A NOTE ON THE THEORY OF COMPLETE MEREOTOPOLOGIES * †

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Abstract. We investigate theories of Boolean algebras of regular sets of topological spaces. By $RC(X)$, we denote the complete Boolean algebra of regular closed sets over a topological space X. By a *mereotopology* $\mathcal{M}$ over a topological space X, we denote every dense Boolean sub-algebra of $RC(X)$; $\mathcal{M}$ is called a *complete mereotopology* if it is a complete Boolean algebra.

In this paper we consider mereotopologies as $\mathcal{L}$-structures, where $\mathcal{L}$ is the language of Boolean algebras extended with the binary relational symbol C interpreted as the *contact* relation. We show that the $\mathcal{L}-$theories of complete mereotopologies and all mereotopologies are different. We also show that no complete mereotopology $\mathcal{M}$, over a connected, compact, Hausdorff topological space X, is elementarily equivalent to a mereotopology $\mathcal{M}'$, over X, that is a closed base for X and is finitely decomposable — i.e. every region in $\mathcal{M}'$ has only finitely many connected components.

1. Introduction

Formal systems for reasoning about space can be classified as point-based or region-based, depending on whether the variables of their formal languages are interpreted as points or sets of points. A notable example of a point-based theory of space is the decidable and complete theory of the Euclidean plane axiomatized by Tarski in (Tarski 1959). An early example of a region-based theory of space can be seen in another work of Tarski. In (Tarski 1956), he axiomatized the second-order theory of the regular closed sets of the 3-dimensional Euclidean space, with respect to the language consisting of the two predicates for the binary relation *part-of* and the property of *being a sphere*.

Authors usually motivate their interest in region-based spatial logics by arguing that they are more natural in comparison with point-based spatial logics, for people think in terms of objects, rather than in terms of the sets of points that these objects occupy. There are also practical advantages: greater expressive power, as noted in (Aiello, et al. 2007); ability to reason with incomplete knowledge, which was argued in (Renz & Nebel 2007); spatial reasoning free of numerical calculations.

In most region-based spatial logics, variables range over sets of a topological space, but it is a matter of choice whether arbitrary sets should count as regions. An example of a formal system for reasoning about arbitrary sets of topological spaces is given by McKinsey and Tarski in (McKinsey & Tarski 1944). The regular sets of a topological space are widely accepted as an appropriate choice for regions when it comes to spatial reasoning about real world objects. Regular closed (open) sets of a topological space X are those equal to the closure of their interior (the interior of their closure), and the set of all regular closed (open) sets is denoted by $RC(X)$ ($RO(X)$). $RC(X)$ and $RO(X)$ form complete

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Boolean algebras, as can be seen in (Koppelberg, et al. 1989). The aforementioned work of Tarski (Tarski 1956), is an example of a formal system in which variables range over all regular closed sets of the Euclidean space. Pratt and Schoop in (Pratt & Schoop 1998), restricted the variables of their formal system to range over the $ROP(\mathbb{R}^2)$ — the Boolean algebra of regular open polygons of $\mathbb{R}^2$, which is a dense Boolean sub-algebra of $RO(\mathbb{R}^2)$.

A *mereotopology* over a topological space X, is any dense Boolean sub-algebra of $RC(X)$; $\mathcal{M}$ is a *complete mereotopology* if it is a complete Boolean algebra — i.e. $\mathcal{M}$ is $RC(X)$. We consider this slightly weaker definition when compared to the one proposed by Pratt-Hartmann in (Pratt-Hartmann 2007), in order to allow mereotopologies over topological spaces that are not semi-regular.

In the recent years, the $\mathcal{L}-$theories of different classes of mereotopologies have been axiomatized, where $\mathcal{L}$ is the language of Boolean algebras extended with the binary relational symbol C interpreted as the *contact* relation. Roeper in (Roeper 1997) axiomatized the $\mathcal{L}-$theory of the mereotopologies over compact, Hausdorff topological spaces. Düntsch and Winter in (Düntsch & Winter 2005) established an axiomatization of the $\mathcal{L}-$theory of the mereotopologies over weakly regular, T_1 topological spaces. The $\mathcal{L}-$theory of the class of all mereotopologies was axiomatized by Dimov and Vakarelov in (Dimov & Vakarelov 2006).

To the best of our knowledge, there are no published results about the $\mathcal{L}-$theory of complete mereotopologies. In this paper we show that this theory is different from the theory of all mereotopologies. In particular, we introduce a sentence that is true in every complete mereotopology, but is not true in the incomplete mereotopology of the regular closed polygons of the real plane. As a corollary of our main result, we show that no complete mereotopology $\mathcal{M}$, over a Hausdorff topological space X, is elementarily equivalent to a mereotopology $\mathcal{M}'$, over X, that is a closed base for X and is finitely decomposable — i.e. every region in $\mathcal{M}'$ has only finitely many connected components.

