$\mathbf{Alt}_n$ in a Strictly Positive Context

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A *strictly positive term* (or *$\mathcal{SP}$-term*) is a modal formula constructed from propositional variables $p_0, p_1, \dots$, constants $\top$ and $\bot$, conjunction $\wedge$, and the unary diamond operator $\Diamond$. An *$\mathcal{SP}$-implication* takes the form $\sigma \to \tau$, where σ, τ are $\mathcal{SP}$-terms, and an *$\mathcal{SP}$-logic* is a set of $\mathcal{SP}$-implications. (An $\mathcal{SP}$-implication $\sigma \to \tau$ can be regarded as an algebraic equation $\sigma \wedge \tau \equiv \sigma$, while $\sigma \equiv \tau$ as a shorthand for '$\sigma \to \tau$ and $\tau \to \sigma$'.) In various contexts, $\mathcal{SP}$-logics were investigated in [3, 7, 2, 1, 8, 6, 5, 4].

We consider two consequence relations. For an $\mathcal{SP}$-logic $\mathcal{L}$ and $\mathcal{SP}$-implication φ, we write $\mathcal{L} \models_{\mathsf{Kr}} \varphi$ if φ is valid in all Kripke frames for $\mathcal{L}$, and we write $\mathcal{L} \models_{\mathsf{SLO}} \varphi$ if φ is valid in all *bounded meet-semilattices with normal monotone operators* (or SLOs) that validate $\mathcal{L}$. We call $\mathcal{L}$ (*Kripke*) *complete* in case $\mathcal{L} \models_{\mathsf{Kr}} \varphi$ iff $\mathcal{L} \models_{\mathsf{SLO}} \varphi$, for all φ. Since $\mathcal{SP}$-implications are Sahlqvist formulas, $\mathcal{L} \models_{\mathsf{Kr}} \varphi$ iff $\mathcal{L} \models_{\mathsf{BAO}} \varphi$, where BAO stands for Boolean algebras with operators. Thus, completeness is equivalent to (purely algebraic) *conservativity* of $\models_{\mathsf{BAO}}$ over $\models_{\mathsf{SLO}}$. Completeness of an $\mathcal{SP}$-logic $\mathcal{L}$ also means that its $\mathcal{SP}$-implications *axiomatise* the *$\mathcal{SP}$-fragment* of $\mathcal{L}$ regarded as a standard modal logic. A simple example of an incomplete $\mathcal{SP}$-logic is $\mathcal{L} = \{\Diamond p \to p\}$; indeed, for $\varphi = (p \wedge \Diamond\top \to \Diamond p)$, we have $\mathcal{L} \models_{\mathsf{Kr}} \varphi$ and $\mathcal{L} \not\models_{\mathsf{SLO}} \varphi$.

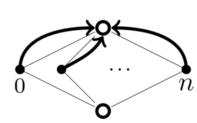
A classical method of showing completeness of a modal logic $\mathcal{L}$ is to prove its canonicity, which can be done by establishing that every BAO for $\mathcal{L}$ is embeddable into the full complex BAO $\mathfrak{F}^+$ of some Kripke frame $\mathfrak{F}$ for $\mathcal{L}$. We call an $\mathcal{SP}$-theory $\mathcal{L}$ *complex* if every SLO for $\mathcal{L}$ is embeddable into the SLO-type reduct of $\mathfrak{F}^+$ of some Kripke frame $\mathfrak{F}$ for $\mathcal{L}$. Examples of complex, and so complete $\mathcal{SP}$-logics include $\{p \to \Diamond p\}$ (reflexivity), $\{\Diamond\Diamond p \to \Diamond p\}$ (transitivity), $\{q \wedge \Diamond p \to \Diamond(p \wedge \Diamond q)\}$ (symmetry), $\{\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)\}$ (functionality), and their unions. By Sahlqvist's theorem, all $\mathcal{SP}$-logics have first-order correspondents. A number of general results linking complexity of $\mathcal{SP}$-logics to the form of their correspondents have been obtained in [4].

