Extending *DL-Lite* Sometime in the Future

A. Artale,[1] R. Kontchakov,[2] V. Ryzhikov,[1] and M. Zakharyaschev[2]

[1] KRDB Research Centre
Free University of Bozen-Bolzano
I-39100 Bolzano, Italy
lastname@inf.unibz.it

[2] School of Comp. Science and Inf. Sys.
Birkbeck College
London WC1E 7HX, UK
{roman,michael}@dcs.bbk.ac.uk

1 Introduction

Many types of temporalised description logics (DLs) have been suggested and investigated in the past 15 years. We refer the reader to the survey papers and monograph [9, 14, 3, 16], where the history of the development of both interval and point-based temporal extensions of DLs is discussed in full detail.

Temporal operators can be applied in various ways in order to introduce a temporal dimension to a DL. In particular, they can be used as constructors for concepts, roles, TBox and ABox axioms (such concepts, roles or axioms are called temporalised). Alternatively, one may declare a certain concept, role or axiom to be regarded as *rigid* in the sense that its interpretation does not change in time—usually, the rigidity can be expressed if temporal operators are allowed to be applied to the respective construct. A number of complexity results have been obtained for different combinations of temporal operators and DLs. For instance, the following is known for combinations of $\mathcal{ALC}$ with the linear-time temporal logic $\mathcal{LTL}$: the satisfiability problem for the temporal $\mathcal{ALC}$ is

1. *undecidable* if temporalised concepts with rigid axioms and roles are allowed in the language (actually, a single rigid role is enough); see [14] and references therein;
2. 2ExpTime-complete if the language allows rigid concepts and roles with temporalised axioms [10];
3. ExpSpace-complete if the language allows temporalised concepts and axioms (but no rigid or temporalised roles) [14];
4. ExpTime-complete if the language allows only temporalised concepts and rigid axioms (but no rigid or temporalised roles) [17, 4].

In other words, as long as one wants to express the temporal behaviour of only axioms and concepts (but not roles), then the resulting combination is likely to be decidable. As soon as the combination allows reasoning about the temporal behaviour of binary relations it becomes undecidable, unless we limit the means to describe the temporal behaviour of concepts. Furthermore, we notice that better computational behaviour is exhibited in cases where rigid axioms are used instead of more general temporalised ones.

In this paper, we are interested in the scenario where axioms are rigid, concepts are temporalised and roles may be rigid or local (i.e., can change arbitrarily). To regain decidability in this case one has to restrict either the temporal [8] or the DL component [7]. The decidable (in fact 2ExpTime-complete) logic $\mathbf{S5}_{\mathcal{ALCQI}}$ [8] is obtained by combining the modal logic **S5** with $\mathcal{ALCQI}$. This approach weakens the temporal dimension to the much simpler **S5**, which can nevertheless represent rigid concepts and roles, and allows one to state that concept and role memberships change in time (however, without discriminating between changes in the past and in the future).

Temporal extensions of various dialects of *DL-Lite* have also been studied [7]. The most interesting result of [7] is the combination $\mathit{TDL\text{-}Lite}_{bool}$ of $\mathit{DL\text{-}Lite}^{\mathcal{N}}_{bool}$ [1, 2]—i.e., *DL-Lite* extended with full Booleans over concepts and cardinality restrictions on roles—with $\mathcal{LTL}$, which allows rigid roles and temporalised axioms and concepts and which was shown to be ExpSpace-complete.

In this paper, we consider another temporal extension $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$ of the logic $\mathit{DL\text{-}Lite}^{\mathcal{N}}_{bool}$. The logic $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$ weakens $\mathit{TDL\text{-}Lite}_{bool}$ of [7] in two ways: (i) axioms can be only rigid, and (ii) the temporal component is limited to the operators $\Diamond$ (*sometime in the future*) and $\Box$ (*always in the future*). We show that reasoning in $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$ and $\mathit{TDL\text{-}Lite}^{\Diamond}_{core}$ (a sub-language of $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$ that allows only very primitive concept inclusions) is NP-complete. Thus, allowing only $\Diamond$ and $\Box$ as temporal operators, and forbidding temporalised axioms reduces the complexity from ExpSpace—for $\mathit{TDL\text{-}Lite}_{bool}$ as in [7]—to NP. This result matches the minimal complexity of the two components: in case of $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$ both components ($\mathit{DL\text{-}Lite}^{\mathcal{N}}_{bool}$ and $\mathcal{LTL}$ with $\Diamond$ only) are NP-complete; in case of $\mathit{TDL\text{-}Lite}^{\Diamond}_{core}$ one component, $\mathit{DL\text{-}Lite}^{\mathcal{N}}_{core}$, is NLogSpace-complete, while the other is NP-complete. It should be noted, however, that $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$ is not simply a fusion (or independent join) of its components.

2 $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$: a Simple Temporal Description Logic

We begin by defining the description logic $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$ as a temporalisation of $\mathit{DL\text{-}Lite}^{\mathcal{N}}_{bool}$ [1, 2], which extends the original $\mathit{DL\text{-}Lite}_{\sqcap,\mathcal{F}}$ language [11–13] with full Booleans between concepts and cardinality restrictions on roles.

The language of $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$ contains *object names* $a_0, a_1, \ldots$, *concept names* $A_0, A_1, \ldots$, *local role names* $P_0, P_1, \ldots$ and *rigid role names* $G_0, G_1, \ldots$; *roles* R, *basic concepts* B and *concepts* C, D are defined as follows:

$$\begin{array}{rcl} R & ::= & P_i \quad | \quad P_i^- \quad | \quad G_i \quad | \quad G_i^-, \\ B & ::= & \bot \quad | \quad A_i \quad | \quad \geq q\, R, \\ C, D & ::= & B \quad | \quad \neg C \quad | \quad C \sqcap D \quad | \quad \Diamond C, \end{array}$$

where $q \geq 1$ is a natural number. A $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$ *TBox* $\mathcal{T}$ consists of *concept inclusions* of the form $C \sqsubseteq D$, and an *ABox* $\mathcal{A}$ of the assertions of the form: $\bigcirc^n B(a)$, $\bigcirc^n R(a, b)$, $\Box B(a)$ and $\Box R(a, b)$, where B is a basic concept, R a role, a, b object names and $\bigcirc^n$ denotes the sequence of n *next-time operators* $\bigcirc$, for $n \geq 0$. The TBox and ABox together form the *knowledge base* (KB) $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. A $\mathit{TDL\text{-}Lite}^{\Diamond}_{bool}$ *interpretation* $\mathcal{I}$ is a function on natural numbers $\mathbb{N}$:

$$\mathcal{I}(n) \;=\; \big(\Delta^{\mathcal{I}}, a_0^{\mathcal{I}}, \ldots, A_0^{\mathcal{I}(n)}, \ldots, P_0^{\mathcal{I}(n)}, \ldots, G_0^{\mathcal{I}(n)}, \ldots\big),$$

