On dynamic topological and metric logics

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Abstract

The first result of this paper shows that some dynamic topological logics interpreted in various topological spaces with homeomorphisms are not recursively enumerable (and so are not recursively axiomatisable). This gives a 'negative' solution to a conjecture of Kremer and Mints [12]. Second, we prove the non-elementary decidability of the dynamic metric logic with distance operators of the form 'somewhere in the ball of radius a,' for $a \in \mathbb{Q}^+$, interpreted in arbitrary metric spaces with distance preserving automorphisms.

1 Introduction

Dynamic topological logics were first introduced in 1997 (see, e.g., [10, 11, 13, 2, 12]) as a logical formalism for describing the behaviour of *dynamical systems*, e.g., in order to specify liveness and safety properties of hybrid systems [6]. Dynamical systems [4, 9] are usually represented by some 'mathematical' space W (modelling possible system states) and a function f on W (modelling the evolution of the system), with one of the main research problems being the study of iterations of f, in particular, the orbits $O(w) = \{w, f(w), f^2(w), \dots\}$ of states $w \in W$.

A natural logical formalism for speaking about such iterations is a variant of temporal logic. For example, given a subset V of W, we can introduce the standard temporal operators $\bigcirc$ ('at the next moment'), $\Box_F$ ('always in the future'), and its dual $\Diamond_F$ ('eventually') by taking

$$\bigcirc V = f^{-1}(V), \qquad \Box_F V = \bigcap_{0<n<\omega} f^{-n}(V) \qquad \text{and} \qquad \Diamond_F V = \bigcup_{0<n<\omega} f^{-n}(V).$$

Using this language we can describe in a succinct and transparent way properties like

- starting from a state in some region P, one will never leave a region Q: $P \to \Box_F Q$;
- starting from a state in a region P, one will eventually reach a state in Q: $P \to \Diamond_F Q$;
- w 'visits' P ever and ever again: $w \in \Box_F \Diamond_F P$.

To speak about the structure of the underlying space W—important examples are (subspaces of) the Euclidean spaces $\mathbb{R}^n$, general topological spaces, metric spaces, and measure spaces—as well as the type of the intended functions f, one may require different non-temporal operators. So far, research has mainly been focused on topological spaces with continuous mappings. The corresponding logical constructors are those of modal logic **S4** which can be regarded also as the topological closure and interior operators—we denote them by **C** and **I**, respectively. For example, a property similar to Poincaré's recurrence theorem corresponds in this language to the validity of the formula $\mathbf{C}(\mathbf{I}p \to \bigcirc\Diamond_F \mathbf{I}p)$ in spaces based on the unit disc with measure preserving continuous mappings.

Metric operators were suggested in [16] in order to formulate safety properties. For example, using the operator $\exists^{\leq a}$, where a is a positive rational number, the formula $P \to \Box_F \neg\exists^{\leq a} Q$ states that, having started from a point in P, one can never reach the a-neighbourhood of some 'unsafe' area Q.

The resulting combinations of temporal and topological/metric logics are of a clear 'two-dimensional character,' which makes it very difficult to analyse their computational properties (see, e.g., [7]). Perhaps this is the main reason why in the field of dynamic topological systems no (un)decidability or axiomatisability results have been obtained yet for the full language containing both $\bigcirc$ and the infinitary $\Box_F$.

This note provides answers to some of the open problems. First, we show that some dynamic topological logics introduced in [12] and interpreted in various topological spaces with homeomorphisms are not recursively enumerable (and so are not recursively axiomatisable). This result gives negative solutions to Conjectures 2.7 (ii) and 2.7 (iv) from [12]. Second, we prove the non-elementary decidability of the dynamic metric logic with distance operators of the form $\exists^{\leq a}$ from [14] interpreted in arbitrary metric spaces with distance preserving automorphisms.

Although numerous problems remain open, the obtained results clearly indicate that the logics for dynamic systems are very sensitive to the available operators (say, topological vs metric) as well as the constraints imposed on the spaces $\langle W, f\rangle$ (e.g., the proof of the undecidability result mentioned above does not go through for continuous functions, while the decidability proof only works for *arbitrary* metric spaces, but not for, say, compact ones).

2 Definitions

Syntax. The language $\mathcal{DTL}$ of *dynamic topological logic* (or *dynamic topo-logic*, for short) [2, 12] is constructed from a countably infinite set of propositional variables using the Booleans $\wedge$ and $\neg$, the modal operators **I** and **C** (for topological interior and closure), and the temporal operators $\bigcirc$ (for 'next'), $\Box_F$ and $\Diamond_F$ (for 'always' and 'eventually'). By $\mathcal{DTL}_{\bigcirc}$ we denote the fragment of $\mathcal{DTL}$ which does not use $\Box_F$ and $\Diamond_F$. We write $\Box_F^+\varphi$ for $\varphi \wedge \Box_F\varphi$ and dually $\Diamond_F^+\varphi = \varphi \vee \Diamond_F\varphi$, for every $\mathcal{DTL}$-formula φ.

Semantics. In this paper, by a *dynamic topological structure* (or DTS, for short) we understand a pair of the form $\mathfrak{F} = \langle \mathfrak{T}, f\rangle$, where $\mathfrak{T} = \langle T, \mathbb{I}\rangle$ is a topological space with an interior operator $\mathbb{I}$ (satisfying the standard Kuratowski axioms) and f is a homeomorphism[1] (i.e., a bijective continuous and open mapping) on $\mathfrak{T}$. A *dynamic topological model* (or DTM) is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$, where $\mathfrak{F}$ is a DTS and $\mathfrak{V}$, a *valuation*, associates with each propositional variable p a subset $\mathfrak{V}(p)$ of T. The *truth-relation* $(\mathfrak{M}, w) \models \varphi$, for a $\mathcal{DTL}$-formula φ, is defined as follows:

$$(\mathfrak{M}, w) \models p \quad \text{iff} \quad w \in \mathfrak{V}(p),$$
$$(\mathfrak{M}, w) \models \mathbf{I}\varphi \quad \text{iff} \quad w \in \mathbb{I}\{v \in T \mid (\mathfrak{M}, v) \models \varphi\},$$

[1] In a more general setting, f can be a continuous mapping.

$$
\begin{array}{lll}
(\mathfrak{M},w) \models \mathbf{C}\varphi & \text{iff} & w \in \mathbb{C}\{v \in T \mid (\mathfrak{M},v) \models \varphi\}, \\
(\mathfrak{M},w) \models \bigcirc\varphi & \text{iff} & (\mathfrak{M}, f(w)) \models \varphi, \\
(\mathfrak{M},w) \models \Box_F\varphi & \text{iff} & (\mathfrak{M}, f^n(w)) \models \varphi \text{ for every } n > 0, \\
(\mathfrak{M},w) \models \Diamond_F\varphi & \text{iff} & (\mathfrak{M}, f^n(w)) \models \varphi \text{ for some } n > 0.
\end{array}
$$

Here $f^n(w) = \overbrace{f \dots f}^{n}(w)$. If $(\mathfrak{M},w) \models \varphi$ for some $w \in T$, then we say that φ is *satisfied* in $\mathfrak{M}$. A $\mathcal{DTL}$-formula φ is *satisfiable* in a DTS $\mathfrak{F}$ if φ is satisfied in a DTM based on $\mathfrak{F}$.

Given a class $\mathcal{K}$ of dynamic topological structures, we denote by $\mathsf{Log}\,\mathcal{K}$ (respectively, $\mathsf{Log}_{\bigcirc}\mathcal{K}$) the *logic* of $\mathcal{K}$ *in the language* $\mathcal{DTL}$ (or $\mathcal{DTL}_{\bigcirc}$), i.e., the set of all $\mathcal{DTL}$-formulas (respectively, $\mathcal{DTL}_{\bigcirc}$-formulas) φ such that $(\mathfrak{M},w) \models \varphi$ holds for every model $\mathfrak{M}$ based on a structure in $\mathcal{K}$ and every point w in $\mathfrak{M}$.

We remind the reader that every quasi-order $\mathfrak{G} = \langle W, R\rangle$ (R is a reflexive and transitive relation on W) gives rise to a topological space $\mathfrak{T}_{\mathfrak{G}} = \langle W, \mathbb{I}_{\mathfrak{G}}\rangle$, where, for every $X \subseteq W$,

$$\mathbb{I}_{\mathfrak{G}}X = \{x \in X \mid \forall y \in W\ (xRy \to y \in X)\}.$$

Such spaces are known as *Aleksandrov spaces*. Alternatively they can be defined as topological spaces where arbitrary (not only finite) intersections of open sets are open; for details see [1, 3]. Clearly, for $\mathfrak{M} = \langle\langle\mathfrak{T}_{\mathfrak{G}}, f\rangle, \mathfrak{V}\rangle$ we have

$$
\begin{array}{lll}
(\mathfrak{M},w) \models \mathbf{I}\varphi & \text{iff} & (\mathfrak{M},v) \models \varphi \text{ for every } v \in W \text{ with } wRv, \\
(\mathfrak{M},w) \models \mathbf{C}\varphi & \text{iff} & \text{there is } v \in W \text{ such that } wRv \text{ and } (\mathfrak{M},v) \models \varphi.
\end{array}
$$

It should be also clear that a function $f\colon W \to W$ is a continuous mapping on $\mathfrak{T}_{\mathfrak{G}}$ if, for all $w, v \in W$,

$$wRv \quad \text{implies} \quad f(w)Rf(v).$$

The function f is a *homeomorphism* on $\mathfrak{T}_{\mathfrak{G}}$ if f is bijective and the converse implication holds as well.

Let $\mathbb{R}^n$ denote the standard Euclidean space of dimension n and $\mathbb{R}$ is the real line. For $n \geq 2$, a *unit ball* is a DTS $\mathfrak{B}^n = \langle B^n, f\rangle$, where B^n is a ball in $\mathbb{R}^n$ of radius 1, and f is a *measure preserving* homeomorphism on B^n.

