Comparative similarity, tree automata, and Diophantine equations

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Abstract. The notion of comparative similarity 'X is more similar or closer to Y than to Z' has been investigated in both foundational and applied areas of knowledge representation and reasoning, e.g., in concept formation, similarity-based reasoning and areas of bioinformatics such as protein sequence alignment. In this paper we analyse the computational behaviour of the 'propositional' logic with the binary operator 'closer to a set τ_1 than to a set τ_2' and nominals interpreted over various classes of distance (or similarity) spaces. In particular, using a reduction to the emptiness problem for certain tree automata, we show that the satisfiability problem for this logic is ExpTime-complete for the classes of all finite symmetric and all finite (possibly non-symmetric) distance spaces. For finite subspaces of the real line (and higher dimensional Euclidean spaces) we prove the undecidability of satisfiability by a reduction of the solvability problem for Diophantine equations. As our 'closer' operator has the same expressive power as the standard operator $>$ of conditional logic, these results may have interesting implications for conditional logic as well.

1 Introduction

There are two main approaches to defining and classifying concepts in computer science and artificial intelligence. One of them is *logic based*. It uses formalisms like description logics to define concepts by establishing relationships between them, for example,

$$\mathsf{Mother} \quad \equiv \quad \mathsf{Woman} \sqcap \exists \mathsf{hasChild.Person}$$

The main tool for analysing and using such definitions (e.g., to compute the concept hierarchy based on the subsumption relation) is *reasoning*.

Another approach is based on *similarity*.[1] Using various techniques (such as alignment algorithms) we compute suitable similarity measures on (part of) the application domain and then define concepts in terms of similarity, for example,

$$\mathsf{Reddish} \quad \equiv \quad \{Red\} \leftleftarrows \{Green, \ldots, Black\}$$

which reads 'a colour is reddish iff it is more similar (with respect to the RGB, HSL or some other explicit or implicit colour model) to the prototypical colour *Red* than to the prototypical colours *Green*, ..., *Black*.' The established tools for dealing with concepts of this sort are *numerical computations* (say, with the help of Voronoi tessellations, nearest neighbour or clustering algorithms).

As more and more application areas—like bioinformatics and linguistics—use both of these ways of defining concepts, we are facing the problem of integrating them. In particular, we need formalisms that are capable of *reasoning* about concepts defined in terms of (explicit or implicit) similarity in the same way as this is done in description logic (DL).

In [6, 16, 8, 17] we presented and investigated rudimentary DL-like formalisms for reasoning about concepts and similarity with concept constructors of the form $\exists^{<a}\tau$, that is, 'in the a-neighbourhood of τ,' where $a \in \mathbb{Q}^{\geq 0}$. The apparent limitation of these languages is that they can only operate with *concrete* degrees of similarity $a \in \mathbb{Q}^{\geq 0}$, and so require substantial expert knowledge in order to define concepts.

In this paper we propose a purely *qualitative* logic $\mathcal{CSL}$ for knowledge representation and reasoning about *comparative similarity*. Its main ingredients are the binary *closer* operator $\leftleftarrows$ as in the example above and individual constants (nominals) for representing prototypical objects (we refer the reader to [7, 15] for a discussion of relations like 'X more similar to Y than to Z'). The logic is interpreted in various natural classes of distance (or similarity) spaces such as finite metric spaces, finite metric spaces without symmetry (see, e.g., [13] for an argumentation that similarity measures are not necessarily symmetric) as well as the finite subspaces of the Euclidean space $\mathbb{R}^n$, $n \geq 1$ (natural similarity measures for weight, length, etc.).

The computational behaviour of $\mathcal{CSL}$ over the class of finite metric spaces (with or without symmetry) turns out be similar to the behaviour of standard description logics: the satisfiability problem is ExpTime-complete which can be established by a reduction to the emptiness problem for certain tree automata. However, it was a great surprise for us to discover that over finite subspaces of the real line $\mathbb{R}$ (as well as any higher dimensional Euclidean space or any $\mathbb{Z}^n$) the logic turns out to be undecidable. This result is proved by a reduction of the solvability problem for Diophantine equations.

[1] "There is nothing more basic to thought and language than our sense of similarity; our sorting of things into kinds." Quine (1969)

2 The logic of comparative similarity

The *logic* $\mathcal{CSL}$ *of comparative similarity* we consider in this paper is based on the following language:

$$\tau \quad ::= \quad p_i \quad | \quad \{\ell_i\} \quad | \quad \neg\tau \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \tau_1 \leftleftarrows \tau_2$$

where the p_i are *atomic terms*, the ℓ_i are *object names*, and $\leftleftarrows$ is the *closer operator*. We call $\{\ell_i\}$ a *nominal* and τ a $\mathcal{CSL}$*-term* or simply a *term*.

The intended models for $\mathcal{CSL}$ are based on *distance* (or rather *similarity*) *spaces* $\mathfrak{D} = (\Delta, d)$, where Δ is a nonempty set and d is a map from $\Delta \times \Delta$ to the set $\mathbb{R}^{\geq 0}$ of nonnegative real numbers such that, for all $x, y \in \Delta$, we have $d(x, y) = 0$ iff $x = y$. If the *distance function* d satisfies two additional properties

$$d(x,y) \;=\; d(y,x) \qquad \textbf{(sym)}$$
$$d(x,z) \;\leq\; d(x,y) + d(y,z) \qquad \textbf{(tr)}$$

then $\mathfrak{D}$ is a standard *metric space*. The *distance* $d(X, Y)$ between two nonempty sets X and Y of Δ is defined by taking

$$d(X,Y) \;=\; \inf\{d(x,y) \mid x \in X,\ y \in Y\}.$$

If one of X, Y is empty then $d(X, Y) = \infty$. Finally, if we actually have

$$d(X,Y) \;=\; \min\{d(x,y) \mid x \in X,\ y \in Y\} \qquad \textbf{(min)}$$

for any nonempty X and Y, then the distance space $\mathfrak{D}$ is called a *min-space*. Every finite distance space is clearly a min-space.

$\mathcal{CSL}$*-models* are structures of the form

$$\mathfrak{I} \;=\; \left(\Delta^{\mathfrak{I}}, d^{\mathfrak{I}}, \ell_1^{\mathfrak{I}}, \ell_2^{\mathfrak{I}}, \dots, p_1^{\mathfrak{I}}, p_2^{\mathfrak{I}}, \dots\right), \qquad (1)$$

where $\left(\Delta^{\mathfrak{I}}, d^{\mathfrak{I}}\right)$ is a distance space, the $p_i^{\mathfrak{I}}$ are subsets of $\Delta^{\mathfrak{I}}$, and $\ell_i^{\mathfrak{I}} \in \Delta^{\mathfrak{I}}$ for every i. We call such models *min-models*, *symmetric* or satisfying the *triangle inequality* if the underlying distance space satisfies (min), (sym) or (tr), respectively. If both (sym) and (tr) are satisfied then $\mathfrak{I}$ is called a *metric* $\mathcal{CSL}$*-model*.

The interpretation of the Boolean operators $\neg$ and $\sqcap$ in $\mathfrak{I}$ is as usual (we will use $\sqcup$, $\rightarrow$, $\bot$ (for $\emptyset$), and $\top$ (for the whole space) as standard abbreviations), $\{\ell\}^{\mathfrak{I}} = \{\ell^{\mathfrak{I}}\}$, and

$$(\tau_1 \leftleftarrows \tau_2)^{\mathfrak{I}} \;=\; \{x \in \Delta^{\mathfrak{I}} \mid d^{\mathfrak{I}}(x, \tau_1^{\mathfrak{I}}) < d^{\mathfrak{I}}(x, \tau_2^{\mathfrak{I}})\}. \qquad (2)$$

A term τ is *satisfied* in $\mathfrak{I}$ if $\tau^{\mathfrak{I}} \neq \emptyset$; τ is *satisfiable* in a class $\mathcal{C}$ of models if it is satisfied in some model from $\mathcal{C}$. Finally, τ is *valid* in $\mathfrak{I}$ if $\tau^{\mathfrak{I}} = \Delta^{\mathfrak{I}}$.

The seemingly simple logic $\mathcal{CSL}$ turns out to be quite expressive. First, the operator $\exists\tau = (\tau \leftleftarrows \bot)$ is interpreted by the *existential modality* (in the sense that $\exists\tau$ is the whole space iff τ is not empty); its dual, the *universal modality*, will be denoted by $\forall$. Thus the term $\forall(\tau_1 \to \tau_2)$ expresses in $\mathcal{CSL}$ the *subsumption relation* $\tau_1 \sqsubseteq \tau_2$ which is usually used in description logic knowledge bases.

Second, in metric models the operator $\Box$ defined by taking $\Box\tau = (\top \leftleftarrows \neg\tau)$ is actually interpreted by the *interior operator* of the induced topology. Thus, $\mathcal{CSL}$ contains the full logic $\mathbf{S4}_u$ of topological spaces, and so can be used for spatial representation and reasoning (see, e.g., [1]). The topological aspects of $\mathcal{CSL}$ will be considered elsewhere.

Finally, it is to be noted that the operator $\leftleftarrows$ is closely related to the 'implication' $>$ of conditional logics. According to Lewis' [7] semantics for conditionals, propositions are interpreted in a set W of possible worlds that come together with orderings $\preceq_w \subseteq W \times W$, for $w \in W$, which can be understood as follows: $w' \preceq_w w''$ if w' is more similar or closer to w than w''. A formula $\varphi > \psi$ is true at w iff, for every $\preceq_w$-minimal v with $v \models \varphi$, we have $v \models \psi$. Various authors (see, for example, [3, 10]) have considered the case where the relations $\preceq_w$ are induced by min-spaces (Δ, d) (in conditional logic, the requirement (min) is often called the *limit assumption*) by setting

$$w' \preceq_w w'' \quad \text{iff} \quad d(w, w') \leq d(w, w'').$$

Under this interpretation the operators $\leftleftarrows$ and $>$ have exactly the same expressive power: for every min-model $\mathfrak{I} = \left(\Delta^{\mathfrak{I}}, d^{\mathfrak{I}}, p_1^{\mathfrak{I}}, p_2^{\mathfrak{I}}, \ldots\right)$ we have

$$(p_1 > p_2)^{\mathfrak{I}} \;=\; \left((p_1 \leftleftarrows (p_1 \sqcap \neg p_2)) \sqcup \forall\neg p_1\right)^{\mathfrak{I}}$$

and, conversely,

$$(p_1 \leftleftarrows p_2)^{\mathfrak{I}} \;=\; \left(((p_1 \sqcup p_2) > p_1) \sqcap (p_1 > \neg p_2) \sqcap \neg(p_1 > \bot)\right)^{\mathfrak{I}}.$$

Relations $\preceq_w$ induced by *symmetric* distance spaces have not been considered in the conditional logic literature. According to the classification of [5], our (nominal-free) logic of arbitrary min-spaces corresponds to the conditional logic of frames satisfying the normality, reflexivity, strict centering, uniformity and connectedness conditions.