On the other hand, there are many $\mathcal{SP}$-logics that define standard frame properties, but are not complex. In this note, we aim to develop a new method for proving completeness of such logics. First, we axiomatise the $\mathcal{SP}$-fragment of the (Kripke complete) modal logic $\mathbf{Alt}_n$ whose Kripke frames are *n-functional*, i.e., satisfy $\forall x, y_0, \dots, y_n \, \big(\bigwedge_{i \le n} R(x, y_i) \to \bigvee_{i \ne j}(y_i = y_j)\big)$. We set $\mathbf{Alt}_n^+ = \{\varphi^n_{fun}\}$, where $P = \{p_0, \dots, p_n\}$ and

$$\varphi^n_{fun} = \Big(\bigwedge_{Q \subseteq P,\, |Q| = n} \Diamond \bigwedge Q \;\to\; \Diamond \bigwedge P \Big).$$

Note that Kripke frames for φ^n_{fun} are exactly n-functional frames. Here we sketch the proof of

Theorem 1. *For any $n \ge 1$, the $\mathcal{SP}$-logic $\mathbf{Alt}_n^+$ is complete, though not complex if $n \ge 2$.*



To prove that $\mathbf{Alt}_n^+$ ($n \ge 2$) is not complex, one can show that the SLO on the right (where $\Diamond\top = \top$, $\Diamond\bot = \bot$, and the arrows define $\Diamond$ in other cases) validates φ^n_{fun} but is not embeddable into $\mathfrak{F}^+$, for any n-functional $\mathfrak{F}$.

To show completeness, we require *n-terms* that are defined by induction: (i) all propositional variables, $\bot$ and $\top$ are n-terms; (ii) if $\tau_1, \dots, \tau_n$ are n-terms, then so is $\Diamond(\tau_1 \wedge \dots \wedge \tau_n)$.

Lemma 2. *For any $\mathcal{SP}$-term ϱ, there is conjunction ϱ' of n-terms with $\mathbf{Alt}_n^+ \models_{\mathsf{SLO}} (\varrho \equiv \varrho')$.*

The proof is by induction on the modal depth d of ϱ. The basis $d = 0$ is trivial. Suppose now that ϱ is of depth $d > 0$. Then $\varrho = \bigwedge P_\varrho \wedge \Diamond\varrho_1 \wedge \dots \wedge \Diamond\varrho_k$, where P_ϱ is a set of propositional variables, $\bot$ and $\top$, and each ϱ_i is of depth $\leq d-1$. By IH, $\mathbf{Alt}_n^+ \models_{\mathsf{SLO}} (\varrho_i \equiv \bigwedge A_i)$, for some set A_i of n-terms. Then $\mathbf{Alt}_n^+ \models_{\mathsf{SLO}} \big(\varrho \equiv (\bigwedge P_\varrho \wedge \bigwedge_{i=1}^k \Diamond \bigwedge A_i)\big)$. If $|A_i| \leq n$, then we are done. So fix some i and suppose that $|A_i| = k > n$. Then we always have $\models_{\mathsf{SLO}} \big((\Diamond \bigwedge A_i) \to (\bigwedge_{Q \subseteq A_i,\, |Q|=n} \Diamond \bigwedge Q)\big)$. We show that

$$\mathbf{Alt}_n^+ \models_{\mathsf{SLO}} \big(\bigwedge_{Q \subseteq A_i, |Q|=n} \Diamond \bigwedge Q \;\to\; \Diamond \bigwedge A_i \big). \tag{1}$$

Indeed, by a syntactic argument, we have $\mathbf{Alt}_n^+ \models_{\mathsf{SLO}} \varphi_{\mathit{fun}}^m$, for every $m > n$, from which we obtain (1) as a substitution instance of φ_{fun}^k.

Lemma 3. *For any $\mathcal{SP}$-term σ and any n-term τ, $\mathbf{Alt}_n^+ \models_{\mathsf{Kr}} \sigma \to \tau$ implies $\models_{\mathsf{Kr}} \sigma \to \tau$.*

The proof is by induction on the modal depth d of τ. The basis is again trivial. Now assume inductively that the lemma holds for d and the depth of τ is $d+1$. Let $\sigma = \bigwedge P_\sigma \wedge \Diamond\sigma_1 \wedge \ldots \wedge \Diamond\sigma_k$, where P_σ is some set of propositional variables, $\bot$, $\top$, and each σ_i is an $\mathcal{SP}$-term. Suppose $\tau = \Diamond(\tau_1 \wedge \ldots \wedge \tau_n)$, where each τ_i is either a variable, $\top$, $\bot$, or of the form $\Diamond(\tau_1^i \wedge \cdots \wedge \tau_n^i)$.