where $\Delta^{\mathcal{I}}$ is a non-empty set, the domain of $\mathcal{I}$, $a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, $A_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}}$ and $P_i^{\mathcal{I}(n)}, G_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, for all i and all $n \in \mathbb{N}$. Furthermore, $a_i^{\mathcal{I}} \neq a_j^{\mathcal{I}}$ for $i \neq j$ (which means that we adopt the *unique name assumption*) and $G_i^{\mathcal{I}(n)} = G_i^{\mathcal{I}(m)}$, for all $n, m \in \mathbb{N}$. The role and concept constructs are interpreted in $\mathcal{I}$ as follows: for each moment of time $n \in \mathbb{N}$,

$$
\begin{aligned}
(R_i^-)^{\mathcal{I}(n)} &= \{(y,x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (x,y) \in R_i^{\mathcal{I}(n)}\}, & \bot^{\mathcal{I}(n)} &= \emptyset,\\
(\geq q\, R)^{\mathcal{I}(n)} &= \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \mid (x,y) \in R^{\mathcal{I}(n)}\} \geq q\}, & (\neg C)^{\mathcal{I}(n)} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}(n)},\\
(C \sqcap D)^{\mathcal{I}(n)} &= C^{\mathcal{I}(n)} \cap D^{\mathcal{I}(n)}, & (\Diamond C)^{\mathcal{I}(n)} &= \bigcup\nolimits_{k>n} C^{\mathcal{I}(k)},
\end{aligned}
$$

where $\sharp X$ is the cardinality of X; note that the $\Diamond$ is interpreted in the strong sense, i.e., it does not include the present. We will use standard abbreviations such as $C_1 \sqcup C_2 = \neg(\neg C_1 \sqcap \neg C_2)$, $\top = \neg\bot$, $\exists R = (\geq 1\, R)$ and $\Box C = \neg\Diamond\neg C$. The satisfaction relation $\models$ is defined as follows:

$$
\begin{aligned}
\mathcal{I} \models C \sqsubseteq D \quad & \text{iff} \quad C^{\mathcal{I}(n)} \subseteq D^{\mathcal{I}(n)} \text{ for all } n \geq 0,\\
\mathcal{I} \models \bigcirc^n B(a) \quad & \text{iff} \quad a^{\mathcal{I}} \in B^{\mathcal{I}(n)},\\
\mathcal{I} \models \Box B(a) \quad & \text{iff} \quad a^{\mathcal{I}} \in B^{\mathcal{I}(n)} \text{ for all } n > 0,\\
\mathcal{I} \models \bigcirc^n R(a,b) \quad & \text{iff} \quad (a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}(n)},\\
\mathcal{I} \models \Box R(a,b) \quad & \text{iff} \quad (a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}(n)} \text{ for all } n > 0.
\end{aligned}
$$

We say an interpretation $\mathcal{I}$ is a *model* of a KB $\mathcal{K}$ if $\mathcal{I} \models \alpha$ for all α in $\mathcal{K}$. In this case we also say that $\mathcal{K}$ is *consistent* and we write $\mathcal{I} \models \mathcal{K}$. A concept A (role R) is *satisfiable w.r.t.* $\mathcal{K}$ if there exists a model $\mathcal{I}$ of $\mathcal{K}$ and $n \geq 0$ such that $A^{\mathcal{I}(n)} \neq \emptyset$ ($R^{\mathcal{I}(n)} \neq \emptyset$).

Note that *TDL-Lite*$_{bool}^{\Diamond}$ is not a simple fusion of the two component logics, *DL-Lite*$_{bool}^{\mathcal{N}}$ and $\mathcal{LTL}$. Indeed, let $\mathcal{K} = (\{\Diamond\exists R^- \sqsubseteq \bot, \exists R \sqsubseteq \Diamond\exists R\}, \{\exists R(a)\})$. It is easy to see that $\mathcal{K}$ is not satisfiable in *TDL-Lite*$_{bool}^{\Diamond}$. However, it is satisfiable both in *DL-Lite*$_{bool}^{\mathcal{N}}$ (if we substitute the temporal concepts by fresh *DL-Lite*$_{bool}^{\mathcal{N}}$ concepts) and in $\mathcal{LTL}$ (by substituting $\exists R$ concepts with fresh atomic propositions).

3 Satisfiability of *TDL-Lite*$_{bool}^{\Diamond}$ KBs is NP-complete

To prove the NP complexity result we first establish in Section 3.1 a relation between *TDL-Lite*$_{bool}^{\Diamond}$ and the one-variable fragment $\mathcal{QTL}^1$ of first-order temporal logic. This will allow us to polynomially reduce the satisfiability problem in *TDL-Lite*$_{bool}^{\Diamond}$ to that in *TDL-Lite*$_{0}^{\Diamond}$, a language that has neither rigid roles nor role assertions. Next, in Section 3.2, we show that a *TDL-Lite*$_{0}^{\Diamond}$ KB $\mathcal{K}$ is satisfiable iff there exists a *quasimodel* for it. Then we show that if there is a quasimodel for $\mathcal{K}$ then there exists an *ultimately periodic quasimodel* for it such that both the length of the prefix and the length of the period are polynomial in the length of $\mathcal{K}$. Finally, in Section 3.3, we describe an algorithm that checks (in non-deterministic polynomial time) the existence of an ultimately periodic quasimodel for a given *TDL-Lite*$_{0}^{\Diamond}$ KB.

3.1 *TDL-Lite*$^{\Diamond}_{bool}$ in the context of First-Order Temporal Logic

For a *TDL-Lite*$^{\Diamond}_{bool}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, let $ob(\mathcal{A})$ be the set of all object names occurring in $\mathcal{A}$. Let $role^{\pm}(\mathcal{K})$ be the set of all (local and rigid) role names, together with their inverses, occurring in $\mathcal{K}$, and $grole^{\pm}(\mathcal{K})$ the set of rigid role names, together with their inverses, occurring in $\mathcal{K}$. For $R \in role^{\pm}(\mathcal{K})$, let $Q^R_{\mathcal{K}}$ be the set of natural numbers containing 1 and all the numerical parameters q for which $\geq q\, R$ occurs in $\mathcal{K}$. Denote by $ev(\mathcal{K})$ the set of all concepts of the form $\Diamond C$ occurring in $\mathcal{K}$ and, finally, let $N_{\mathcal{K}} = \{n \mid \bigcirc^n B(a) \in \mathcal{A} \text{ or } \bigcirc^n R(a, b) \in \mathcal{A}\}$; without loss of generality, we assume that $N_{\mathcal{K}}$ is non-empty.