3 Main results

In this paper, our main concern is the computational behaviour of $\mathcal{CSL}$ over natural classes of *min-models*, in particular, *finite* models.

Theorem 1. *Let $\mathcal{C}$ be the class of all min-models satisfying any combination of the properties (sym) and (tr), in particular, neither of them. Then the satisfiability problem for $\mathcal{CSL}$-terms in $\mathcal{C}$ is ExpTime-complete. Moreover, a term is satisfiable in $\mathcal{C}$ iff it is satisfiable in a finite model from $\mathcal{C}$.*

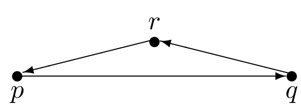
Remark 1. For the nominal-free fragment of $\mathcal{CSL}$ over arbitrary min-models, Theorem 1 was essentially proved in [5] in the framework of conditional logic. We provide a new proof here because it serves as a preparation for the much more sophisticated proof for the class of *symmetric* min-models.

Remark 2. It is to be noted that in fact the language of $\mathcal{CSL}$ cannot distinguish between models with and without (tr). To see this, let us suppose that τ is satisfied in a model $\mathfrak{I}$ of the form (1) which does not satisfy (tr). Take any strictly monotonic function $f : \mathbb{R}^{\geq 0} \to (9, 10)$, where $(9, 10)$ is the open interval between 9 and 10. Define a new model $\mathfrak{I}'$ which differs from $\mathfrak{I}$ only in the distance function: $d^{\mathfrak{I}'}(x, y) = f(d^{\mathfrak{I}}(x, y))$ for all $x \neq y$ and $d^{\mathfrak{I}'}(x, x) = 0$ for all x. Clearly, $\mathfrak{I}'$ satisfies the triangle inequality. It is easily checked that τ is satisfied in $\mathfrak{I}'$.

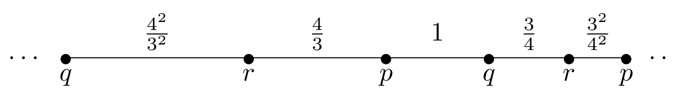
Remark 3. On the other hand, $\mathcal{CSL}$ can distinguish between models with and without (sym). Consider, for example the term

$$p \sqcap \forall\big[(p \to (q \leftleftarrows r)) \sqcap (q \to (r \leftleftarrows p)) \sqcap (r \to (p \leftleftarrows q))\big]$$

One can readily check that it is satisfiable in a three-point model without (sym), say, in the one depicted below where the distance from x to y is the length of the shortest directed path from x to y.



However, this term is not satisfiable in any symmetric min-model. On the other hand, it can be satisfied in the following subspace of $\mathbb{R}$ which is not a min-space:



Our second main result is quite surprising: $\mathcal{CSL}$ turns out to be undecidable when interpreted in finite subspaces of $\mathbb{R}$. More precisely, we are going to prove the following:

Theorem 2. *For each $n \geq 1$, the satisfiability problem for $\mathcal{CSL}$-terms is undecidable in the following classes of models:*

1. *models based on finite subspaces of $\mathbb{R}^n$,*
2. *models based on finite subspaces of $\mathbb{Z}^n$,*
3. *models based on min-subspaces of $\mathbb{R}^n$,*
4. *models based on min-subspaces of $\mathbb{Z}^n$.*

Theorem 1 will be proved in the next section: for the lower bound we use a reduction of the global consequence relation for the modal logic **K**, while the upper bound is established by reduction to the emptiness problem for tree automata. Theorem 2 is proved in Section 5 by reduction of the solvability problem for Diophantine equations (and, for $n > 1$, the $\mathbb{Z} \times \mathbb{Z}$ tiling problem).

4 Proof of Theorem 1

We begin by establishing the lower ExpTime bound. The proof is by reduction of the global **K**-consequence relation that is known to be ExpTime-hard [11]. We remind the reader that the language $\mathcal{L}_{\mathbf{K}}$ of modal logic **K** extends propositional logic (with propositional variables $p_1, p_2, \dots$) by means of one unary operator $\Diamond$. $\mathcal{L}_{\mathbf{K}}$ is interpreted in models of the form

$$\mathfrak{N} \;=\; \big(W, R, p_1^{\mathfrak{N}}, p_2^{\mathfrak{N}}, \dots\big), \tag{3}$$

where W is a nonempty set, $S \subseteq W \times W$ and $p_i^{\mathfrak{N}} \subseteq W$. The *value* $\varphi^{\mathfrak{N}} \subseteq W$ of an $\mathcal{L}_{\mathbf{K}}$-formula φ in $\mathfrak{N}$ is defined inductively as follows:

- $(\varphi \wedge \psi)^{\mathfrak{N}} = \varphi^{\mathfrak{N}} \cap \psi^{\mathfrak{N}}$;
- $(\neg\varphi)^{\mathfrak{N}} = W \setminus \varphi^{\mathfrak{N}}$;
- $(\Diamond\varphi)^{\mathfrak{N}} = \{v \in W \mid \exists w\ (vRw \wedge w \in \varphi^{\mathfrak{N}})\}$.

Say that φ_1 *follows globally from* φ_2 and write $\varphi_2 \vdash \varphi_1$ if, for every model $\mathfrak{N}$, $\varphi_1^{\mathfrak{N}} = W$ whenever $\varphi_2^{\mathfrak{N}} = W$. The problem of deciding whether $\varphi_2 \vdash \varphi_1$ holds is ExpTime-hard [11].

Lemma 1. *The satisfiability problem for (nominal-free) $\mathcal{CSL}$-terms in any class $\mathcal{C}$ of models mentioned in Theorem 1 is ExpTime-hard.*

Proof. We define inductively a translation $^\#$ from $\mathcal{L}_{\mathbf{K}}$ into the set of $\mathcal{CSL}$-terms. Let $\kappa_0 = q_0$, $\kappa_1 = \neg q_0 \sqcap q_1$, $\kappa_2 = \neg q_0 \sqcap \neg q_1$, for some fresh variables q_i. Then we set $p_i^\# = p_i$, $(\neg\varphi)^\# = \neg\varphi^\#$, $(\varphi_1 \wedge \varphi_2)^\# = \varphi_1^\# \sqcap \varphi_2^\#$, and

$$(\Diamond\varphi)^\# \;=\; \bigsqcup_{i<3} \Big(\kappa_i \sqcap \exists\kappa_{i\oplus 1} \sqcap \big((\kappa_{i\oplus 1} \sqcap \varphi^\#) \leftrightarrows \kappa_{i\oplus 1}\big)\Big),$$

where $\oplus$ is addition modulo 3 and $\leftrightarrows$ means 'at the same distance,' i.e.,

$$\tau_1 \leftrightarrows \tau_2 \;=\; \neg(\tau_1 \leftleftarrows \tau_2) \sqcap \neg(\tau_2 \leftleftarrows \tau_1). \tag{4}$$

We show now that, for any $\varphi, \psi \in \mathcal{L}_{\mathbf{K}}$,

$$\varphi \vdash \psi \qquad \text{iff} \qquad \forall\varphi^\# \to \psi^\# \quad \text{is valid in } \mathcal{C}.$$

Suppose first that $\varphi \not\vdash \psi$. This means that there is a **K**-model $\mathfrak{N}$ of the form (3) such that $\varphi^{\mathfrak{N}} = W$ and $r \notin \psi^{\mathfrak{N}}$ for some $r \in W$. As is well-known from modal logic, without loss of generality we may assume that (W, R) is an irreflexive intransitive tree with root r. Consider the tree metric model

$$\mathfrak{I} \;=\; \big(W, d, p_1^{\mathfrak{N}}, p_2^{\mathfrak{N}}, \dots, q_0^{\mathfrak{I}}, q_1^{\mathfrak{I}}\big),$$

where d is the standard tree metric on (W, R) (that is, $d(u, v) = d(v, u)$ is the length of the shortest *undirected* path from u to v in (W, R)) and $q_0^{\mathfrak{I}}$ and $q_1^{\mathfrak{I}}$ consist of all points $u \in W$ such that $d(u, r) = 3n$ for some $n \in \mathbb{N}$ and

$d(u,r) = 3n+1$ for some $n \in \mathbb{N}$, respectively. Clearly, $\mathfrak{I}$ is a min-model. Now, it is easily checked by induction that, for every formula $\chi \in \mathcal{L}_{\mathbf{K}}$, we have $\chi^{\mathfrak{N}} = (\chi^{\#})^{\mathfrak{I}}$, whence $(\varphi^{\#})^{\mathfrak{I}} = W$ and $r \notin (\psi^{\#})^{\mathfrak{I}}$, and so $\forall\varphi^{\#} \to \psi^{\#}$ is not valid in $\mathcal{C}$.

Conversely, suppose that $\forall\varphi^{\#} \sqcap \neg\psi^{\#}$ is satisfied in some model

$$\mathfrak{I} \;=\; \bigl(\Delta^{\mathfrak{I}}, d^{\mathfrak{I}}, p_1^{\mathfrak{I}}, p_2^{\mathfrak{I}}, \dots, q_0^{\mathfrak{I}}, q_1^{\mathfrak{I}}\bigr)$$

from $\mathcal{C}$. Consider a $\mathbf{K}$-model

$$\mathfrak{N} \;=\; \bigl(\Delta^{\mathfrak{I}}, R^{\mathfrak{I}}, p_1^{\mathfrak{I}}, p_2^{\mathfrak{I}}, \dots\bigr)$$

where $uR^{\mathfrak{I}}v$ for $u, v \in \Delta^{\mathfrak{I}}$ iff, for some $i \leq 2$,

$$u \in \kappa_i^{\mathfrak{I}}, \qquad v \in \kappa_{i\oplus 1}^{\mathfrak{I}}, \qquad \text{and} \qquad d^{\mathfrak{I}}(u,v) = d^{\mathfrak{I}}(u, \kappa_{i\oplus 1}^{\mathfrak{I}}).$$

Again, it is easily checked by induction that, for every formula $\chi \in \mathcal{L}_{\mathbf{K}}$, we have $\chi^{\mathfrak{N}} = (\chi^{\#})^{\mathfrak{I}}$. It follows that $\varphi \not\models \psi$.

Our next task is to prove the ExpTime upper bound and the finite model property with respect to the given class $\mathcal{C}$ of models. We will require a number of definitions.