The results of the theorem below were explicitly proved in or easily follow from [2, 13, 12].

Theorem 1. *The four dynamic topo-logics listed below coincide, have the finite model property, are finitely axiomatisable, and so decidable:*

1. $\mathsf{Log}_{\bigcirc}\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS*$\}$,
2. $\mathsf{Log}_{\bigcirc}\{\langle\mathbb{R}^n, f\rangle \mid \langle\mathbb{R}^n, f\rangle$ *a DTS*, $n \geq 1\}$,
3. $\mathsf{Log}_{\bigcirc}\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS*, $\mathfrak{T}$ *an Aleksandrov space*$\}$,
4. $\mathsf{Log}_{\bigcirc}\{\mathfrak{B}^n \mid \mathfrak{B}^n$ *a unit ball*, $n \geq 2\}$.

Later on we will use the fact that $\mathsf{Log}_{\bigcirc}\{\langle\mathbb{R}, x \mapsto x+1\rangle\}$ coincides with all of the logics above as well (see [12]).

We show now that the computational behaviour of dynamic topo-logics becomes completely different if we allow the use of the operators $\Box_F$ and $\Diamond_F$.

3 Undecidability and non-axiomatisability

Theorem 2. *No logic from the list below is recursively enumerable:*

1. $\mathsf{Log}\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS*$\}$,
2. $\mathsf{Log}\{\langle\mathbb{R}^n, f\rangle \mid \langle\mathbb{R}^n, f\rangle$ *a DTS*, $n \geq 1\}$,
3. $\mathsf{Log}\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS*, $\mathfrak{T}$ *an Aleksandrov space*$\}$,
4. $\mathsf{Log}\{\mathfrak{B}^n \mid \mathfrak{B}^n$ *a unit ball*, $n \geq 2\}$.

Remark 3. Before proceeding to the proof, we note that all logics mentioned in this theorem are different. As was shown in [18], the formula $\mathbf{I}\Diamond_F(p \wedge \mathbf{C}\mathbf{I}\neg p)$ is not satisfiable in any DTS of the form $\langle\mathbb{R}^n, f\rangle$, while it is clearly satisfiable. According to [12], the formula $\mathbf{C}(\mathbf{I}p \to \bigcirc\Diamond_F\mathbf{I}p)$ is valid in all unit balls, but refuted in a DTS based on both an Aleksandrov space and $\langle\mathbb{R}^n, x \mapsto x+1\rangle$. Finally, the formula $\Diamond_F\mathbf{C}p \leftrightarrow \mathbf{C}\Diamond_F p$ is valid in DTSs based on Aleksandrov spaces, but refuted both in $\langle\mathbb{R}^n, f\rangle$ and in all unit balls.

We prove Theorem 2 by reduction of *Post's correspondence problem* or PCP, for short [17]. (Cases (2) and (4) will only be proved for $n = 1$ and $n = 2$, respectively.) The idea of the proof is taken from [7]. Let A be a finite alphabet and P a finite set of pairs $\langle\mathbf{v}_1, \mathbf{u}_1\rangle, \dots, \langle\mathbf{v}_k, \mathbf{u}_k\rangle$ of nonempty finite words

$$\mathbf{v}_i = \left\langle b^i_1, \dots, b^i_{l_i}\right\rangle, \qquad \mathbf{u}_i = \left\langle c^i_1, \dots, c^i_{r_i}\right\rangle \qquad (i = 1, \dots, k)$$

over A. For a sequence of indices $i_1, \dots, i_N$ ranging over $1, \dots, k$, let

$$m_j = l_{i_1} + \cdots + l_{i_j} \qquad \text{and} \qquad n_j = r_{i_1} + \cdots + r_{i_j},$$

for $1 \leq j \leq N$. The following problem is undecidable (for a proof see, e.g., [8]): *given a set P of pairs of words as above, decide whether there exist an $N \geq 1$ and a sequence $i_1, \dots, i_N$ of indices such that*

$$\mathbf{v}_{i_1} * \cdots * \mathbf{v}_{i_N} = \mathbf{u}_{i_1} * \cdots * \mathbf{u}_{i_N}, \tag{1}$$

where $*$ is the concatenation operation. If the condition above holds for a PCP instance P then we say that P *has a solution*. Later, we will use a consequence of this result, namely, that the set of PCP instances without solutions *is not recursively enumerable*.

The reduction formula $\phi_{A,P}$ is constructed using the propositional variables r, s, *left* and *right*, left_a and right_a, for every $a \in A$, as well as pair_i, for every pair $\langle\mathbf{v}_i, \mathbf{u}_i\rangle$ in P, $1 \leq i \leq k$.

The variable s is used to introduce a new operator $\mathbf{S}$ (which can be interpreted as a 'strict diamond' in Kripke quasi-ordered frames). Namely, for every $\mathcal{DTL}$-formula ψ, we put

$$\mathbf{S}\psi = \big(s \to \mathbf{C}(\neg s \wedge \mathbf{C}\psi)\big) \wedge \big(\neg s \to \mathbf{C}(s \wedge \mathbf{C}\psi)\big).$$

Denote by $\mathbf{S}^m$ a string of m operators $\mathbf{S}$ (so that $\mathbf{S}^0\psi = \psi$). The variable r is used to 'relativise' $\Box_F$ in the following ways: $\Box_F^{<\mathbf{r}}\psi = \Box_F^+(\Diamond_F r \to \psi)$ and $\Box_F^{\leq\mathbf{r}}\psi = \Box_F^+(\Diamond_F^+ r \to \psi)$.

Now $\phi_{A,P}$ is defined as the conjunction

$$\phi_{A,P} = \varphi_{eq} \wedge \varphi_{pair} \wedge \varphi_{stripe} \wedge \varphi_{left} \wedge \varphi_{right},$$

where

$$\varphi_{eq} = \Diamond_F\big(r \wedge \bigwedge_{a \in A}\mathbf{I}(\mathit{left}_a \leftrightarrow \mathit{right}_a)\big),$$

$$\varphi_{pair} = \Box_F^{<\mathbf{r}}\Big(\bigvee_{1 \leq i \leq k}\mathit{pair}_i \wedge \bigwedge_{1 \leq i < j \leq k}\neg(\mathit{pair}_i \wedge \mathit{pair}_j)\Big),$$

$$\varphi_{stripe} = \Box_F^{<\mathbf{r}} \mathbf{I}(s \leftrightarrow \bigcirc s),$$

φ_{left} is the conjunction of (2)–(8), for $1 \le i \le k$,

$$\bigwedge_{\substack{a \neq b \\ a,b \in A}} \Box_F^{\leq \mathbf{r}} \mathbf{I} \neg \big(left_a \wedge left_b \big) \ \wedge \ \Box_F^{\leq \mathbf{r}} \mathbf{I} \big(left \leftrightarrow \bigvee_{a \in A} left_a \big), \tag{2}$$

$$\bigwedge_{a \in A} \Box_F^{<\mathbf{r}} \mathbf{I}(left_a \to \bigcirc left_a), \tag{3}$$

$$\mathbf{I} \neg left \ \wedge \ \Box_F^{\leq \mathbf{r}} \mathbf{I}(\neg left \to \neg \mathbf{S} left), \tag{4}$$

$$\Box_F^{<\mathbf{r}} \big(pair_i \to \mathbf{I}(\neg left \to \bigcirc \neg \mathbf{S}^{l_i} left) \big), \tag{5}$$

$$\Box_F^{<\mathbf{r}} \big(pair_i \to \bigwedge_{j < l_i} \bigcirc \mathbf{I} \big((\mathbf{S}^j left \wedge \neg \mathbf{S}^{j+1} left) \to left_{b^i_{l_i - j}} \big) \big), \tag{6}$$

$$pair_i \to \bigcirc lw_i, \tag{7}$$

$$\Box_F^{<\mathbf{r}} \big(pair_i \to \mathbf{I}((left \wedge \neg \mathbf{S} left) \to \mathbf{S} \bigcirc lw_i) \big), \tag{8}$$

where

$$lw_i = left_{b^i_1} \wedge \mathbf{S} \big(left_{b^i_2} \wedge \mathbf{S}(left_{b^i_3} \wedge \cdots \wedge \mathbf{S} left_{b^i_{l_i}}) \dots \big)$$

(remember that l_i is the length of the word $\mathbf{v}_i = \langle b^i_1, \dots, b^i_{l_i} \rangle$), and the conjunct φ_{right}—the 'right counterpart' of φ_{left}—is defined by replacing in φ_{left} all occurrences of *left* with *right*, $left_a$ with $right_a$, l_i with r_i, etc.

We also require $\mathcal{DTL}_{\bigcirc}$-formulas $\phi^n_{A,P}$, $n > 0$, which are defined similarly to $\phi_{A,P}$ by replacing

- φ_{eq} with $\bigcirc^n \bigwedge_{a \in A} \mathbf{I}(left_a \leftrightarrow right_a)$, and
- every occurrence of $\Box_F^{\leq \mathbf{r}} \psi$ (or $\Box_F^{<\mathbf{r}} \psi$) with $\Box_F^{\leq n} \psi$ (respectively, $\Box_F^{<n} \psi$), where

$$\Box_F^{\leq n} \psi = \psi \wedge \bigcirc \psi \wedge \bigcirc\bigcirc \psi \wedge \cdots \wedge \bigcirc^n \psi \qquad \text{and} \qquad \Box_F^{<n} \psi = \psi \wedge \bigcirc \psi \wedge \bigcirc\bigcirc \psi \wedge \cdots \wedge \bigcirc^{n-1} \psi.$$

We denote the subformula of $\phi^n_{A,P}$ corresponding to a subformula φ of $\phi_{A,P}$ by φ^n.