With every object name $a \in ob(\mathcal{A})$ we associate the individual constant a of $\mathcal{QTL}^1$, the one variable fragment of first-order temporal logic over $(\mathbb{N}, <)$, and with every concept name A the unary predicate $A(x)$ from the signature of $\mathcal{QTL}^1$. For each $R \in role^{\pm}(\mathcal{K})$, we also introduce $|Q^R_{\mathcal{K}}|$ fresh unary predicates $E_q R(x)$, for $q \in Q^R_{\mathcal{K}}$. Intuitively, for each $n \geq 0$, $E_1 R(x)$ and $E_1 R^-(x)$ represent the domain and range of R at moment n (i.e., $E_1 R(x)$ and $E_1 R^-(x)$ are interpreted by the sets of points with *at least one* R-successor and *at least one* R-predecessor at moment n, respectively), while $E_q R(x)$ and $E_q R^-(x)$ represent the sets of points with *at least* q distinct R-successors and *at least* q distinct R-predecessors at moment n.

By induction on the construction of a *TDL-Lite*$^{\Diamond}_{bool}$ concept C we define the $\mathcal{QTL}^1$- formula C^*:

$$\begin{aligned}
\bot^* &= \bot, & (A)^* &= A(x),\\
(\geq q\, R)^* &= E_q R(x), & (\neg C)^* &= \neg C^*(x),\\
(C_1 \sqcap C_2)^* &= C_1^*(x) \wedge C_2^*(x), & (\Diamond C)^* &= \Diamond C^*(x),
\end{aligned}$$

and then extend this translation to *TDL-Lite*$^{\Diamond}_{bool}$ TBoxes $\mathcal{T}$:

$$\mathcal{T}^* = \bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}} \Box^+ \forall x\, (C_1^*(x) \to C_2^*(x)),$$

where $\Box^+\varphi = \varphi \wedge \Box\varphi$. The following formulas express some natural properties of the role domains and ranges. For $R \in role^{\pm}(\mathcal{K})$, we need two $\mathcal{QTL}^1$-sentences:

$$\varepsilon_R \;=\; \exists x\, E_1 R(x) \;\to\; \exists x\, inv(E_1 R)(x), \tag{1}$$

$$\delta_R = \bigwedge_{\substack{q, q' \in Q^R_{\mathcal{K}},\;\; q' > q\\ q' > q'' > q \text{ for no } q'' \in Q^R_{\mathcal{K}}}} \forall x\, \big(E_{q'} R(x) \to E_q R(x)\big), \tag{2}$$

where $inv(E_1 R)$ is the predicate $E_1 P^-(x)$ if $R = P$ and $E_1 P(x)$ if $R = P^-$. Sentence (1) says that if the domain of R is non-empty then its range is non-empty either.

Without loss of generality we may assume that if R is a rigid role and $\mathcal{A}$ contains $\bigcirc^n R(a, b)$ or $\Box R(a, b)$ then it also contains both $R(a, b)$ and $\Box R(a, b)$.

Then we define 'temporal slices' of the ABox $\mathcal{A}$ by taking:

$$\begin{aligned}
\mathcal{A}_{\Box} &= \big\{R(a,b) \mid \Box R(a,b) \in \mathcal{A} \text{ or } \Box \mathit{inv}(R)(b,a) \in \mathcal{A}\big\},\\
\mathcal{A}_{n} &= \big\{R(a,b) \mid \bigcirc^n R(a,b) \in \mathcal{A} \text{ or } \bigcirc^n \mathit{inv}(R)(b,a) \in \mathcal{A}\big\} \cup\\
&\quad\ \big\{R(a,b) \mid n > 0 \text{ and either } \Box R(a,b) \in \mathcal{A} \text{ or } \Box \mathit{inv}(R)(b,a) \in \mathcal{A}\big\}.
\end{aligned}$$

The $\mathcal{QTL}^1$ translation of the ABox $\mathcal{A}$ is defined as follows:

$$\mathcal{A}^* = \bigwedge_{\bigcirc^n B(a_i) \in \mathcal{A}} \bigcirc^n B^*(a_i) \ \wedge \bigwedge_{R(a,b)\in\mathcal{A}_n} \bigcirc^n E_{q_{R,a,\mathcal{A}_n}} R(a) \ \wedge \bigwedge_{R(a,b)\in\mathcal{A}_{\Box}} \Box E_{q_{R,a,\mathcal{A}_{\Box}}} R(a),$$

where, for a role R, $a \in \mathit{ob}(\mathcal{A})$ and any ABox $\mathcal{A}'$,

$$q_{R,a,\mathcal{A}'} = \max(\{0\} \cup \{q \in Q^R_{\mathcal{K}} \mid R(a,a_i) \in \mathcal{A}', 1 \leq i \leq q \ \&\ a_{i_1} \neq a_{i_2} \text{ if } i_1 \neq i_2\}).$$

Finally, we set

$$\mathcal{K}^{\ddagger} = \mathcal{T}^* \wedge \bigwedge_{R \in \mathit{role}^{\pm}(\mathcal{K})} \Box^+ \big(\varepsilon_R \wedge \delta_R\big) \wedge \bigwedge_{T \in \mathit{grole}^{\pm}(\mathcal{K})} \bigwedge_{q \in Q^T_{\mathcal{K}}} \Box^+ \forall x \, \big(E_q T(x) \leftrightarrow \Box E_q T(x)\big) \ \wedge\ \mathcal{A}^*.$$

Observe that the length of $\mathcal{K}^{\ddagger}$ is polynomial in the length of $\mathcal{K}$. It can be shown (for details see [7, Theorem 2 and Corollary 3]) that we have:

Theorem 1. *A TDL-Lite$^{\Diamond}_{bool}$ KB $\mathcal{K}$ is satisfiable iff the $\mathcal{QTL}^1$-sentence $\mathcal{K}^{\ddagger}$ is satisfiable.*

Denote by *TDL-Lite*$^{\Diamond}_{0}$ the fragment of *TDL-Lite*$^{\Diamond}_{bool}$ without rigid roles and ABox assertions of the form $\Box R(a,b)$ or $\bigcirc^n R(a,b)$. By Theorem 1, this fragment is of the same complexity as *TDL-Lite*$^{\Diamond}_{bool}$:

Lemma 1. *Given a TDL-Lite$^{\Diamond}_{bool}$ KB $\mathcal{K}$ one can construct* (*in polynomial time*) *a TDL-Lite$^{\Diamond}_{0}$ KB $\mathcal{K}'$ such that $\mathcal{K}$ and $\mathcal{K}'$ are equisatisfiable.*