Given a term τ, denote by $cl\tau$ the closure under single negation of the set consisting of all subterms of τ, the term $\bot$, and the term $\exists\rho = \rho \leftleftarrows \bot$ for every subterm ρ of τ. A *type* t for τ is a subset of $cl\tau$ such that $\bot \notin t$ and the following Boolean closure conditions are satisfied:

- $\tau_1 \sqcap \tau_2 \in t$ iff $\tau_1, \tau_2 \in t$, for every $\tau_1 \sqcap \tau_2 \in cl\tau$,
- $\neg\rho \in t$ iff $\rho \notin t$, for every $\neg\rho \in cl\tau$.

Clearly, $|cl\tau|$ is a linear function of the *length* $|\tau|$ (say, the number of subterms) of τ.

A 'typical' type is given by an element $w \in \Delta^{\mathfrak{I}}$ from a model $\mathfrak{I}$ of the form (1), namely,

$$t^{\mathfrak{I}}(w) \;=\; \{\rho \in cl\tau \mid w \in \rho^{\mathfrak{I}}\}.$$

A τ-*bouquet* is a pair $\mathfrak{B} = (T_{\mathfrak{B}}, \leq_{\mathfrak{B}})$, where $T_{\mathfrak{B}}$ is a set of types for τ such that $2 \leq |T_{\mathfrak{B}}| \leq |cl\tau|$, and $\leq_{\mathfrak{B}}$ is a transitive, reflexive, and connected relation on $T_{\mathfrak{B}}$ with a unique minimal element $t_{\mathfrak{B}} \in T_{\mathfrak{B}}$ for which the following conditions hold:

- $\tau_1 \leftleftarrows \tau_2 \in t_{\mathfrak{B}}$ iff there exists some $t \in T_{\mathfrak{B}}$ such that $\tau_1 \in t$ and $\tau_2 \notin t'$ for any $t' \leq_{\mathfrak{B}} t$,
- $\exists\rho \in t$ for some $t \in T_{\mathfrak{B}}$ iff $\exists\rho \in t$ for all $t \in T_{\mathfrak{B}}$.

We use the following notation:

$$\begin{array}{lllll} t \sim_{\mathfrak{B}} t' & \text{iff} & t \leq_{\mathfrak{B}} t' & \text{and} & t' \leq_{\mathfrak{B}} t \\ t <_{\mathfrak{B}} t' & \text{iff} & t \leq_{\mathfrak{B}} t' & \text{and} & t \not\sim_{\mathfrak{B}} t'. \end{array}$$

The intended meaning of a τ-bouquet $\mathfrak{B}$ is to encode the local requirements in order to realise the type $t_{\mathfrak{B}}$. A 'typical' τ-bouquet can be obtained by taking a point w from $\mathfrak{I}$ above and then selecting, for every term $\tau_1 \leftleftarrows \tau_2$ from $t^{\mathfrak{I}}(w)$, a point w' such that $d^{\mathfrak{I}}(w, w')$ is minimal with $w' \in \tau_1^{\mathfrak{I}}$. Denote by V the set of all selected points. Clearly, $|V| < |cl\tau|$ and we can assume that $t^{\mathfrak{I}}(w_1) \neq t^{\mathfrak{I}}(w_2)$ for any two distinct w_1, w_2 from V. If $|V| \geq 1$, then we define the *τ-bouquet* $(T_V^{\mathfrak{I}}(w), \leq_w)$ *induced by w and V in $\mathfrak{I}$* by taking

$$T_V^{\mathfrak{I}}(w) \;=\; \{t^{\mathfrak{I}}(w)\} \cup \{t^{\mathfrak{I}}(w') \mid w' \in V\},$$
$$t^{\mathfrak{I}}(w') \leq_w t^{\mathfrak{I}}(w'') \quad \text{iff} \quad d^{\mathfrak{I}}(w, w') \leq d^{\mathfrak{I}}(w, w'').$$

Notice that if we require a certain type t satisfied in $\mathfrak{I}$ to be a member of the bouquet then we can add to V a point w' such that $d(w, w')$ is minimal with $t = t^{\mathfrak{I}}(w')$ and form the bouquet induced by w and $V \cup \{w'\}$. In particular, if $\mathfrak{I}$ satisfies at least two distinct types, then we can always find a set V such that w and V induce a bouquet. In what follows we will only be working with models satisfying at least two distinct types. This is the interesting case because the problem of checking satisfiability in a model with only one type is clearly decidable in NP.

4.1 Non-symmetric case

First we establish the finite model property and the ExpTime upper bound for satisfiability in min-models that are not necessarily symmetric. Let N be the set of nominals occurring in τ. A set B τ-bouquets is said to be *nominal ready* if there is a set $\{t_\ell \mid \ell \in N\}$ of types for τ such that whenever $\{\ell\} \in t \in T_{\mathfrak{B}}$, for some $\mathfrak{B} \in B$, then $t = t_\ell$.

Let $k = |cl\tau|$. We remind the reader that the *full k-ary tree over* the set $\{1, \ldots, k\}^*$ (of finite sequences of elements of $\{1, \ldots, k\}$) contains the empty sequence ϵ as its root, and the immediate successors (children) of each node α are precisely the nodes αi, where $1 \leq i \leq k$. Given some set L (of labels), a function $K : \{1, \ldots, k\}^* \to L$ will be called an *L-labelled tree over* $\{1, \ldots, k\}^*$.

A *Hintikka tree satisfying* τ is a B-labelled tree K over $\{1, \ldots, k\}^*$, for some nominal ready set B of τ-bouquets, such that the following conditions are satisfied (where, as before, $t_{K(\alpha)}$ denotes the unique $\leq_{K(\alpha)}$-minimal element of the set of types $T_{K(\alpha)}$ in the bouquet $K(\alpha)$):

- $\tau \in t_{K(\epsilon)}$,
- for every nominal $\ell \in N$, there exists a type in $K(\epsilon)$ containing $\{\ell\}$,
- for every $\alpha \in \{1, \ldots, k\}^*$, $K(\alpha)$ is a bouquet such that

$$T_{K(\alpha)} \setminus \{t_{K(\alpha)}\} \;=\; \{t_{K(\alpha i)} \mid 1 \leq i \leq k\}$$

  and $t_{K(\alpha)} \in T_{K(\alpha i)}$, for $1 \leq i \leq k$.

Lemma 2. *For every term τ, the following conditions are equivalent:*

(a) *τ is satisfiable in some min-model (with at least two distinct types);*
(b) *there exists a Hintikka tree satisfying τ over $\{1,\dots,k\}^*$, where $k = |cl\,\tau|$;*
(c) *τ is satisfiable in a finite model (with at least two distinct types).*

Proof. (a) $\Rightarrow$ (b) Suppose that $\tau^{\mathfrak{I}} \neq \emptyset$ in some model $\mathfrak{I} \in \mathcal{C}$ of the form (1) with at least two distinct types. We define a Hintikka tree K satisfying τ by induction as follows. First take some $w \in \tau^{\mathfrak{I}}$ and set

$$K(\epsilon) \;=\; (T^{\mathfrak{I}}_{V_\epsilon}(w), \leq_w),$$

where $(T^{\mathfrak{I}}_{V_\epsilon}(w), \leq_w)$ is a bouquet induced by w and a suitable set $V_\epsilon \subseteq W$ containing $\{\ell\}^{\mathfrak{I}}$ for all ℓ that occur in τ. Here and in what follows we assume that we construct the underlying sets of the bouquet as described above in the introduction of bouquets.

Suppose now that we have already defined $K(\alpha)$, for some $\alpha \in \{1,\dots,k\}^*$:

$$K(\alpha) \;=\; (T^{\mathfrak{I}}_{V_\alpha}(w_\alpha), \leq_{w_\alpha}),$$

where $(T^{\mathfrak{I}}_{V_\alpha}(w_\alpha), \leq_{w_\alpha})$ is induced by w_α and a suitable set V_α. Take some surjective map $s : \{1,\dots,k\} \to V_\alpha$. For each j, $1 \leq j \leq k$, let

$$K(\alpha j) \;=\; (T^{\mathfrak{I}}_{V_{\alpha j}}(s(j)), \leq_{s(j)})$$

where $(T^{\mathfrak{I}}_{V_{\alpha j}}(s(j)), \leq_{s(j)})$ is the bouquet induced by $s(j)$ and a suitable set $V_{\alpha j}$ which contains a w' such that $t^{\mathfrak{I}}(w') = t^{\mathfrak{I}}(w_\alpha)$.

It is easy to see that the resulting K is a Hintikka tree satisfying τ.

(b) $\Rightarrow$ (c) Suppose that $K : \{1,\dots,k\}^* \to B$ with

$$K(\alpha) \;=\; (T_\alpha, \leq_\alpha)$$

is a Hintikka tree satisfying τ over a nominal ready set B of τ-bouquets. First we define a distance space (Δ_0, d_0) with the domain $\Delta_0 = \{1,\dots,k\}^*$ in the following way. Take a finite subset I of the interval $(0,1)$ and, for each $\alpha \in \Delta_0$, a map

$$f_\alpha : (T_{K(\alpha)} \setminus \{t_{K(\alpha)}\}) \to I$$

for which $t <_{K(\alpha)} t'$ iff $f_\alpha(t) < f_\alpha(t')$. Now set

- $d_0(\alpha, \alpha i) = f_\alpha(t_{K(\alpha i)})$ for all $\alpha \in \Delta_0$ and $1 \leq i \leq k$,
- $d_0(\alpha, \alpha) = 0$ and,
- $d_0(\alpha, \beta) = 1$ for $\beta \notin \{\alpha, \alpha 1, \dots, \alpha k\}$.

It is not difficult to see that (Δ_0, d_0) is a (non-symmetric) min-space.

For every type t such that $t = t_{K(\alpha)}$ for some $\alpha \in \Delta_0$, we fix exactly one α with this property. Let Δ be the set of the selected α. Construct a finite distance model from $\mathcal{C}$

$$\mathfrak{I} \;=\; (\Delta, d, \ell_1^{\mathfrak{I}}, \dots, p_1^{\mathfrak{I}}, \dots)$$

by taking $p_i^{\mathfrak{I}} = \{\alpha \in \Delta \mid p_i \in t_{K(\alpha)}\}$, $\ell_i^{\mathfrak{I}} = \alpha$ for the unique $\alpha \in \Delta$ with $\{\ell_i\} \in t_{K(\alpha)}$, and, for $\alpha, \beta \in \Delta$,

$$d(\alpha, \beta) = d_0(\alpha, \{\beta' \in \Delta_0 \mid t_{K(\beta')} = t_{K(\beta)}\}).$$

Now, given a subterm ρ of τ, one can prove by induction on the construction of ρ that $\alpha \in \rho^{\mathfrak{I}}$ iff $\rho \in t_{K(\alpha)}$. Therefore, τ is satisfied in $\mathfrak{I}$.

