp(\alpha),\\ e(\alpha)/2 & \text{if } \ \varphi_i \in \min tp(\alpha) \setminus tp(\alpha). \end{cases}
\end{aligned}$$

Note that (49) is satisfied and $0 < e(\alpha) \leq 1/4$.

Finally, we define distances between arbitrary nodes in Δ as follows:

$$\begin{gathered}
d(\alpha,\alpha) = 0, \qquad d(\alpha,\alpha') = 1 \ \text{ if } \alpha' \text{ is the parent of } \alpha,\\
d(\alpha,\beta) = d(\alpha,\alpha_1) + \dots + d(\alpha_n,\beta) \ \text{ if } \alpha, \alpha_1, \dots, \alpha_n, \beta \text{ is the shortest}\\
\text{undirected path between } \alpha \text{ and } \beta, \text{ and } n \geq 1.
\end{gathered}$$

Then d is a distance function on Δ satisfying (tri) (but not (sym), which will be essentially used in the proof below).

Now we can prove an analogue of Lemma 6 for the case of models with (tri), where as before we let $\varphi^\Delta = \{\alpha \in \Delta \mid \varphi \in tp(\alpha)\}$.

Lemma 15. *Let $\alpha \in \Delta$ and $i < k$. Then $d(\alpha, \varphi_i^\Delta) = d^{\alpha+i}$. More precisely,*

- *if $\varphi_i \in \varrho_{tp(\alpha)} \setminus tp(\alpha)$ then $d(\alpha, \varphi_i^\Delta) = d^{\alpha(i,0)}$,*
- *if $\varphi_i \notin \varrho_{tp(\alpha)}$ then $d(\alpha, \varphi_i^\Delta) = \lim_{j\to\infty} d^{\alpha(i,j)}$ and $d(\alpha, \varphi_i^\Delta)$ is not realised.*

Proof. Again it is enough to show that

$$\forall \beta \in \varphi_i^\Delta \ \ \exists \beta_1 \in \alpha + i \ \ d(\alpha, \beta_1) \leq d(\alpha, \beta). \tag{51}$$

Suppose that $\beta \in \psi^\Delta$ is not a successor of α. Then $d(\alpha, \beta) \geq 1$ and $d(\alpha, \beta_1) < 1 - e(\alpha) < 1$, for all $\beta_1 \in \alpha + i$. Now let β be a successor of α. We prove (51) by induction on the number of nodes between α and β. Let α' be the parent of β.

Induction basis: Suppose first that $n = 0$, i.e., α is the parent of β. Then we simply repeat the argument from the proof of Lemma 6.

Suppose now that $n = 1$. Then $\alpha' \in \alpha{+}l$, for some $l < k$, and we consider three cases:

Case 1: $\varphi_l \notin \min tp(\alpha)$ and $\varphi_i \notin \min tp(\alpha')$. Then $e(\alpha') = e(\alpha)$ and $d(\alpha, \beta) = d(\alpha, \alpha') + d(\alpha', \beta) \geq 2(1 - 2e(\alpha)) \geq 1$, while $d(\alpha, \beta_1) < 1$ for all $\beta_1 \in \alpha + i$.

Case 2: $\varphi_l \in \min tp(\alpha)$ and $\varphi_i \notin \min tp(\alpha')$. Then $e(\alpha') = e(\alpha)/2$ and $d(\alpha, \beta) > d(\alpha', \beta) \geq 1 - e(\alpha)$, while $d(\alpha, \beta_1) < 1 - e(\alpha)$ for all $\beta_1 \in \alpha + i$.

Case 3: $\varphi_i \in \min tp(\alpha')$. Then $(tp(\alpha), tp(\alpha'))$ is a φ_l-link by construction. Hence $\varphi_i \leq_{tp(\alpha)} \varphi_l$ by the definition of a link, $d^{\alpha+i} \leq d^{\alpha+l}$ by (50), and therefore $d(\alpha, \beta) > d(\alpha, \alpha') \geq d^{\alpha+l} \geq d^{\alpha+i}$. As $d^{\alpha+i} = \inf\{d^{\beta_1} \mid \beta_1 \in \alpha{+}i\}$, we obtain $d(\alpha, \beta) > d(\alpha, \beta_1)$ for some $\beta_1 \in \alpha + i$.

Induction step: suppose that the parent α'' of α' is still a successor of α. By the induction basis, there exists $\beta_2 \in \alpha'' + i$ such that $d(\alpha'', \beta_2) \leq d(\alpha'', \beta)$. Then $d(\alpha, \beta_2) \leq d(\alpha, \beta)$ and the number of nodes between α and β_2 is less than that between α and β. So we can apply the induction hypothesis. ❏

Define now a model $\mathfrak{I}$ by setting $\Delta^\mathfrak{I} = \Delta$, $d^\mathfrak{I} = d$, and $p^\mathfrak{I} = p^\Delta$, for all atomic terms p. Then the following lemma is proved in precisely the same way as Lemma 7 (we only need to use Lemma 15 instead of Lemma 6).

Lemma 16. *For all $\varphi \in \mathsf{cl}\,\tau$, we have $\varphi^\mathfrak{I} = \varphi^\Delta$.*

Thus, $\mathfrak{I}$ is a distance model satisfying (tri), and $\tau^\mathfrak{I} \neq \emptyset$ because $\tau \in tp(\lambda)$ by the construction. ❏

The proof of the next theorem is almost the same as the proof of Theorem 8 (we use Theorem 14 instead of Theorem 5, and only consider models with (tri), not all distance models; the proof of the lower bound remains without changes).

Theorem 17. *The satisfiability problem for $\mathcal{CSL}$- and $\mathcal{QML}$-terms in models with the triangle inequality is* ExpTime*-complete.*

4.2 Axiomatisation

As we observed at the beginning of Section 4.1, the term $\neg(\Diamond\varphi \leftleftarrows \varphi)$, that is,

$$\neg(\neg(\top \leftleftarrows \varphi) \leftleftarrows \varphi) \tag{52}$$

is valid in the class of distance models with (tri). Let us add (52) as an axiom schema to the axiomatic system from Section 3.2. Then it is easy to see that (34)–(46) and Lemmas 9–11 hold true for the extended system as well.

To show that this new axiomatic system is complete with respect to the class of distance models with (tri) it suffices to prove that Lemma 12 holds for the new definition of links.

As before, we write $\varphi_0, \dots, \varphi_{n-1} \vdash \varphi_n$ if there exists a derivation of φ_n from the premises $\varphi_0, \dots, \varphi_{n-1}$ in which (Gen) is not applied to terms that depend on $\varphi_0, \dots, \varphi_{n-1}$. We also use the notation $s \circ_\varphi t = \bigsqcap s \sqcap \neg(\varphi \leftleftarrows \bigsqcap t) \sqcap (ⓡ\varphi \to ⓡ\bigsqcap t)$ for finite sets s, t of terms and a term φ.

Lemma 18. *Let s be a consistent τ-type and $\varphi <_s \bot$, $\varphi \notin s$. Then there exists a consistent τ-type t such that (s,t) is a φ-link.*

Proof. Taking into account the proof of Lemma 12, we only need to show that the pair (s,t) from that proof satisfies the second condition in (48), that is, $\psi \leq_s \varphi$, for all $\psi \in \min t$. So let $\psi \in \min t$, i.e., $\neg(\top \leftleftarrows \psi) \in t$. Then $s \circ_\varphi t \vdash \bigsqcap s \sqcap \neg(\varphi \leftleftarrows \neg(\top \leftleftarrows \psi))$ by (38), $\vdash \neg(\neg(\top \leftleftarrows \psi) \leftleftarrows \psi)$ by (52), and so $s \circ_\varphi t \vdash \bigsqcap s \sqcap \neg(\varphi \leftleftarrows \psi)$ by (23). Hence $s \cup \{\neg(\varphi \leftleftarrows \psi)\}$ is consistent, which means that $\neg(\varphi \leftleftarrows \psi) \in s$, i.e., $\psi \leq_s \varphi$, because s is a τ-type and $\neg(\varphi \leftleftarrows \psi) \in \mathsf{cl}\,\tau$. ❑

Theorem 19. *A $\mathcal{CSL}$-term τ is valid in the class of distance models satisfying* (tri) *iff* $\vdash \tau$.

Proof. As we know, all the axioms are valid in the class of distance models with (tri) and the inference rules preserve the validity. Conversely, as we have $\vdash \varphi \leftrightarrow \neg\neg\varphi$, it suffices to show that every consistent term is satisfiable. Suppose that τ is consistent. Then τ is contained in some consistent τ-type and the set of all consistent τ-types is a τ-diagram by Lemma 18. But then, by Theorem 14, τ is satisfiable in a distance model with (tri). ❑

5 $\mathcal{CSL}$ over metric models

A typical $\mathcal{CSL}$-term which distinguishes between metric and non-metric models is

$$(\varphi \leftleftarrows \psi) \ \rightarrow \ (\top \leftleftarrows \neg(\varphi \leftleftarrows \psi)). \tag{53}$$

Interpreted over metric models, it says in fact that $(\varphi \leftleftarrows \psi) \leftrightarrow \Box(\varphi \leftleftarrows \psi)$. Indeed, let $\mathfrak{I}$ be a metric model and $u \in (\varphi \leftleftarrows \psi)^{\mathfrak{I}}$. Then $\varepsilon = d^{\mathfrak{I}}(u,\psi^{\mathfrak{I}}) - d^{\mathfrak{I}}(u,\varphi^{\mathfrak{I}}) > 0$. Take any v with $d^{\mathfrak{I}}(u,v) < \varepsilon/2$. By (tri), we have $d(v,\varphi^{\mathfrak{I}}) \leq d^{\mathfrak{I}}(v,u) + d^{\mathfrak{I}}(u,\varphi^{\mathfrak{I}})$ and $d(u,\psi^{\mathfrak{I}}) \leq d^{\mathfrak{I}}(u,v) + d^{\mathfrak{I}}(v,\psi^{\mathfrak{I}})$. It follows, by (sim), that

$$\begin{aligned} d^{\mathfrak{I}}(v,\psi^{\mathfrak{I}}) - d^{\mathfrak{I}}(v,\varphi^{\mathfrak{I}}) &\geq d^{\mathfrak{I}}(u,\psi^{\mathfrak{I}}) - d^{\mathfrak{I}}(u,v) - \big(d^{\mathfrak{I}}(v,u) + d^{\mathfrak{I}}(u,\varphi^{\mathfrak{I}})\big) \\ &\geq \varepsilon - 2d^{\mathfrak{I}}(u,v) > 0, \end{aligned}$$

from which $v \in (\varphi \leftleftarrows \psi)^{\mathfrak{I}}$, and so $u \in (\Box(\varphi \leftleftarrows \psi))^{\mathfrak{I}}$.

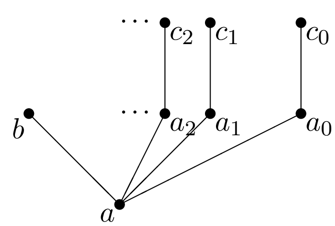
On the other hand, $(p \leftleftarrows q) \rightarrow \Box(p \leftleftarrows q)$ is not valid in the following non-symmetric model $\mathfrak{T}$ with (tri), where

$$\Delta^{\mathfrak{T}} = \{a, a_i, b, c_i \mid i \in \mathbb{N}\},$$

$$p^{\mathfrak{T}} = \{b\}, \quad q^{\mathfrak{T}} = \{c_i \mid i \in \mathbb{N}\},$$

$$d^{\mathfrak{T}}(a,b) = d^{\mathfrak{T}}(b,a) = 1, \quad d^{\mathfrak{T}}(a,a_i) = 1/2^i,$$

$$d^{\mathfrak{T}}(a_i,a) = 1, \quad d^{\mathfrak{T}}(a_i,c_i) = d^{\mathfrak{T}}(c_i,a_i) = 3/2, \quad i \in \mathbb{N},$$



and the other distances are computed as the lengths of the corresponding paths in the graph above. It is easy to see that $a \in (p \leftleftarrows q)^{\mathfrak{T}}$ but $a \notin (\Box(p \leftleftarrows q))^{\mathfrak{T}}$ (in fact, $d^{\mathfrak{T}}(a, \{a_i \mid i \in \mathbb{N}\}) = 0$ and $\{a_i \mid i \in \mathbb{N}\} \subseteq (q \leftleftarrows p)^{\mathfrak{T}}$).

5.1 Decidability and complexity

In metric models, every sequence converging to a given point should eventually satisfy all the 'strict inequalities' satisfied by this point. Therefore, we have to consider two essentially different cases when defining a link (s, t): if the distance from an s-point to the t-set is positive, we have the usual constraints on s and t; but if the t-set is infinitely close to the s-point, then s and t should agree on terms of the form $\varphi \leftleftarrows \psi$.