We provide our main results in Section 4. The necessary definitions and basic facts about topological spaces and mereotopologies we give in Section 2. In Section 3, we summarize the main axiomatization results of classes of mereotopologies, established in (Dimov & Vakarelov 2006, Düntsch & Winter 2005, Roeper 1997), and some related results provided in (Pratt-Hartmann 2007). We discuss possible future work in Section 5.

2. Preliminary Notions

In this section we recall the definition and some examples of mereotopologies. We also prove a result about topological spaces that we use in Section 4. We assume that the reader is familiar with the basic definitions and results about Boolean algebra (see e.g. (Koppelberg et al. 1989)), and topological spaces (see e.g. (Kelley 1975)).

We start by defining the Boolean algebra of the regular closed sets over a topological space X.

Definition 1. Let X be a topological space with $\cdot^-$ and $\cdot^\circ$ the closure and interior operations in X. A subset A of X is called *regular closed* if it equals the closure of its interior, i.e. $A = A^{\circ -}$. The set of all regular closed sets in X is denoted by $RC(X)$. The Boolean operations, relations and constants can be defined in $RC(X)$ in the following way: for $a, b \in RC(X)$, $a + b = a \cup b$, $a \cdot b = (a \cap b)^{\circ -}$, $-a = (X \setminus a)^-$, $a \leq b$ iff $a \cdot b = a$, $0 = \emptyset$ and $1 = X$. The topological *contact* relation $C(x, y)$, is defined by: $C(a, b)$ iff $a \cap b \neq \emptyset$.

Recall that $\mathcal{B}$ is a *complete* Boolean algebra, if each set of elements of $\mathcal{B}$ has an infimum and a supremum. It is a well-know fact, that the structure $(RC(X), +, \cdot, -, 0, 1, \leq)$ is a complete Boolean algebra (see e.g. (Koppelberg et al. 1989)). For the definition of mereotopology, recall that, a Boolean sub-algebra $\mathcal{B}'$ of $\mathcal{B}$ is dense in $\mathcal{B}$, if for every non-zero element $a \in \mathcal{B}$ there is some non zero element $a' \in \mathcal{B}'$ such that $a' \leq a$.

Definition 2. A *mereotopology* over a topological space X is any dense Boolean sub-algebra, $\mathcal{M}$, of the complete Boolean algebra $RC(X)$.

In a dual way, one can define a *mereotopology of regular open sets*. Note that Definition 2 is weaker than the one given by Pratt-Hartmann in (Pratt-Hartmann 2007). We do not require the mereotopology to form a base for the topological space, in order to have mereotopologies over arbitrary topological spaces.

A well-studied example of an incomplete mereotopology of regular open sets, is that of the regular open polygons in the real plane (see (Pratt & Schoop 1998, Pratt & Schoop 2000, Pratt-Hartmann 2007)). The dual mereotopology, $RCP(\mathbb{R}^2)$, of the regular closed polygons of the real plane, plays an important role in proving our main result in Section 4. The formal definition of $RCP(\mathbb{R}^2)$ follows.

Definition 3. Each line in $\mathbb{R}^2$ divides the real plane into two regular open sets called open half-planes. The closure of an open half-plane is regular closed, and is called half-plane. The product in $RC(\mathbb{R}^2)$ of finitely many half planes is called a *basic polygon*. The sum of finitely many basic polygons is called a *polygon*. The set off all polygons is denoted by $RCP(\mathbb{R}^2)$.

We need the following lemma for Section 4. Recall that in a topological space X the non-empty sets $A, B \subseteq X$ are said to separate the set $C \subseteq X$ iff $C = A \cup B$, $A^- \cap B = \emptyset$ and $A \cap B^- = \emptyset$; a set $C \subseteq X$ is connected iff no pair of non-empty sets separates it; a *connected component* of a set $A \subseteq X$ is a maximal connected subset of A.

Lemma 4. Let X be a topological space and A, A_1 and A_2 be subsets of X such that A_1 and A_2 separate A. Then the following are true:

$i)$ A is closed iff A_1 and A_2 are closed;

$ii)$ A is regular closed iff A_1 and A_2 are regular closed.

Proof. First notice, that the right to left implications are obvious, since the union of two (regular) closed sets is a (regular) closed set.