The implication (c) $\Rightarrow$ (a) is clear.

We are now in a position to prove the ExpTime upper bound by a reduction to the emptiness problem for finite looping tree automata; see [14, 12]. Recall that a *finite looping tree automaton $\mathcal{A}$ for infinite k-ary trees* is a quadruple (Σ, Q, Γ, Q_0), where

- Σ is a (nonempty) finite alphabet,
- Q is a (nonempty) finite set of states of the automaton,
- $\Gamma \subseteq \Sigma \times Q \times Q^k$ is a transition relation,
- $Q_0 \subseteq Q$ is a (nonempty) set of start states of the automaton.

Let T be a Σ-labelled tree over $\{1, \dots, k\}^*$. A *run* of $\mathcal{A}$ on T is a function $R : \{1, \dots, k\}^* \to Q$ such that

- $R(\epsilon) \in Q_0$, and
- $\big(T(\alpha), R(\alpha), (R(\alpha 1), \dots, R(\alpha k))\big) \in \Gamma$ for all nodes α of T.

$\mathcal{A}$ *accepts* T if there exists a run R of $\mathcal{A}$ on T. The following *emptiness problem* for looping automata is decidable in polynomial time [12]: given a looping automaton for k-ary trees, decide whether the set of trees it accepts is empty.

To reduce the satisfiability problem for $\mathcal{CSL}$-terms in $\mathcal{C}$, we associate with every term τ and every nominal ready set B of τ-bouquets a finite looping automaton $\mathcal{A}_\tau^B = (\Sigma, Q, \Gamma, Q_0)$ which is defined as follows:

- Σ is the set of types occurring in bouquets from B,
- $Q = B$,
- $Q_0 = \{\mathfrak{B} \in B \mid \tau \in t_{\mathfrak{B}}$, $\mathfrak{B}$ contains a type containing ℓ, for every ℓ in $\tau\}$,
- $\big(t, \mathfrak{B}_0, (\mathfrak{B}_1, \dots, \mathfrak{B}_k)\big) \in \Gamma$ iff $t_{\mathfrak{B}_0} = t$, $T_{\mathfrak{B}_0} \setminus \{t_{\mathfrak{B}_0}\}$ coincides with the set $\{t_{\mathfrak{B}_i} \mid 1 \le i \le k\}$, and $t_{\mathfrak{B}_0} \in T_{\mathfrak{B}_i}$, for $1 \le i \le k$.

It follows immediately from Lemma 2 and the given definitions that the runs of $\mathcal{A}_\tau^B$ on Σ-labelled trees are exactly the B-labelled Hintikka-trees satisfying τ.

Lemma 3. *A term τ is satisfiable in a min-model* (*with at least two types*) *iff there exists a nominal ready set B such that $\mathcal{A}_\tau^B$ accepts at least one tree.*

As there are only exponentially many different nominal ready sets B and as $\mathcal{A}_\tau^B$ is only exponential in $|cl\tau|$, the satisfiability problem in min- (and finite) models is decidable in ExpTime.

4.2 Symmetric case

The construction is more involved if we deal with the class of symmetric $\mathcal{CSL}$-models. Suppose that B is a nominal ready set of τ-bouquets, $|cl\tau| = k$, and $K : \{1,\dots,k\}^* \to B$ is a B-labelled Hintikka tree with $K(\alpha) = (T_\alpha, \leq_\alpha)$ and $t_\alpha = t_{K(\alpha)}$, for $\alpha \in \{1,\dots,k\}^*$.

We 'paint' each node of K in one of three 'colours:' inc (for increasing), const (for constant), and dec (for decreasing). The colour of a node α will be denoted by $c(\alpha)$. It is defined by induction as follows. The root ϵ and its immediate successors are painted with the same colour, say, $c(\epsilon) = c(1) = \cdots = c(k) = \mathsf{inc}$. Suppose now that we have already defined $c(\alpha i)$. Then, for $1 \leq j \leq k$, we set

- $c(\alpha ij) = \mathsf{const}$ iff $t_{\alpha ij} \sim_{\alpha i} t_\alpha$,
- $c(\alpha ij) = \mathsf{dec}$ iff $t_\alpha >_{\alpha i} t_{\alpha ij}$,
- $c(\alpha ij) = \mathsf{inc}$ iff $t_\alpha <_{\alpha i} t_{\alpha ij}$.

Intuitively, the colours determine whether in the symmetric space (Δ_0, d_0) to be constructed from $\{1,\dots,k\}^*$ we have $d_0(\alpha,\alpha i) = d_0(\alpha i, \alpha ij)$ (the constant case), $d_0(\alpha,\alpha i) < d_0(\alpha i,\alpha ij)$ (the increasing case), or $d_0(\alpha,\alpha i) > d_0(\alpha i,\alpha ij)$ (the decreasing case).

We call K a *min-tree* if its every branch with infinitely many dec nodes also contains infinitely many inc nodes.

We require two simple observations. First, we claim that the Hintikka tree K constructed in the proof of Lemma 2 starting from a *symmetric* min-model $\mathfrak{I}$ is a min-tree. Indeed, suppose otherwise. Then we have an infinite branch $\bar{\alpha}$ in K starting from some α all of whose nodes are dec or const and such that dec occurs infinitely many times in $\bar{\alpha}$. As the set $\{t \mid t = t_\beta,\ \beta \in \bar{\alpha}\}$ is finite and $t_\gamma \neq t_{\gamma i}$, for any i, we must have two distinct types t, t' for τ such that the set of pairs

$$A \;=\; \big\{(\beta,\beta') \in \bar{\alpha}^2 \mid \beta' = \beta i,\ 1 \leq i \leq k,\ \text{and } (t_\beta, t_{\beta'}) = (t,t')\big\}$$

is infinite. Take the points $w_\beta \in \Delta^{\mathfrak{I}}$, $\beta \in \bar{\alpha}$, which induce the nodes from $\bar{\alpha}$. As $\mathfrak{I}$ is symmetric and by the construction of bouquets, we have $d(w_\beta, w_{\beta i}) \geq d(w_{\beta i}, w_{\beta ij})$ whenever $c(\beta ij) = \mathsf{const}$ and $d(w_\beta, w_{\beta i}) > d(w_{\beta i}, w_{\beta ij})$ whenever $c(\beta ij) = \mathsf{dec}$.

Now let $T \subseteq \Delta^{\mathfrak{I}}$ and $T' \subseteq \Delta^{\mathfrak{I}}$ be defined by

$$T = \{w_\beta \mid (\beta,\beta') \in A\} \quad \text{and} \quad T' = \{w_{\beta'} \mid (\beta,\beta') \in A\}.$$

As $\mathfrak{I}$ is a min-model, there are points $u \in T$ and $u' \in T'$ such that $d^{\mathfrak{I}}(u,u') = d^{\mathfrak{I}}(T,T')$. By the definition of bouquets induced by points of $\mathfrak{I}$, we may assume that (u,u') gives rise to a $(\beta,\beta') \in A$. Now take some $(\gamma,\gamma') \in A$ that occurs in the branch $\bar{\alpha}$ later than (β,β') and such that some dec node occurs between β' and γ. Let v and v' be the points from $\mathfrak{I}$ inducing γ and γ'. Since all nodes in $\bar{\alpha}$ are dec or const and as there is at least one dec node between β' and γ, we come to the conclusion that $d^{\mathfrak{I}}(v,v') < d^{\mathfrak{I}}(u,u') = d^{\mathfrak{I}}(T,T')$, which is a contradiction.

Our second observation is that if there is a sequence $\alpha, \alpha i_1, \ldots, \alpha i_1 \cdots i_{n+1}$ (for $1 \leq i_j \leq k$) of nodes of K such that

$$(K(\alpha), K(\alpha i_1)) = (K(\alpha i_1 \cdots i_n), K(\alpha i_1 \cdots i_{n+1}))$$

then by 'cutting off' the nodes $\alpha, \ldots, \alpha i_1 \cdots i_{n-1}$ we obtain again a B-labelled Hintikka tree such that the colours of the (renamed) nodes do not change.

We are now in a position to prove a symmetric analogue of Lemma 2.

Lemma 4. *For every term τ, the following conditions are equivalent:*

(a) *τ is satisfiable in some symmetric model* (*with at least two distinct types*)*;*
(b) *there exists a Hintikka min-tree satisfying τ over $\{1, \ldots, k\}^*$, where $k = |cl\tau|$;*
(c) *τ is satisfiable in a finite symmetric model* (*with at least two different types*)*.*

Proof. (a) $\Rightarrow$ (b) is established in precisely the same way as in the proof of Lemma 2 using the first observation above that if we start with a symmetric model then the resulting Hintikka tree is a min-tree. (c) $\Rightarrow$ (a) is again trivial.

(b) $\Rightarrow$ (c) Suppose that $K : \{1, \ldots, k\}^* \to B$ is a Hintikka min-tree satisfying τ with

$$K(\alpha) = (T_\alpha, \leq_\alpha) \quad \text{and} \quad t_\alpha = t_{K(\alpha)}.$$

By the second observation above, without loss of generality we may assume that if no node in a path of the form $\alpha, \alpha i_1, \ldots, \alpha i_1 \cdots i_n$ is inc then no two dec nodes βij and $\beta' ij$ in it can have predecessors $(\beta, \beta i)$ and $(\beta', \beta' j)$ such that $(K(\beta), K(\beta i)) = (K(\beta'), K(\beta' j))$. It follows that there is a number n_τ (exponential in $|cl\tau|$) which bounds the numbers of dec nodes in each such path.

Now we define a symmetric distance space (Δ_0, d_0) with $\Delta_0 = \{1, \ldots, k\}^*$ (symmetry means that $d_0(\alpha, \alpha i) = d_0(\alpha i, \alpha)$ for $\alpha \in \Delta_0$ and $1 \leq i \leq k$). First we take a set $D \subset (9, 10)$ of cardinality $n_\tau \times |cl\tau|$. For all $1 \leq i \leq k$ we define $d_0(\epsilon, i)$ to be the maximal numbers in D such that we can satisfy the constraint: $d_0(\epsilon, i) < d_0(\epsilon, j)$ iff $t_{K(i)} <_\epsilon t_{K(j)}$, for $1 \leq i, j \leq k$. Suppose now that $d_0(\alpha, \alpha i) \in D$ is defined. Then we define $d_0(\alpha i, \alpha ij)$ to be the maximal number in D such that we can satisfy the constraints

- $d_0(\alpha i, \alpha ij) = d_0(\alpha, \alpha i)$ for $t_{K(\alpha ij)} = t_{K(\alpha)}$ and
- $d_0(\alpha i, \alpha ij) < d_0(\alpha i, \alpha ij')$ iff $t_{K(\alpha ij)} <_{\alpha i} t_{K(\alpha ij')}$, for $1 \leq j, j' \leq k$.