Lemma 20. *Let $\mathfrak{I}$ be a metric model, $u \in \Delta^{\mathfrak{I}}$, and $d^{\mathfrak{I}}(u, \varphi^{\mathfrak{I}}) = 0$ for some $\varphi \in \mathsf{com}\,\tau$. Then there is a τ-type t such that $\varphi \in t$ and $d^{\mathfrak{I}}(u, t^{\mathfrak{I}}) = 0$. Moreover, for any such t we have $<_{t^{\mathfrak{I}}(u)} \subseteq <_t$, that is, $\chi <_{t^{\mathfrak{I}}(u)} \psi$ implies $\chi <_t \psi$, for all $\chi, \psi \in \mathsf{com}\,\tau$.*

Proof. If $u \in \varphi^{\mathfrak{I}}$ then we take $t = t^{\mathfrak{I}}(u)$ and everything is trivial. So assume that $u \notin \varphi^{\mathfrak{I}}$. Then, in $\varphi^{\mathfrak{I}}$, there exists a sequence z_i, $i \in \mathbb{N}$, converging to x, with all the z_i being of the same type t. The remaining part of the proof is similar to the argument for (53). ❑

Now, let s, t be τ-types and $\varphi <_s \bot$. Suppose first that $\varphi \notin \min s$. We say that (s, t) is a *φ-link* if conditions (48) hold for all $\psi \in \mathsf{com}\,\tau$. In this case we also say that (s, t) is a *long link*. Suppose now that $\varphi \in \min s \setminus s$. Then we call (s, t) a *φ-link* if $\varphi \in t$ and

$$\psi <_s \bot \;\text{ iff }\; \psi <_t \bot, \qquad \chi <_s \psi \;\text{ implies }\; \chi <_t \psi, \tag{54}$$

for all $\psi, \chi \in \mathsf{com}\,\tau$. In this case we also call (s, t) a *short link*.

Note that φ must be a $\leq_t$-minimal element in (48). So the second condition there is equivalent to the following one: $\varphi <_s \psi$ implies $\varphi <_t \psi$, which is a special case of the second condition in (54). In particular, we have $\min t \subseteq \min s$ for every short link (s, t). We also observe that the third condition in (48) trivially holds for any short φ-link (s, t), as we have $\varphi \notin \varrho_s$ in this case.

The following lemma is proved similarly to Lemma 4 wit the help of Lemma 20:

Lemma 21. *Let $\mathfrak{I}$ be a metric model, $u \in \Delta^{\mathfrak{I}}$ and $u \notin \varphi^{\mathfrak{I}} \neq \emptyset$, for some $\varphi \in \mathsf{com}\,\tau$. Then there is a type t such that $\varphi \in t$ and $d^{\mathfrak{I}}(u, \varphi^{\mathfrak{I}}) = d^{\mathfrak{I}}(u, t^{\mathfrak{I}})$, with $d(u, \varphi^{\mathfrak{I}})$ and $d(u, t^{\mathfrak{I}})$ being realised or not realised simultaneously. Moreover, $(t^{\mathfrak{I}}(u), t)$ is a long φ-link if $d^{\mathfrak{I}}(u, \varphi^{\mathfrak{I}}) > 0$, and a short φ-link if $d^{\mathfrak{I}}(u, \varphi^{\mathfrak{I}}) = 0$.*

Unfortunately, the notion of a link does not take into account a possible interaction of two (or more) short links. To be more specific, consider the following situation. Suppose that t_0 is a type and $\varphi \in \min t_0 \setminus t_0$. Then we need a short φ-link (t_0, t_1). Assume further that $\psi \in \min t_0 \setminus t_1$. This means that we also need a (long or short) ψ-link (t_1, t_2). In a model, say $\mathfrak{I}$, this corresponds to the following situation: we have $u \in t_0^{\mathfrak{I}}(u)$ such that $d^{\mathfrak{I}}(u, t_1^{\mathfrak{I}}) = 0 = d^{\mathfrak{I}}(u, \psi^{\mathfrak{I}})$ and $d^{\mathfrak{I}}(v, t_2^{\mathfrak{I}}) = d(v, \psi^{\mathfrak{I}})$, for all $v \in t_1^{\mathfrak{I}}$. Then we have $d^{\mathfrak{I}}(u, t_2^{\mathfrak{I}}) = 0$. Indeed, take an arbitrary $\varepsilon > 0$ and choose $v \in t_1^{\mathfrak{I}}$ such that $d^{\mathfrak{I}}(u, v) < \varepsilon/2$. Then $d^{\mathfrak{I}}(v, t_2^{\mathfrak{I}}) = d^{\mathfrak{I}}(v, \psi^{\mathfrak{I}}) \leq d^{\mathfrak{I}}(v, u) + d^{\mathfrak{I}}(u, \psi^{\mathfrak{I}}) < \varepsilon/2$, and so $d(u, t^{\mathfrak{I}}) \leq d(u, v) + d(v, \psi^{\mathfrak{I}}) < \varepsilon$. Therefore, we must have $<_{t_0} \subseteq <_{t_2}$, which by no means follows from the definition of a link.

Thus, we should be careful when constructing sequences of links starting with a short one in the sense that sometimes we should remember some previous links in the sequence. Let us consider possible scenarios when we start with a short link (t_0, t_1).

1. Suppose that $<_{t_0} = <_{t_1}$ and we need a φ-link (t_1, t_2) for some $\varphi \in \mathsf{com}\,\tau$. In this case the types t_0 and t_1 contain precisely the same terms of the form $\chi_1 \leftleftarrows \chi_2$ and can only differ in

Boolean terms. It follows that (t_1, t_2) is a φ-link iff (t_0, t_2) is a φ-link. This means that the choice of t_2 does not depend on the link (t_0, t_1).

2. Suppose that $<_{t_0} \subsetneq <_{t_1}$ and we need a φ-link (t_1, t_2) for some $\varphi \in \mathsf{com}\,\tau$. As we have $\min t_1 \subseteq \min t_0$, three cases are possible.

2.1: $\varphi \in \min t_1$. Then for any φ-link (t_1, t_2) we have $<_{t_0} \subset <_{t_1} \subseteq <_{t_2}$, and therefore no additional requirement should be imposed on (t_1, t_2).

2.2: $\varphi \in \min t_0 \setminus \min t_1$. In this case, when choosing a (long) φ-link (t_1, t_2), we must also ensure that (t_0, t_2) is a short φ-link.

2.3: $\varphi \notin \min t_0$, and so $\varphi \notin \min t_1$. In this case (t_0, t_1) does not have any influence on subsequent links at all.

3. Suppose that $<_{t_0} \subsetneq <_{t_1}$ and (t_1, t_2) is a φ-link, for $\varphi \in \min t_0 \setminus \min t_1$ (as in **2.2**), and so $<_{t_0} \subseteq <_{t_2}$. Suppose also that we are looking for a ψ-link (t_2, t_3). As (t_1, t_2) is a long link, t_1 has no influence on the choice of t_3. However, (t_0, t_2) should be taken into account. We again have three cases.

3.1: $\psi \in \min t_2$. Then the inclusion is satisfied for any ψ-link (t_2, t_3).

3.2: $\varphi \in \min t_0 \setminus \min t_2$. Then, when choosing a long φ-link (t_2, t_3), we must also ensure that $<_{t_0} \subseteq <_{t_3}$.

3.3: $\varphi \notin \min t_0$. No additional requirement is needed in this case.

This analysis suggests the following definitions. A sequence $\mathbf{t} = (t_0, \dots, t_n)$ of τ-types is called a *block* if we have $<_{t_0} \subset \cdots \subset <_{t_{n-1}} \subseteq <_{t_n}$. We call t_n the *type* of $\mathbf{t}$, while $(t_0, \dots, t_{n-1})$ is understood as its 'history' or 'heredity.' We say that $\mathbf{t}$ is *realised* in a model $\mathfrak{I}$ if there exist subsets $U_0 \subseteq t_0^{\mathfrak{I}}, \dots, U_n \subseteq t_n^{\mathfrak{I}}$ such that $d^{\mathfrak{I}}(u_i, U_{i+1}) = 0$ for all $u_i \in U_i$ and $i < n$.

It is easy to see that the size of $\mathsf{com}\,\tau$, and so the length of any block, is bounded by $|\tau|$. Therefore, by Lemma 2, we have

Lemma 22. *The number of distinct blocks does not exceed* $2^{|\mathsf{at}\,\tau| \cdot |\mathsf{com}\,\tau|^4}$.

Now, for $\varphi \in \mathsf{com}\,\tau$, we introduce a notion of a *φ-link of blocks*, which specialises the notion of a φ-link of types. Let $\mathbf{s}$ and $\mathbf{t}$ be blocks with $\mathbf{s} = (s_0, \dots, s_m)$. Consider four cases.

- Suppose that $\varphi \notin \min s_0$. Then $(\mathbf{s}, \mathbf{t})$ is called a *φ-link* (of blocks) if $\mathbf{t} = (t)$ and (s_m, t) is a φ-link of types. In this case the long link (s_m, t) allows us to 'forget' everything that happened before t.
- Suppose that $\varphi \in \min s_{n-1} \setminus \min s_n$, for some $n \leq m$. Then $(\mathbf{s}, \mathbf{t})$ is a *φ-link* (of blocks) if $\mathbf{t} = (s_0, \dots, s_{n-1}, t)$ and (s_m, t) is a φ-link of types. In this case (s_n, t) is a long link, while (s_{n-1}, t) is a short one, and so s_{n-1} and its 'heredity' should be kept.
- Suppose that $\varphi \in \min s_m \setminus s_m$ and $<_{s_{m-1}} = <_{s_m}$. Then $(\mathbf{s}, \mathbf{t})$ is a *φ-link* (of blocks) if $\mathbf{t} = (s_0, \dots, s_{m-1}, t)$ and (s_m, t) is a φ-link of types. In this case s_{m-1} and s_m carry the same information on 'heredity' of t, so we can drop s_m.
- Suppose that $\varphi \in \min s_m \setminus s_m$ and $<_{s_{m-1}} \subset <_{s_m}$. Then $(\mathbf{s}, \mathbf{t})$ is a *φ-link* (of blocks) if $\mathbf{t} = (s_0, \dots, s_m, t)$ and (s_m, t) is a φ-link of types.

Let D be a set of blocks and T the set of all types occurring in blocks from D. We call D a *τ-diagram* if the following conditions hold:

$$\text{there exists } (t) \in D \text{ with } \tau \in t, \tag{55}$$

$$\text{for all } s, t \in T \text{ and } \varphi \in \mathsf{com}\,\tau, \text{ we have } \varphi <_s \bot \text{ iff } \varphi <_t \bot, \tag{56}$$

$$\begin{array}{l}\text{for all } \mathbf{s} = (s_0, \dots, s_n) \in D \text{ and } \varphi <_{s_n} \bot,\ \varphi \notin s_n, \text{ there exists } \mathbf{t} \in D \\ \text{such that } (\mathbf{s}, \mathbf{t}) \text{ is a } \varphi\text{-link.}\end{array} \tag{57}$$

Theorem 23. *A $\mathcal{CSL}$-term τ is satisfied in a metric model iff there exists a τ-diagram.*

Proof. ($\Rightarrow$) Suppose first that $\mathfrak{I}$ is a metric model with $\tau^{\mathfrak{I}} \neq \emptyset$. Let D be the set of blocks realised in $\mathfrak{I}$. We show that D is a τ-diagram.