$i)$ $(\rightarrow)$ From $A_1^- \subseteq A^- = A = A_1 \cup A_2$, we get $A_1^- \subseteq A_1^- \cap (A_1 \cup A_2) = (A_1^- \cap A_1) \cup (A_1^- \cap A_2) = A_1^- \cap A_1 = A_1$, so A_1 is closed. Similarly for A_2.

$ii)$ $(\rightarrow)$ From $i)$ it follows that A_1 and A_2 are closed. We want to show that $A^\circ = A_1^\circ \cup A_2^\circ$ because this implies $A_1 = A \cap X \setminus A_2 = A^{\circ-} \cap X \setminus A_2 = (A_1^{\circ-} \cup A_2^{\circ-}) \cap X \setminus A_2 = A_1^{\circ-} \cap X \setminus A_2 = A_1^{\circ-}$. The inclusion $A^\circ \supseteq A_1^\circ \cup A_2^\circ$ is trivial. Suppose $p \in A^\circ$ and w.l.o.g. let $p \in A_1$. Then $p \in X \setminus A_2$ since $A_1 \cap A_2 = \emptyset$. We get $p \in A^\circ \cap X \setminus A_2$. This set is open because A_2 is closed and subset of A_1, and, hence, $p \in A_1^\circ$.

□

3. Representation Theorems for Mereotopologies

We consider mereotopologies as $\mathcal{L}-$structures, where $\mathcal{L}$ is the language $\{C, +, \cdot, -, 0, 1, \leq\}$ (see Definition 1). The $\mathcal{L}-$theories of different classes of mereotopologies were axiomatized in (Dimov & Vakarelov 2006, Düntsch & Winter 2005, Roeper 1997), although different terminology was used. In this section we give a translation of the original results in terms of mereotopologies in a way almost identical to the one in (Pratt-Hartmann 2007). Nice discussions on the algebraic approach taken in (Dimov & Vakarelov 2006, Düntsch & Winter 2005, Roeper 1997), can be seen in (Bennett & Düntsch 2007, Vakarelov 2007).

We assume the reader is familiar with some basic notions in Model Theory (see e.g. (Marker 2002)). Before we continue, we recall definitions of semi-regular and weakly regular topological spaces.

Definition 5. A topological space X is called *semi-regular*, if the set of all regular closed sets in X form a closed base for X. A semi-regular topological space is called *weakly regular* (Düntsch & Winter 2005), if for each nonempty open set $A \subseteq X$, there exists a nonempty open set B such that $B^- \subseteq A$.

Definition 6. We denote by Φ_{CA} the set of axioms for Boolean algebra, together with the following sentences:

$\psi_1 := \forall x \forall y (C(x, y) \rightarrow x \neq 0);$

$\psi_2 := \forall x \forall y (C(x, y) \rightarrow C(y, x));$

$\psi_3 := \forall x \forall y \forall z (C(x, y + z) \leftrightarrow C(x, y) \vee C(x, z));$

$\psi_4 := \forall x \forall y (x \cdot y \neq 0 \rightarrow C(x, y)).$

As we will see in Theorem 9, Φ_{CA} is an axiomatization for the class of all mereotopologies. Extending Φ_{CA} with different combinations of the axioms ψ_{ext}, ψ_{int} and ψ_{conn} (see bellow), leads to axiomatizations for mereotopologies over different classes of topological spaces.

In the following definition we abbreviate $\neg C(x, -y)$, by $x \ll y$.

Definition 7. Consider the following sentences:

$\psi_{ext} := \forall x (x \neq 0 \rightarrow \exists y (y \neq 0 \wedge y \ll x))$ - extensionality axiom;

$\psi_{int} := \forall x \forall y (x \ll y \rightarrow \exists z (x \ll z \wedge z \ll y))$ - interpolation axiom;

$\psi_{conn} := \forall x (x \neq 1 \wedge x \neq 0 \rightarrow C(x, -x))$ - connectedness axiom.

Theorem 8. *(Pratt-Hartmann 2007) Let $\mathcal{M}$ be a mereotopology over a topological space X, considered as an $\mathcal{L}-$structure.*

i) $\mathcal{M} \models \Phi_{CA}$.

ii) If X is weakly regular, then $\mathcal{M} \models \psi_{ext}$.

iii) If X is compact and Hausdorff and the elements of $\mathcal{M}$ form a closed base for X, then $\mathcal{M} \models \psi_{int}$.