Notice that this is possible by the definition of n_τ. Finally, set $d_0(\alpha, \alpha) = 0$ and $d_0(\alpha, \beta) = 10$ for all remaining $\alpha \neq \beta$.

Now construct a finite symmetric model $\mathfrak{I} = (\Delta, d, p_1^{\mathfrak{I}}, \ldots, \ell_1^{\mathfrak{I}}, \ldots)$ as follows. Let $\sim$ be the equivalence relation on Δ_0 defined by taking $\alpha \sim \beta$ iff $t_{K(\alpha)} = t_{K(\beta)}$. Then we set

$$[\alpha] = \{\beta \in \Delta_0 \mid \alpha \sim \beta\}, \quad \Delta = \{[\alpha] \mid \alpha \in \{1, \ldots, k\}^*\}, \quad d([\alpha], [\beta]) = d_0([\alpha], [\beta])$$

and $[\alpha] \in p_i^{\mathfrak{i}}$ iff $p_i \in t_{K(\alpha)}$, and $\ell_i^{\mathfrak{I}} = [\alpha]$ for the uniquely determined $[\alpha]$ such that $\{\ell_i\} \in t_{K(\alpha)}$. We leave it to the reader to check that this model is as required.

A *single complemented pair automaton* $\mathcal{A}$ on infinite k-ary trees is a tuple $(\Sigma, Q, \Gamma, Q_0, F)$, where

- (Σ, Q, Γ, Q_0) is a looping tree automaton as defined in Section 4.1,
- F is a pair of disjoint sets of states from Q; it will be convenient for us to assume that $F = (\mathsf{dec}, \mathsf{inc})$ and $\mathsf{dec}, \mathsf{inc} \subseteq Q$.

$\mathcal{A}$ *accepts* a Σ-labelled tree T over $\{1, \dots, k\}^*$ iff there exists a run R of $\mathcal{A}$ on T such that, for every path $i_0 i_1 \dots$ in T, if $R(i_0 i_1 \dots i_j) \in \mathsf{dec}$ for infinitely many j, then $R(i_0 i_1 \dots i_j) \in \mathsf{inc}$ for infinitely many j as well.

As was shown in [4], the emptiness problem for single complemented pair automata is decidable in polynomial time. We show now how to reduce the satisfiability problem for $\mathcal{CSL}$-terms in symmetric models to the emptiness problem for these automata.

A *coloured τ-bouquet* is a pair $(\mathfrak{B}, c)$ where $\mathfrak{B} = (T_{\mathfrak{B}}, \leq_{\mathfrak{B}})$ is a τ-bouquet and c is a function from $T_{\mathfrak{B}}$ to $\{\mathsf{dec}, \mathsf{inc}, \mathsf{const}\}$.

With every term τ and every nominal ready set B of coloured τ-bouquets we associate a single complemented pair automaton $\mathcal{A}_\tau^B = (\Sigma, Q, \Delta, Q_0, F)$ by taking

- Σ to be the set of types occurring in coloured bouquets of B,
- $Q = B$,
- $Q_0 = \{(\mathfrak{B}, c) \in B \mid \tau \in t_{\mathfrak{B}},\ \mathfrak{B}$ contains a type with ℓ for every ℓ in $\tau\}$,
- $\mathsf{dec} = \{(\mathfrak{B}, c) \in B \mid c(t_{\mathfrak{B}}) = \mathsf{dec}\}$,
- $\mathsf{inc} = \{(\mathfrak{B}, c) \in B \mid c(t_{\mathfrak{B}}) = \mathsf{inc}\}$,
- $\big(t, (\mathfrak{B}_0, c_0), (\mathfrak{B}_1, c_1), \dots, (\mathfrak{B}_k, c_k)\big) \in \Gamma$ iff $t_{\mathfrak{B}_0} = t$,

$$T_{\mathfrak{B}_0} \setminus \{t_{\mathfrak{B}_0}\} = \{t_{\mathfrak{B}_i} \mid 1 \leq i \leq k\}$$

  $t_{\mathfrak{B}_0} \in T_{\mathfrak{B}_i} \setminus \{t_{\mathfrak{B}_i}\}$, $c_i(t_{\mathfrak{B}_i}) = c_0(t_{\mathfrak{B}_i})$ for $1 \leq i \leq k$, and for all $t' \in T_{\mathfrak{B}_i} \setminus \{t_{\mathfrak{B}_i}\}$,
  - $c_i(t') = \mathsf{inc}$ iff $t <_{\mathfrak{B}_i} t'$,
  - $c_i(t') = \mathsf{const}$ iff $t' \sim_{\mathfrak{B}_i} t$,
  - $c_i(t') = \mathsf{dec}$ iff $t' <_{\mathfrak{B}_i} t$.

It follows immediately from Lemma 4 and the given definitions that the runs of $\mathcal{A}_\tau^B$ on Σ-labelled trees are exactly the B-labelled Hintikka-trees satisfying τ.

Lemma 5. *A term τ is satisfiable in a symmetric min-model (with at least two distinct types) iff there exists a nominal ready set B of coloured τ-bouquets such that $\mathcal{A}_\tau^B$ accepts at least one tree.*

As there are only exponentially many different nominal ready sets B of coloured τ-bouquets and as $\mathcal{A}_\tau^B$ is only exponential in $|cl\,\tau|$, the satisfiability problem in symmetric min-models is decidable in ExpTime.

This completes the proof of Theorem 1.

5 Proof of Theorem 2

First we show that one can always deal with models based on *one-dimensional* spaces. Suppose that a $\mathcal{CSL}$-model $\mathfrak{I}$ is based on $\mathbb{R}^n$ for $n > 1$, and that a, b are object names. The term

$$(\{a\} \leftrightarrows \{b\}) \;\sqcap\; \forall\neg(\{a\} \sqcap \{b\})$$

is satisfied in $\mathfrak{I}$ iff $a^{\mathfrak{I}} \neq b^{\mathfrak{I}}$, and so it defines in $\mathfrak{I}$ an affine subspace of dimension $n-1$ (the mediating hyperplane for the line segment $a^{\mathfrak{I}}b^{\mathfrak{I}}$). Now we can iterate this construction: take distinct object names $a_0, b_0, \dots, a_{n-2}, b_{n-2}$ and consider the term

$$\bigsqcap_{i<n-1} \big(\{a_i\} \leftrightarrows \{b_i\}\big) \;\sqcap\; \bigsqcap_{i<n-1} \forall\neg\big(\{a_i\} \sqcap \{b_i\}\big) \;\sqcap\; \bigsqcap_{j<i<n-1} \Big(\{a_i\} \sqcup \{b_i\} \;\rightarrow\; \big(\{a_j\} \leftrightarrows \{b_j\}\big)\Big).$$

Clearly, if this term is satisfied in $\mathfrak{I}$ then it defines a one-dimensional subspace.

We show now how to prove the undecidability of the satisfiability problem for $\mathcal{CSL}$-terms in finite subspaces of $\mathbb{R}$ (in particular, those of the form $\{1, \dots, m\}$ where $m \geq 1$), and then discuss how to deal with other cases. Our proof is by reduction of the decision problem for Diophantine equations (Hilbert's 10th problem) which was shown to be undecidable by Matiyasevich–Robinson–Davis–Putnam; see [9, 2] and references therein. More precisely, we will use the following (still undecidable) variant of this problem:

given arbitrary polynomials g and h with coefficients from $\mathbb{N} \setminus \{0, 1\}$, decide whether the equation $g = h$ has a solution in the set $\mathbb{N} \setminus \{0, 1\}$.

Denote by $\mathcal{F}$ the class of finite metric models $\mathfrak{I}$ of the form (1) such that $(\Delta^{\mathfrak{I}}, d^{\mathfrak{I}})$ is a finite subspace of $(\mathbb{R}, d)$, where d is the standard metric on $\mathbb{R}$. Our aim is to provide an algorithm that constructs, for every polynomial equation $g = h$ over $\mathbb{N} \setminus \{0, 1\}$, a $\mathcal{CSL}$-term which is satisfiable in $\mathcal{F}$ iff $g = h$ is solvable in $\mathbb{N} \setminus \{0, 1\}$. It is easy to see that each polynomial equation can be rewritten equivalently as a set of elementary equations of the form

$$x = y + z, \quad x = y \cdot z, \quad x = y, \quad x = n, \tag{5}$$

where x, y, z are variables and $n \in \mathbb{N} \setminus \{0, 1\}$. Thus, it suffices to encode satisfiability of such elementary equations via satisfiability of $\mathcal{CSL}$-terms in $\mathcal{F}$. This will be done in three steps:

1. first we ensure that the underlying space of a given model coincides (modulo an affine transformation) with $\{0, 1, \dots, n\}$, for some $n \in \mathbb{N}$, and define the operations '$+1$' and '-1' on this space;
2. then we define, in this model, sets of the form $\{j, l+j, 2l+j, \dots\}$ with $j < l$ that are used to represent the (possibly unknown) number l;
3. finally, we encode addition and multiplication on such sets.

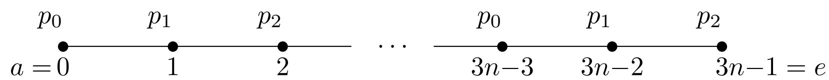
Step 1. Say that models $\mathfrak{I}, \mathfrak{L} \in \mathcal{F}$ are *affine isomorphic* and write $\mathfrak{I} \simeq \mathfrak{L}$ if there exists an affine transformation $f(x) = ax + b$ from $\Delta^{\mathfrak{I}}$ onto $\Delta^{\mathfrak{L}}$ such that $f(\ell^{\mathfrak{I}}) = \ell^{\mathfrak{L}}$ and $x \in p^{\mathfrak{I}}$ iff $f(x) \in p^{\mathfrak{L}}$, for all $x \in \Delta^{\mathfrak{I}}$, nominals ℓ and atomic terms p. In this case we clearly have $f(\tau^{\mathfrak{I}}) = \tau^{\mathfrak{L}}$ for every term τ.

Take object names a, e, atomic terms p_0, p_1, p_2, and set $\mathsf{base}(a, e; p_0, p_1, p_2)$ to be the following term:

$$\begin{aligned}\forall\Big(\prod_{i<j<3} \neg(p_i \sqcap p_j) \ \sqcap \ \big(\{a\} \ \to \ p_0 \sqcap (p_1 \leftleftarrows p_2)\big) \ \sqcap \\ \prod_{i<3}\big(p_i \sqcap \neg\{a\} \sqcap \neg\{e\} \ \to \ (p_{i\oplus 1} \leftrightarrows p_{i\ominus 1})\big) \ \sqcap \\ \big(\{e\} \to p_2 \sqcap (p_1 \leftleftarrows p_0)\big)\Big),\end{aligned}$$

where $\oplus$ and $\ominus$ denote $+$ and $-$ modulo 3.