Clearly, D satisfies (55) and (56). Let us prove (57). Suppose that a block $\mathbf{s} = (s_0, \dots, s_m)$ is realised in $\mathfrak{I}$ and $U_0 \subseteq s_0^{\mathfrak{I}}, \dots, U_m \subseteq s_m^{\mathfrak{I}}$ are such that $d^{\mathfrak{I}}(u_i, U_{i+1}) = 0$ for all $i < m$ and all $u_i \in U_i$. Let $\varphi <_{s_m} \bot$ and $\varphi \notin s_m$. We first prove that there exist a type t and subsets $V_0 \subseteq U_0, \dots, V_m \subseteq U_m$ with the following properties:

$$\begin{array}{l} d^{\mathfrak{I}}(v, t^{\mathfrak{I}}) = d^{\mathfrak{I}}(v, \varphi^{\mathfrak{I}}), \quad \text{for all } v \in V_m, \\ d^{\mathfrak{I}}(v, V_{l+1}) = 0, \quad \text{for all } v \in V_l \text{ and } l < m. \end{array} \tag{58}$$

Choose an arbitrary $u_\lambda \in U_0$, where λ denotes the empty sequence. By our assumption, we can further choose elements $u_\alpha \in U_l$, for all $\alpha \in \mathbb{N}^l$ with $l \leq m$, so that each u_β is the limit of the sequence $u_{(\beta,0)}, u_{(\beta,1)}, \dots$. And for every $\alpha \in \mathbb{N}^m$, there exists a type t_α such that $\varphi \in t_\alpha$ and $d^{\mathfrak{I}}(u_\alpha, \varphi^{\mathfrak{I}}) = d^{\mathfrak{I}}(u_\alpha, t_\alpha^{\mathfrak{I}})$. Thus we obtain a finite partition $\mathbb{N}^m = N_0 \cup \dots \cup N_n$, where $t_\alpha = t_{\alpha'}$ iff $\alpha, \alpha' \in N_r$ for some common $r \leq n$. It is not hard to check that there exists $r \leq n$ such that the set $N = N_r$ satisfies the following condition:

$$\exists^\infty a_0 \dots \exists^\infty a_{m-1}\ (a_0, \dots, a_{m-1}) \in N, \tag{59}$$

where $\exists^\infty$ means 'there exist infinitely many.' Let $t = t_\alpha$ (all the t_α, for $\alpha \in N$, coincide) and, for every $l \leq m$, let

$$N|_l \ = \ \{\alpha \in \mathbb{N}^l \mid (\alpha, \beta) \in N \text{ for some } \beta \in \mathbb{N}^{m-l}\}$$

(in particular, $N|_0 = \{\lambda\}$ and $N|_m = N$). Then (59) implies that, for all $\alpha \in N|_l$ with $l < m$, there are infinitely many $a \in \mathbb{N}$ such that $(\alpha, a) \in N|_{l+1}$. So by setting $V_l = \{u_\alpha \mid \alpha \in N|_l\}$, for all $l \leq m$, we obtain that each $u_\alpha \in V_l$ (where $l < m$) is a limit of some sequence in V_{l+1}. Therefore, t and $V_0, \dots, V_m$ satisfy (58). Note that the second line in (58) implies that $d^{\mathfrak{I}}(v, V_{l'}) = 0$, for all $v \in V_l$ and $l < l' \leq m$.

Our aim now is to present a block $\mathbf{t}$ such that $(\mathbf{s}, \mathbf{t})$ is a φ-link and $\mathbf{t}$ is realised in $\mathfrak{I}$. Four cases are possible.

Case 1: $\varphi \notin \min s_0$. Then the block $\mathbf{t} = (t)$ is realised in $\mathfrak{I}$ since $t^{\mathfrak{I}} \neq \emptyset$ and $(\mathbf{s}, \mathbf{t})$ is a φ-link by construction.

Case 2: $\varphi \in \min s_{n-1} \setminus \min s_n$ for some $n \leq m$. Let us show that $\mathbf{t} = (s_0, \dots, s_{n-1}, t)$ is a block realised in $\mathfrak{I}$. Take any $u \in V_{n-1}$ and $v \in V_m$. Then $d^{\mathfrak{I}}(u, t^{\mathfrak{I}}) \leq d^{\mathfrak{I}}(u, v) + d^{\mathfrak{I}}(v, t^{\mathfrak{I}}) = d^{\mathfrak{I}}(u, v) + d^{\mathfrak{I}}(v, \varphi^{\mathfrak{I}}) \leq d^{\mathfrak{I}}(u, v) + d^{\mathfrak{I}}(v, u) + d^{\mathfrak{I}}(u, \varphi^{\mathfrak{I}}) = 2\, d^{\mathfrak{I}}(u, v)$. Since $d^{\mathfrak{I}}(u, V_m) = 0$, we obtain $d^{\mathfrak{I}}(u, t^{\mathfrak{I}}) = 0$ and hence (s_{n-1}, t) is a short link, as $u \in V_{n-1} \subseteq s_{n-1}^{\mathfrak{I}}$. Thus, $\mathbf{t}$ is a block. By considering the sets $V_0, \dots, V_{n-1}, t^{\mathfrak{I}}$, we see that $\mathbf{t}$ is realised in $\mathfrak{I}$. Finally, $(\mathbf{s}, \mathbf{t})$ is a φ-link by construction.

Case 3: $\varphi \in \min s_m \setminus s_m$ and $<_{s_{m-1}} = <_{s_m}$. Similarly to the previous case we obtain that $d^{\mathfrak{I}}(u, t^{\mathfrak{I}}) = 0$ for all $u \in V_{m-1}$ and hence (s_{m-1}, t) is a short link. Thus, $\mathbf{t} = (s_0 \dots, s_{m-1}, t)$ is a block, $\mathbf{t}$ is realised in $\mathfrak{I}$ (consider $V_0, \dots, V_{m-1}, t^{\mathfrak{I}}$), and $(\mathbf{s}, \mathbf{t})$ is a φ-link.

Case 4: $\varphi \in \min s_m \setminus s_m$ and $<_{s_{m-1}} \subset <_{s_m}$. Then $d(u, t^{\mathfrak{I}}) = d(u, \varphi^{\mathfrak{I}}) = 0$ for all $u \in V_m$. Therefore $\mathbf{t} = (s_0 \dots, s_m, t)$ is a block, $\mathbf{t}$ is realised in $\mathfrak{I}$ (consider $V_0, \dots, V_m, t^{\mathfrak{I}}$), and $(\mathbf{s}, \mathbf{t})$ is a φ-link.

($\Leftarrow$) To prove our theorem in the other direction, suppose that D is a τ-diagram and construct a metric model $\mathfrak{I}$ with $\tau^{\mathfrak{I}} \neq \emptyset$. Roughly, we proceed according to the same plan as in the proofs of Theorems 5 and 14. Let T be the set of all types from blocks in D. Let $\varphi_0, \dots, \varphi_{k-1}$ be all different elements of the set $\{\varphi \in \mathsf{com}\,\tau \mid \varphi <_t \bot \text{ for all } t \in T\}$. The first goal is to unravel D into a tree $\Delta \subseteq (\{0, \dots, k-1\} \times \mathbb{N})^*$ endowed with *three* labelling functions: $tp : \Delta \to T$, $bl : \Delta \to D$ and $hr : \Delta \to \Delta^*$. The intended meaning of the labellings is as follows: $bl(\alpha)$ is some block in D of the type $tp(\alpha)$, and if $bl(\alpha) = (t_0, \dots, t_n)$ then $hr(\alpha) = (\alpha_0, \dots, \alpha_{n-1})$, where α_m is the node 'responsible' for the presence of t_m in $bl(\alpha)$.

We proceed by induction. First we choose some $(t_*) \in D$ with $\tau \in t_*$, and set

$$\lambda \in \Delta, \quad tp(\lambda) = t_*, \quad bl(\lambda) = (t_*), \quad hr(\lambda) = \lambda$$

(recall that λ is the empty word). Suppose now that at some step α is the shortest word in Δ which has no children in Δ yet. Suppose also that

$$tp(\alpha) = s_m, \quad bl(\alpha) = (s_0, \dots, s_m), \quad hr(\alpha) = (\alpha_0, \dots, \alpha_{m-1})$$

(so $hr(\alpha) = \lambda$ when $m = 0$). As D is a diagram, for each $i < k$ with $\varphi_i \notin tp(\alpha)$, there exists some $\mathbf{t} = (t_0, \dots, t_n)$ in D such that $(bl(\alpha), \mathbf{t})$ is a φ_i-link (note that $n \leq m + 1$). We extend Δ according to the following rules:

- if $\varphi_i \in tp(\alpha)$ then $\alpha(i, j) \notin \Delta$, for all $j \in \mathbb{N}$,
- if $\varphi_i \in \varrho_{tp(\alpha)} \setminus tp(\alpha)$, then $\alpha(i, 0) \in \Delta$ and $\alpha(i, j) \notin \Delta$, for all $j > 0$,
- if $\varphi_i \notin \varrho_{tp(\alpha)}$, then $\alpha(i, j) \in \Delta$, for all $j \in \mathbb{N}$.

Now, for all i, j with $\alpha(i, j) \in \Delta$ we set

$$tp(\alpha(i, j)) = t_n, \quad bl(\alpha(i, j)) = \mathbf{t}, \quad hr(\alpha(i, j)) = (\alpha_0, \dots, \alpha_{n-1}),$$

where α_m stands for α if $n = m + 1$. Clearly, we have the following:

$$\textit{if } \alpha \in \Delta,\ hr(\alpha) = (\alpha_0, \dots, \alpha_{n-1}) \textit{ and } bl(\alpha) = (t_0, \dots, t_n) \textit{ then, for all } m < n, \qquad hr(\alpha_m) = (\alpha_0, \dots, \alpha_{m-1}) \quad \textit{and} \quad bl(\alpha_m) = (t_0, \dots, t_m). \tag{60}$$

Our next goal is to define a metric function d on Δ. We again use the notation introduced in the proof of Theorem 5. For $\alpha \in \Delta$ and $i < k$, we set $\alpha{+}i = \{\alpha(i, j) \in \Delta \mid j \in \mathbb{N}\}$ if $\varphi_i \notin tp(\varphi_i)$, and $\alpha{+}i = \{\alpha\}$ otherwise. Further, we let $\alpha{+} = \bigcup_{i<k}(\alpha{+}i)$. The distance $d(\alpha', \alpha)$, where α' is the parent of α, is denoted by d^{α}.

Recall that, by the construction of Δ, if $\varphi \in tp(\beta)$, for some $\varphi \in \mathsf{com}\,\tau$ and $\beta \in \Delta$, then every $\alpha \in \Delta$ with $\varphi \notin tp(\alpha)$ has a child β' with $\varphi \in tp(\beta')$. The main idea behind the construction of d is to ensure that such a β' can always be chosen so that $d(\alpha, \beta') \leq d(\alpha, \beta)$. For this purpose we introduce a number of numerical parameters that will be defined simultaneously with the distances d^{α}:

- A sequence $c(\alpha)$ of the same length as $bl(\alpha)$. The distances d^β, for all children β of α, will be distributed among several disjoint segments of the form $[2c/3, c)$ within the interval $(0, 1)$, and $c(\alpha)$ stores the upper bounds c of these segments.
- Numbers $d^{\alpha+i}$, for all $i < k$, that provide some landmarks for concrete distances in the sense that the condition $d(\alpha, \alpha + i) = d^{\alpha+i}$ is to be satisfied.
- A 'sufficiently small' number $\varepsilon(\alpha)$ which is defined as follows. Suppose $c(\alpha) = (c_0, \dots, c_n)$. Then
$$\begin{aligned}\varepsilon(\alpha) \;=\; \min\big(&\{d^{\alpha+i} - d^{\alpha+j} \mid i, j < k,\ d^{\alpha+i} > d^{\alpha+j}\} \cup \\ &\{c_m - d^{\alpha+i} \mid m \le n,\ i < k,\ c_m > d^{\alpha+i}\}\big).\end{aligned}$$
Roughly speaking, $\varepsilon(\alpha)$ measures the space available for 'splitting' the values $d^{\alpha+i} = d^{\alpha+l}$, where $i \neq l$, with respect to the existing strict inequalities.

We now list the principal conditions (61)–(66) that determine the choice of distances:

- *For all $\gamma \in \Delta$ and $i, l < k$,*
$$d^{\gamma+i} \le d^{\gamma+l} \ \ \textit{iff} \ \ \varphi_i \le_{tp(\gamma)} \varphi_l, \qquad d^{\gamma+i} = 0 \ \ \textit{iff} \ \ \varphi_i \in \min tp(\gamma). \tag{61}$$
- *Let $\gamma \in \Delta$ be such that $hr(\gamma) = \lambda$, $bl(\gamma) = (t)$, $c(\gamma) = (c)$. Then, for all $i < k$, $j \in \mathbb{N}$,*
$$d^{\gamma+i} \in [2c/3, c) \quad \textit{if} \ \ \varphi_i \notin \min t, \tag{62}$$
$$d^{\gamma(i,j)} \in (0, \varepsilon(\gamma)/2] \quad \textit{if} \ \ \varphi_i \in \min t \setminus t. \tag{63}$$
- *Let $\gamma \in \Delta$ be such that $hr(\gamma) = (\gamma_0, \dots, \gamma_{n-1})$, $bl(\gamma) = (t_0, \dots, t_n)$, $c(\gamma) = (c_0, \dots, c_n)$, where $n > 0$. Then, for all $i < k$, $j \in \mathbb{N}$,*
$$d^{\gamma+i} \in [d^{\gamma_{n-1}+i}, d^{\gamma_{n-1}+i} + c_n/3) \quad \textit{if} \ \ \varphi_i \notin \min t_{n-1}, \tag{64}$$
$$d^{\gamma+i} \in [2c_n/3, c_n) \quad \textit{if} \ \ \varphi_i \in \min t_{n-1} \setminus \min t_n, \tag{65}$$
$$d^{\gamma(i,j)} \in (0, \varepsilon(\gamma)/2] \quad \textit{if} \ \ \varphi_i \in \min t_n \setminus t_n. \tag{66}$$

And in the process of construction we will prove that the following property is satisfied as well:

- *Let $\gamma \in \Delta$, $hr(\gamma) = (\gamma_0, \dots, \gamma_{n-1})$, $bl(\gamma) = (t_0, \dots, t_n)$, and $c(\gamma) = (c_0, \dots, c_n)$. Then, for all $m < n$,*
$$c_{m+1} \le \varepsilon(\gamma_m)/2, \quad c_{m+1} \le c_m/2, \quad c(\gamma_m) = (c_0, \dots, c_m). \tag{67}$$

Let us now turn to the construction. First, let $c(\lambda) = (1)$ and $d^\lambda = 2/3$ (the latter is introduced for convenience). Then (67) holds trivially for $\gamma = \lambda$.