Theorem 9. *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure.*

$i)$ *If $\mathfrak{A} \models \Phi_{CA}$, then $\mathfrak{A}$ is isomorphic to a mereotopology over a compact semi-regular T_0 topological space X. (Dimov & Vakarelov 2006)*

$ii)$ *If $\mathfrak{A} \models \Phi_{CA} \cup \{\psi_{ext}\}$, then $\mathfrak{A}$ is isomorphic to a mereotopology over a weakly regular and T_1. (Düntsch & Winter 2005)*

$iii)$ *If $\mathfrak{A} \models \Phi_{CA} \cup \{\psi_{ext,}\psi_{int}\}$, then $\mathfrak{A}$ is isomorphic to a mereotopology over a compact and Hausdorff. (Roeper 1997)*

Additionally, $\mathfrak{A} \models \psi_{conn}$ implies X is connected.

To the best of our knowledge, there are no results in the literature about the $\mathcal{L}-$theory of complete mereotopologies. It turns out that this $\mathcal{L}-$theory is different from the $\mathcal{L}-$theory of all mereotopologies. We devote the next section to establish this result.

4. The $\mathcal{L}-$theory of Complete Mereotopologies

In this section we show that the theory of complete mereotopologies differs from the theory of all mereotopologies. We accomplish this by introducing a first-order sentence that is true in each complete mereotopology but that is not true in $RCP(\mathbb{R}^2)$, which is an incomplete mereotopology. This result relies on the fact, that every non-trivial complete mereotopology satisfying $\{\psi_{ext}, \psi_{int,}\psi_{conn}\}$, has a pair of regions that are in contact, such that neither connected component of the first region is in contact with the second. The latter, however, is false for all finitely decomposable mereotopologies, including $RCP(\mathbb{R}^2)$, which on the other hand, is a non-trivial incomplete mereotopology satisfying the set of axioms $\{\psi_{ext}, \psi_{int,}\psi_{conn}\}$.

Connected regions play an important role in the proof of the main result, so we start by introducing a formula, which defines the set of connected regions in $RCP(\mathbb{R}^2)$ and each complete mereotopology $\mathcal{M}$. We make use of the fact that regular closed sets can be separated only by regular closed sets (Lemma 4).

Lemma 10. Let $\mathcal{M}$ be a complete mereotopology. Then for all $a \in \mathcal{M}$, a is connected iff $\mathcal{M} \models \psi_c[a]$, where

$$\psi_c(x) := (\forall y)(\forall z)(y \neq 0 \wedge z \neq 0 \wedge y + z = x \rightarrow C(y, z)).$$

Proof. $(\rightarrow)$ From $\mathcal{M} \not\models \psi_c[a]$ it follows that there are regular closed sets b, c that separate a, thus a is not connected.

$(\leftarrow)$ From a is not connected and Lemma 4, it follows that there are nonempty regular closed sets b, c such that $a = b + c$ and $\neg C(b, c)$. So b and c are witnesses for $\mathcal{M} \not\models \psi_c[a]$. □

In order to establish the same result for $RCP(\mathbb{R}^2)$, we have to show that a regular closed polygon can be separated only by regular closed polygons.

Lemma 11. Consider the mereotopologies $RC(\mathbb{R}^2)$ and $RCP(\mathbb{R}^2)$. For each $a \in RCP(\mathbb{R}^2)$ and $b, c \in RC(\mathbb{R}^2)$, if $a = b + c$ and $\neg C(b, c)$, then $b, c \in RCP(\mathbb{R}^2)$.

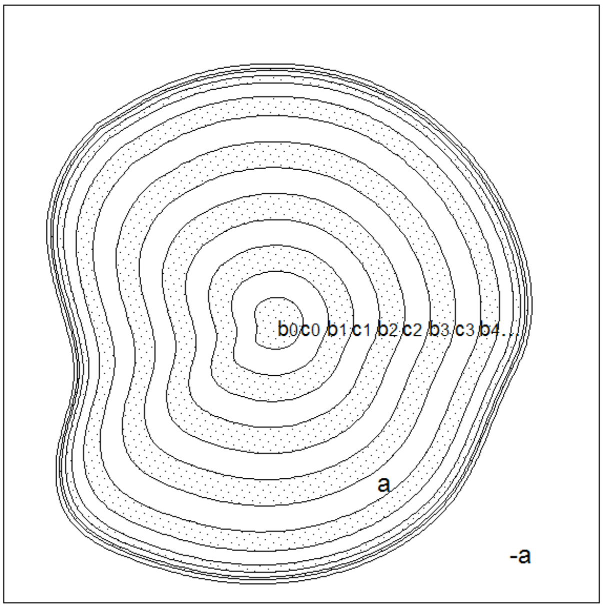

Figure 1: (Lemma 14) At least one of $b := \sum_{i\in\omega} b_i$ and $c := \sum_{i\in\omega} c_i$ is in contact with $(-a)$, but none of $\{b_i\}_{i\in\omega}$ and $\{c_i\}_{i\in\omega}$ is.