Proof. Let $\mathcal{A}_0$ be the part of $\mathcal{A}$ that contains no assertions of the form $\Box R(a,b)$ or $\bigcirc^n R(a,b)$. Then we set $\mathcal{K}' = (\mathcal{T} \cup \mathcal{T}', \mathcal{A}_0 \cup \mathcal{A}')$, where

$$\begin{aligned}
\mathcal{T}' &= \big\{\Box(\geq q\, T) \sqsubseteq (\geq q\, T),\ (\geq q\, T) \sqsubseteq \Box(\geq q\, T) \mid q \in Q^T_{\mathcal{K}}, T \in \mathit{grole}^{\pm}(\mathcal{K})\big\},\\
\mathcal{A}' &= \{\bigcirc^n (\geq q_{R,a,\mathcal{A}_n}\, R)(a) \mid R(a,b) \in \mathcal{A}_n\} \cup \{\Box(\geq q_{R,a,\mathcal{A}_{\Box}}\, R)(a) \mid R(a,b) \in \mathcal{A}_{\Box}\}.
\end{aligned}$$

Clearly, $\mathcal{K}^{\ddagger} = (\mathcal{K}')^{\ddagger}$. Then the claim immediately follows from Theorem 1. ❑

3.2 Quasimodels for *TDL-Lite*$^{\Diamond}_{0}$

In this section, we define a notion of a quasimodel for a *TDL-Lite*$^{\Diamond}_{0}$ KB and show that a *TDL-Lite*$^{\Diamond}_{0}$ KB is satisfiable iff there is an ultimately periodical quasimodel with the length of both the prefix and period bounded by a polynomial function in the length of $\mathcal{K}$. It will follow then that the satisfiability problem for *TDL-Lite*$^{\Diamond}_{0}$, and thus for *TDL-Lite*$^{\Diamond}_{bool}$, is in NP.

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a *TDL-Lite*$_0^{\Diamond}$ KB. We introduce, for every concept of the form $\Diamond C$, a fresh concept name F_C, the *surrogate* of $\Diamond C$, and then, for a concept D, denote by $\overline{D}$ the result of replacing each $\Diamond C$ in D with the surrogate F_C. For a *TDL-Lite*$_0^{\Diamond}$ TBox $\mathcal{T}$, denote by $\overline{\mathcal{T}}$ the *DL-Lite*$_{bool}^{\mathcal{N}}$ TBox obtained by replacing every concept C in $\mathcal{T}$ with $\overline{C}$.

Let $cl(\mathcal{K})$ be the closure under negation of all concepts occurring in $\mathcal{T}$ together with the $\exists R$, for $R \in role^{\pm}(\mathcal{K})$, and the B, for $\bigcirc^n B(a) \in \mathcal{A}$ or $\Box B(a) \in \mathcal{A}$. A *type* for $\mathcal{K}$ is a subset $\mathbf{t}$ of $cl(\mathcal{K})$ such that

– $C \sqcap D \in \mathbf{t}$ iff $C, D \in \mathbf{t}$, for every $C \sqcap D \in cl(\mathcal{K})$;
– $\neg C \in \mathbf{t}$ iff $C \notin \mathbf{t}$, for every $C \in cl(\mathcal{K})$.

A type $\mathbf{t}$ for $\mathcal{K}$ is *realisable* if the concept $\bigsqcap_{C \in \mathbf{t}} \overline{C}$ is satisfiable w.r.t. $\overline{\mathcal{T}}$.

A function r mapping $\mathbb{N}$ to types for $\mathcal{K}$ is called a *coherent* and *saturated run* for $\mathcal{K}$ if the following conditions are satisfied:

(real) $r(i)$ is realisable, for all $i \geq 0$;
(coh) for all $i \geq 0$ and $\Diamond C \in ev(\mathcal{K})$, if $C \in r(i)$ then $\Diamond C \in r(j)$, for all j with $0 \leq j < i$;
(sat) for all $i \geq 0$ and $\Diamond C \in ev(\mathcal{K})$, if $\Diamond C \in r(i)$ then there is $j > i$ such that $C \in r(j)$.

A *witness* for $\mathcal{K}$ is a pair of the form $(r, \varXi)$, where r is a coherent and saturated run for $\mathcal{K}$, $\varXi \subseteq \mathbb{N}$ and $|\varXi| \leq 1$.

Given a run r and a finite sequence $s = (s(0), \dots, s(n))$ of types for $\mathcal{K}$, we set:

$$r^{<i} = (r(0), \dots, r(i-1)), \qquad r^{\geq i} = (r(i), r(i+1), \dots),$$
$$s^{\omega} = (s(0), \dots, s(n), s(0), \dots, s(n), \dots), \quad s \cdot r = (s(0), \dots, s(n), r(0), r(1), \dots),$$

We say that a type $\mathbf{t}$ for $\mathcal{K}$ is *stutter-invariant* if $\neg\Diamond C \in \mathbf{t}$ implies $\neg C \in \mathbf{t}$, for each $\Diamond C \in ev(\mathcal{K})$.

A *quasimodel* for $\mathcal{K}$ is a triple $\mathfrak{Q} = \langle W, K, L\rangle$, where W is a set of witnesses for $\mathcal{K}$ and K, L are natural numbers with $0 \leq K \leq L$ such that:

(runs) $W = \{(r_a, \emptyset) \mid a \in ob(\mathcal{A})\} \cup \{(r_R, \{i_R\}) \mid R \in \varOmega\}$, for some $\varOmega \subseteq role^{\pm}(\mathcal{K})$;
(stuttr) $r(K)$ and the $r(i)$, for $i \geq L$, are stutter-invariant, for each $(r, \varXi) \in W$;
(obj) if $\bigcirc^n B(a) \in \mathcal{A}$ then $B \in r_a(n)$; if $\Box B(a) \in \mathcal{A}$ then $B \in r_a(i)$ for all $i > 0$;
(role) for all $i \geq 0$ and $R \in role^{\pm}(\mathcal{K})$, if $\exists R^- \in r(i)$, for some $(r, \varXi) \in W$, then $(r_R, \{i_R\}) \in W$, $\exists R \in r_R(i_R)$ and either $i \leq i_R < K$ or $K \leq i_R < L$.