Lemma 4. *If $\phi_{A,P}$ is satisfiable in $\langle \mathfrak{T}, f \rangle$ then there is $n > 0$ such that $\phi^n_{A,P}$ is satisfiable in $\langle \mathfrak{T}, f \rangle$.*

Proof. Suppose $(\mathfrak{M}, w) \models \phi_{A,P}$. Then $(\mathfrak{M}, w) \models \Diamond_F (r \wedge \bigwedge_{a \in A} \mathbf{I}(left_a \leftrightarrow right_a))$, that is, there exists $m > 0$ such that $(\mathfrak{M}, f^m(w)) \models r \wedge \bigwedge_{a \in A} \mathbf{I}(left_a \leftrightarrow right_a)$. Let n be the minimal such m. One can easily check that $(\mathfrak{M}, w) \models \phi^n_{A,P}$. □

Lemma 5. *If P has a solution, then the following hold:*

(i) *$\phi_{A,P}$ is satisfiable in a DTS $\langle \mathfrak{T}, f \rangle$, where $\mathfrak{T}$ is an Aleksandrov space;*

(ii) *$\phi_{A,P}$ is satisfiable in a DTS;*

(iii) *$\phi_{A,P}$ is satisfiable in $\mathfrak{B}^2$;*

(iv) *$\phi_{A,P}$ is satisfiable in $\langle \mathbb{R}, f \rangle$, where $f : x \mapsto x + 1$.*

Proof. Suppose that P has a solution

$$\mathbf{v}_{i_1} * \cdots * \mathbf{v}_{i_N} = \mathbf{u}_{i_1} * \cdots * \mathbf{u}_{i_N}. \tag{9}$$

Let $\mathbf{v}_{i_1} * \cdots * \mathbf{v}_{i_N} = \langle b_0, \ldots, b_{m_N - 1} \rangle$.

(i) Define a quasi-order $\mathfrak{G} = \langle W, R \rangle$ by taking $W = \{0, \ldots, 2m_N\} \times \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, and $(x,y)R(x',y')$ iff $x \leq x'$ and $y = y'$. Define $f : W \to W$ by taking $f(x,y) = (x, y+1)$. Clearly, f is a homeomorphism on $\mathfrak{T}_{\mathfrak{G}}$. Finally, define $\mathfrak{V}$ by taking

$$
\begin{aligned}
\mathfrak{V}(s) &= \{(2n,z) \mid 0 \leq 2n \leq 2m_N,\ z \in \mathbb{Z}\}, & \mathfrak{V}(r) &= \{(0,N)\},\\
\mathfrak{V}(pair_i) &= \{(0, j-1) \mid i = i_j, \text{ for some } j \leq N\}, & &\\
\mathfrak{V}(left_a) &= \{(2k, j) \mid 0 < j \leq N,\ k < m_j,\ b_k = a\}, & \mathfrak{V}(left) &= \textstyle\bigcup_{a \in A} \mathfrak{V}(left_a),\\
\mathfrak{V}(right_a) &= \{(2k, j) \mid 0 < j \leq N,\ k < n_j,\ b_k = a\}, & \mathfrak{V}(right) &= \textstyle\bigcup_{a \in A} \mathfrak{V}(right_a).
\end{aligned}
$$

Let $\mathfrak{M} = \langle \mathfrak{T}_{\mathfrak{G}}, \mathfrak{V} \rangle$. It is an easy exercise to show $(\mathfrak{M}, (0,0)) \models \phi_{A,P}$. We leave this to the reader.

(ii) follows from (i).

(iii) We only consider the two-dimensional unit ball $\mathfrak{B}^2 = \langle B^2, g \rangle$, where g is the *rotation* of B^2 clockwise by some *rational* angle α such that $0 < \alpha < 2\pi/N+1$ and N is given by (9) (for $n > 2$, the construction is similar: we rotate the ball around a fixed axis by the same angle α). Obviously, g is a measure preserving homeomorphism.

Let E be an open set, say, a smaller open ball contained in the sector $[-\alpha/2, \alpha/2]$ of B^2 and let $E_i = g^i(E)$, for $i < \omega$. Note that $E = E_0, E_1, \ldots, E_N$ are pairwise disjoint sets. Let $\mathfrak{H} = \langle \{0, \ldots, 2m_N\}, \leq \rangle$. By the main result of McKinsey and Tarski [15], there exists a continuous mapping h_i from E_i *onto* $\mathfrak{T}_{\mathfrak{H}}$. Moreover, one may assume that, for every $x \in E_i$, we have $h_i(x) = h_{i+1}(g(x))$ and that $h_i(e_i) = 0$, where e_i is the center of the ball E_i. Define a valuation $\mathfrak{V}$ on $\mathfrak{B}^2$ by taking

$$
\begin{aligned}
\mathfrak{V}(s) &= \textstyle\bigcup_{i=0}^{N} \{h_i^{-1}(2n) \mid 0 \leq 2n \leq 2m_N\}, & \mathfrak{V}(r) &= \{e_N\},\\
\mathfrak{V}(pair_i) &= \{e_{j-1} \mid i = i_j, \text{ for some } j \leq N\}, & &\\
\mathfrak{V}(left_a) &= \textstyle\bigcup_{j=1}^{N} \{h_j^{-1}(2k) \mid k < m_j,\ b_k = a\}, & \mathfrak{V}(left) &= \textstyle\bigcup_{a \in A} \mathfrak{V}(left_a),\\
\mathfrak{V}(right_a) &= \textstyle\bigcup_{j=1}^{N} \{h_j^{-1}(2k) \mid k < n_j,\ b_k = a\}, & \mathfrak{V}(right) &= \textstyle\bigcup_{a \in A} \mathfrak{V}(right_a).
\end{aligned}
$$

As α is rational, we have $g^j(e_0) = e_N$ iff $j = N$. It is not hard to check now that $(\langle \mathfrak{B}, \mathfrak{V} \rangle, e_0) \models \phi_{A,P}$.

(iv) We know from (i) and Lemma 4 that there exists $n > 0$ such that $\phi_{A,P}^n$ is satisfiable in a DTS $\langle \mathfrak{T}, f \rangle$. Then, by the remark following Theorem 1 and since $\phi_{A,P}^n$ is a $\mathcal{DTL}_{\bigcirc}$-formula, $(\mathfrak{M}, w) \models \phi_{A,P}^n$, for some model $\mathfrak{M} = \langle \langle \mathbb{R}, f \rangle, \mathfrak{V} \rangle$ and $f : x \mapsto x+1$. Define a new valuation $\mathfrak{V}'$ on $\mathbb{R}$ which coincides with $\mathfrak{V}$ except for only one case: now we set $\mathfrak{V}'(r) = \{f^n(w)\}$. (Note that r does not occur in $\phi_{A,P}^n$.) Let $\mathfrak{M}' = \langle \langle \mathbb{R}, f \rangle, \mathfrak{V}' \rangle$. Then clearly $(\mathfrak{M}', w) \models \phi_{A,P}$. □

Lemma 6. *Suppose that there exists $n > 0$ such that $\phi_{A,P}^n$ is satisfiable in a DTS based on an Aleksandrov space. Then P has a solution.*

Proof. Suppose that $(\mathfrak{M}, w_1^0) \models \phi_{A,P}^N$ for some DTM $\mathfrak{M} = \langle \langle \mathfrak{T}_{\mathfrak{G}}, f \rangle, \mathfrak{V} \rangle$, where $\mathfrak{G} = \langle W, R \rangle$ is a quasi-order, f a homeomorphism on $\mathfrak{T}_{\mathfrak{G}}$ and $w_1^0 \in W$. For $j < \omega$, let

$$W_j = \{w \in W \mid f^j(w_1^0) R w\}.$$

As $(\mathfrak{M}, w_1^0) \models \bigcirc^N \bigwedge_{a \in A} \mathbf{I}(left_a \leftrightarrow right_a)$, we have

$$(\mathfrak{M}, f^N(w_1^0)) \models \bigwedge_{a \in A} \mathbf{I}(left_a \leftrightarrow right_a). \tag{10}$$

Since $(\mathfrak{M}, w_1^0) \models \varphi_{stripe}^N$, for each $w \in W_0$ and each $j \leq N$, we have $(\mathfrak{M}, w) \models s$ iff $(\mathfrak{M}, f^j(w)) \models s$.

Denote by S_j, $j \leq N$, the transitive binary relation on W_j defined by taking wS_jv iff there is $u \in W_j$ such that $wRuRv$ and $(\mathfrak{M}, w) \models s$ iff $(\mathfrak{M}, u) \not\models s$. Then we clearly have that, for every $j \leq N$ and every $w \in W_j$,

$$(\mathfrak{M}, w) \models \mathbf{S}\psi \quad \text{iff} \quad \text{there is } v \in W_j \text{ such that } wS_jv \text{ and } (\mathfrak{M}, v) \models \psi.$$

Note that, since f is a homeomorphism and in view of $(\mathfrak{M}, w_1^0) \models \varphi^N_{stripe}$, for all $w, v \in W_0$ and $i \leq N$, we have wS_0v iff $f^i(w)S_if^i(v)$.

Let $i_1, \dots, i_N$ be the sequence of indices such that, for $1 \leq j \leq N$, we have $(\mathfrak{M}, f^{j-1}(w_1^0)) \models pair_{i_j}$ (φ^N_{pair} ensures that there is a unique sequence of this sort). We claim that (1) holds for this sequence.

For every j with $1 \leq j \leq N$, let

$$W_j^L = \{w \in W_j \mid (\mathfrak{M}, w) \models \mathit{left}\}.$$

Call a sequence $\langle w_1, \dots, w_l \rangle$ of (not necessarily distinct) points from W_j^L an *S_j-path in W_j^L of length l* if $w_1S_jw_2S_j \dots S_jw_l$, and set

$$\mathit{leftword}_j(w_1, \dots, w_l) = \langle a_1, \dots, a_l \rangle,$$

where the a_i are the (uniquely determined by (2)) symbols from A such that $(\mathfrak{M}, w_i) \models \mathit{left}_{a_i}$.