Suppose now that d^α and $c(\alpha) = (c_0, \dots, c_n)$ are defined for some $\alpha \in \Delta$, conditions (61)–(67) are satisfied for every ancestor γ of α, and (67) is satisfied for $\gamma = \alpha$ as well. Let $hr(\alpha) = (\alpha_0, \dots, \alpha_{n-1})$ and $bl(\alpha) = (t_0, \dots, t_n)$. Two cases are possible.

Case 1: $n = 0$, i.e., $hr(\alpha) = \lambda$, $bl(\alpha) = (t_0)$, and $c(\alpha) = (c_0)$. Then we can choose values $d^{\alpha+i}$, $i < k$, that satisfy (61) and (62) for $\gamma = \alpha$. Thus $\varepsilon(\alpha)$ is defined, and we set, for all $i < k$, $j \in \mathbb{N}$,
$$\begin{aligned} &d^{\alpha(i,0)} = d^{\alpha+i} \quad \text{if} \ \ \varphi_i \in \varrho_{t_0} \setminus t_0, \\ &d^{\alpha(i,j)} = d^{\alpha+i} + \varepsilon(\alpha)/(j+2) \quad \text{if} \ \ \varphi_i \notin \varrho_{t_0}. \end{aligned} \tag{68}$$

This makes (63) satisfied for $\gamma = \alpha$, while (64)–(66) do not apply to this case. We further set

$$c(\alpha(i,j)) = (c_0/2) \quad \text{if} \ \ \varphi_i \notin \min t_0,$$
$$c(\alpha(i,j)) = (c_0, d^{\alpha(i,j)}) \quad \text{if} \ \ \varphi_i \in \min t_0 \setminus t_0,$$

for all $\alpha(i,j) \in \Delta$. This makes (67) satisfied on the children of α, as $\varepsilon(\alpha) \leq c_n$ and $d^{\alpha(i,j)} \leq \varepsilon(\alpha)/2$, for $\varphi_i \in \min t_0 \setminus t_0$, by definition.

Case 2: $n > 0$, i.e., $hr(\alpha)$ is a nonempty sequence. Since $(t_0, \dots, t_n)$ is a block, we have $<_{t_{n-1}} \subseteq <_{t_n}$. And for all $i < k$ with $\varphi_i \notin \min t_{n-1}$, we have $c_n < \varepsilon(\alpha_{n-1}) \leq d^{\alpha_{n-1}+i}$ in view of (67), (61), and the definition of $\varepsilon(\alpha_{n-1})$. Therefore we can choose values $d^{\alpha+i}$, $i < k$, that satisfy (61) and (64)–(65). Now $\varepsilon(\alpha)$ is defined, and we also define the distances $d^{\alpha(i,j)}$ according to (68). This makes (66) satisfied for $\gamma = \alpha$, while (62)–(63) do not apply to this case.

It remains to define $c(\gamma)$, for all children γ of α, so that (67) holds. Consider any $\alpha(i,j) \in \Delta$. We have four possibilities. First, let $\varphi_i \notin \min t_0$. Then $hr(\alpha(i,j)) = \lambda$, and we set $c(\alpha(i,j)) = (c_0/2)$. Clearly, (67) holds for $\gamma = \alpha(i,j)$.

Let $\varphi_i \in \min t_{m-1} \setminus \min t_m$ for some $1 \leq m \leq n$. Then $hr(\alpha(i,j)) = (\alpha_0, \dots, \alpha_{m-1})$, and we set $c(\alpha(i,j)) = (c_0, \dots, c_{m-1}, c_m/2)$. Now (67) holds for $\gamma = \alpha(i,j)$ in view of the induction hypothesis.

Let $\varphi_i \in \min t_n$ and $<_{t_{n-1}} = <_{t_n}$. Then $hr(\alpha(i,j)) = (\alpha_0, \dots, \alpha_{n-1})$, and we set $c(\alpha(i,j)) = (c_0, \dots, c_{n-1}, d^{\alpha(i,j)})$. Recall that $d^{\alpha(i,j)} \leq \varepsilon(\alpha)/2$ and $\varepsilon(\alpha) \leq c_n$ by definition. Therefore (67) holds for $\gamma = \alpha(i,j)$ by the induction hypothesis.

Let finally $\varphi_i \in \min t_n$ and $<_{t_{n-1}} \subset <_{t_n}$. Then $hr(\alpha(i,j)) = (\alpha_0, \dots, \alpha_n)$, and we set $c(\alpha(i,j)) = (c_0, \dots, c_n, d^{\alpha(i,j)})$. Again, (67) holds for $\gamma = \alpha(i,j)$ as in the previous case.

Thus we define all the distances $d^\beta = d(\alpha, \beta)$, where α is the parent of β. Then we extend d to all the pairs in Δ by setting

$$d(\alpha,\alpha) = 0, \quad \text{for all } \alpha \in \Delta,$$
$$d(\beta,\alpha) = d(\alpha,\beta), \quad \text{if } \alpha \text{ is the parent of } \beta,$$
$$d(\alpha,\beta) = d(\alpha,\alpha_1) + \cdots + d(\alpha_n,\beta), \quad \text{if } \alpha, \alpha_1, \dots, \alpha_n, \beta \text{ is the shortest path from } \alpha \text{ to } \beta.$$

This distance function satisfies the following properties:

Lemma 24. *Let $\alpha \in \Delta$ and $hr(\alpha) = (\alpha_0, \dots, \alpha_{n-1})$, $bl(\alpha) = (t_0, \dots, t_n)$, $c(\alpha) = (c_0, \dots, c_n)$.*

(i) *Let $m < n$. Then, for all $i < k$ with $\varphi_i \notin \min t_m$, we have*

$$0 \leq d^{\alpha+i} - d^{\alpha_m+i} < (c_{m+1} + \cdots + c_n)/3 < 2c_{m+1}/3. \tag{69}$$

(ii) *For all $i < k$ and $1 \leq m \leq n$, we have*

$$\begin{aligned} &d^{\alpha+i} \in [2c_0/3, c_0), \quad \text{if} \ \ \varphi_i \notin \min t_0, \\ &d^{\alpha+i} \in [2c_m/3, c_m), \quad \text{if} \ \ \varphi_i \in \min t_{m-1} \setminus \min t_m, \end{aligned} \tag{70}$$

Proof. Let us prove (69) first. Note that, by (67), we have,

$$c_{m+1} + \cdots + c_n \leq (1 + 1/2 + \cdots + 1/2^{n-m-1})\, c_{m+1} < 2c_{m+1},$$

for any $m < n$. This proves the right-hand side inequality in (69). We then proceed by induction on $n - m$.

For $m = n-1$, (69) follows directly from (64). Let now $m \le n-2$ and $\varphi_i \notin \min t_m$, for some $i < k$. By (60) and (67), we have $hr(\alpha_{n-1}) = (\alpha_0, \dots, \alpha_{n-2})$, $bl(\alpha_{n-1}) = (t_0, \dots, t_{n-1})$ and $c(\alpha_{n-1}) = (c_0, \dots, c_{n-1})$. Therefore, by the induction hypothesis, we have

$$0 \le d^{\alpha_{n-1}+i} - d^{\alpha_m+i} < (c_{m+1} + \dots + c_{n-1})/3.$$

Combining this with (64) we obtain the required inequalities.

We now prove (70). Let $1 \le m \le n$ and $\varphi_i \in \min t_{m-1} \setminus \min t_m$, or $m = 0$ and $\varphi_i \notin \min t_0$. By (60) and (67) we have $hr(\alpha_m) = (\alpha_0, \dots, \alpha_{m-1})$, $bl(\alpha_m) = (t_0, \dots, t_m)$ and $c(\alpha_m) = (c_0, \dots, c_m)$. Therefore, by (65), we have $2c_m/3 \le d^{\alpha_m+i} < c_m$, and moreover $d^{\alpha_m+i} \le c_m - \varepsilon(\alpha_m)$ by the definition of $\varepsilon(\alpha_m)$. But then, by applying (69) and (67), we obtain $2c_m/3 \le d^{\alpha+i} < c_m - \varepsilon(\alpha_m) + 2c_{m+1}/3 < c_m$. ❑

For $\varphi \in \mathsf{cl}\,\tau$, let again $\varphi^\Delta = \{\alpha \in \Delta \mid \varphi \in tp(\alpha)\}$ (so $\varphi^\Delta = \emptyset$ if $\varphi \in \mathsf{com}\,\tau \setminus \{\varphi_0, \dots, \varphi_{k-1}\}$, and $\alpha + i \subseteq \varphi_i^\Delta$ for all $i < k$ and $\alpha \in \Delta$).

Lemma 25. *Let $\alpha \in \Delta$ and $i < k$. Then $d(\alpha, \varphi_i^\Delta) = d^{\alpha+i}$. More precisely,*

- *if $\varphi_i \in \varrho_{tp(\alpha)} \setminus tp(\alpha)$ then $d(\alpha, \varphi_i^\Delta) = d^{\alpha(i,0)}$,*
- *if $\varphi_i \notin \varrho_{tp(\alpha)}$ then $d(\alpha, \varphi_i^\Delta) = \lim_{j \to \infty} d^{\alpha(i,j)}$ and $d(\alpha, \varphi_i^\Delta)$ is not realised.*

Proof. As before, we have to show that for every $\beta \in \varphi_i^\Delta$ there exists $\beta_1 \in \alpha + i$ such that $d(\alpha, \beta_1) \le d(\alpha, \beta)$. The latter, in turn, is implied by the following:

Claim 26. *Let $\alpha, \beta \in \Delta$ and $i < k$. Then, for every $\beta_1 \in \beta + i$, there exists $\alpha_1 \in \alpha + i$ such that $d(\alpha, \alpha_1) - d(\beta, \beta_1) \le d(\alpha, \beta)$.*

We proceed by induction on the length N of the shortest path between α and β.

Induction basis. If $N = 0$ (that is, $\alpha = \beta$), there is nothing to prove. To handle the case $N = 1$ (which means $\beta \in \alpha+$ or $\alpha \in \beta+$) we suppose that α is the parent of β and prove the assertion of Claim 26 together with the symmetrical one: *for every $\alpha_1 \in \alpha + i$, there exists $\beta_1 \in \beta + i$ such that $d(\beta, \beta_1) - d(\alpha, \alpha_1) \le d(\alpha, \beta)$.*

Recall that, by (68), we have $d^{\alpha+i} = \inf\{d(\alpha, \gamma) \mid \gamma \in \alpha + i\}$, and similarly for β.

Let $hr(\alpha) = (\alpha_0, \dots, \alpha_{n-1})$, $bl(\alpha) = (s_0, \dots, s_n)$, and $c(\alpha) = (c_0, \dots, c_n)$. Let also $\beta \in \alpha + l$, where $l < k$, and $t = tp(\beta)$. Six cases are possible.

Case 1: $\varphi_i \in t$. Then $\beta + i = \{\beta\}$, $d^{\beta+i} = 0$, and $d(\beta, \beta) - d^{\alpha+i} \le 0 < d^\beta$. On the other hand, we have $\varphi_i \le_{s_n} \varphi_l$, since $\varphi_i \in t$ and (s_n, t) is a φ_l-link. Hence $d^{\alpha+i} \le d^{\alpha+l}$ by (60).

If $\varphi_i \in \varrho_{s_n}$, then $\alpha + i = \{\alpha_1\}$, for some α_1, and $d(\alpha, \alpha_1) = d^{\alpha+i} \le d^{\alpha+l} \le d^\beta$ in view of (68). Thus, $d(\alpha, \alpha_1) - d^{\beta+i} \le d^\beta$.

And if $\varphi_i \notin \varrho_{s_n}$, then, by the definition of a link [see the third condition in (48)] we have either $\varphi_l \notin \varrho_{s_n}$, or $\varphi_l \not\le_{s_n} \varphi_i$. Hence we have either $d^{\alpha+i} \le d^{\alpha+l} < d^\beta$, or $d^{\alpha+i} < d^{\alpha+l} \le d^\beta$. Thus, $d^{\alpha+i} - d^{\beta+i} = d^{\alpha+i} < d^\beta$ and therefore $d^{\alpha_1} - d^{\beta+i} < d^\beta$, for some $\alpha_1 \in \alpha + i$.