Theorem 2. *A TDL-Lite*$_0^{\Diamond}$ *KB $\mathcal{K}$ is satisfiable iff there exists a quasimodel $\mathfrak{Q} = \langle W, K, L\rangle$ for $\mathcal{K}$ such that $L \leq \max N_{\mathcal{K}} + |ev(\mathcal{K})| \cdot (|role^{\pm}(\mathcal{K})| + 2) + 3$.*

Proof. ($\Rightarrow$) Suppose $\mathcal{I} \models \mathcal{K}$. For $m \geq 0$, let

$$\mathbf{F}^m = \{R \in role^{\pm}(\mathcal{K}) \mid \text{ there is } i \geq m \text{ with } R^{\mathcal{I}(i)} \neq \emptyset\}.$$

Lemma 2. *For all $n, v \geq 0$, there exists m such that $n \leq m \leq n + v \cdot |\mathbf{F}^0|$ and, for every role $R \in \mathbf{F}^0$, either $R \in \mathbf{F}^{m+v+1}$ or $R \notin \mathbf{F}^{m+1}$.*

Proof. If a role R is non-empty infinitely often then $R \in \mathbf{F}^{m+v+1}$, for any m. So we have to consider only those roles that are non-empty finitely many times. Let

$$\mathbf{FG} = \{R \in \mathit{role}^{\pm}(\mathcal{K}) \mid \text{ there is } i \geq 0 \text{ such that } R \notin \mathbf{F}^i\}.$$

For $R \in \mathbf{FG} \cap \mathbf{F}^0$, let $i_R = \min\{i \mid R \notin \mathbf{F}^{i+1}\}$ (i.e., i_R is the last moment when R is non-empty). If $\max\{i_R \mid R \in \mathbf{FG}\} \leq n + v \cdot |\mathbf{F}^0|$, we take $m = \max(\{n\} \cup \{i_R \mid R \in \mathbf{FG}\})$. Clearly, $\mathbf{FG} \cap \mathbf{F}^{m+1} = \emptyset$ (so all roles in $\mathbf{FG}$ are empty after m). Otherwise, $\mathbf{FG} \cap \mathbf{F}^0 \neq \emptyset$ and without loss of generality we may assume that $\mathbf{FG} \cap \mathbf{F}^0 = \{R_1, \dots, R_s\}$ and $i_{R_1} \leq i_{R_2} \leq \dots \leq i_{R_s}$. If $i_{R_1} > n + v$, we take $m = n$; then $\mathbf{FG} \cap \mathbf{F}^0 \subseteq \mathbf{F}^{m+v+1}$ (all roles in $\mathbf{FG} \cap \mathbf{F}^0$ are non-empty after $m+v$). Otherwise, $i_{R_1} \leq n + v$ and $i_{R_s} > n + v \cdot |\mathbf{F}^0|$, whence $i_{R_s} - i_{R_1} > (v-1) \cdot |\mathbf{F}^0|$. Let j_0 be the smallest j, $1 \leq j < s$, such that $i_{R_j} \geq n$ and $i_{R_{j+1}} - i_{R_j} > v$ (it exists as $s \leq |\mathbf{F}^0|$) and set $m = i_{R_{j_0}}$. We then have $R_1, \dots, R_{j_0} \notin \mathbf{F}^{m+1}$ and $R_{j_0+1}, \dots, R_s \in \mathbf{F}^{m+v+1}$. ❑

Let $N = \max N_{\mathcal{K}}$ and $V = |\mathit{ev}(\mathcal{K})|$. By Lemma 2, there exists M with $N \leq M \leq N + V \cdot |\mathbf{F}^0|$ such that, for every role $R \in \mathbf{F}^0$, either $R \in \mathbf{F}^K$ or $R \notin \mathbf{F}^{M+1}$, where $K = M + V + 1$. We then set $i_R = \min\{i \geq K \mid R^{\mathcal{I}(i)} \neq \emptyset\}$, for each $R \in \mathbf{F}^K$, and $i_R = \max\{i \mid R^{\mathcal{I}(i)} \neq \emptyset\}$, for each $R \in \mathbf{F}^0 \setminus \mathbf{F}^{M+1}$. Clearly, for each $R \in \mathbf{F}^0$, either $i_R \leq M$ or $i_R \geq K$. For $d \in \Delta^{\mathcal{I}}$, denote $r_d \colon i \mapsto \{C \in \mathit{cl}(\mathcal{K}) \mid d \in C^{\mathcal{I}(i)}\}$ (it evidently is a coherent and saturated run). For each $R \in \mathbf{F}^0$, we fix some $d_R \in (\exists R)^{\mathcal{I}(i_R)}$ and set $r_R = r_{d_R}$. Let

$$W = \{(r_R, \{i_R\}) \mid R \in \mathbf{F}^0\} \cup \{(r_{a^{\mathcal{I}}}, \emptyset) \mid a \in \mathit{ob}(\mathcal{A})\}.$$

Clearly, both **(runs)** and **(obj)** hold. We also have $\exists R^- \in r(i)$ iff $\exists R \in r_R(i_R)$ and $(r_R, \{i_R\}) \in W$, for all $(r, \Xi) \in W$ and $i \geq 0$.

We now transform W by expanding and pruning runs in such a way that the $r(i)$ are never thrown out, for $(r, \Xi) \in W$ and $i \in \Xi$.

Lemma 3. *For each coherent and saturated run r,*

$$|\{i \mid r(i) \textit{ is not stutter-invariant}\}| \leq |\mathit{ev}(\mathcal{K})|.$$

Proof. Suppose there are $0 \leq i_1 < \dots < i_n$ such that $n > |\mathit{ev}(\mathcal{K})|$ and $r(i_1), \dots, r(i_n)$ are not stutter-invariant, i.e., for each $1 \leq j \leq n$, there are $\Diamond C_j \in \mathit{ev}(\mathcal{K})$ with $\neg\Diamond C_j, C_j \in r(i_j)$. Then there is $\Diamond C \in \mathit{ev}(\mathcal{K})$ such that $\neg\Diamond C, C \in r(i_j)$ and $\neg\Diamond C, C \in r(i_{j'})$ for some $0 \leq i_j < i_{j'}$. As $C \in r(i_{j'})$, we obtain, by **(coh)**, $\Diamond C \in r(i_j)$, contrary to $\neg\Diamond C \in r(i_j)$. ❑

Step 1. By Lemma 3, for each $(r, \Xi) \in W$, there is j_r, $M < j_r \leq K$, such that $r(j_r)$ is stutter-invariant. Set

$$r' = r^{<j_r} \cdot \underbrace{r(j_r) \cdot \ldots \cdot r(j_r)}_{K - j_r \text{ times}} \cdot r^{\geq j_r},$$

$$\Xi' = \{i \mid i \in \Xi,\ i \leq j_r\} \cup \{i + K - j_r \mid i \in \Xi,\ i > j_r\}.$$

It should be clear that r' is a coherent and saturated run. Denote by W' the set of all (r', Ξ') constructed as above. Clearly, $r'(K)$ is stutter-invariant, for each $(r', \Xi') \in W'$. It is easy to see that, for each $R \in \mathbf{F}^0$, we have $(r'_R, \{i'_R\}) \in W'$ and either $i'_R \leq M$ or $i'_R \geq K$.