We show now that there is a sequence $\pi_1, \dots, \pi_N$ such that, for every j with $1 \leq j \leq N$, the following hold:

(a) $\pi_j = \langle w_1^j, \dots, w_{m_j}^j \rangle$ is an S_j-path in W_j^L of length m_j, and there is no S_j-path in W_j^L of length $> m_j$;

(b) $f(w_1^0) = w_1^1$ and if $j > 1$ then $w_m^j = f(w_m^{j-1})$, for all m, $1 \leq m \leq m_{j-1}$;

(c) $\mathit{leftword}_j(w_1^j, \dots, w_{m_j}^j) = \mathbf{v}_{i_1} * \dots * \mathbf{v}_{i_j}$;

(d) for every S_j-path $\langle w_1, \dots, w_{m_j} \rangle$ in W_j^L of length m_j, we have $\mathit{leftword}_j(w_1, \dots, w_{m_j}) = \mathbf{v}_{i_1} * \dots * \mathbf{v}_{i_j}$.

Indeed, by $(\mathfrak{M}, w_1^0) \models pair_{i_1}$, (7), (4) and (5), there exists an S_1-path π_1 in W_1^L such that (a)–(c) hold. Condition (d) follows from (6).

Now assume inductively that conditions (a)–(d) hold for some $j-1$ with $1 \leq j-1 < N$. Let $\pi_{j-1} = \langle w_1^{j-1}, \dots, w_{m_{j-1}}^{j-1} \rangle$ be an S_{j-1}-path in W_{j-1}^L for which (a)–(d) hold. By (3), the sequence $\langle f(w_1^{j-1}), \dots, f(w_{m_{j-1}}^{j-1}) \rangle$ is an S_j-path in W_j^L. Since $(\mathfrak{M}, w_{m_{j-1}}^{j-1}) \models \mathit{left} \wedge \neg \mathbf{S}\mathit{left}$ and $(\mathfrak{M}, w_1^{j-1}) \models pair_{i_j}$, (8) means that there exists a sequence $w_{m_{j-1}+1}^j, \dots, w_{m_{j-1}+l_{i_j}}^j$ of points in W_j^L such that

$$\pi_j = \langle f(w_1^{j-1}), \dots, f(w_{m_{j-1}}^{j-1}), w_{m_{j-1}+1}^j, \dots, w_{m_{j-1}+l_{i_j}}^j \rangle$$

is an S_j-path in W_j^L of length $m_j = m_{j-1} + l_{i_j}$ such that $\mathit{leftword}_j(w_{n_{j-1}+1}^j, \dots, w_{m_{j-1}+l_{i_j}}^j) = \mathbf{v}_{i_j}$. By (5) and the induction hypothesis, there is no S_j-path in W_j^L of length $> m_j$. Thus, (a) and (b) hold for π_j, (c) follows from (3), and (d) from (6) and the induction hypothesis.

Now define sets W_j^R in the same way as W_j^L, but with *left* replaced by *right*, introduce the notion of an S_j-path in W_j^R, and, for every sequence $w_1, \dots, w_l$ of points from W_j^R, set

$$\mathit{rightword}_j(w_1, \dots, w_l) = \langle a_1, \dots, a_l \rangle,$$

where the a_i are the uniquely determined (by 'right analogue' of (2)) elements of A such that $(\mathfrak{M}, w_i) \models \mathit{right}_{a_i}$. In precisely the same way as above we show now that there is a sequence $\pi'_1, \dots, \pi'_N$ such that, for every j with $1 \leq j \leq N$, (a′) $\pi'_j = \langle w^j_1, \dots, w^j_{n_j} \rangle$ is an S_j-path in W^R_j of length n_j, and there is no S_j-path in W^R_j of length $> n_j$;

(b′) $f(w^0_1) = w^1_1$ and if $j > 1$ then $w^j_n = f(w^{j-1}_n)$, for all n with $1 \le n \le n_{j-1}$;

(c′) $rightword_j(w^j_1, \dots, w^j_{n_j}) = \mathbf{u}_{i_1} * \dots * \mathbf{u}_{i_j}$;

(d′) for every S_j-path $\langle w_1, \dots, w_{n_j} \rangle$ in W^R_j of length n_j, we have $rightword_j(w_1, \dots, w_{n_j}) = \mathbf{u}_{i_1} * \dots * \mathbf{u}_{i_j}$.

Now it is easy to see that (10) means that

$$\mathbf{v}_{i_1} * \dots * \mathbf{v}_{i_N} = leftword_N(w^N_1, \dots, w^N_{m_N}) = rightword_N(w^N_1, \dots, w^N_{n_N}) = \mathbf{u}_{i_1} * \dots * \mathbf{u}_{i_N},$$

as required. □

Theorem 2 now follows immediately. Just observe that we have proved that, for any of the classes $\mathcal{K}$ of DTSs listed in Theorem 2, $\phi_{A,P}$ is satisfiable in $\mathcal{K}$ iff P has a solution. Indeed, the direction from right to left is Lemma 5. The direction from left to right for DTSs based on Aleksandrov spaces follows from Lemmas 4 and 6. For the remaining classes this direction follows from Lemmas 4, 6, and Theorem 1, since the $\phi^n_{A,P}$ are $\mathcal{DTL}_{\bigcirc}$-formulas.

4 Dynamic metric logic

The language $\mathcal{DML}$ of *dynamic metric logic* is defined in the same way as $\mathcal{DTL}$ with the exception that the topological operators are replaced by the *metric operators* $\exists^{\le a}$, for $a \in \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the set of positive rational numbers. The intended semantics of this logic is defined as follows.

A *dynamic metric structure* (DMS, for short) is a pair $\mathfrak{F} = \langle \langle W, d \rangle, f \rangle$, where $\langle W, d \rangle$ is a metric space (with a metric d) and $f \colon W \to W$ is a *metric automorphism*, i.e., a bijection on W such that $d(x,y) = d(f(x), f(y))$ for all $x, y \in W$. For example, the map $x \mapsto x+1$ on $\mathbb{R}$ and the rotation g on B^2 considered above are metric automorphisms on the respective spaces with the Euclidean metric.

A *dynamic metric model* (or DMM) is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, where $\mathfrak{F}$ is a DMS and $\mathfrak{V}$ a valuation defined in precisely the same way as in the topological case. The truth-relation is also defined in the same manner as for DTMs with the exception that the truth-conditions for the topological operators **I** and **C** are replaced by

$$(\mathfrak{M}, x) \models \exists^{\le a}\varphi \quad \text{iff} \quad \text{there exists } y \in W \text{ such that } d(x,y) \le a \text{ and } (\mathfrak{M}, y) \models \varphi.$$

In contrast to the topological case, now we have the following:

Theorem 7. *The set of $\mathcal{DML}$-formulas that are valid in all DMSs is decidable. However, the decision problem is not elementary.*

Roughly, the idea of the decidability proof is similar to that of Theorem 13.6 from [7]: first we represents DMMs in the form of *quasimodels* and then show that quasimodels can be encoded in monadic second-order logic. The main novelty of this proof is the rather involved notion of a quasimodel.

Given a $\mathcal{DML}$-formula φ, denote by γ_φ the maximal numerical parameter occurring in φ and by $M[\varphi] \subseteq \mathbb{Q}^+$ the smallest set containing all numerical parameters in φ and closed under the rule

$(+)$ if $a, b \in M[\varphi]$ and $a + b \le \gamma_\varphi$, then $a + b \in M[\varphi]$.

Clearly, $M[\varphi]$ is finite. Let $M^+[\varphi] = M[\varphi] \cup \{2 \cdot \gamma_\varphi\}$.

Define the *metric depth* $mtd(\varphi)$ of φ inductively by taking:

$$
\begin{aligned}
mtd(p) &= 0, & mtd(\exists^{\leq a}\varphi) &= mtd(\varphi) + a,\\
mtd(\neg\varphi) &= mtd(\varphi), & mtd(\bigcirc\varphi) &= mtd(\varphi),\\
mtd(\varphi_1 \wedge \varphi_2) &= \max(mtd(\varphi_1), mtd(\varphi_2)), & mtd(\Box_F\varphi) &= mtd(\varphi).
\end{aligned}
$$

Our first observation is that every satisfiable $\mathcal{DML}$-formula φ can be satisfied in a DMS that is based on the metric space generated by some intransitive labelled tree.

Given an intransitive tree $\langle T, < \rangle$ and a function δ labelling the edges of $\langle T, < \rangle$ with positive real numbers, we denote by $\langle T, \delta^* \rangle$ the *tree metric space induced by* $\langle T, < \rangle$ and δ, i.e., for any $x \neq y$ in T, $\delta^*(x,y)$ is the sum of labels on the edges occurring in the shortest path from x to y in $\langle T, < \rangle$, and $\delta^*(x,x) = 0$. If $\delta^*(x,y)$ is bounded, then the number

$$\max\{\delta^*(r,x) \mid r \text{ the root and } x \in T\}$$

is called it the *radius* of the tree metric space $\langle T, \delta^* \rangle$.