Case 2: $\varphi_i \in \min t \setminus t$. Then $d^{\beta+i} = 0 < d^\beta$ and therefore $d^{\beta_1} - d^{\alpha+i} < d^\beta$, for some $\beta_1 \in \beta + i$. On the other hand, we have $\varphi_i \le_{s_n} \varphi_l$, since $\varphi_i \in \min t$ and (s_n, t) is a φ_l-link. This implies $d^{\alpha+i} \le d^{\alpha+l} \le d^\beta$. And, for each $\beta_1 \in \beta + i$, we have $d^{\beta_1} > 0$; hence there exists $\alpha_1 \in \alpha + i$ such that $d^{\alpha+i} - d^{\beta_1} < d^\beta$.

So we assume that $\varphi_i \notin \min t$ in all the remaining cases. Note that the strict inequality $|d^{\alpha+i} - d^{\beta+i}| < d^\beta$ ensures the properties we are aiming to prove.

Case 3: $\varphi_l \notin \min s_0$. Then $d^\beta \in [2c_0/3, c_0)$, $c(\beta) = (c_0/2)$, $d^{\beta+i} \in [c_0/3, c_0/2)$, and $d^{\alpha+i} \in [0, c_0)$. Therefore we have $|d^{\alpha+i} - d^{\beta+i}| < c_0 - c_0/3 \le d^\beta$.

Case 4: $\varphi_l \in \min s_{m-1} \setminus s_m$, for some $m \in \{1, \dots, n\}$. Then $d^\beta \in [2c_m/3, c_m)$, $hr(\beta) = (\alpha_0, \dots, \alpha_{m-1})$, $bl(\beta) = (s_0, \dots, s_{m-1}, t)$, and $c(\beta) = (c_0, \dots, c_{m-1}, c_m/2)$. Suppose first that $\varphi_i \in \min s_{m-1} \setminus \min t$. Then $d^{\beta+i} \in [c_m/3, c_m/2)$ and $d^{\alpha+i} \in [0, c_m)$, hence $|d^{\alpha+i} - d^{\beta+i}| < c_m - c_m/3 \le d^\beta$.

Suppose now that $\varphi_i \notin \min s_{m-1}$. Then $0 \le d^{\beta+i} - d^{\alpha_{m-1}+i} < c_m/6$ by (64) and $0 \le d^{\alpha+i} - d^{\alpha_{m-1}+i} < 2c_m/3$ by (69). Thus, we again have $|d^{\alpha+i} - d^{\beta+i}| < 2c_m/3 \le d^\beta$.

Case 5: $\varphi_l \in \min s_m$ and $\le_{s_{n-1}} = \le_{s_n}$. Then $hr(\beta) = (\alpha_0, \dots, \alpha_{n-1})$ and $c(\beta) = (c_0, \dots, c_{n-1}, d^\beta)$. Suppose first that $\varphi_i \in \min s_n$. Then $d^{\alpha+i} = 0$ and $d^{\beta+i} < d^\beta$ by construction; hence $|d^{\alpha+i} - d^{\beta+i}| < d^\beta$.

Suppose now that $\varphi_i \notin \min s_n$. Then $\varphi_i \notin \min s_{n-1}$ and $d^{\alpha+i}, d^{\beta+i} \in [d^{\alpha_{n-1}+i}, d^{\alpha_{n-1}+i} + d^\beta/3)$ by (64); hence $|d^{\alpha+i} - d^{\beta+i}| < d^\beta$.

Case 6: $\varphi_l \in \min s_m$ and $\le_{s_{n-1}} \subset \le_{s_n}$ is similar to the previous one.

Induction step: $N \ge 2$. Consider $\gamma \in \Delta$ such that $\alpha, \dots, \gamma, \beta$ is the shortest path between α and β. Take an arbitrary $\beta_1 \in \beta + i$. Then, by the induction basis, there exists $\gamma_1 \in \gamma + i$ such that $d(\gamma, \gamma_1) \le d(\gamma, \beta_1)$. Hence $d(\alpha, \gamma_1) \le d(\alpha, \beta_1)$. Now, by the induction hypothesis, there exists $\alpha_1 \in \alpha + i$ such that $d(\alpha, \alpha_1) \le d(\alpha, \gamma_1)$. ❑

Define a metric model $\mathfrak{I}$ by setting $\Delta^\mathfrak{I} = \Delta$, $d^\mathfrak{I} = d$, and $p^\mathfrak{I} = p^\Delta$, for all atomic terms p.

Lemma 27. *For all $\varphi \in \mathsf{cl}\,\tau$, we have $\varphi^\mathfrak{I} = \varphi^\Delta$.*

Proof. We proceed by induction on the structure of $\varphi \in \mathsf{cl}\,\tau$. If φ is an atomic term, then we simply have the definition of $\mathfrak{I}$. If $\varphi = \neg\psi_0$ or $\varphi = \psi_0 \sqcap \psi_1$, then our assertion for φ follows easily from the induction hypothesis.

So, let now $\varphi = \psi_0 \leftleftarrows \psi_1$. Recall that D is the initial diagram and T is the set of types occurring in blocks from D (thus, $tp(\alpha) \in T$ for all $\alpha \in \Delta$). Suppose first that $\psi_0 = \varphi_i$ and $\psi_1 = \varphi_l$ for some $i, l < k$. Then, by Lemma 25 and (61), we have

$$\alpha \in (\varphi_i \leftleftarrows \varphi_l)^\mathfrak{I} \ \text{ iff } \ d^{\alpha+i} < d^{\alpha+l} \ \text{ iff } \ \alpha \in (\varphi_i \leftleftarrows \varphi_l)^\Delta.$$

Suppose now that $\psi_0 = \varphi_i$ and $\psi_1 \notin \{\varphi_0, \dots, \varphi_{k-1}\}$. Then $\psi_1^\Delta = \emptyset \ne \psi_1^\Delta$. Moreover, for all $t \in T$, ψ_1 is a $\le_t$-maximal element, while ψ_0 is not; hence $(\psi_0 \leftleftarrows \psi_1) \in t$. Therefore $(\psi_0 \leftleftarrows \psi_1)^\Delta = \Delta$. On the other hand, $\psi_1^\mathfrak{I} = \emptyset \ne \psi_1^\mathfrak{I}$ by the induction hypothesis. Hence $(\psi_0 \leftleftarrows \psi_1)^\mathfrak{I} = \Delta$ as well.

The cases with $\psi_0 \notin \{\varphi_0, \dots, \varphi_{k-1}\} \ni \psi_1$ or $\psi_0, \psi_1 \notin \{\varphi_0, \dots, \varphi_{k-1}\}$ are considered similarly.

Let finally $\varphi = ⓡ\psi$. Suppose first that $\psi = \varphi_i$ for some $i < k$. Then, by Lemma 25, we have

$$\alpha \in (ⓡ\varphi_i)^\mathfrak{I} \ \text{ iff } \ d(\alpha, \alpha + i) \text{ is realised } \ \text{ iff } \ \alpha \in (ⓡ\psi)^\Delta.$$

Suppose now that $\psi \notin \{\varphi_0, \dots, \varphi_{k-1}\}$. Then $\psi^\Delta = \emptyset$ and ψ is a $\le_t$-maximal element, for all $t \in T$. Hence $\psi^\mathfrak{I} = \emptyset$ by the induction hypothesis, and $\varphi \notin \varrho_t$ $(t \in T)$ by the definition of a link. This implies $\varphi^\mathfrak{I} = \emptyset = \varphi^\Delta$. ❑

Thus, by Lemma 27, $\lambda \in \tau^\mathfrak{I}$ which completes the proof of Theorem 23. ❑

Theorem 28. *The satisfiability problem for $\mathcal{CSL}$- and $\mathcal{QML}$-terms in metric models is* ExpTime*-complete.*

Proof. To prove the upper bound, we use basically the same elimination procedure as in the proof of Theorem 8. The only difference is that we now apply it to the set of blocks rather than the set of types. So satisfiability of τ in metric models can be checked in time $\leq 2^{O(|\mathsf{at}\,\tau|\cdot|\mathsf{com}\,\tau|^4)} \leq 2^{O(|\tau|^5)}$. The proof of the lower bound remains the same as in the proof of Theorem 8. ❑

5.2 Axiomatisation

Recall that term (53) corresponds to a property of metric models which follows neither from (sym) nor from (tri). Similarly, (52) is a consequence of (tri), but not (sym). Let us now add both (52) and (53) as axiom schemas to the axiomatic system from Section 3.2. In this section we prove that the extended axiomatic system is complete with respect to the class of metric models.

Note first that (34)–(45) and Lemmas 9–11 still hold true. So we now need to find terms that reflect the newly introduced or modified notions (links of types, blocks, links of blocks).

As before, we write $\varphi_0, \ldots, \varphi_{n-1} \vdash \varphi_n$ to say that here is a derivation of φ_n from the premises $\varphi_0, \ldots, \varphi_{n-1}$ in which (Gen) is not applied to terms that depend on $\varphi_0, \ldots, \varphi_{n-1}$, and $s \circ_\varphi t = \bigsqcap s \sqcap \neg(\varphi \leftleftarrows \bigsqcap t) \sqcap (ⓡ\varphi \to ⓡ\bigsqcap t)$ for finite sets s, t of terms and a term φ.

Lemma 29. *Suppose that s, t are τ-types, $\varphi \in \min s$, and $\bigsqcap s \sqcap \neg(\varphi \leftleftarrows \bigsqcap t)$ is consistent. Then $<_s \subseteq <_t$.*

Proof. Let $\xi = \bigsqcap s \sqcap \neg(\varphi \leftleftarrows \bigsqcap t)$. Take an arbitrary $\neg(\psi_0 \leftleftarrows \psi_1) \in t$. We have to show that $\neg(\psi_0 \leftleftarrows \psi_1) \in s$. We have $\vdash s \circ_\varphi t \to \neg(\top \leftleftarrows \varphi)$ and $\vdash s \circ_\varphi t \to \neg(\varphi \leftleftarrows \neg(\psi_0 \leftleftarrows \psi_1))$. Therefore $\vdash s \circ_\varphi t \to \neg(\top \leftleftarrows \neg(\psi_0 \leftleftarrows \psi_1))$ by (23), and so $\vdash \xi \to \neg(\psi_0 \leftleftarrows \psi_1)$ in view of (53). Thus, $s \cup \{\neg(\psi_0 \leftleftarrows \psi_1)\}$ is consistent, which means that $\neg(\psi_0 \leftleftarrows \psi_1) \in s$, as s is a τ-type and $\neg(\psi_0 \leftleftarrows \psi_1) \in \mathsf{cl}\,\tau$. ❑

Lemma 30. *Let s be a consistent τ-type and $\varphi <_s \bot$, $\varphi \notin s$. Then there exists a consistent τ-type t such that (s, t) is a φ-link.*

Proof. By Lemma 12, there exists a consistent τ-type t such that the pair (s, t) satisfies (16). And by Lemma 18, (s, t) satisfies (48). Finally, if $\varphi \in \min s$ then (s, t) satisfies (54) in view of Lemma 29. ❑

For a sequence $\mathbf{t} = (t_0, \ldots, t_n)$ of sets of terms, let $\mathbf{t}^{\&}$ denote a term defined inductively by the following rules:

- if $n = 0$ then $\mathbf{t}^{\&} = \bigsqcap t_0$,
- if $n > 0$ then $\mathbf{t}^{\&} = \bigsqcap t_0 \sqcap \neg(\top \leftleftarrows \mathbf{t}_1^{\&})$, where $\mathbf{t}_1 = (t_1, \ldots, t_n)$.

So we have $(t_0, \ldots, t_n)^{\&} = (t_0, \ldots, t_m, (t_{m+1}, \ldots, t_n)^{\&})^{\&}$, for any $m < n$. We say that $\mathbf{t}$ is *consistent* if $\mathbf{t}^{\&}$ is consistent.

Lemma 31. *Let $\mathbf{t} = (t_0, \ldots, t_n)$ be a sequence of sets of terms.*

(i) *If* $t_0, \ldots, t_n$ *are* τ*-types and* $\mathbf{t}$ *is consistent then* $<_{t_0} \subseteq \cdots \subseteq <_{t_n}$.

(ii) *If* $\mathbf{s} = (s_0, \ldots, s_m)$ *is a subsequence of* $\mathbf{t}$ *then* $\mathbf{t}^{\&} \vdash \neg(\top \leftleftarrows \mathbf{s}^{\&})$*; moreover,* $\mathbf{t}^{\&} \vdash \mathbf{s}^{\&}$ *provided that* $s_0 = t_0$.