A typical model from $\mathcal{F}$ satisfying $\mathsf{base}(a, e; p_0, p_1, p_2) \sqcap \forall(p_0 \sqcup p_1 \sqcup p_2)$ is depicted below:



(6)

More precisely, we have the following:

Lemma 6. *A model $\mathfrak{L} \in \mathcal{F}$ satisfies $\mathsf{base}(a, e; p_0, p_1, p_2) \sqcap \forall(p_0 \sqcup p_1 \sqcup p_2)$ iff there exist $\mathfrak{I} \in \mathcal{F}$ and $n > 0$ such that $\mathfrak{I} \simeq \mathfrak{L}$ and*

$$\begin{aligned}\Delta^{\mathfrak{I}} &= \{0, 1, \dots, 3n-1\}, \quad a^{\mathfrak{I}} = 0, \quad e^{\mathfrak{I}} = 3n-1, \\ p_i^{\mathfrak{I}} &= \{3k + i \mid k \in \mathbb{N},\ k < n\},\ i < 3.\end{aligned} \tag{7}$$

Proof. Given $x \in \Delta^{\mathfrak{L}}$ and $y \in Y \subseteq \Delta^{\mathfrak{L}}$, we say that y is a Y-*neighbour* of x if $d(x, y) = d(x, Y)$. In particular,

- for $Y = \{y \in \Delta^{\mathfrak{L}} \mid y < x\}$, we call y the *left neighbour* of x,
- for $Y = \{y \in \Delta^{\mathfrak{L}} \mid y > x\}$, y is called a *right neighbour* of x, and
- for $Y = \tau^{\mathfrak{L}}$, we say that y is the τ-*neighbour* of x.

Clearly, the left and right neighbours of a given x are always unique.

($\Rightarrow$) Suppose $\mathfrak{L}$ satisfies $\mathsf{base}(a, e; p_0, p_1, p_2) \sqcap \forall(p_0 \sqcup p_1 \sqcup p_2)$. Then $p_0^{\mathfrak{L}}$, $p_1^{\mathfrak{L}}$, $p_2^{\mathfrak{L}}$ are pairwise disjoint, and so the endpoints of $\Delta^{\mathfrak{L}}$ cannot satisfy $p_i \leftrightarrows p_j$ for $i \neq j$. It follows that $\{a^{\mathfrak{L}}, e^{\mathfrak{L}}\} = \{\min \Delta^{\mathfrak{L}}, \max \Delta^{\mathfrak{L}}\}$.

Let x be a p_1-neighbour of $a^{\mathfrak{L}}$. Then $x \notin a^{\mathfrak{L}}$, and so there exist some $\mathfrak{I} \in \mathcal{F}$ and an affine isomorphism $f : \mathfrak{L} \to \mathfrak{I}$ such that $a^{\mathfrak{I}} = 0 = \min \Delta^{\mathfrak{I}}$, $f(x) = 1$, and $e^{\mathfrak{I}} = \max \Delta^{\mathfrak{I}}$.

Note that if $i < 3$, $y \in p_i^{\mathfrak{I}} \setminus \{0, e^{\mathfrak{I}}\}$ and y_-, y_+ are, respectively, $p_{i\ominus 1}$- and $p_{i\oplus 1}$-neighbours of y, then y_- is the left neighbour of y, y_+ its right neighbour, and $y_+ - y = y - y_-$.

Suppose now that x' is a p_2-neighbour of 0. Then $x' > 1$ and, therefore, $p_1 \leftleftarrows p_2$ holds everywhere (strictly) between 0 and 1. It follows that 0 and x' are, respectively, the p_0- and p_2-neighbours of 1. As we saw above, this means that $x' - 1 = 1 - 0$ (i.e., $x' = 2$) and $(0, 1) \cap \Delta^{\mathfrak{I}} = (1, 2) \cap \Delta^{\mathfrak{I}} = \emptyset$ (as in (6)).

In exactly the same way we can show that if $k \in \mathbb{N}$, $k < e^{\mathfrak{I}}$ and $i = k \bmod 3$, then $k \in p_i^{\mathfrak{I}}$, $k+1 \in p_{i\oplus 1}^{\mathfrak{I}}$ and $(k, k+1) \cap \Delta^{\mathfrak{I}} = \emptyset$. Therefore, since $e^{\mathfrak{I}} = \max \Delta^{\mathfrak{I}}$, we have $e^{\mathfrak{I}} = \max\{y \in \mathbb{N} \mid y < e^{\mathfrak{I}}\} + 1$, that is, $e^{\mathfrak{I}} \in \mathbb{N}$ and $e^{\mathfrak{I}} \equiv 2 (\bmod 3)$.

($\Leftarrow$) Let $\mathfrak{I}$ be a model defined by (7). Then the term $\mathsf{base}(a, e; p_0, p_1, p_2) \sqcap \forall(p_0 \sqcup p_1 \sqcup p_2)$ is clearly satisfied in $\mathfrak{I}$, and so in every $\mathfrak{L} \simeq \mathfrak{I}$.

Let $\mathcal{I}_n$ be the class of models from $\mathcal{F}$ satisfying (7). In what follows we will only consider models of this form. Define the following analogues of the temporal 'next-time' operators simulating the functions '$+1$' and '-1':

$$\begin{aligned} \bigcirc\rho &= \bigsqcup_{i<3} \big(\neg\{e\} \sqcap p_i \sqcap (p_{i\oplus 1} \leftrightarrows (p_{i\oplus 1} \sqcap \rho))\big), \\ \bigcirc^{-1}\rho &= \bigsqcup_{i<3} \big(\neg\{a\} \sqcap p_i \sqcap (p_{i\ominus 1} \leftrightarrows (p_{i\ominus 1} \sqcap \rho))\big) \end{aligned}$$

and set

$$\bigcirc^0\rho = \rho, \quad \bigcirc^{k+1}\rho = \bigcirc\bigcirc^k\rho, \quad \bigcirc^{-k-1}\rho = \bigcirc^{-1}\bigcirc^{-k}\varphi, \quad \text{for all } k \in \mathbb{N}.$$

One can readily check that we have the following:

Lemma 7. *For all $k \in \mathbb{N}$ and $x \in \Delta^{\mathfrak{I}_n}$,*

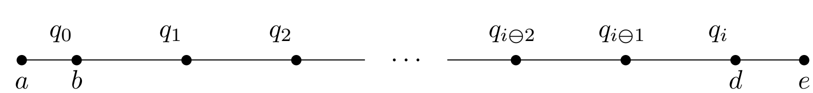
$$\begin{aligned} x \in (\bigcirc^k\rho)^{\mathfrak{I}_n} \quad &\textit{iff} \quad x < 3n-k \quad \textit{and} \quad x+k \in \rho^{\mathfrak{I}_n}, \\ x \in (\bigcirc^{-k}\rho)^{\mathfrak{I}_n} \quad &\textit{iff} \quad k \le x < 3n \quad \textit{and} \quad x-k \in \rho^{\mathfrak{I}_n}. \end{aligned}$$

Step 2. Let $\mathfrak{I} \in \mathcal{I}_n$ for some n, and $l \in \mathbb{N}$. As a 'standard' representation of l in $\mathfrak{I}$ we use the subset of $\mathfrak{I}$ of the form $\{0, l, 2l, \dots\}$. However, 'indented' subsets of the form $\{j, j+l, j+2l\}$ with $j < l$ will also be required at Step 3. Our next term defines subsets of this form.

To simplify notation, we denote lists like p_0, p_1, p_2 by $\mathbf{p}$, and terms of the form $p_0 \sqcup p_1 \sqcup p_2$ by p_*. Take fresh object names b, d, atomic terms q_0, q_1, q_2, and define $\mathsf{seq}(b, d; \mathbf{q})$ to be the term

$$\begin{aligned} \forall\Big(\bigsqcap_{i<j<3} \big(q_i \ \rightarrow\ \neg q_j \sqcap \neg\bigcirc q_j\big) \ \sqcap\ \big(\{b\} \ \rightarrow\ q_0 \sqcap (\{a\} \leftleftarrows q_1) \sqcap (q_1 \leftleftarrows q_2)\big) \ \sqcap \\ \bigsqcap_{i<3} \big(q_i \sqcap \neg\{b\} \sqcap \neg\{d\} \ \rightarrow\ (q_{i\ominus 1} \leftrightarrows q_{i\oplus 1})\big) \ \sqcap \\ \bigsqcup_{i<3} \big(\{d\} \ \rightarrow\ q_i \sqcap (\{e\} \leftleftarrows q_{i\ominus 1}) \sqcap (q_{i\ominus 1} \leftleftarrows q_{i\ominus 2})\big)\Big). \end{aligned}$$

This term is supposed to describe the following structure:



That is, points of the sets q_0, q_1, q_2 are periodically placed within equal distances (greater than one) between b and d, with $d(a, b)$ and $d(d, e)$ being smaller than the distance between different points satisfying q_*. Indeed, similarly to the proof of Lemma 6 one can easily show the following:

Lemma 8. *Let $\mathfrak{I} \in \mathcal{I}_n$. Then* $\mathsf{seq}(b, d; \mathbf{q})$ *is satisfied in $\mathfrak{I}$ iff there exists $j < l$ such that, for all $i < 3$,*

$$q_i^{\mathfrak{I}} \;=\; \{lk + j \mid k \equiv i (\mathrm{mod}\, 3),\ lk + j < 3n\}, \quad b^{\mathfrak{I}} = \min q_*^{\mathfrak{I}}, \quad d^{\mathfrak{I}} = \max q_*^{\mathfrak{I}}. \tag{8}$$

If (8) holds, we say that the triple $(b, d; \mathbf{q})$—or just $\mathbf{q}$—*encodes* in $\mathfrak{I}$ the number l with *indent* j. If $j = 0$, then we say that this encoding is *standard*.