Suppose $\not\models_{\mathsf{Kr}} \sigma \to \tau$. Then, for every j $(1 \leq j \leq k)$, there is i $(1 \leq i \leq n)$ such that $\not\models_{\mathsf{Kr}} \sigma_j \to \tau_i$, and so $\bigcup_{i=1}^n K_i = \{1, \ldots, k\}$, for $K_i = \{1 \leq j \leq k \mid \not\models_{\mathsf{SLO}} \sigma_j \to \tau_i\}$. It is not hard to see that, for any i with $K_i \neq \emptyset$, we have $\not\models_{\mathsf{Kr}} (\bigwedge_{j \in K_i} \sigma_j) \to \tau_i$. By IH, for any such i, there is a Kripke model $\mathfrak{M}_i$ based on an n-functional frame with root r_i where $\bigwedge_{j \in K_i} \sigma_j$ holds, but τ_i does not. Now take a fresh node r, make $\bigwedge P_\sigma$ true in r, and connect r to r_i of each $\mathfrak{M}_i$. The constructed model is based on an n-functional frame and refutes $\sigma \to \tau$ at r, showing that $\mathbf{Alt}_n^+ \not\models_{\mathsf{Kr}} \sigma \to \tau$ as required. That $\mathbf{Alt}_n^+$ is complete follows now from Lemmas 2, 3 and the completeness of the empty $\mathcal{SP}$-logic [7].

Using a similar (but more involved) technique, we can also show (see [4] for details) that the $\mathcal{SP}$-logic $\mathbf{S4.3}^+ = \{p \to \Diamond p, \Diamond\Diamond p \to \Diamond p, \Diamond(p \wedge q) \wedge \Diamond(p \wedge r) \to \Diamond(p \wedge \Diamond q \wedge \Diamond r)\}$ is complete, has exactly the same frames as **S4.3**, and is decidable in polynomial time. However, this does not generalise to **K4.3** whose class of Kripke frames is not $\mathcal{SP}$-definable [4]. Svyatlovski has recently shown that the $\mathcal{SP}$-logic $\mathcal{L}_s = \{\Diamond\Diamond p \to \Diamond p,\ \Diamond(p \wedge \Diamond q) \wedge \Diamond(p \wedge \Diamond r) \to \Diamond(p \wedge \Diamond q \wedge \Diamond r)\}$ is complete, tractable, and, for any $\mathcal{SP}$-implication φ, we have $\mathcal{L}_s \models \varphi$ iff φ is valid in all frames for **K4.3** (although $\mathcal{L}_s$ has non-**K4.3** frames).

References

[1] L. Beklemishev. Positive provability logic for uniform reflection principles. *Ann. Pure Appl. Logic*, 165:82–105, 2014.

[2] E. Dashkov. On the positive fragment of the polymodal provability logic **GLP**. *Mathematical Notes*, 91:318–333, 2012.

[3] M. Jackson. Semilattices with closure. *Algebra Universalis*, 52:1–37, 2004.

[4] S. Kikot, A. Kurucz, Y. Tanaka, F. Wolter, and M. Zakharyaschev. Kripke completeness of strictly positive modal logics over meet-semilattices with operators. Submitted, 2017.

[5] A. Kurucz, Y. Tanaka, F. Wolter, and M. Zakharyaschev. Conservativity of boolean algebras with operators over semilattices with operators. In *Proceeding of TACL 2011*. Marseille, 2011.

[6] A. Kurucz, F. Wolter, and M. Zakharyaschev. Islands of tractability for relational constraints: towards dichotomy results for the description logic $\mathcal{EL}$. In *AiML 8*. College Publications, 2010.

[7] V. Sofronie-Stokkermans. Locality and subsumption testing in $\mathcal{EL}$ and some of its extensions. In *AiML 7*. College Publications, 2008.

[8] M. Svyatlovski. Positive fragments of modal logics. BSc Thesis (in Russian), 2014.