(iii) *If* $\mathbf{r} = (r_0, \ldots, r_n)$ *and* $\bigsqcap t_i \vdash \bigsqcap r_i$*, for all* $i \leq n$*, then* $\mathbf{t}^{\&} \vdash \mathbf{r}^{\&}$.

(iv) *For any* ψ*, we have* $\mathbf{t} \vdash (t_0, \ldots, t_{n-1}, t_n \cup \{\psi\})^{\&} \sqcup (t_0, \ldots, t_{n+1}, t_n \cup \{\neg\psi\})^{\&}$.

Proof. Note first that the following properties hold for all terms φ, ψ, ψ_0, and ψ_1:

$$\vdash \psi_0 \to \psi_1 \;\text{ implies }\; \vdash \neg(\varphi \leftleftarrows \psi_0) \to \neg(\varphi \leftleftarrows \psi_1), \tag{71}$$

$$\vdash \neg(\top \leftleftarrows \varphi) \to \neg(\top \leftleftarrows (\varphi \sqcap \psi)) \sqcup \neg(\top \leftleftarrows (\varphi \sqcap \neg\psi)). \tag{72}$$

Indeed, (71) is proved using (38) and (Gen), while (72) is shown similarly to (46). Now (ii), (iii) and (iv) are easily proved by induction using (71) and (72), and (i) follows from Lemma 29 and (ii). ❑

Lemma 32. *Let* $\mathbf{s} = (s_0, \ldots, s_m)$ *be a consistent block and* $\varphi <_{s_m} \bot$, $\varphi \notin s_m$*. Then there exists a consistent block* $\mathbf{t}$ *such that* $(\mathbf{s}, \mathbf{t})$ *is* φ*-link.*

Proof. For a set of terms t, let $\mathbf{s} \circ_\varphi t = (s_0, \ldots, s_{m-1}, s_m \circ_\varphi t)$. So $\mathbf{s} \circ_\varphi \varphi$ is consistent. Hence we can find some $t \subseteq \mathsf{cl}\,\tau$ that is maximal with the properties $\varphi \in t$ and $\mathbf{s} \circ_\varphi t$ is consistent. By Lemma 31, t is a maximal consistent subset of $\mathsf{cl}\,\tau$ and $s_m \circ_\varphi t$ is consistent. So by the proof of Lemma 29, (s_m, t) is a φ-link of consistent types. Two cases are now possible.

Case 1: $(\top \leftleftarrows \varphi) \in s_0$. Then $\mathbf{t} = (t)$ is a consistent block and $(\mathbf{s}, \mathbf{t})$ is a φ-link of blocks.

Case 2: $\neg(\top \leftleftarrows \varphi) \in s_n$ for some $n \leq m$. Then we may assume that n is chosen to be maximal with this property. We have $(\mathbf{s} \circ_\varphi t)^{\&} \vdash (s_0, \ldots, s_n, \neg(\varphi \leftleftarrows \bigsqcap t))^{\&}$ by Lemma 31, where the latter term can be represented as $(s_0, \ldots, s_{n-1}, (s_n, \neg(\varphi \leftleftarrows \bigsqcap t))^{\&})^{\&}$. Further, $(s_n, \neg(\varphi \leftleftarrows \bigsqcap t))^{\&} = \bigsqcap s_n \sqcap \neg(\top \leftleftarrows \neg(\varphi \leftleftarrows \bigsqcap t))$, and so

$$(s_n, \neg(\varphi \leftleftarrows \textstyle\bigsqcap t))^{\&} \;\vdash\; \textstyle\bigsqcap s_n \sqcap \neg(\varphi \leftleftarrows \bigsqcap t), \quad \text{in view of (53)},$$

$$(s_n, \neg(\varphi \leftleftarrows \textstyle\bigsqcap t))^{\&} \;\vdash\; \textstyle\bigsqcap s_n \sqcap \neg(\top \leftleftarrows \bigsqcap t), \quad \text{by (36), since } \neg(\top \leftleftarrows \varphi) \in s_n,$$

that is $(s_n, \neg(\varphi \leftleftarrows \bigsqcap t))^{\&} \vdash (s_n, t)^{\&}$. Let $\mathbf{t} = (s_0, \ldots, s_n, t)$. Then we obtain $\mathbf{s} \vdash \mathbf{t}$ by Lemma 31. Hence $\mathbf{t}$ is a consistent block and $(\mathbf{s}, \mathbf{t})$ is a φ-link of blocks. ❑

Theorem 33. *A* $\mathcal{CSL}$*-term* τ *is valid in the class of metric models iff* $\vdash \tau$.

Proof. As before, it suffices to show that an arbitrary consistent term, say, τ, is satisfiable. Then τ is contained in some consistent τ-type t_*, and hence (t_*) is a consistent block. Let T be the set of all τ-types t such that $\psi <_t \bot$ iff $\psi <_{t_*} \bot$, for all $\psi \in \mathsf{cl}\,\tau$. Take the set D of all consistent blocks which only contain types from T. Then D is a diagram: (55) and (56) are satisfied by the construction, and (57) by Lemma 32. Thus, τ is satisfiable by Theorem 23, which completes the proof. ❑

6 Non-axiomatisability of $\mathcal{CSL}$ over $\mathbb{R}$

Despite the decidability and axiomatisability results obtained in the previous sections, $\mathcal{CSL}$ turns out to be undecidable and non-axiomatisable when interpreted over models based on $\mathbb{R}$ or its metric subspaces (perhaps at this point it is worth recalling Tarski's theorem [32] according to which the first-order theory of $(\mathbb{R}, +, \times, =)$ is decidable). It follows, in particular, that the set of $\mathcal{CSL}$-terms valid in models based on $\mathbb{R}$ is a proper superset of the set of $\mathcal{CSL}$ formulas valid in all metric models.

Theorem 34. *Let $\mathbf{D}$ be the class of all models based on $\mathbb{D}$, or all models based on metric subspaces of $\mathbb{D}$, where $\mathbb{D}$ is $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Z}$ (with the standard Euclidean metric). Then the set of $\mathcal{CSL}$-terms valid in $\mathbf{D}$ is not recursively enumerable.*

The remainder of this section is devoted to the proof of Theorem 34. The proof is by reduction of the decision problem for Diophantine equations (Hilbert's 10th problem) which is known to be undecidable; see [19, 5] and references therein. More precisely, we will use the following (still undecidable) variant of this problem:

given arbitrary polynomials g and h with coefficients from $\mathbb{N} \setminus \{0, 1\}$, decide whether the equation $g = h$ has a solution in the set $\mathbb{N} \setminus \{0, 1\}$.

We give an algorithm that constructs, for every polynomial equation $g = h$ over $\mathbb{N} \setminus \{0, 1\}$, a $\mathcal{CSL}$-term $\tau_{g,h}$ such that the following conditions are equivalent:

- $\tau_{g,h}$ is satisfiable in a model $\mathfrak{I} \in \mathbf{D}$;
- $\tau_{g,h}$ is satisfiable in a model based on $\mathbb{Z}$;
- $g = h$ is solvable in $\mathbb{N} \setminus \{0, 1\}$.

As set of equations without solutions is not recursively enumerable, we immediately obtain Theorem 34.

Each polynomial equation can be rewritten equivalently as a set of elementary equations of the form

$$x = y + z, \quad x = y \cdot z, \quad x = y, \quad x = n, \tag{73}$$

where x, y, z are variables and $n \in \mathbb{N} \setminus \{0, 1\}$. Thus, it suffices to reduce solvability of such sets of elementary equations to satisfiability of $\mathcal{CSL}$-terms. This will be done in three steps:

1. first we ensure that (modulo an affine transformation) the underlying space of a given model contains $\mathbb{Z}$, and define the operations '$+1$' and '-1' on $\mathbb{Z}$;
2. then we define, in this model, sets of the form $\{kl + j \mid k \in \mathbb{Z}\}$ that are used to represent the (possibly unknown) number $l \in \mathbb{N}$;
3. finally, we encode addition and multiplication on such sets.

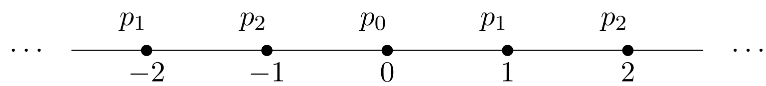
In what follows, we use $\tau_1 \leftrightarrows \tau_2$ as an abbreviation for $\neg(\tau_1 \leftleftarrows \tau_2) \sqcap \neg(\tau_2 \leftleftarrows \tau_1)$.

Step 1. Say that models $\mathfrak{I}, \mathfrak{L} \in \mathbf{D}$ are *affine isomorphic* and (in symbols, $\mathfrak{I} \simeq \mathfrak{L}$) if there exists an affine transformation $f(x) = ax + b$ from $\Delta^{\mathfrak{I}}$ onto $\Delta^{\mathfrak{L}}$ such that $x \in p^{\mathfrak{I}}$ iff $f(x) \in p^{\mathfrak{L}}$, for all $x \in \Delta^{\mathfrak{I}}$ and atomic terms p. In this case we clearly have $f(\tau^{\mathfrak{I}}) = \tau^{\mathfrak{L}}$ for every term τ.

Take atomic terms p_0, p_1, p_2 and set $\mathsf{Base}(p_0, p_1, p_2)$ to be the following term:

$$\prod_{i<3} \forall \textcircled{r} p_i \ \sqcap \prod_{i<j<3} \forall \neg (p_i \sqcap p_j) \ \sqcap \prod_{i<3} \forall \big(p_i \to (p_{i \oplus 1} \leftrightarrows p_{i \ominus 1})\big),$$

where $\oplus$ and $\ominus$ denote $+$ and $-$ modulo 3. A typical model satisfying $\mathsf{Base}(p_0, p_1, p_2)$ is depicted below:



More precisely, we have the following:

Lemma 35. *A model $\mathfrak{L} \in \mathbf{D}$ satisfies $\mathsf{Base}(p_0, p_1, p_2)$ iff there exists $\mathfrak{I} \in \mathbf{D}$ such that $\mathfrak{I} \simeq \mathfrak{L}$ and*

$$p_i^{\mathfrak{I}} \ = \ \{3k + i \mid k \in \mathbb{Z}, \ k < n\}, \quad i < 3. \tag{74}$$

Proof. Given $x, y \in \Delta^{\mathfrak{L}}$ and a term τ, we say that y is a *τ-neighbour* of x if $y \in \tau^{\mathfrak{L}}$ and $d(x, y) = d(x, \tau^{\mathfrak{L}})$. If y is a τ-neighbour of x and $y \leq x$, then y is called the *left τ-neighbour* of x (observe that there exists at most one such neighbour); the *right τ-neighbour* of x is defined dually.

($\Rightarrow$) Suppose $\mathfrak{L}$ satisfies $\mathsf{Base}(p_0, p_1, p_2)$; then, in particular, $p_0^{\mathfrak{L}}$, $p_1^{\mathfrak{L}}$, and $p_2^{\mathfrak{L}}$ are nonempty and pairwise disjoint. Suppose that $\{i, j, k\} = \{0, 1, 2\}$ and take any $x \in p_i^{\mathfrak{L}}$. According to $\mathsf{Base}(p_0, p_1, p_2)$, there exist, for x, a p_j-neighbour y and a p_k-neighbour z, and we have $y \neq z$ and $d(x, y) = d(x, z)$. Hence y and z lie on the different sides of x, that is, the interval between y and z does not intersect with $p_j^{\mathfrak{L}} \cup p_k^{\mathfrak{L}}$, which implies that $p_j \leftrightarrows p_k$ is not true anywhere strictly between x and y or x and z. Thus, x, y, and z are the only points in $p_i^{\mathfrak{L}}$, $p_j^{\mathfrak{L}}$, and $p_k^{\mathfrak{L}}$, respectively, in the segment between y and z; and $z - x = x - y$.

Using an appropriate affine transformation, we may assume that $0 \in p_0^{\mathfrak{L}}$, $1 \in p_1^{\mathfrak{L}}$, and $-1 \in p_2^{\mathfrak{L}}$. Then 0 is the left $p_0 \sqcup p_2$-neighbour of 1 and belongs to $p_0^{\mathfrak{L}}$. By the reasoning above, we obtain that the right $p_0 \sqcup p_2$-neighbour of 1 is equal to 2 and belongs to $p_2^{\mathfrak{L}}$; moreover, the interval $(1, 2)$ does not contain points from $p_0^{\mathfrak{L}} \cup p_1^{\mathfrak{L}} \cup p_2^{\mathfrak{L}}$. Similarly, we have that $-2 \in p_1^{\mathfrak{L}}$ and $(-2, 1)$ does not intersect with $p_0^{\mathfrak{L}} \cup p_1^{\mathfrak{L}} \cup p_2^{\mathfrak{L}}$.