Lemma 8. *A $\mathcal{DML}$-formula φ (with $mtd(\varphi) > 0$) is satisfiable iff it is satisfiable in a DMS of the form $\mathfrak{F}' = \langle \langle T \times \mathbb{Z}, d' \rangle, f' \rangle$, where*

- *$\langle \langle T, < \rangle, \delta \rangle$ is a labelled intransitive tree such that $\delta(x,y) \in M^+[\varphi]$, for all $x, y \in T$ with $x < y$, and $\langle T, \delta^* \rangle$ is of radius $\leq mtd(\varphi)$;*
- *$\langle T \times \mathbb{Z}, d' \rangle$ is the metric space with*

$$d'(\langle x, i \rangle, \langle y, j \rangle) = \begin{cases} \delta^*(x,y), & \text{if } i = j,\\ 3 \cdot mtd(\varphi), & \text{otherwise}; \end{cases}$$

- *$f'(\langle x, i \rangle) = \langle x, i+1 \rangle$, for all $\langle x, i \rangle \in T \times \mathbb{Z}$.*

Proof. Suppose that $(\mathfrak{M}, u_0) \models \varphi$ for some model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ with $\mathfrak{F} = \langle \langle W, d \rangle, f \rangle$ and some point $u_0 \in W$. For any $u, v \in W$, put

$$d_0(u,v) = \min\bigl(\{2 \cdot \gamma_\varphi\} \cup \{a \in M[\varphi] \mid d(u,v) \leq a\}\bigr) \in M^+[\varphi].$$

Now define the required labelled tree $\langle \langle T, < \rangle, \delta \rangle$ by taking

$$
\begin{aligned}
&T = \{\langle u_0, u_1, \ldots, u_n \rangle \mid \sum_{i=1}^{n} d_0(u_{i-1}, u_i) \leq mtd(\varphi),\ u_1, \ldots, u_n \in W\},\\
&x < y \quad \text{iff} \quad x = \langle u_0, \ldots, u_n \rangle \text{ and } y = \langle u_0, \ldots, u_n, u_{n+1} \rangle,\\
&\delta(\langle u_0, \ldots, u_n \rangle, \langle u_0, \ldots, u_n, u_{n+1} \rangle) = d_0(u_n, u_{n+1}).
\end{aligned}
$$

Clearly, the radius of $\langle T, \delta^* \rangle$ is $\leq mtd(\varphi)$.

Let $\mathfrak{F}' = \langle \langle T \times \mathbb{Z}, d' \rangle, f' \rangle$, where d' and f' are defined as above. It is easy to see that $\langle T \times \mathbb{Z}, d' \rangle$ is indeed a metric space and $f' \colon T \times \mathbb{Z} \to T \times \mathbb{Z}$ a metric automorphism. Put, for every propositional variable p,

$$\mathfrak{V}'(p) = \{\langle \langle u_0, \ldots, u_n \rangle, i \rangle \in T \times \mathbb{Z} \mid (\mathfrak{M}, f^i(u_n)) \models p\}$$

and $\mathfrak{M}' = \langle \mathfrak{F}', \mathfrak{V}' \rangle$.

Denote by x_0 the root $\langle u_0\rangle$ of $\langle T,<\rangle$. We show now by induction that, for every $\psi\in sub\,\varphi$, every $i\in\mathbb{Z}$ and every $x=\langle u_0,\dots,u_n\rangle\in T$ such that

$$\delta^*(x_0,x)\leq mtd(\varphi)-mtd(\psi), \tag{11}$$

we have

$$(\mathfrak{M},f^i(u_n))\models\psi \quad\text{iff}\quad (\mathfrak{M}',\langle x,i\rangle)\models\psi. \tag{12}$$

The basis of induction (propositional variables) follows immediately from the definition of $\mathfrak{M}'$. The case of the Booleans is trivial.

Case $\psi=\exists^{\leq a}\psi'$. Let $(\mathfrak{M},f^i(u_n))\models\exists^{\leq a}\psi'$. Since f is a metric automorphism, there is $u_{n+1}\in W$ such that $d(u_n,u_{n+1})=d(f^i(u_n),f^i(u_{n+1}))\leq a$ and $(\mathfrak{M},f^i(u_{n+1}))\models\psi'$. Take $y=\langle u_0,\dots,u_n,u_{n+1}\rangle$. By (11), $y\in T$. By the definition of δ, we have $\delta^*(x,y)\leq a$. By the triangle inequality,

$$\delta^*(x_0,y)\quad\leq\quad\delta^*(x_0,x)+\delta^*(x,y)\quad\leq\quad mtd(\varphi)-mtd(\psi').$$

By IH, $(\mathfrak{M}',\langle y,i\rangle)\models\psi'$ and, since $d'(\langle x,i\rangle,\langle y,i\rangle)\leq a$, we finally obtain $(\mathfrak{M}',\langle x,i\rangle)\models\exists^{\leq a}\psi'$.

Conversely, suppose $(\mathfrak{M}',\langle x,i\rangle)\models\exists^{\leq a}\psi'$. Then there is $\langle y,j\rangle\in T\times\mathbb{Z}$ with $d'(\langle x,i\rangle,\langle y,j\rangle)\leq a$ and $(\mathfrak{M}',\langle y,j\rangle)\models\psi'$. By the definition of d', we must have $j=i$, and so $\delta^*(x,y)\leq a$. By the triangle inequality, $\delta^*(x_0,y)\leq\delta^*(x_0,x)+\delta^*(x,y)\leq mtd(\varphi)-mtd(\psi')$. By IH, $(\mathfrak{M},f^i(v_m))\models\psi'$ for $\langle u_0,v_1,\dots,v_m\rangle=y$. As f is a metric automorphism, $d(f^i(u_n),f^i(v_m))=d(u_n,v_m)$ and, by the definition of δ^*, we have $d(u_n,v_m)\leq\delta^*(x,y)$. Therefore, $d(f^i(u_n),f^i(v_m))\leq a$ and $(\mathfrak{M},f^i(u_n))\models\exists^{\leq a}\psi'$.

Case $\psi=\bigcirc\psi'$.

$$\begin{array}{lll}(\mathfrak{M},f^i(u_n))\models\bigcirc\psi' & \text{iff} & (\mathfrak{M},f^{i+1}(u_n))\models\psi'\\ & \text{iff} & (\mathfrak{M}',\langle x,i+1\rangle)\models\psi' \quad\text{[by IH]}\\ & \text{iff} & (\mathfrak{M}',\langle x,i\rangle)\models\bigcirc\psi' \quad\text{[by definition of } f'\text{]}.\end{array}$$

Case $\psi=\Box_F\psi'$ is similar.

It follows that $(\mathfrak{M}',\langle x_0,0\rangle)\models\varphi$. □

Define the *closure* $cl\,\varphi$ of φ to be the set

$$\{\psi,\neg\psi\mid\psi\in sub\,\varphi\}\cup\{\exists^{\leq a}\psi,\neg\exists^{\leq a}\psi\mid\psi\in sub\,\varphi\text{ and }a\in M[\varphi]\},$$

where $sub\,\varphi$ is the set of all subformulas of φ. A *type* for φ is a subset $\boldsymbol{t}$ of $cl\,\varphi$ such that

- for every $\neg\psi\in cl\,\varphi$, $\psi\in\boldsymbol{t}$ iff $\neg\psi\notin\boldsymbol{t}$;
- for every $\psi_1\wedge\psi_2\in cl\,\varphi$, $\psi_1\wedge\psi_2\in\boldsymbol{t}$ iff $\psi_1\in\boldsymbol{t}$ and $\psi_2\in\boldsymbol{t}$.

We are now in a position to define the notion of a quasimodel for a given formula φ as a set of quasistates connected by runs.

A *quasistate* $\boldsymbol{q}$ for φ is a triple $\boldsymbol{q}=\langle\langle T_{\boldsymbol{q}},<_{\boldsymbol{q}}\rangle,\delta_{\boldsymbol{q}},\boldsymbol{t}_{\boldsymbol{q}}\rangle$, where

- $\langle T_{\boldsymbol{q}},<_{\boldsymbol{q}}\rangle$ is an intransitive tree with a labelling function $\delta_{\boldsymbol{q}}:\{(u,v)\in T_{\boldsymbol{q}}\times T_{\boldsymbol{q}}\mid u<_{\boldsymbol{q}}v\}\to M^+[\varphi]$ such that the radius of $\langle T_{\boldsymbol{q}},\delta^*_{\boldsymbol{q}}\rangle$ is bounded by $mtd(\varphi)$;
- $\boldsymbol{t}_{\boldsymbol{q}}$ is a function associating with each $u\in T_{\boldsymbol{q}}$ a type $\boldsymbol{t}_{\boldsymbol{q}}(u)$ for φ, and

(qs1) for every $u \in T_{\boldsymbol{q}}$ and every $\exists^{\leq a}\psi \in cl\,\varphi$, we have $\exists^{\leq a}\psi \in \boldsymbol{t}_{\boldsymbol{q}}(u)$ iff there is $v \in T_{\boldsymbol{q}}$ such that $\delta^*_{\boldsymbol{q}}(u,v) \leq a$ and $\psi \in \boldsymbol{t}_{\boldsymbol{q}}(v)$;

(qs2) for no $u \in T_{\boldsymbol{q}}$, there exist two isomorphic substructures generated by immediate $<_{\boldsymbol{q}}$-successors v_1 and v_2 of u and such that $\delta_{\boldsymbol{q}}(u,v_1) = \delta_{\boldsymbol{q}}(u,v_2)$.

We say that a point $u \in T_{\boldsymbol{q}}$ is of *index* $\langle a_1,\dots,a_n\rangle$ if $u_0 <_{\boldsymbol{q}} u_1 <_{\boldsymbol{q}} \cdots <_{\boldsymbol{q}} u_n = u$, where u_0 is the root of $\left\langle T_{\boldsymbol{q}}, <_{\boldsymbol{q}}\right\rangle$, and $a_i = \delta_{\boldsymbol{q}}(u_{i-1},u_i)$, for all i, $1 \leq i \leq n$. The index of the root, u_0, is $\langle\rangle$.