Step 2. For $(r', \Xi') \in W'$, let $\Xi^0 = \{i > K \mid r'(i) \text{ not stutter-invariant}\}$. By Lemma 3, $|\Xi^0| \leq |ev(\mathcal{K})|$. We prune the run r', if $\Xi^0 \cup \Xi' \neq \emptyset$, by removing all stutter-invariant $r'(i)$ with $K < i < \max(\Xi^0 \cup \Xi')$. Denote the resulting run by r''. It should be clear that r'' is coherent and saturated. Set

$$\Xi'' = \{i \mid i \in \Xi',\ i \leq K\} \cup \{K + |\{j \in \Xi^0 \cup \Xi' \mid j \leq i\}| \mid i \in \Xi',\ i > K\}.$$

Let W'' be the set of all witnesses (r'', Ξ'') constructed as above and $L = K + V + 2$. It follows that, for each $(r'', \Xi'') \in W''$, all the types $r''(i)$ are stutter-invariant, for $i \geq L$, and thus **(stuttr)** holds. It is also easy to see that, for each $R \in \mathbf{F}^0$, we have $(r''_R, \{i''_R\}) \in W''$ and $K \leq i''_R < L$, if $R \in \mathbf{F}^K$, and $i''_R \leq M$, if $R \notin \mathbf{F}^{M+1}$. Therefore, we have **(role)**. So, $\mathfrak{Q} = \langle W'', K, L\rangle$ is as required.

($\Leftarrow$) Let $\mathfrak{Q} = \langle W, K, L\rangle$ be a quasimodel for $\mathcal{K}$. We construct a model for $\mathcal{K}^\ddagger$ which, by Theorem 1, is enough to show that $\mathcal{K}$ is satisfiable. Let

$$\begin{aligned}\mathfrak{R} \quad = \quad & \big\{r_a \mid (r_a, \emptyset) \in W\big\} \ \cup \ \big\{r_R^{\geq i} \mid (r_R, \{i_R\}) \in W,\ 0 \leq i \leq i_R\big\} \ \cup \\ & \big\{r_R^{<K} \cdot (r_R(K))^{i - i_R} \cdot r_R^{\geq K} \mid (r_R, \{i_R\}) \in W,\ i > i_R \geq K\big\}.\end{aligned}$$

Clearly, each $r \in \mathfrak{R}$ is a coherent and saturated run for $\mathcal{K}$. Moreover, if we have $(r_R, \{i_R\}) \in W$ and $i_R < K$ then there is $r' \in \mathfrak{R}$ with $\exists R \in r'(i)$, for all i, $0 \leq i \leq i_R$. And if $(r_R, \{i_R\}) \in W$ and $i_R \geq K$ then there is $r' \in \mathfrak{R}$ with $\exists R \in r'(i)$, for all $i \geq 0$. As follows from **(role)**, for each $R \in \Omega$, we have either $i_R \geq K$ and $i_{R^-} \geq K$ or $i_R = i_{R^-} < K$. So, for all $i \geq 0$ and $r \in \mathfrak{R}$,

$$\text{if } \exists R^- \in r(i) \text{ then there is } r' \in \mathfrak{R} \text{ such that } \exists R \in r'(i).$$

We construct a first-order temporal model $\mathfrak{M}$ based on the domain $D = \mathfrak{R}$ by taking $a^{\mathfrak{M}} = r_a$, for each $a \in ob(\mathcal{A})$, and $(B^*)^{\mathfrak{M},i} = \{r \in \mathfrak{R} \mid B \in r(i)\}$, for each $B \in cl(\mathcal{K})$ and $i \geq 0$. It should be clear that $(\mathfrak{M}, 0) \models \mathcal{K}^\ddagger$. ❑

Theorem 3. *If there is a quasimodel $\mathfrak{Q} = \langle W, K, L\rangle$ for $\mathcal{K}$ then there is an ultimately periodical quasimodel $\mathfrak{Q}' = \langle W', K, L\rangle$, that is, there is $P \leq |ev(\mathcal{K})|$ such that $r'(i + P) = r'(i)$, for all $i > L$ and $(r', \Xi') \in W'$.*

Proof. We begin the proof with the following observation:

Lemma 4. *Let r be a coherent and saturated run and let $l \geq 0$ be such that every $r(i)$ is stutter-invariant, $i \geq l$. Then there are $i_1, \dots, i_{|ev(\mathcal{K})|} \geq l$ such that $r' = r^{\leq l} \cdot \big(r(i_1) \cdot \ldots \cdot r(i_{|ev(\mathcal{K})|})\big)^\omega$ is a coherent and saturated run.*

Proof. First we show that

$$r(l) \cap ev(\mathcal{K}) = r(j) \cap ev(\mathcal{K}), \qquad \text{for all } j > l. \tag{3}$$

Assume there is $j > l$ and $\Diamond C \in r(l)$ such that $\Diamond C \notin r(j)$. As $r(j)$ is stutter-invariant, we have $C \notin r(j)$ and, by **(coh)**, $\Diamond C \notin r(j-1)$. By repeating this argument sufficiently many times, we obtain $\Diamond C \notin r(l)$, contrary to our assumption. The converse direction—i.e., for each $j > l$, if $\Diamond C \in r(j)$ then $\Diamond C \in r(l)$—follows immediately from **(coh)**.

For each $\Diamond C \in ev(\mathcal{K})$, we can select an i, $i \geq l$, such that $C \in r(i)$ whenever $\Diamond C \in r(l)$. Let $i_1, \dots, i_{|ev(\mathcal{K})|}$ be all such i. It remains to show that r' is coherent and saturated.

For coherency, suppose that $C \in r'(i)$, for $i \geq 0$. By **(coh)** for r, we have $\Diamond C \in r'(j)$, for each $0 \leq j < i$ such that $j \leq l$. It remains to consider j with $l < j < i$. It follows that $r'(i) = r(i_k)$, for some $1 \leq k \leq |ev(\mathcal{K})|$, from which, by **(coh)** for r, $\Diamond C \in r(l) = r'(l)$ and, by (3), $\Diamond C \in r'(j)$.

For saturation of r', suppose $\Diamond C \in r'(i)$, for $i \geq 0$. If $\Diamond C \in r(l)$ then $C \in r(i_k)$ for $1 \leq k \leq |ev(\mathcal{K})|$ and, by the construction of r', there is $j > i$ such that $r'(j) = r(i_k)$. Thus $C \in r'(j)$. If $\Diamond C \notin r(l)$ then, by (3), $i < l$, from which $\Diamond C \in r(i)$. By **(sat)** of r, there is $j > i$ with $C \in r(j)$ and, by (3), $j \leq l$. Thus $C \in r(j) = r'(j)$. ❑

Let $P = |ev(\mathcal{K})|$. For $(r, \Xi) \in W$, take $r' = r^{\leq L} \cdot (r(i_1) \cdot \ldots r(i_P))^\omega$ provided by Lemma 4. Denote the set of all such (r', Ξ) by W'. It follows that $\mathfrak{Q}' = \langle W', K, L \rangle$ is an ultimately periodical quasimodel for $\mathcal{K}$ (with period P). ❑

3.3 Decision Procedure for *TDL-Lite*$^{\Diamond}_{bool}$

As shown in Section 3.1, there is a polynomial-time reduction of the satisfiability problem for *TDL-Lite*$^{\Diamond}_{bool}$ KBs to the satisfiability problem for *TDL-Lite*$^{\Diamond}_{0}$ KBs. So it suffices to present an NP decision algorithm for the latter problem.