By induction, one can now show that $p_0^{\mathfrak{L}} \cup p_1^{\mathfrak{L}} \cup p_2^{\mathfrak{L}} = \mathbb{Z}$, and $k \in p_i^{\mathfrak{L}}$ iff $k \equiv i \pmod 3$.

($\Leftarrow$) Let $\mathfrak{I}$ be a model satisfying (74). Then $\mathsf{Base}(p_0, p_1, p_2)$ is clearly satisfied in $\mathfrak{I}$, and so in every $\mathfrak{L} \simeq \mathfrak{I}$. ❑

In any model satisfying (74) we can now define the following analogues of the temporal 'next-time' operators simulating the functions '$+1$' and '-1':

$$\bigcirc \varrho \ = \ \prod_{i<3} (p_i \to (p_{i \oplus 1} \leftrightarrows p_{i \oplus 1} \sqcap \varrho)), \quad \bigcirc^{-1} \varrho \ = \ \prod_{i<3} (p_i \to (p_{i \ominus 1} \leftrightarrows p_{i \ominus 1} \sqcap \varrho)).$$

and set

$$\bigcirc^0 \varrho \ = \ \varrho, \quad \bigcirc^{k+1} \varrho \ = \ \bigcirc \bigcirc^k \varrho, \quad \bigcirc^{-k-1} \varrho \ = \ \bigcirc^{-1} \bigcirc^{-k} \varphi, \quad \text{for all } k \in \mathbb{N}.$$

We immediately obtain:

Lemma 36. *Let $\mathfrak{I} \in \mathbf{D}$ satisfy (74). Then for all $k \in \mathbb{Z}$ and $x \in p_0^{\mathfrak{I}} \cup p_1^{\mathfrak{I}} \cup p_2^{\mathfrak{I}}$,*

$$x \in (\bigcirc^k \varrho)^{\mathfrak{I}} \quad \textit{iff} \quad x + k \in \varrho^{\mathfrak{I}}.$$

To fix the origin and orientation of our model, we take a fresh variable p and set

$$\mathsf{Ori}(p) \;=\; \exists(p_0 \sqcap p \sqcap \neg\bigcirc^{-1}p) \,\sqcap\, \forall(p \rightarrow \bigcirc p).$$

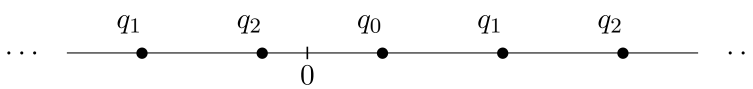
Then, for a model $\mathfrak{I} \in \mathbf{D}$, (74) and $\mathsf{Ori}(p)$ imply $p^{\mathfrak{I}} \cap (p_0^{\mathfrak{I}} \cup p_1^{\mathfrak{I}} \cup p_2^{\mathfrak{I}}) = \{k, k+1, \dots\}$, for some $k \in \mathbb{Z}$, $k \equiv 0 \,(\mathrm{mod}\, 3)$. We call a model $\mathfrak{I}$ *standard* if $\mathfrak{I} \in \mathcal{R}$, (74) holds, and $p^{\mathfrak{I}} = \mathbb{N}$. Thus, every model in $\mathcal{R}$ satisfying $\mathsf{Base}(p_0, p_1, p_2)$ and $\mathsf{Ori}(p)$ is affine isomorphic to a standard model. Note that $\{0\}$ is defined by $p \sqcap \neg\bigcirc^{-1}p$ in a standard model.

Step 2. Let $\mathfrak{I}$ be a standard model. As the representation of l in $\mathfrak{I}$ we use the subset $\{kl \mid k \in \mathbb{Z}\}$ of $\mathfrak{I}$. However, subsets of the form $\{kl + j \mid k \in \mathbb{Z}\}$ with $0 < j < l$ will also be required in Step 3. To define these we introduce our next term.

To simplify notation, we denote lists of the form p_0, p_1, p_2 by $\mathbf{p}$, and terms of the form $p_0 \sqcup p_1 \sqcup p_2$ by p_*. Take fresh atomic terms q_0, q_1, q_2, and define $\mathsf{Seq}(\mathbf{q})$ to be the term

$$\forall(q_* \rightarrow p_*) \;\sqcap\; \prod_{i<3} \forall\big(q_i \rightarrow (\neg q_{i\oplus 1} \sqcap \neg\bigcirc q_{i\oplus 1} \sqcap (q_{i\ominus 1} \leftrightarrows q_{i\oplus 1}))\big) \;\sqcap\; \exists(q_0 \sqcap p \sqcap (q_1 \leftrightarrows q_1 \sqcap p)).$$

This term is supposed to describe the following structure:



That is, q_0, q_1, q_2 are subsets of $\mathbb{Z}$, their points are periodically placed within equal distances greater than one, and the least non-negative q_*-point belongs to q_0.

Indeed, similarly to the proof of Lemma 35 one can show the following:

Lemma 37. *Let $\mathfrak{I}$ be a standard model. Then $\mathsf{Seq}(\mathbf{q})$ is satisfied in $\mathfrak{I}$ iff there exist j and l with $l > j \geq 0$ and $l > 1$ such that*

$$q_i^{\mathfrak{I}} \;=\; \{lk + j \mid k \equiv i \ (\mathrm{mod}\, 3)\}, \quad i < 3. \tag{75}$$

If (75) holds, we say that $\mathbf{q}$ *encodes* in $\mathfrak{I}$ the number l with *indent* j. If $j = 0$, then we say that this encoding is *standard*.

Let $\mathbf{q}$ and $\mathbf{q}'$ encode in $\mathfrak{I}$ some numbers l and l', respectively. If these encodings are standard, then the relations $<$, $=$, and $>$ between l and l' are easily expressed; for example the term

$$\forall\big(\neg p \rightarrow ((q_1 \sqcap p) \leftleftarrows (q_1' \sqcap p))\big)$$

ensures that $l < l'$. Thus, it remains to understand how to express $l = l'$ when the encodings are not necessarily standard. First, observe that l is equal to l' iff the sets defined by q_* and q_*' either coincide or are strictly alternating. More precisely, if $l, l', j, j' \in \mathbb{Z}$ satisfy

$$kl + j \;<\; kl' + j' \;<\; (k+1)l + j, \quad k \in \mathbb{Z}, \tag{76}$$

then $l = l'$. Now, let $\mathsf{Alt}(\mathbf{q}, \mathbf{q}')$ denote the term

$$\prod_{i<3} \forall\Big(\big(q_i \rightarrow ((q_{i\ominus 1}' \leftleftarrows q_{i\ominus 1}) \sqcap (q_i' \leftleftarrows q_{i\oplus 1}))\big) \;\sqcap\; \big(q_i' \rightarrow ((q_i \leftleftarrows q_{i\ominus 1}') \sqcap (q_i' \leftleftarrows q_{i\oplus 1}))\big)\Big).$$

Lemma 38. *Let $\mathfrak{I}$ be a standard model. Suppose that $\mathbf{q}$ and $\mathbf{q}'$ encode in $\mathfrak{I}$ some numbers l and l' with indents j and j', respectively. Then $\mathfrak{I}$ satisfies $\mathsf{Alt}(\mathbf{q}, \mathbf{q}')$ iff (76) holds.*

Proof. Assume that $\mathfrak{I}$ satisfies $\mathsf{Alt}(\mathbf{q},\mathbf{q}')$ and consider $j' \in (q_0')^{\mathfrak{I}}$. By the second conjunct of $\mathsf{Alt}(\mathbf{q},\mathbf{q}')$, we have

$$-l' + j' \ < \ kl + j \ < \ j' \ < \ (k+1)l + j \ < \ l' + j'$$

for some $kl + j \in q_0^{\mathfrak{I}}$, i.e., $k \equiv 0 \,(\mathrm{mod}\, 3)$. Then by the first conjunct of $\mathsf{Alt}(\mathbf{q},\mathbf{q}')$, we have

$$(k-1)l + j \ < \ -l' + j' \ < \ kl + j \ < \ j' \ < \ (k+1)l + j,$$

whence $k - 1 < 0$ and $k + 1 \geq 0$, as $l > j \geq 0$ and $l' > j' \geq 0$. Hence $k = 0$ and we have

$$-l + j < -l' + j' < j < j' < l + j < l' + j'.$$

By applying $\mathsf{Alt}(\mathbf{q},\mathbf{q}')$ to the 'end points' of the expanding chain of inequalities, we obtain (76). The other direction is straightforward. ❑

Let $\mathsf{Equ}(\mathbf{q},\mathbf{q}')$ denote the term $\mathsf{Alt}(\mathbf{q},\mathbf{q}') \sqcup \forall(q_* \leftrightarrow q_*')$. Then we readily obtain:

Lemma 39. *Suppose that $\mathfrak{I}$ is a standard model satisfying $\mathsf{Seq}(\mathbf{q})$ and $\mathsf{Seq}(\mathbf{q}')$. Then $\mathfrak{I}$ satisfies $\mathsf{Equ}(\mathbf{q},\mathbf{q}')$ iff $l = l'$, for the numbers l and l' encoded by $\mathbf{q}$ and $\mathbf{q}'$ in $\mathfrak{I}$.*

Step 3. Now we encode addition and multiplication. Let $\mathbf{q}$, $\mathbf{r}$, and $\mathbf{s}$ be standard encodings of some numbers u, v and w, respectively. Suppose we want to say that $u = v + w$. Consider first the case $v < w$, which can be expressed by $\forall\big(\neg p \to ((r_1 \sqcap p) \leftleftarrows (s_1 \sqcap p))\big)$. Take a fresh $\mathbf{s}'$ and state:

$$\begin{array}{ll}
\mathsf{Seq}(\mathbf{s}') \ \sqcap \ \mathsf{Alt}(\mathbf{s},\mathbf{s}') \ \sqcap & \qquad \mathbf{s}' \text{ encodes } w \\
\forall\big(\neg p \to ((s_0' \sqcap p) \leftrightarrows (r_1 \sqcap p))\big) \ \sqcap & \qquad \text{with indent } v, \\
\forall\big(\neg p \to ((q_1 \sqcap p) \leftrightarrows (s_1' \sqcap p))\big). & \qquad \text{and } u = v + w
\end{array}$$

The case $v > w$ is the mirror image. And to say that $v = w$ and $u = v + v$ we can use the terms $\forall(r_* \leftrightarrow s_*)$ and $\forall\big(\neg p \to ((q_1 \sqcap p) \leftrightarrows (r_2 \sqcap p))\big)$.

To encode multiplication we use the following observation.

Fact 1. *Let v, w be integer numbers with $0 < v < w - 1$. Then*

(i) *$u = vw$ is the least solution to $u \equiv 0 \,(\mathrm{mod}\, w)$, $u \equiv v \,(\mathrm{mod}\,(w-1))$, $u \geq 0$.*

(ii) *$u = (w-1)w$ is the least solution to $u \equiv 0 \,(\mathrm{mod}\, w)$, $u \equiv 0 \,(\mathrm{mod}\,(w-1))$, $u > 0$.*

(iii) *$u \in \{w, w^2\}$ are the least two solutions to $u \equiv 0 \,(\mathrm{mod}\, w)$, $u \equiv 1 \,(\mathrm{mod}\,(w-1))$, $u > 0$.*

Suppose we want to say that $u = v \cdot w$. Consider first the case $v < w - 1$, which can be expressed as $\forall\big(\neg p \to ((r_1 \sqcap p) \leftleftarrows \bigcirc(s_1 \sqcap p))\big)$. Take fresh $\mathbf{t}$ and $\mathbf{t}'$, and state:

$$\begin{array}{ll}
\mathsf{Seq}(\mathbf{t}) \ \sqcap \ \forall\big(\neg p \to ((t_0 \sqcap p) \leftrightarrows p)\big) \ \sqcap & \qquad \mathbf{t} \text{ is a standard encoding of} \\
\forall\big(\neg p \to ((t_1 \sqcap p) \leftrightarrows \bigcirc(s_1 \sqcap p))\big) \ \sqcap & \qquad \text{the number } w - 1 \\
\mathsf{Seq}(\mathbf{t}') \ \sqcap \ \mathsf{Alt}(\mathbf{t},\mathbf{t}') \ \sqcap & \qquad \text{and } \mathbf{t}' \text{ encodes } w - 1 \text{ as well} \\
\forall\big(\neg p \to ((t_0' \sqcap p) \leftrightarrows (r_1 \sqcap p))\big). & \qquad \text{with indent } v
\end{array} \tag{77}$$

Then, in view of Fact 1 (i), term (77) means that $v \cdot w$ is the least point satisfying $p \sqcap t_* \sqcap t_*'$. Therefore, $\forall\big(\neg p \to ((q_1 \sqcap p) \leftrightarrows (t_* \sqcap t_*' \sqcap p))\big)$ in conjunction with (77) ensures that $u = v \cdot w$.