Let $\mathbf{q}$ be a function associating with each $i \in \mathbb{Z}$ a quasistate $\mathbf{q}(i) = \langle\langle T_i, <_i\rangle, \delta_i, \boldsymbol{t}_i\rangle$ for φ. A *run of index* $\langle a_1,\dots,a_n\rangle$ through $\mathbf{q}$ is a function r mapping each $i \in \mathbb{Z}$ to a point $r(i) \in T_i$ of index $\langle a_1,\dots,a_n\rangle$ such that, for every $i \in \mathbb{Z}$

- and every $\bigcirc\psi \in cl\,\varphi$, $\quad \bigcirc\psi \in \boldsymbol{t}_i(r(i))$ iff $\psi \in \boldsymbol{t}_{i+1}(r(i+1))$;
- and every $\Box_F\psi \in cl\,\varphi$, $\quad \Box_F\psi \in \boldsymbol{t}_i(r(i))$ iff $\psi \in \boldsymbol{t}_j(r(j))$ for all $j > i$.

Given a set $\mathfrak{R}$ of runs, we denote by $\mathfrak{R}_{\langle a_1,\dots,a_n\rangle}$ its subset of all runs of index $\langle a_1,\dots,a_n\rangle$.

A *quasimodel* for φ is a pair $\langle \mathbf{q}, \mathfrak{R}\rangle$, where for every $i \in \mathbb{Z}$, $\mathbf{q}(i) = \langle\langle T_i, <_i\rangle, \delta_i, \boldsymbol{t}_i\rangle$ is a quasistate for φ such that

(qm2) $\varphi \in \boldsymbol{t}_0(u_0)$, where u_0 is the root of $\langle T_0, <_0\rangle$,

and $\mathfrak{R}$ is a set of runs through $\mathbf{q}$ satisfying the following condition

(qm3) $\mathfrak{R}_{\langle\rangle} \neq \emptyset$ and, for all $r \in \mathfrak{R}_{\langle a_1,\dots,a_n\rangle}$, $i \in \mathbb{Z}$ and $u \in T_i$, if $r(i) <_i u$ and $\delta_i(r(i),u) = a_{n+1}$ then there is a run $r' \in \mathfrak{R}_{\langle a_1,\dots,a_n,a_{n+1}\rangle}$ such that $r'(i) = u$ and $r(i) <_i r'(i)$ for all $i \in \mathbb{Z}$.

Lemma 9. *A $\mathcal{DML}$-formula φ (with $mtd(\varphi) > 0$) is satisfiable in a DMM iff there is a quasimodel for φ.*

Proof. ($\Rightarrow$) Suppose φ is satisfiable. Take some model $\mathfrak{M}' = \langle \mathfrak{F}', \mathfrak{V}'\rangle$ with $\mathfrak{F}' = \langle\langle T \times \mathbb{Z}, d'\rangle, f'\rangle$ provided by Lemma 8. Let $(\mathfrak{M}', \langle x_0, 0\rangle) \models \varphi$, where x_0 is the root of $\langle T, <\rangle$.

For every pair $\langle x, i\rangle \in T \times \mathbb{Z}$, we define a type $\boldsymbol{t}(x,i)$ for φ by taking

$$\boldsymbol{t}(x,i) = \{\psi \in cl\,\varphi \mid (\mathfrak{M}', \langle x,i\rangle) \models \psi\}.$$

Fix now $i \in \mathbb{Z}$ and define a binary relation $\sim_i$ on T by taking

$$\begin{aligned} x \sim_i y \quad \text{iff} \quad & \boldsymbol{t}(x,i) = \boldsymbol{t}(y,i) \\ & \wedge\ \forall z \in T\,\big(x < z \to \exists z' \in T\,\big(y < z' \ \wedge\ \delta(x,z) = \delta(y,z') \ \wedge\ z \sim_i z'\big)\big) \\ & \wedge\ \forall z \in T\,\big(y < z' \to \exists z \in T\,\big(x < z \ \wedge\ \delta(x,z) = \delta(y,z') \ \wedge\ z \sim_i z'\big)\big). \end{aligned}$$

Clearly, $\sim_i$ is an equivalence relation on T. Denote by $[x]_i$ the $\sim_i$-equivalence class of $x \in T$ and put, for $a \in M^+[\varphi]$,

$$[x]_i S_i^a [y]_i \quad \text{iff} \quad \exists y' \in [y]_i\ \big(x < y' \ \wedge\ \delta(x,y') = a\big).$$

By the definition of $\sim_i$, the S_i^a are well-defined. Let

$$\begin{aligned} & T_i = \{\langle [x_0]_i, a_1, [x_1]_i, a_2, \dots, a_n, [x_n]_i\rangle \ \mid\ [x_0]_i S_i^{a_1} [x_1]_i S_i^{a_2} \dots S_i^{a_n} [x_n]_i\}, \\ & u <_i v \quad \text{iff} \quad u = \langle [x_0]_i, a_1, \dots, a_n, [x_n]_i\rangle, \quad v = \langle [x_0]_i, a_1, \dots, a_n, [x_n]_i, a_{n+1}, [x_{n+1}]_i\rangle \quad \text{and} \\ & \qquad\qquad [x_n]_i S_i^{a_{n+1}} [x_{n+1}]_i, \\ & \delta_i(\langle [x_0]_i, a_1, \dots, a_n, [x_n]_i\rangle, \langle [x_0]_i, a_1, \dots, a_n, [x_n]_i, a_{n+1}, [x_{n+1}]_i\rangle) = \delta(x_n, x_{n+1}) = a_{n+1}, \\ & \boldsymbol{t}_i(\langle [x_0]_i, a_1, \dots, a_n, [x_n]_i\rangle) = \boldsymbol{t}(x_n, i). \end{aligned}$$

We show that $\langle\langle T_i,<_i\rangle,\delta_i,\boldsymbol{t}\rangle$ is a quasistate for φ. It is easy to see that $\langle T_i,<_i\rangle$ is an intransitive tree, the tree metric space $\langle T_i,\delta_i^*\rangle$ induced by $\langle\langle T_i,<_i\rangle,\delta_i\rangle$ is of radius $\leq mtd(\varphi)$ and that **(qs2)** holds.

To show **(qs1)**, suppose first that $\exists^{\leq a}\psi\in\boldsymbol{t}_i(u)$, for $u=\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i\rangle\in T_i$. By the definition of $\boldsymbol{t}_i$, we have $\exists^{\leq a}\psi\in\boldsymbol{t}(x_n,i)$, and so $(\mathfrak{M}',\langle x_n,i\rangle)\models\exists^{\leq a}\psi$. Then there is $\langle y_m,j\rangle\in T\times\mathbb{Z}$ such that $d'(\langle x_n,i\rangle,\langle y_m,j\rangle)\leq a$ and $(\mathfrak{M}',\langle y_m,j\rangle)\models\psi$. By the definition of d', $j=i$ and $\delta^*(x_n,y_m)\leq a$. Then we have $\psi\in\boldsymbol{t}(y_m,i)$. Take the sequence $x_0=y_0<y_1<\dots<y_m$. Let $v=\langle[y_0]_i,b_1,\dots,b_m,[y_m]_i\rangle\in T_i$. By the definition of $\boldsymbol{t}_i$, $\psi\in\boldsymbol{t}_i(v)$ and clearly we have $\delta_i^*(u,v)\leq\delta^*(x_n,y_m)\leq a$.

Conversely, consider $u=\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i\rangle\in T_i$ and $v=\langle[y_0]_i,b_1,\dots,b_m,[y_m]_i\rangle\in T_i$ such that $\delta_i^*(u,v)\leq a$ and $\psi\in\boldsymbol{t}_i(v)$. Clearly, $x_0=y_0$. Let $k\geq 0$ be the maximal number such that $a_j=b_j$ and $x_j\sim_i y_j$, for all j, $1\leq j\leq k$. Denote by w the common prefix $\langle[x_0]_i,a_1,\dots,a_k,[x_k]_i\rangle$ of u and v. By the definition of δ_i^*, we have $\delta_i^*(w,u)=\delta^*(x_k,x_n)$ and $\delta_i^*(w,v)=\delta^*(y_k,y_m)$. Let $b=\delta^*(y_k,y_m)$. Since $\langle T_i,\delta_i^*\rangle$ is a tree metric space, $\delta_i^*(u,v)=\delta_i^*(w,u)+\delta_i^*(w,v)$. Then $b\leq a$ follows from $\delta_i^*(u,v)\leq a$ and, as a sum of elements of $M^+[\varphi]$ not exceeding γ_φ, $b\in M[\varphi]$.

By the definition of $\boldsymbol{t}_i$, $\psi\in\boldsymbol{t}(y_m,i)$ and therefore, $(\mathfrak{M}',\langle y_m,i\rangle)\models\psi$. Then $(\mathfrak{M}',\langle y_k,i\rangle)\models\exists^{\leq b}\psi$ and, as $\exists^{\leq b}\psi\in cl\,\varphi$, $\exists^{\leq b}\psi\in\boldsymbol{t}(y_k,i)$. Since $\boldsymbol{t}(x_k,i)=\boldsymbol{t}(y_k,i)$, we obtain $(\mathfrak{M}',\langle x_k,i\rangle)\models\exists^{\leq b}\psi$. Then, there is $z\in T$ such that $\delta^*(x_k,z)\leq b$ and $(\mathfrak{M}',\langle z,i\rangle)\models\psi$. Now, by the triangle inequality,

$$\delta^*(x_n,z)\leq\underbrace{\delta^*(x_n,x_k)}_{\leq a-b}+\underbrace{\delta^*(x_k,z)}_{\leq b}.$$

Then $\delta^*(x_n,z)\leq a$ and so $(\mathfrak{M}',\langle x_n,i\rangle)\models\exists^{\leq a}\psi$. Therefore, $\exists^{\leq a}\psi\in\boldsymbol{t}(x_n,i)$ and $\exists^{\leq a}\psi\in\boldsymbol{t}_i(u)$.