Let $\mathbf{q}$ and $\mathbf{q}'$ encode in $\mathfrak{I}$ some numbers l and l'. If the encodings are standard, then it is easy to express that $l < l'$ or $l = l'$ (take the terms $\forall(\{a\} \to (q_1 \leftleftarrows q_1'))$ and $\forall(\{a\} \to (q_1 \leftrightarrows q_1'))$, for example). Thus, it remains to understand how to express $l = l'$ if, say, $\mathbf{q}'$ encodes l' with indent $j' \neq 0$. Obviously, the sets defined by q_* and q_*' must either coincide or have an empty intersection. And this condition turns out to be sufficient as well, provided that the members of these sets alternate within a sufficiently long segment:

Fact 1 *Let l, l' and j' be integer numbers such that $kl < kl' + j' < (k+1)l$ for every $k < l$. Then $l = l'$.*

We show now how to ensure that $0 \le j < l \le l' + j' < 2l \le 2l' + j' < \dots$ and $l^2 \le \max \Delta^{\mathfrak{I}}$, for such l and l'. First, using Lemmas 7 and 8 we can express the property that the common difference of one arithmetic progression is greater by 1 than the common difference of the other progression:

Lemma 9. *Let $\mathfrak{I} \in \mathcal{I}_n$, $\bar{c}$ be an object name and $\bar{q}_0, \bar{q}_1, \bar{q}_2$ atomic terms. Then the term*

$$\mathsf{seq}(a, c; \mathbf{q}) \;\sqcap\; \mathsf{seq}(a, \bar{c}; \bar{\mathbf{q}}) \;\sqcap\; \forall(\{a\} \to (\bar{q}_1 \leftrightarrows \bigcirc q_1))$$

is satisfied in $\mathfrak{I}$ iff there exists $l > 1$ such that, for all $i < 3$,

$$\begin{aligned} q_i^{\mathfrak{I}} &= \{lk \mid k \equiv i (\mathrm{mod}\, 3),\ lk < 3n\}, \quad c^{\mathfrak{I}} = \max q_*^{\mathfrak{I}} \quad \textit{and} \\ \bar{q}_i^{\mathfrak{I}} &= \{(l-1)k \mid k \equiv i (\mathrm{mod}\, 3),\ (l-1)k < 3n\}, \quad \bar{c}^{\mathfrak{I}} = \max \bar{q}_*^{\mathfrak{I}}. \end{aligned} \tag{9}$$

Next, Lemma 9 and the fact that $x = l(l-1)$ is the least positive solution to

$$x \equiv 0 (\mathrm{mod}\, l) \quad \text{and} \quad x \equiv 0 (\mathrm{mod}\, (l-1))$$

can be used to say that $\mathsf{seq}(a, c; \mathbf{q})$ defines a set with $l^2 < \max \Delta^{\mathfrak{I}}$.

Lemma 10. *The term*

$$\begin{aligned} \mathsf{seq}(a, c; \mathbf{q}) \;\sqcap\; \mathsf{seq}(a, \bar{c}; \bar{\mathbf{q}}) \;\sqcap\; \forall(\{a\} \to (\bar{q}_1 \leftrightarrows \bigcirc q_1)) \sqcap \\ \exists(q_* \sqcap \bar{q}_* \sqcap \neg\{a\} \sqcap \neg\{c\}) \end{aligned}$$

is satisfied in $\mathfrak{I} \in \mathcal{I}_n$ iff (9) holds for some $l > 1$ with $l^2 < 3n$ and all $i < 3$.

Further, let $\mathsf{alt}(a, c, b, d; \mathbf{q}, \mathbf{r})$ denote the term

$$\begin{aligned} \forall \bigsqcap_{i<3} \big((\{a\} \;\to\; (b \leftleftarrows q_1)) \sqcap \\ (q_i \sqcap \neg\{a\} \sqcap \neg\{c\} \;\to\; \neg(q_{i\ominus 1} \leftleftarrows r_{i\ominus 1}) \sqcap (r_i \leftleftarrows q_i)) \sqcap \\ (r_i \sqcap \neg\{d\} \;\to\; (q_i \leftleftarrows r_{i\ominus 1}) \sqcap \neg(r_{i\oplus 1} \leftleftarrows q_i)) \big). \end{aligned}$$

Lemma 11. *Suppose that* $\mathsf{seq}(a, c; \mathbf{q}) \sqcap \mathsf{seq}(b, d; \mathbf{r})$ *is satisfied in* $\mathfrak{I} \in \mathcal{I}_n$*, and let* $q_*^{\mathfrak{I}} = \{x_0, \dots, x_k\}$*,* $r_*^{\mathfrak{I}} = \{y_0, \dots, y_m\}$ *be such that* $x_i < x_{i+1}$ *and* $y_i < y_{i+1}$*. Then* $\mathfrak{I}$ *satisfies* $\mathsf{alt}(a, c, b, d; \mathbf{q}, \mathbf{r})$ *iff* $k - 1 \le m \le k$ *and* $x_0 \le y_0 < x_1 \le y_1 < \dots$ *.*

Proof. Suppose $\mathfrak{I}$ satisfies $\mathsf{alt}(a, c, b, d; \mathbf{q}, \mathbf{r})$. Then, by the first conjunct of alt, we obtain $x_0 \le y_0 < x_1$. The third one yields $y_1 - y_0 \ge x_1 - y_0$, i.e., $x_1 \le y_1$. And using the second conjunct we obtain $x_2 - x_1 < y_1 - x_1$, i.e., $y_1 < x_2$. By the same argument, we see that the sequence $x_0 \le y_0 < x_1 \le y_1 < \dots$ terminates either with $x_{k-1} \le y_{k-1} < x_k$, $c^{\mathfrak{I}} = x_k$, $d^{\mathfrak{I}} = y_{k-1}$, or with $y_{k-1} < x_k \le y_k$, $c^{\mathfrak{I}} = x_k$, $d^{\mathfrak{I}} = y_k$.

The other direction is straightforward and left to the reader.

We are now in a position to express that two tuples of terms represent the same number. Let $\mathsf{equ}(a, c, b, d; \mathbf{q}, \mathbf{r})$ be the term

$$\mathsf{alt}(a, c, b, d; \mathbf{q}, \mathbf{r}) \ \sqcap \ \big(\forall(q_* \leftrightarrow r_*) \sqcup \forall(q_* \to \neg r_*)\big).$$

Lemma 12. *Suppose that* $\mathfrak{I} \in \mathcal{I}_n$*,* $(a, c; \mathbf{q})$ *encodes standardly some* l *with* $l^2 < 3n$*, and* $(b, d; \mathbf{r})$ *encodes some* l'*. Then* $\mathfrak{I}$ *satisfies* $\mathsf{equ}(a, c, b, d; \mathbf{q}, \mathbf{r})$ *iff* $l = l'$*.*

Proof. Let $\mathfrak{I}$ satisfy $\mathsf{equ}(a, c, b, d; \mathbf{q}, \mathbf{r})$. By the second conjunct of equ we know that our sequences either coincide or are disjoint. In the former case we are done. In the latter one it suffices to use Lemma 11 and Fact 1. The other direction is clear.

Step 3. Now we encode addition and multiplication. Let $(a, c; \mathbf{q})$, $(a, c', \mathbf{r})$, $(a, c'', \mathbf{s})$ encode standardly some numbers u, v and w respectively. Suppose we want to say that $u = v + w$. Consider first the case $v < w$, which can be expressed as $\forall(\{a\} \to (r_1 \leftleftarrows s_1))$. Take fresh $(b, d; \mathbf{s}')$ and state:

$$\begin{array}{ll} \mathsf{seq}(b, d; \mathbf{s}') \ \sqcap & (b, d; \mathbf{s}') \text{ encodes} \\ \mathsf{alt}(a, c', b, d; \mathbf{s}, \mathbf{s}') \ \sqcap & \text{the number } w \\ \forall(\{a\} \to (r_1 \leftrightarrows s'_0)) \ \sqcap & \text{with indent } v, \\ \forall(\{a\} \to (q_1 \leftrightarrows s'_1)) & \text{and } u = v + w. \end{array}$$

The case $v > w$ is a mirror image. And to say that $v = w$ and $u = v + v$ we can use the term $\forall\big(\{a\} \to (r_1 \leftrightarrows s_1)\big) \ \sqcap \ \forall\big(\{a\} \to (q_1 \leftrightarrows s_2)\big)$.

To encode multiplication we use the following observation.

Fact 2 *Let* v*,* w *be integer numbers with* $0 < v < w - 1$*. Then*

1) $u = vw$ *is the least solution to* $u \equiv 0 (\mathrm{mod}\, w)$*,* $u \equiv v (\mathrm{mod}\, (w{-}1))$*.*
2) $u = (w{-}1)w$ *is the least positive solution to* $u \equiv 0 (\mathrm{mod}\, w)$*,* $u \equiv 0 (\mathrm{mod}\, (w{-}1))$*.*
3) $u = w^2$ *is the least solution to* $u \equiv 0 (\mathrm{mod}\, w)$*,* $u \equiv 1 (\mathrm{mod}\, (w{-}1))$*,* $x > w$*.*

Suppose we want to say that $u = v \cdot w$ Consider first the case $v < w - 1$. Take fresh $(a, \bar{c}; \bar{\mathbf{s}})$, $(b, d; \bar{\bar{\mathbf{s}}})$ and state:

$$\begin{array}{ll} \mathsf{seq}(a, \bar{c}; \bar{\mathbf{s}}) \ \sqcap \ \forall(\{a\} \to (\bar{s}_1 \leftrightarrows \bigcirc s_1)) \ \sqcap & (a, \bar{c}; \bar{\mathbf{s}}) \text{ encodes standardly } w - 1, \\ \mathsf{seq}(b, d; \bar{\bar{\mathbf{s}}}) \ \sqcap \ \mathsf{alt}(a, \bar{c}, b, d; \bar{\mathbf{s}}, \bar{\bar{\mathbf{s}}}) \ \sqcap & \text{and } \bar{\bar{\mathbf{s}}} \text{ encodes } w - 1 \\ \forall(\{a\} \to (r_1 \leftrightarrows \bar{\bar{s}}_0)) & \text{with indent } v. \end{array} \tag{10}$$

Then, in view of Fact 2 (1), term (10) means that $v \cdot w$ is the least point satisfying $s_* \sqcap \bar{\bar{s}}_*$. Therefore, $\forall\big(\{a\} \to (q_1 \leftrightarrows (s_* \sqcap \bar{\bar{s}}_*))\big)$ in conjunction with (10) ensures that $u = v \cdot w$.

The case $v = w-1$ is similar (we use Fact 2 (2)), and we can deal by symmetry with $w < v$, $w = v - 1$. Assume now that $v = w$. Take fresh $(b, d; \bar{\mathbf{s}})$. The term

$$\mathsf{seq}(b, d; \bar{\mathbf{s}}) \;\sqcap\; \forall(\{a\} \to \bigcirc\bar{\mathbf{s}} \sqcap (\bar{s}_1 \leftrightarrows s_1)) \tag{11}$$

means that $(b, d; \bar{\mathbf{s}})$ encodes $w - 1$ with indent 1. Then, in view of Fact 2 (3), term (11) implies that w and w^2 are two smallest points satisfying $s_* \sqcap \bar{s}_*$. But w^2 also satisfies $s_0 \leftleftarrows \{a\}$, while w does not. Hence the term

$$\forall\Big(\{a\} \;\to\; \big(q_1 \leftrightarrows (s_* \sqcap \bar{s}_* \sqcap (s_0 \leftleftarrows \{a\}))\big)\Big)$$

in conjunction with (11) ensures that $u = w^2$.