Our algorithm, given a *TDL-Lite*$^{\Diamond}_{0}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, guesses the 'prefix' of length $L + 1$ and the period of length P of an ultimately periodical quasimodel $\mathfrak{Q}' = \langle W', K, L \rangle$ for $\mathcal{K}$ as in Theorem 3, and then checks whether conditions **(runs)**, **(stuttr)**, **(obj)** in Section 3.2 hold and whether the types in positions $L + 1$ and $L + P + 1$ of the prefix coincide for every run.

More precisely, first we guess and store numbers K, L and P such that $K \leq L$, $L \leq \max N_{\mathcal{K}} + |ev(\mathcal{K})| \cdot (|role^{\pm}(\mathcal{K})| + 2) + 3$ and $P \leq |ev(\mathcal{K})|$. Then we guess a set $\Omega \subseteq role^{\pm}(\mathcal{K})$ and numbers $\{i_R < L \mid R \in \Omega\}$. For every $R \in \Omega$, we also guess a sequence r_R of length $L + P + 2$ of types for $\mathcal{K}$ and, for every $a \in ob(\mathcal{K})$, a sequence r_a of length $L + P + 2$ of types for $\mathcal{K}$.

Let $W_0 = \{(r_R, \{i_R\}) \mid R \in \Omega\} \cup \{(r_a, \emptyset) \mid a \in ob(\mathcal{K})\}$. The set W_0 can be regarded as a finite representation of the witnesses W' from $\mathfrak{Q}'$. Now we check that the following conditions hold:

1. $r(K)$ and the $r(i)$, for $L \leq i \leq L + P + 1$, are stutter-invariant, for each $(r, \Xi) \in W_0$;
2. if $\bigcirc^n B(a) \in \mathcal{A}$ then $B \in r_a(n)$; if $\Box B(a) \in \mathcal{A}$ then $B \in r_a(i)$, for all $0 < i \leq L + P + 1$;
3. for all $i \leq L + P + 1$ and $R \in role^{\pm}(\mathcal{K})$, if $\exists R^- \in r(i)$, for some $(r, \Xi) \in W_0$, then $(r_R, \{i_R\}) \in W_0$, $\exists R \in r_R(i_R)$ and either $i \leq i_R < K$ or $K \leq i_R < L$;

4. $r(L+1) = r(L+P+1)$, for all $(r, \Xi) \in W_0$;
5. $r(i)$ is realisable, for all $(r, \Xi) \in W_0$ and $i \leq L+P+1$;
6. for all $(r, \Xi) \in W_0$, $i \leq L+P+1$ and $\Diamond C \in r(i)$
   - if $i \leq L$ then there is j, $i < j \leq L+P+1$, with $C \in r(j)$;
   - if $L < i \leq L+P+1$ then there is j, $L < j \leq L+P+1$, with $C \in r(j)$;
7. $\Diamond C \in r(j)$, for all $(r, \Xi) \in W_0$, $i \leq L+P+1$, $C \in r(i)$ and $j < i$.

The algorithm returns 'yes' iff all the conditions above are satisfied.

The presented algorithm is sound: indeed, if conditions 1–7 are satisfied we can construct an ultimately periodical quasimodel for $\mathcal{K}$ which, by Theorem 2, means that $\mathcal{K}$ is satisfiable. The algorithm is also complete: if $\mathcal{K}$ is satisfiable then, by Theorems 2 and 3, there exists an ultimately periodical quasimodel $\mathfrak{Q} = \langle W', K, L\rangle$ with period P and K, L, P bounded by polynomial functions in $|\mathcal{K}|$ as above; then W_0 consisting of the prefixes of length $L+P+2$ of runs in W' satisfies conditions 1–7 and thus the algorithm returns 'yes.'

Finally, it is easy to see that L, K, P and W_0 can be constructed and conditions 1–7 checked by a non-deterministic polynomial-time algorithm in $|\mathcal{K}|$. In particular, condition 5 can be verified by calling, for each r with $(r, \Xi) \in W_0$ and $i \leq L+P+1$, a $\textit{DL-Lite}^{\mathcal{N}}_{\textit{bool}}$ satisfiability checking algorithm for the concept $\bigsqcap_{C \in r(i)} \overline{C}$ w.r.t. the TBox $\overline{\mathcal{T}}$, which can be done in NP [1, 2].

Then, by Lemma 1 and because $\textit{TDL-Lite}^{\Diamond}_{\textit{bool}}$ 'contains' propositional logic, we obtain the following:

Theorem 4. *The satisfiability problem for* $\textit{TDL-Lite}^{\Diamond}_{\textit{bool}}$ *KBs is* NP*-complete.*

3.4 NP-hardness of $\textit{TDL-Lite}^{\Diamond}_{\textit{core}}$

Now we show NP-hardness of satisfiability in the fragment $\textit{TDL-Lite}^{\Diamond}_{\textit{core}}$ of $\textit{TDL-Lite}^{\Diamond}_{\textit{bool}}$ that allows only concept inclusions of the form $A_1 \sqsubseteq A_2$, $A_1 \sqsubseteq \neg A_2$, $\Diamond A_1 \sqsubseteq A_2$ or $A_1 \sqsubseteq \Diamond A_2$, where A_1 and A_2 are concept names.

Lemma 5. *The satisfiability problem for* $\textit{TDL-Lite}^{\Diamond}_{\textit{core}}$ *KBs is* NP*-hard.*

Proof. We prove this by reduction of the graph 3-colourability (3-Col) problem, which is formulated as follows: given a graph $G = (V, E)$, decide whether there is an assignment of colours $\{1, 2, 3\}$ to vertices V such that no two vertices $a_i, a_j \in V$ sharing the same edge, $(a_i, a_j) \in E$, have the same colour. Let A_i, for $A_i \in V$, X_i, for $0 \leq i \leq 3$, and V, V' be concept names and a an object name. Consider the KB $\mathcal{K}_G = (\mathcal{T}_G, \{V(a)\})$, where $\mathcal{T}_G$ consists of the following axioms:

$$
\begin{array}{lll}
V \sqsubseteq \Diamond A_i, & A_i \sqsubseteq X_3, & \text{for all } A_i \in V,\\
A_i \sqsubseteq \neg A_j, & \text{for all } (A_i, A_j) \in E, & \\
V \sqsubseteq \neg V', & \Diamond X_0 \sqsubseteq V', & \\
\Diamond X_3 \sqsubseteq X_2, & \Diamond X_2 \sqsubseteq X_1, & \Diamond X_1 \sqsubseteq X_0.
\end{array}
$$