So, the $\langle\langle T_i,<_i\rangle,\delta_i,\boldsymbol{t}_i\rangle$ are quasistates for φ. Let $\mathbf{q}(i)=\langle\langle T_i,<_i\rangle,\delta_i,\boldsymbol{t}_i\rangle$, for $i\in\mathbb{Z}$. As $\varphi\in\boldsymbol{t}_0([x_0]_0)$, **(qm2)** holds for $\mathbf{q}$. It remains to define runs through $\mathbf{q}$. For each sequence $\langle x_0,x_1,\dots,x_n\rangle$ of points in T such that $x_0<x_1<\dots<x_n$, take the map

$$r\colon i\mapsto\langle[x_0]_i,a_1,[x_1]_i,a_2,\dots,a_n,[x_n]_i\rangle,$$

where $a_j=\delta(x_{j-1},x_j)$, for all j, $1\leq j\leq n$. It is easy to see that r is a run of index $\langle a_1,\dots,a_n\rangle$. Let $\mathfrak{R}$ be the set of all such runs. Observe that $r_0\in\mathfrak{R}_{\langle\rangle}$, where $r_0\colon i\mapsto\langle[x_0]_i\rangle$. Take $r\in\mathfrak{R}_{\langle a_1,\dots,a_n\rangle}$, $i\in\mathbb{Z}$ and $u\in T_i$ such that $r(i)<_i u$ and $\delta_i(r(i),u)=a_{n+1}$. Clearly, r is of the form $r\colon i\mapsto\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i\rangle$, for $x_1,\dots,x_n\in T$ such that $x_0<\dots<x_n$ and $a_j=\delta(x_{j-1},x_j)$, for all j, $1\leq j\leq n$, and u is of the form $\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i,a_{n+1},[x_{n+1}]_i\rangle$, for $x_{n+1}\in T$ such that $x_n<x_{n+1}$ and $a_{n+1}=\delta(x_n,x_{n+1})$. It is easy to check that the run

$$r'\colon i\mapsto\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i,a_{n+1},[x_{n+1}]_i\rangle$$

is as required by **(qm3)**.

($\Leftarrow$) Suppose that $\langle\mathbf{q},\mathfrak{R}\rangle$ is a quasimodel for φ. First, we construct a tree metric space $\langle T_{\mathfrak{R}},\delta_{\mathfrak{R}}^*\rangle$, elements of which are sequences of runs from $\mathfrak{R}$. For each $a\in M[\varphi]$, define a binary relation $\lhd_a$ on $\mathfrak{R}$ as follows: for $r_1,r_2\in\mathfrak{R}$,

$$r_1\lhd_a r_2\quad\text{iff}\quad r_1(i)<_i r_2(i)\text{ and }\delta_i(r_1(i),r_2(i))=a,\text{ for all }i\in\mathbb{Z}.$$

Let $r_0\in\mathfrak{R}_{\langle\rangle}$ (by **(qm3)** such a run exists) and set

$$\begin{aligned}&T_{\mathfrak{R}}=\{\langle r_0,r_1,\dots r_n\rangle\mid r_0\lhd_{a_1}r_1\lhd_{a_2}\dots\lhd_{a_n}r_n,\quad a_1,\dots,a_n\in M[\varphi]\};\\&u<_{\mathfrak{R}}v\quad\text{iff}\quad u=\langle r_0,\dots,r_n\rangle,v=\langle r_0,\dots,r_n,r_{n+1}\rangle\text{ and }r_n\lhd_a r_{n+1};\\&\delta_{\mathfrak{R}}(\langle r_0,\dots,r_n\rangle,\langle r_0,\dots,r_n,r_{n+1}\rangle)=a\quad\text{iff}\quad r_n\lhd_a r_{n+1}.\end{aligned}$$

Clearly, $\langle\langle T_{\mathfrak{R}}, <_{\mathfrak{R}}\rangle, \delta_{\mathfrak{R}}\rangle$ is a labelled intransitive tree inducing a tree metric space $\langle T_{\mathfrak{R}}, \delta^*_{\mathfrak{R}}\rangle$ of radius $\leq mtd(\varphi)$. Now we construct a model $\mathfrak{M}'' = \langle \mathfrak{F}'', \mathfrak{V}''\rangle$ with $\mathfrak{F}'' = \langle\langle T_{\mathfrak{R}} \times \mathbb{Z}, d''\rangle, f''\rangle$ by taking

$$d''(\langle u,i\rangle, \langle v,j\rangle) = \begin{cases} \delta^*_{\mathfrak{R}}(u,v), & \text{if } i = j, \\ 3 \cdot mtd(\varphi), & \text{otherwise,} \end{cases}$$

$f''(\langle u,i\rangle) = \langle u, i+1\rangle$, for all $\langle u,i\rangle \in T_{\mathfrak{R}} \times \mathbb{Z}$, and $\mathfrak{V}''(p) = \{\langle\langle r_0, \dots r_n\rangle, i\rangle \in T_{\mathfrak{R}} \times \mathbb{Z} \mid p \in \boldsymbol{t}_i(r_n(i))\}$, for every propositional variable p. By induction on the construction of $\psi \in cl\,\varphi$ we show that, for every point $\langle\langle r_0, \dots, r_n\rangle, i\rangle$ in $\mathfrak{M}''$, we have

$$(\mathfrak{M}'', \langle\langle r_0, \dots, r_n\rangle, i\rangle) \models \psi \qquad \text{iff} \qquad \psi \in \boldsymbol{t}_i(r_n(i)),$$

The basis of induction follows from the definition of $\mathfrak{V}''$. The case of the Booleans follows from the definition of type.

Case $\psi = \exists^{\leq a}\psi'$. Let $u = \langle r_0, r_1, \dots, r_n\rangle$ and $(\mathfrak{M}'', \langle u,i\rangle) \models \exists^{\leq a}\psi'$. Then there is $\langle v,j\rangle \in T_{\mathfrak{R}} \times \mathbb{Z}$, $v = \langle r_0, r'_1, \dots, r'_m\rangle$, such that $d''(\langle u,i\rangle, \langle v,j\rangle) \leq a$ and $(\mathfrak{M}'', \langle v,j\rangle) \models \psi$. By the definition of d'', $j = i$ and $\delta^*_{\mathfrak{R}}(u,v) \leq a$. By IH, $\psi \in \boldsymbol{t}_i(r'_m(i))$. Let w be the longest common prefix of u and v, and k the maximal number such that $r_j = r'_j$, for all j, $1 \leq j \leq k$. As $\langle T_{\mathfrak{R}}, \delta^*_{\mathfrak{R}}\rangle$ is a tree metric space, we have $\delta^*_{\mathfrak{R}}(u,v) = \delta^*_{\mathfrak{R}}(w,u) + \delta^*_{\mathfrak{R}}(w,v)$. By the definition of $\delta_{\mathfrak{R}}$, $\delta^*_{\mathfrak{R}}(w,u) = \delta_i(r_k(i), r_n(i))$ and $\delta^*_{\mathfrak{R}}(w,v) = \delta_i(r'_k(i), r'_m(i))$. As $r_k = r'_k$, by the triangle inequality, $\delta_i(r_n(i), r'_m(i)) \leq a$ and therefore, by **(qs1)**, $\exists^{\leq a}\psi \in \boldsymbol{t}_i(r_n(i))$.

Conversely, let $\exists^{\leq a}\psi \in \boldsymbol{t}_i(r_n(i))$ for $u = \langle r_0, \dots, r_n\rangle \in T_{\mathfrak{R}}$. Then, by **(qs1)**, there is $x \in T_i$ such that $\delta^*_i(r_n(i), x) \leq a$ and $\psi \in \boldsymbol{t}_i(x)$. Let k be the maximal number such that $r_k(i) = x_k$,

$$x_k <_i x_{k+1} <_i \cdots <_i x_m$$

and $x_m = x$. Clearly, such a $k \geq 0$ exists. As $\langle T_i, \delta^*_i\rangle$ is a tree metric space, we have $\delta^*_i(r_n(i), x) = \delta^*_i(r_n(i), r_k(i)) + \delta^*_i(x_k, x_m)$. Then, by applying **(qm3)** sufficiently many times, one can construct a sequence of runs $\langle r'_k, r'_{k+1}, \dots, r'_m\rangle$ such that $r'_k = r_k$, $r'_j(i) = x_j$ and $r'_{j-1} \lhd_{a_j} r'_j$, for all j, $k < j \leq m$. Consider $v = \langle r_0, \dots r_k, r'_{k+1}, \dots, r'_m\rangle \in T_{\mathfrak{R}}$. As $\delta_{\mathfrak{R}}(\langle r_0, \dots, r_k\rangle, v) = \delta^*_i(x_k, x_m)$ and $\delta_{\mathfrak{R}}(\langle r_0, \dots, r_k\rangle, u) = \delta^*_i(r_k(i), r_n(i))$, we have $\delta^*_{\mathfrak{R}}(u,v) \leq a$. By IH, we have $(\mathfrak{M}'', \langle v,i\rangle) \models \psi$ and thus $(\mathfrak{M}'', \langle u,i\rangle) \models \exists^{\leq a}\psi$.

Case $\psi = \bigcirc\psi'$. Then we have:

$$\begin{aligned} (\mathfrak{M}'', \langle\langle r_0, \dots, r_n\rangle, i\rangle) \models \bigcirc\psi' \quad & \text{iff} \quad (\mathfrak{M}, \langle\langle r_0, \dots, r_n\rangle, i+1\rangle) \models \psi \\ & \text{iff} \quad \psi \in \boldsymbol{t}_{i+1}(r_n(i+1)) \quad [\text{by IH}] \\ & \text{iff} \quad \bigcirc\psi \in \boldsymbol{t}_i(r_n(i)) \quad [r_n \text{ is a run}]. \end{aligned}$$

Case $\psi = \Box_F\psi'$ is similar.