Thus, we can encode elementary equations of the form (5), and so any polynomial equation as well. This proves Theorem 2 (1,2).

Now we outline the proof Theorem 2 (3,4) for $n = 1$. Basically, we follow the same scheme as before. However, some things can be simplified.

Step 1′. Let $\mathcal{R}$ be the class of models based on min-subspaces of $\mathbb{R}$. Consider the term

$$\mathsf{Base}(\mathbf{p}) \;=\; \exists p_0 \sqcap \exists p_1 \;\sqcap\; \forall \bigsqcap_{i<3} \big(p_i \;\to\; \neg p_{i\oplus 1} \sqcap (p_{i\ominus 1} \leftrightarrows p_{i\oplus 1})\big).$$

Then $\mathsf{Base}(\mathbf{p}) \sqcap \forall p_*$ is satisfied in a model $\mathfrak{I} \in \mathcal{R}$ iff $\mathfrak{I} \simeq \mathfrak{Z}$, where $\mathfrak{Z}$ has the following structure:

$$\Delta^{\mathfrak{Z}} \;=\; \mathbb{Z}, \quad p_i^{\mathfrak{Z}} \;=\; \{3k + i \mid k \in \mathbb{Z}\}, \text{ for all } i < 3. \tag{12}$$

The 'next-time' operator and its inverse are now defined in the following way:

$$\bigcirc\rho \;=\; \bigsqcup_{i<3}(p_i \sqcap (p_{i\oplus 1} \leftrightarrows p_{i\oplus 1} \sqcap \rho)), \quad \bigcirc^{-1}\rho \;=\; \bigsqcup_{i<3}(p_i \sqcap (p_{i\ominus 1} \leftrightarrows p_{i\ominus 1} \sqcap \rho)).$$

To fix the intended origin and orientation for our model, take a fresh variable p and set

$$\mathsf{Orig}(\mathbf{p}) \;=\; \exists(p_2 \sqcap \neg p \sqcap \bigcirc p) \;\sqcap\; \forall(p \to \bigcirc p).$$

Then a model $\mathfrak{Z}$ of the form (12) satisfies $\mathsf{Orig}(p)$ iff $p^{\mathfrak{Z}} = \{k, k+1, \dots\}$, for some $k \in \mathbb{Z}$, $k \equiv 0 (\mathrm{mod}\, 3)$. In what follows, we assume that $p^{\mathfrak{Z}} = \mathbb{N}$. Thus $p \sqcap \bigcirc^{-1} p$ defines $\{0\}$.

Step 2′. The term

$$\mathsf{Seq}(q_0, q_1, q_2) \;=\; \forall \bigsqcap_{i<3} \big(q_i \to (q_{i\oplus 1} \leftrightarrows q_{i\ominus 1})\big) \;\sqcap\; \exists\big(q_0 \sqcap p \sqcap (q_2 \leftleftarrows q_2 \sqcap p)\big)$$

is satisfied in $\mathfrak{Z}$ iff there exists $j < l$ in $\mathbb{N}$ such that $q_i^{\mathfrak{Z}} = \{lk + j \mid k \equiv i (\mathrm{mod}\, 3)\}$, for all $i < 3$. Note that in this case j, $j + l$, $j + 2l$ will be the nearest to 0 points satisfying $q_0 \sqcap p$, $q_1 \sqcap p$, $q_2 \sqcap p$ respectively. As before, a set of the form $\{lk + j \mid k \in \mathbb{Z}\}$ is used to encode the number l. Further, we set

$$\begin{aligned}
\mathsf{Alt}(\mathbf{q}, \mathbf{q}') &= \forall \bigsqcap_{i<3} \big((q_i \to (q'_i \leftleftarrows q_{i\oplus 1})) \;\sqcap\; (q'_i \to (q_i \leftleftarrows q'_{i\oplus 1}))\big),\\
\mathsf{Equ}(\mathbf{q}, \mathbf{q}') &= \mathsf{Alt}(\mathbf{q}, \mathbf{q}') \;\sqcap\; \big(\forall(q_* \leftrightarrow q'_*) \sqcup \forall(q_* \to \neg q'_*)\big).
\end{aligned}$$

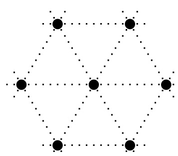
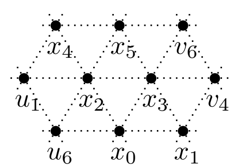
Then $\mathfrak{Z}$ satisfies $\mathsf{Seq}(\mathbf{q}) \sqcap \mathsf{Seq}(\mathbf{q}') \sqcap \mathsf{Equ}(\mathbf{q}, \mathbf{q}')$ iff $\mathbf{q}$ and $\mathbf{q}'$ represent the same number in $\mathfrak{Z}$.

Step 3′. We encode the arithmetic operations using the same ideas as before.

It is to be noted that the undecidability for min-subspaces of $\mathbb{R}^n$, $n \geq 2$, can be proved by reduction of the $\mathbb{Z} \times \mathbb{Z}$ tiling problem. To simulate the $\mathbb{Z} \times \mathbb{Z}$ grid we use the term

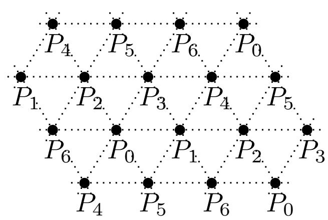
$$\tau_0 \;=\; \exists p_0 \sqcap \exists p_1 \;\sqcap\; \forall \bigsqcap \big\{p_i \to (p_j \leftrightarrows p_{i\oplus 1}) \mid i, j < 7, j \neq i, i\oplus 1\big\},$$

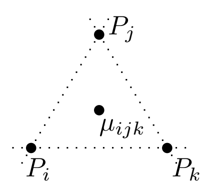
where $\oplus$ is addition modulo 7. Let $\mathfrak{T}$ be a model satisfying τ_0 such that $\Delta^{\mathfrak{T}}$ is a subspace of $\mathbb{R}^2$, and let $P_i = p_i^{\mathfrak{T}}$. Then one can show that $P_0 \cup \dots \cup P_6$ forms an infinite grid as in (13a).

a)  b) 

(13)

To encode tilings, we need to fix some concrete partition of this grid into the sets $P_0, \dots, P_6$. First we note that, in fact, it suffices to fix such a partition for a few points only. Indeed, suppose that $x_0 \in P_0$, …, $x_5 \in P_5$ are located as in (13b). Then either $v_4 \in P_4$ and $v_6 \in P_6$, or $v_4 \in P_6$ and $v_6 \in P_4$, since x_3 must have exactly one neighbour in each of the sets $P_0, \dots, P_6$. Using the same argument inductively, we see that only one partition of the grid into the sets $P_0, \dots, P_6$ can satisfy τ_0 and realise the configuration in (13b) for certain points $x_0 \in P_0$, …, $x_5 \in P_5$; it is shown in (14a).

a) 

b) 

(14)

To ensure that such $x_0, \dots, x_5$ exist, we set, for distinct $i, j, k < 7$,

$$\mu_{ijk} \;=\; (p_i \leftrightarrows p_j) \sqcap (p_j \leftrightarrows p_k) \sqcap \bigsqcap \{p_k \leftleftarrows p_l \mid l < 7,\; l \neq i, j, k\}.$$

Then x belongs to $\mu^{\mathfrak{T}}_{ijk}$ iff there exist $x_i \in P_i$, $x_j \in P_j$, $x_k \in P_k$ that form a small triangle in our grid with centre x; see (14b). Now set

$$\tau_1 \;=\; \exists\Big(\mu_{103} \sqcap \Big(\big(\mu_{032} \sqcap \big((\mu_{325} \sqcap (\mu_{254} \leftrightarrows p_5)) \leftrightarrows p_2\big)\big) \leftrightarrows p_3\Big)\Big).$$

Then $\mathfrak{T}$ satisfies τ_1 iff, for some small triangles $x_1x_0x_3$, $x'_0x'_3x_2$, $x''_3x'_2x_5$, $x''_2x'_5x_4$ with centres x, w, v, u, respectively, we have:

- $x_i, x'_i, x''_i \in P_i$; $x, w, v, u \in \Delta^{\mathfrak{T}}$; and
- $d^{\mathfrak{T}}(x, w) = d^{\mathfrak{T}}(x, x_3)$, $d^{\mathfrak{T}}(w, v) = d^{\mathfrak{T}}(w, x_2)$, $d^{\mathfrak{T}}(v, u) = d^{\mathfrak{T}}(v, x_5)$.

Clearly, this is possible only if $x_0 = x'_0$, $x_3 = x'_3 = x''_3$, $x_2 = x'_2 = x''_2$, $x_5 = x'_5$, which means that the points $x_0, \dots, x_5$ are located as in (13b). Therefore, in this case, the structure of the grid will be as in (14a).

Thus, in any model $\mathfrak{T}$ satisfying $\tau_1 \sqcap \tau_2$, we can interpret the structure of $\mathbb{Z} \times \mathbb{Z}$ and the relations left–right, above–below: for a point $x \in P_i$, its nearest point in $P_{i\oplus 1}$ ($P_{i\oplus 2}$) can be regarded as the right (upper) neighbour of x. Hence $\mathbb{Z} \times \mathbb{Z}$ tilings can be encoded in $\mathfrak{T}$.

6 Outlook

In this paper, we have investigated the computational complexity of the basic logic $\mathcal{CSL}$ for comparative similarity. The final verdict is that this logic behaves similarly to standard description logics (is ExpTime-complete) over general classes of (finite or min-) distance spaces, but becomes undecidable when interpreted over (finite or min-) subspaces of Euclidean spaces.

Starting from the positive results, one can now investigate combinations of $\mathcal{CSL}$ with 'quantitative' similarity logics from [16, 8] as well as with description logics. On the other hand, it would be interesting to find out how one can avoid the 'negative' results for subspaces of $\mathbb{R}^n$. One promising route is to impose restrictions on the interpretations of set variables. For example, in many applications it seems natural to assume that variables are interpreted as intervals in (subspaces of) $\mathbb{R}$. In this case decidability would follow immediately. Another related question is whether the computational behaviour of the logics depends on the 'crisp' truth-conditions. Exploring more relaxed 'non-punctual' truth-conditions could be important as well in order to take into account unprecise measurements, vagueness, and paradoxes of similarity such as the Sorites paradox.

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