It is easy to see that $\mathcal{K}_G$ is satisfiable iff G is 3-colourable. Indeed, if G is 3-colourable, then we take a colouring function $c\colon V \to \{1, 2, 3\}$ and define $\mathcal{I}$ by setting $\Delta^{\mathcal{I}} = \{w\}$, $a^{\mathcal{I}} = w$, $a^{\mathcal{I}} \in A_i^{\mathcal{I}(n)}$ iff $c(A_i) = n$, for all $A_i \in V$, $a^{\mathcal{I}} \in V^{\mathcal{I}(n)}$ iff $n = 0$, $V'^{\mathcal{I}(n)} = \emptyset$, for all $n \geq 0$, and $a^{\mathcal{I}} \in X_i^{\mathcal{I}(n)}$ iff $i < n$. It should be clear that $\mathcal{I} \models \mathcal{K}_G$. For the converse direction, observe that if $\mathcal{K}_G$ is satisfiable then, for all $A_i \in V$, there is $n_i \in \{1, 2, 3\}$ such that $a^{\mathcal{I}} \in A_i^{\mathcal{I}(n_i)}$ and $a^{\mathcal{I}} \notin A_j^{\mathcal{I}(n_i)}$ whenever $(A_i, A_j) \in E$. It is readily seen that $c \colon A_i \mapsto n_i$, for $A_i \in V$, is a colouring function. ❑

As a consequence of Lemma 5 and Theorem 4 we obtain:

Theorem 5. *The satisfiability problem for TDL-Lite*$^{\Diamond}_{core}$ *KBs is* NP*-complete.*

4 Conclusions

The NP complexity result for *TDL-Lite*$^{\Diamond}_{bool}$ is encouraging in view of possible applications of this logic for reasoning about temporal conceptual data models [4]. Indeed, on the one hand, the logic *DL-Lite*$^{\mathcal{N}}_{bool}$ was shown to be adequate for representing different aspects of conceptual models: ISA, disjointness and covering for classes, domain and range of relationships, n-ary relationships, attributes and participation constraints are all expressible in *DL-Lite*$^{\mathcal{N}}_{bool}$ [6]. On the other hand, the approach of [8] shows that rigid axioms and roles with temporalised concepts are enough to capture temporal data models.

The logic *TDL-Lite*$^{\Diamond}_{bool}$ presented in this paper combines a much simpler DL *DL-Lite*$^{\mathcal{N}}_{bool}$ ($\mathcal{ALCQI}$ used in [8] is able to capture ISA between relationships) with a more powerful temporal component and uses rigid axioms and roles with temporalised concepts as proposed in [8]. The resulting logic can capture some form of *evolution constraints* [5, 18, 15] thanks to the $\Diamond$ operator, e.g., to say that students will become alumni we use the rigid axiom $\texttt{Student} \sqsubseteq \Diamond\texttt{Alumni}$. Furthermore, it also captures *snapshot* classes—i.e., classes whose instances do not change over time, e.g., that the extension of the class of human beings remains constant can be represented by $\texttt{Human} \sqsubseteq \Box\texttt{Human}$ and $\Box\texttt{Human} \sqsubseteq \texttt{Human}$. However, by restricting the temporal component only to $\Diamond$ and $\Box$ (in conjunction with rigid axioms), we lose the ability to capture *temporary* entities and relationships, i.e., entities and relationships such that each of their instances has a limited lifespan. To overcome this limitation, we are considering, as a future work, to extend the logic presented here with either past temporal operators or with a special kind of axioms that hold over finite prefix.

References

1. A. Artale, D. Calvanese, R. Kontchakov, and M. Zakharyaschev. *DL-Lite* in the light of first-order logic. In *Proc. of AAAI*, pages 361–366, 2007.
2. A. Artale, D. Calvanese, R. Kontchakov, and M. Zakharyaschev. The *DL-Lite* family and relations. Technical Report BBKCS-09-03, SCSIS, Birkbeck College, London, 2009 (available at `http://www.dcs.bbk.ac.uk/research/techreps/2009/bbkcs-09-03.pdf`).

3. A. Artale and E. Franconi. Temporal description logics. In M. Fisher, D. Gabbay, and L. Vila, editors, *Handbook of Time and Temporal Reasoning in Artificial Intelligence*, pages 375–388. Elsevier, 2005.
4. A. Artale, E. Franconi, F. Wolter, and M. Zakharyaschev. A temporal description logic for reasoning about conceptual schemas and queries. In S. Flesca, S. Greco, N. Leone, and G. Ianni, editors, *Proc. of JELIA-02*, volume 2424 of *LNAI*, pages 98–110. Springer, 2002.
5. A. Artale, C. Parent, and S. Spaccapietra. Evolving objects in temporal information systems. *Annals of Mathematics and Artificial Intelligence*, 50(1-2):5–38, 2007.
6. A. Artale, D. Calvanese, R. Kontchakov, V. Ryzhikov, and M. Zakharyaschev. Reasoning over extended ER models. In *Proc. of ER'07*, volume 4801 of *LNCS*, pages 277–292. Springer, 2007.
7. A. Artale, R. Kontchakov, C. Lutz, F. Wolter, and M. Zakharyaschev. Temporalising tractable description logics. In *Proc. of TIME 07*, pages 11–22. IEEE Computer Society, 2007.
8. A. Artale, C. Lutz, and D. Toman. A description logic of change. In *Proc. of IJCAI-07*, 2007.
9. F. Baader, R. Küsters, and F. Wolter. Extensions to description logics. In *Description Logic Handbook*, pages 219–261. Cambridge University Press, 2003.
10. F. Baader, S. Ghilardi, and C. Lutz. LTL over description logic axioms. In *Proc. of KR 2008*, 2008.
11. D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. *DL-Lite*: Tractable description logics for ontologies. In *Proc. of AAAI 2005*, pages 602–607, 2005.
12. D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. Data complexity of query answering in description logics. In *Proc. of KR 2006*, pages 260–270, 2006.
13. D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. Tractable reasoning and efficient query answering in description logics: The *DL-Lite* family. *J. of Automated Reasoning*, 39(3):385–429, 2007.
14. D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*. Elsevier, 2003.
15. G. Hall and R. Gupta. Modeling transition. In *Proc. of ICDE'91*, pages 540–549, 1991.
16. C. Lutz, F. Wolter, and M. Zakharyaschev. Temporal description logics: A survey. In *Proc. of TIME 08*. IEEE Computer Society Press, 2008.
17. K. Schild. Combining terminological logics with tense logic. In *Proc. of EPIA'93*, October 1993.
18. S. Spaccapietra, C. Parent, and E. Zimanyi. *Conceptual Modeling for Traditional and Spatio-Temporal Applications—The MADS Approach*. Springer, 2006.