The case $v = w - 1$ is similar (we use Fact 1 (ii))), and we can deal by symmetry with $w < v - 1$ and $w = v - 1$. Assume now that $v = w$. Take a fresh $\mathbf{t}$. The term

$$\mathsf{Seq}(\mathbf{t}) \;\sqcap\; \forall\Big(\neg p \to \big((\bigcirc t_0 \sqcap p) \leftrightarrows p) \sqcap ((t_1 \sqcap p) \leftrightarrows (s_1 \sqcap p))\big)\Big) \tag{78}$$

means that $\mathbf{t}$ encodes $w - 1$ with indent 1. Then, in view of Fact 1 (iii), term (78) implies that the least two points satisfying $p \sqcap s_* \sqcap t_*$ are w and w^2. But w satisfies $\bigcirc s_0 \leftrightarrows \neg p$, while for w we have $\bigcirc s_0 \leftleftarrows \neg p$. Hence the term

$$\forall\Big(\neg p \to \big(q_1 \leftrightarrows (s_* \sqcap t_* \sqcap p \sqcap (\bigcirc s_0 \leftleftarrows \neg p))\big)\Big)$$

in conjunction with (78) ensures that $u = w^2$.

It follows that, for each elementary equation $g = h$ of the form (73) one can construct a term $\tau_{h,g}$ such that the following conditions are equivalent:

- $\tau_{g,h}$ is satisfiable in a model $\mathfrak{I} \in \mathbf{D}$;
- $\tau_{g,h}$ is satisfiable in a model based on $\mathbb{Z}$;
- $g = h$ is solvable in $\mathbb{N} \setminus \{0, 1\}$.

Uniform solvability of a set E of such equations is now equivalent to satisfiability of the conjunction of the terms $\tau_{g,h}$, $g = h \in E$ (just ensure that the list $\mathbf{q}$ representing a variable x is the same in each $\tau_{g,h}$). This proves Theorem 34.

7 Discussion and open problems

We have presented a modal logic framework which brings together modal logics for reasoning about topology and relative distances. The topological component of the modalities for relative distances has been pinpointed by introducing the modal operator ⓡ distinguishing between distances $d(x, X)$ that are the minima of $\{d(x, y) \mid y \in X\}$ and those that are not 'realised' by points in X. Here we briefly compare the resulting logics with the logics in the same language, but interpreted over distance spaces satisfying the *limit assumption* (1). As we have seen above, interpreted over models satisfying the limit assumption, the language $\mathcal{QML}$ has the same expressive power as the fragment of the language $\mathcal{CSL}$ with sole non-propositional operator $\leftleftarrows$; in this case the operator ⓡ does not add any expressive power to the language. Observe that none of the logics considered in this paper has the finite model property, whereas the corresponding logics of spaces with the limit assumption enjoy this property. More precisely, the following is shown in [30]:

Theorem 40. *Let $\mathcal{C}$ be the class of all models with the limit assumption satisfying any combination of the conditions 'symmetry' and 'triangle inequality,' in particular, neither of them. Then the satisfiability problem for $\mathcal{CSL}$-terms in $\mathcal{C}$ is* ExpTime*-complete. Moreover, a term is satisfiable in $\mathcal{C}$ iff it is satisfiable in a finite model from $\mathcal{C}$.*

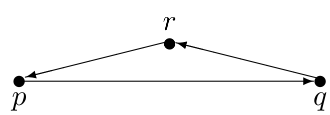
We have seen that $\mathcal{CSL}$ distinguishes between models with and without the triangle inequality, but not between arbitrary and symmetric models. When considering models with the limit assumption only, the situation changes drastically.

- Over models with the limit assumption, the language $\mathcal{CSL}$ cannot distinguish between models with and without the triangle inequality. To see this, let us suppose that τ is satisfied in a model $\mathfrak{I}$ with the limit assumption which does not satisfy the triangle inequality. Take any strictly monotonic function $f : \mathbb{R}^{\geq 0} \to (9, 10)$, where $(9, 10)$ is the open interval between 9 and 10. Define a new model $\mathfrak{I}'$ which differs from $\mathfrak{I}$ only in the distance function: $d^{\mathfrak{I}'}(x, y) = f(d^{\mathfrak{I}}(x, y))$ if $x \neq y$ and $d^{\mathfrak{I}'}(x, x) = 0$. Clearly, $\mathfrak{I}'$ satisfies the triangle inequality. It is easily checked that τ is satisfied in $\mathfrak{I}'$.

- On the other hand, restricted to models with the limit assumption $\mathcal{CSL}$, can distinguish between models with and without (sym). Consider, for example the term

  $$p \sqcap \forall\big[(p \to (q \leftleftarrows r)) \sqcap (q \to (r \leftleftarrows p)) \sqcap (r \to (p \leftleftarrows q))\big].$$

  One can readily check that it is satisfiable in a non-symmetric three-point model, say, in the one depicted below where the distance from x to y is the length of the shortest directed path from x to y.

  However, this term is not satisfiable in any symmetric model with the limit assumption.

The following interesting problems are still open:

- Find finite axiomatisations for logics of spaces satisfying the limit assumption. It is not difficult to see that by adding the axiom $\textcircled{r}\tau \leftrightarrow \exists\tau$ to our axiomatisation of the logic of all distance spaces, one obtains an axiom system for the logic of all distance spaces with the limit assumption. Moreover, by the observation above, this axiomatises the logic of all distance spaces with the limit assumption and the triangle inequality as well. It remains to consider the class of all symmetric distance spaces and all metric spaces with the limit assumption.

- We have considered the parameterised operators $\exists^{<x}$, $\exists^{\leq x}$, and their duals only. It would be desirable to understand as well the behaviour of logics which allow additional operators such as $\exists^{\leq x}_{>0}$ (corresponding to 'derived sets') or $\exists^{>x}$. We conjecture that deciding satisfiability for languages containing those additional operators is much harder than for $\mathcal{QML}$, for the following reason. Our decidability proofs rely on the fact that $\mathcal{QML}$ is determined by tree-like metric spaces and this fails to be the case for the extensions by new parameterised operators. It is well known that many decidability results for modal and monadic second-order logics heavily depend on the 'tree-model property' and fail for more general structures.

- It would also be of interest to characterise the expressive power of $\mathcal{QML}$ model-theoretically: a promising approach might be to introduce a notion of bisimulation between models based on distance spaces and show that $\mathcal{QML}$ is the bisimulation invariant fragment of the canonical two-sorted first-order logic for distance spaces (one sort ranges over the reals and the other over the elements of the distance space). Observe that proving such a result is unlikely to be a straightforward extension of the bisimulation characterisation of modal logic over Kripke models [12] because the parameterisation by reals makes it harder to apply saturation techniques.

Acknowledgements. The work on this paper was partially supported by the U.K. EPSRC research grants GR/S61966, GR/S61973, and EP/E034942.

References

[1] S. Artemov. Modal logic in mathematics. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, pages 927–969. Elsevier, 2007.

[2] B. Bennett. Modal logics for qualitative spatial reasoning. *Logic Journal of the IGPL*, 4:23–45, 1996.

[3] A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford University Press, 1997.

[4] E. Davis. The expressivity of quantifying over regions. *Journal of Logic and Computation*, 16:891–916, 2006.

[5] M. Davis. Unsolvable problems. In J. Barwise, editor, *Handbook of Mathematical Logic*, pages 567–594. North-Holland, Amsterdam, 1977.

[6] T. de Laguna. Point, line and surface as sets of points. *The Journal of Philosophy*, 19:449–461, 1922.

[7] J. Delgrande. Preliminary considerations on the modelling of belief change operators by metric spaces. In *Proceedings of NMR*, pages 118–125, 2004.

[8] M. Egenhofer and R. Franzosa. Point-set topological spatial relations. *International Journal of Geographical Information Systems*, 5:161–174, 1991.

[9] N. Friedman and J. Halpern. On the complexity of conditional logics. In J. Doyle, E. Sandewall, and P. Torasso, editors, *KR'94: Principles of Knowledge Representation and Reasoning*, pages 202–213. Morgan Kaufmann, San Francisco, California, 1994.

[10] M. Giritli. Logics for reasoning with comparative distances. (Manuscript), 2007.

[11] K. Gödel. Eine Interpretation des intuitionistischen Aussagenkalküls. *Ergebnisse eines mathematischen Kolloquiums*, 4:39–40, 1933.

[12] V. Goranko and M. Otto. Model theory of modal logics. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, pages 249–330. Elsevier, 2007.

[13] A. Grzegorczyk. Undecidability of some topological theories. *Fundamenta Mathematicae*, 38:137–152, 1951.

[14] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic Logic*. The MIT Press, 2000.

[15] D. Hilbert. *Grundlagen der Geometrie*. Leipzig, Teubner, 1899.

[16] R. Kontchakov, A. Kurucz, F. Wolter, and M. Zakharyaschev. Spatial logic + temporal logic = ? In M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors, *Handbook of Spatial Logics*, pages 497–564. Springer, 2007.

[17] C. Lewis and C. Langford. *Symbolic Logic*. Appleton-Century-Crofts, New York, 1932.

[18] D. Lewis. *Counterfactuals*. Blackwell, Oxford, 1973.

[19] Yu. V. Matiyasevich. Enumerable sets are Diophantine. *Soviet Mathematics Doklady*, 11:354–358, 1970.

[20] J.C.C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[21] J.J. Meyer and F. Veltman. Intelligent agents and common sense reasoning. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, pages 991–1029. Elsevier, 2007.

[22] I. Orlov. The calculus of compatibility of propositions. *Mathematics of the USSR, Sbornik*, 35:263–286, 1928. (In Russian).

[23] I. Pratt-Hartmann. A topological constraint language with component counting. *Journal of Applied Non-Classical Logics*, 12(3–4):441–467, 2002.

[24] D. Randell, Z. Cui, and A. Cohn. A spatial logic based on regions and connection. In B. Nebel, C. Rich, and W. Swartout, editors, *Proceedings of the 3rd International Conference on Principles of Knowledge Representation and Reasoning (KR'92)*, pages 165–176. Morgan Kaufmann, 1992.

[25] D. Randell, M. Witkowski, and M. Shanahan. From images to bodies: Modelling and exploiting spatial occlusion and motion parallax. In B. Nebel, editor, *Proceedings of the Seventeenth International Joint Conference on Artificial Intelligence (IJCAI 2001)*, pages 57–66. Morgan Kaufmann, 2001.

[26] J. Renz and B. Nebel. Efficient methods for qualitative spatial reasoning. *Journal of Artificial Intelligence Research (JAIR)*, 15:289–318, 2001.

[27] K. Schlechta. *Coherent Systems*. Elsevier, 2004.

[28] W. Schwabhuser, W. Szmielew, and A. Tarski. *Metamathematische Methoden in der Geometrie*. Springer, Berlin, 1983.

[29] V. Shehtman. "Everywhere" and "Here". *Journal of Applied Non-Classical Logics*, 9(2–3), 1999.

[30] M. Sheremet, D. Tishkovsky, F. Wolter, and M. Zakharyaschev. Comparative similarity, tree automata, and Diophantine equations. In G. Sutcliffe and A. Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, 12th International Conference (LPAR 2005)*, volume 3835 of *Lecture Notes in Computer Science*, pages 651–665. Springer, 2005.

[31] E. Spaan. *Complexity of Modal Logics*. PhD thesis, University of Amsterdam, 1993.

[32] A. Tarski. *A Decision Method for Elementary Algebra and Geometry*. University of California Press, 1951. Previous version published as a technical report by the RAND Corporation, 1948; prepared for publication by J.C.C. McKinsey.

[33] J. van Benthem. *The Logic of Time: A Model-Theoretic Investigation into the Varieties of Temporal Ontology and Temporal Discourse*. Reidel, Dordrecht, 1983.

[34] J. van Benthem and G. Bezhanishvili. Modal logics of space. In M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors, *Handbook of Spatial Logics*, pages 217–298. Springer, 2007.

[35] F. Veltman. *Logics of Conditionals*. PhD thesis, University of Amsterdam, 1985.

[36] T. Wilke. $\mathcal{CTL}^+$ is exponentially more succinct than $\mathcal{CTL}$. In *Proceedings of the 19th Conference on Foundations of Software Technology and Theoretical Computer Science*, pages 110–121, London, UK, 1999. Springer.

[37] T. Williamson. First-order logics for comparative similarity. *Notre Dame Journal of Formal Logic*, 29:457–481, 1988.

[38] F. Wolter and M. Zakharyaschev. Spatial reasoning in RCC-8 with Boolean region terms. In W. Horn, editor, *Proceedings of the 14th European Conference on Artificial Intelligence (ECAI 2000)*, pages 244–248. IOS Press, 2000.