It follows from **(qm2)** that $(\mathfrak{M}'', \langle\langle r_0\rangle, 0\rangle) \models \varphi$. □

We can now deduce the decidability of the satisfiability problem for $\mathcal{DML}$-formulas by translating into monadic second-order logic the statement that there exists a quasimodel for a given formula φ. We require a number of auxiliary formulas. Denote by Σ the set of all quasistates for φ. Given a quasistate $\boldsymbol{q} = \langle\langle T_{\boldsymbol{q}}, <_{\boldsymbol{q}}\rangle, \delta_{\boldsymbol{q}}, \boldsymbol{t}_{\boldsymbol{q}}\rangle$ from Σ and a point u in $T_{\boldsymbol{q}}$ we denote the index of u by $idx_{\boldsymbol{q}}(u)$.

Introduce a unary predicate variable $P_{\boldsymbol{q}}$ for each quasistate $\boldsymbol{q} \in \Sigma$ and a unary predicate variable $R_\psi^{\langle a_1, \dots, a_n\rangle}$ for each $\psi \in cl\,\varphi$ and index $\langle a_1, \dots, a_n\rangle$ with $\sum_{i=1}^n a_i \leq mtd(\varphi)$. Given a type $\boldsymbol{t}$ for φ and such an index $\langle a_1, \dots, a_n\rangle$, let

$$\chi_{\boldsymbol{t}}(\overline{R^{\langle a_1, \dots, a_n\rangle}}(x)) = \bigwedge_{\psi \in \boldsymbol{t}} R_\psi^{\langle a_1, \dots, a_n\rangle}(x) \wedge \bigwedge_{\neg\psi \in \boldsymbol{t}} \neg R_\psi^{\langle a_1, \dots, a_n\rangle}(x),$$

saying that the type $\boldsymbol{t}$ at 'moment' x of index $\langle a_1,\dots,a_n\rangle$ is defined with the help of

$$\overline{R^{\langle a_1,\dots,a_n\rangle}}(x) = \Big\langle R_\psi^{\langle a_1,\dots,a_n\rangle}(x) \;\Big|\; \psi \in cl\,\varphi \Big\rangle.$$

For each index $\langle a_1,\dots,a_n\rangle$ with $\sum_{i=0}^n a_i \le mtd(\varphi)$, let $\mathsf{run}_0(\overline{P}(x),\overline{R^{\langle a_1,\dots,a_n\rangle}}(x))$ denote the conjunction of the three formulas

$$\bigwedge_{\boldsymbol{q}\in\Sigma} \forall x\Big[P_{\boldsymbol{q}}(x) \to \bigvee_{\substack{u\in T_{\boldsymbol{q}}\\ idx(u)=\langle a_1,\dots,a_n\rangle}} \chi_{\boldsymbol{t_q}(u)}(\overline{R^{\langle a_1,\dots a_n\rangle}}(x))\Big],$$

$$\bigwedge_{\Box_F\psi\in cl\,\varphi} \forall x\Big[R_{\Box_F\psi}^{\langle a_1,\dots,a_n\rangle}(x) \;\leftrightarrow\; \forall y\big(x<y \to R_\psi^{\langle a_1,\dots,a_n\rangle}(y)\big)\Big],$$

$$\bigwedge_{\bigcirc\psi\in cl\,\varphi} \forall x\Big[R_{\bigcirc\psi}^{\langle a_1,\dots,a_n\rangle}(x) \;\leftrightarrow\; R_\psi^{\langle a_1,\dots,a_n\rangle}(S(x))\Big]$$

—this is intended to say that $\overline{R^{\langle a_1,\dots,a_n\rangle}}(x)$ defines a run of index $\langle a_1,\dots,a_n\rangle$ through a sequence of quasistates defined with the help of

$$\overline{P}(x) = \Big\langle P_{\boldsymbol{q}}(x) \;\Big|\; \boldsymbol{q}\in\Sigma\Big\rangle.$$

However, we have to refine this definition in order to ensure that condition **(qm3)** holds. To this end, we define, by 'backwards' induction on the length of the index, another formula $\mathsf{run}(\overline{P}(x),\overline{R^{\langle a_1,\dots,a_n\rangle}}(x))$ as follows. If $\langle a_1,\dots,a_n\rangle$ is maximal (that is, we are at the 'leaf-level') then take

$$\mathsf{run}(\overline{P}(x),\overline{R^{\langle a_1,\dots,a_n\rangle}}(x)) = \mathsf{run}_0(\overline{P}(x),\overline{R^{\langle a_1,\dots,a_n\rangle}}(x)).$$

Suppose, inductively, that for all proper extensions $\langle a_1,\dots,a_m\rangle$ of $\langle a_1,\dots,a_n\rangle$ (that is, $m>n$) we have already defined $\mathsf{run}(\overline{P}(x),\overline{R^{\langle a_1,\dots,a_m\rangle}}(x))$. Then $\mathsf{run}(\overline{P}(x),\overline{R^{\langle a_1,\dots,a_n\rangle}}(x))$ is the following formula:

$$\begin{aligned}
&\mathsf{run}_0(\overline{P}(x),\overline{R^{\langle a_1,\dots,a_n\rangle}}(x)) \wedge \\
&\bigwedge_{\boldsymbol{q}\in\Sigma} \bigwedge_{\substack{u\in T_{\boldsymbol{q}}\\ idx(u)=\langle a_1,\dots,a_n\rangle}} \forall x\Big[P_{\boldsymbol{q}}(x) \wedge \chi_{\boldsymbol{t_q}(u)}(\overline{R^{\langle a_1,\dots,a_n\rangle}}(x)) \quad\to \\
&\qquad \bigwedge_{\substack{v\in T_q\\ \delta_{\boldsymbol{q}}(u,v)=a}} \exists_{\psi\in cl\,\varphi} R_\psi^{\langle a_1,\dots,a_n,a\rangle}\Big(\mathsf{run}(\overline{P}(x),\overline{R^{\langle a_1,\dots,a_n,a\rangle}}(x)) \wedge \chi_{\boldsymbol{t_q}(v)}(\overline{R^{\langle a_1,\dots,a_n,a\rangle}}(x)) \quad\wedge \\
&\qquad\quad \bigwedge_{\boldsymbol{s}\in\Sigma} \bigwedge_{\substack{u'\in T_{\boldsymbol{s}}\\ idx(u')=\langle a_1,\dots,a_n\rangle}} \forall z\Big(P_{\boldsymbol{s}}(z) \wedge \chi_{\boldsymbol{t_s}(u')}(\overline{R^{\langle a_1,\dots,a_n\rangle}}(z)) \quad\to \bigvee_{\substack{v'\in T_{\boldsymbol{s}}\\ \delta_{\boldsymbol{s}}(u',v')=a}} \chi_{\boldsymbol{t_s}(v')}(\overline{R^{\langle a_1,\dots,a_n,a\rangle}}(z))\Big)\Big)\Big].
\end{aligned}$$

Finally, we define a monadic second-order sentence $\varphi^\sharp$ by taking

$$\begin{aligned}
\varphi^\sharp \quad = \quad & \exists_{\boldsymbol{q}\in\Sigma} P_{\boldsymbol{q}}\Big[\forall x \bigvee_{\boldsymbol{q}\in\Sigma}\Big(P_{\boldsymbol{q}}(x) \wedge \bigwedge_{\substack{\boldsymbol{q}'\in\Sigma\\ \boldsymbol{q}\neq\boldsymbol{q}'}} \neg P_{\boldsymbol{q}'}(x)\Big) \wedge \\
& \bigvee_{\boldsymbol{s}\in\Sigma} \bigvee_{\substack{u\in T_{\boldsymbol{s}}\\ idx(u)=\langle\rangle\\ \varphi\in\boldsymbol{t_s}(u)}} \exists x\Big(P_{\boldsymbol{s}}(x) \wedge \exists_{\psi\in cl\,\varphi} R_\psi^{\langle\rangle}\Big(\mathsf{run}(\overline{P}(x),\overline{R^{\langle\rangle}}(x)) \wedge \chi_{\boldsymbol{t_s}(u)}(\overline{R^{\langle\rangle}}(x))\Big)\Big)\Big].
\end{aligned}$$

Evaluated in $\langle \mathbb{Z}, < \rangle$, the first line of $\varphi^\sharp$ says that the sets $P_{\boldsymbol{q}} \subseteq \mathbb{Z}$ ($\boldsymbol{q} \in \Sigma$) form a partition of $\mathbb{Z}$. By defining the map $\mathbf{q} \colon \mathbb{Z} \to \Sigma$ as

$$\mathbf{q}(i) = \boldsymbol{q} \qquad \text{iff} \qquad i \in P_{\boldsymbol{q}}$$

we obtain a quasimodel $\langle \mathbf{q}, \mathfrak{R} \rangle$ for φ: the second line of $\varphi^\sharp$ states condition **(qm2)**; condition **(qm3)** is satisfied by the formulas $\mathsf{run}(\overline{P}(x), \overline{R^{\langle a_1,\dots,a_n \rangle}}(x))$. Therefore, it is easy to see that the following holds:

Lemma 10. *For every $\mathcal{DML}$-formula φ, $mtd(\varphi) > 0$, $\langle \mathbb{Z}, < \rangle \models \varphi^\sharp$ iff there is a quasimodel for φ.*

Clearly, Σ can be constructed from φ by an algorithm. So we can now apply the result of Büchi [5] stating the decidability of monadic second-order logic over $\langle \mathbb{Z}, < \rangle$.

The non-elementary lower bound can be proved by a trivial polynomial reduction of the satisfiability problem for the product modal logic **PTL** $\times$ **K** (which is non-elementary by Theorem 6.37 from [7]) to the satisfiability problem for $\mathcal{DML}$-formulas in DMSs. We leave this to the reader.

Open problems. Interesting and challenging open problems are (i) the decidability of dynamic topological logics interpreted in various topological spaces with *continuous* mappings, and (ii) the decidability of dynamic metric logics interpreted in various *compact* metric spaces; for a justification and more details see, e.g., [12].

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