Proof. Since a is a regular closed polygon it is the sum of finitely many basic polygons, e.g. $a = \sum_{i=1}^{n} a_i$. Let $b_i := b.a_i$ and $c_i := c.a_i$. Since a_i is connected, $\neg C(b_i, c_i)$ and $a_i = b_i + c_i$, we get that $a_i = b_i$ or $a_i = c_i$. So b_i and c_i are basic polygons (either equal to 0 or to a_i). Since $b = \sum_{i=1}^{n} a_i$ and $c = \sum_{i=1}^{n} c_i$, we get that b and c are polygons, as finite sums of basic polygons. □

Lemma 12. For $a \in RCP(\mathbb{R}^2)$, a is connected iff $RCP(\mathbb{R}^2) \models \psi_c[a]$.

Proof. As in Lemma 10, considering Lemma 11. □

So far, we defined the set of connected regions in $RCP(\mathbb{R}^2)$ and each complete mereotopology by the formula ψ_c. Having shown that, we continue by constructing for every non-trivial complete mereotopology satisfying $\{\psi_{ext}, \psi_{int}, \psi_{conn}\}$, a pair of regions which are in contact, such that no connected component of the first is in contact with the second.

Lemma 13. Let $\mathcal{M}$ be a complete mereotopology such that $\mathcal{M} \models \psi_{ext} \wedge \psi_{int} \wedge \psi_{conn} \wedge \neg\psi_{triv}$, where $\psi_{triv} := (\forall x)(x = 0 \vee x = 1)$. Then there are elements a and $\{a_i\}_{i\in\omega}$ in $\mathcal{M}$ such that:

$i)$	$C(a, -a)$;	$ii)$	$a = \sum_{i\in\omega} a_i$;
$iii)$	$a_i \ll a_{i+1}$, for $i \in \omega$;	$iv)$	$a_i \ll a$, for $i \in \omega$.

Proof. From $\mathcal{M} \models \neg\psi_{triv}$ it follows that there is some $b \in \mathcal{M}$ such that $b \neq 0$ and $b \neq 1$. Now from $\mathcal{M} \models \psi_{ext}$, we get that there is some element $a_0 \in \mathcal{M}$ such that $a_0 \ll b$ and $a_0 \neq 0$. Considering that $\mathcal{M} \models \psi_{int}$, it follows that there is some a_1 such that $a_0 \ll a_1 \ll b$ and again by $\mathcal{M} \models \psi_{int}$, we get that there is some a_2 such that $a_1 \ll a_2 \ll b$. Arguing in a similar way one can construct a sequence $\{a_i\}_{i\in\omega}$ such that

$a_0 \ll a_i \ll a_{i+1} \ll b$ for $i \in \omega$. Now we take $a := \sum_{i\in\omega} a_i$, which is in $\mathcal{M}$, for $\mathcal{M}$ is complete. It is easy to see that $i) - iv)$ hold. We give details only in the case $i)$.

$i)$ From $a_0 \neq 0$ and $a_0 \leq a$, we get $a \neq 0$. On the other hand, $a \leq b$ since b is an upper bound of $\{a_i\}_{i\in\omega}$ and since $b \neq 1$, we get also that $a \neq 1$. Now considering ψ_{conn}, we get that $C(a, -a)$. □

In the following lemma we introduce an $\mathcal{L}-$ sentence, denoted by ψ_{cmp}, and show that it is true in each complete mereotopology.

Lemma 14. For each complete mereotopology $\mathcal{M}$, $\mathcal{M} \models \psi_{cmp}$, where

$$\begin{aligned} \psi_{cmp} :=& \quad \psi_{ext} \wedge \psi_{int} \wedge \psi_{conn} \wedge \neg\psi_{triv} \rightarrow (\exists x)(\exists y)(\psi_{\odot}(x, y)) \text{ and} \\ \psi_{\odot}(x, y) :=& \quad \psi_{\odot}(x, y) := C(x, y) \wedge (\forall x')(x' \leq x \wedge \psi_c(x') \rightarrow \neg C(x', y)). \end{aligned}$$

Proof. If $\mathcal{M} \models \psi_{ext} \wedge \psi_{int} \wedge \psi_{conn} \wedge \neg\psi_{triv}$, it follows from Lemma 13, that there are elements a and $\{a_i\}_{i\in\omega}$ in $\mathcal{M}$, such that $C(a, -a)$, $a = \sum_{i\in\omega} a_i$ and for $i \in \omega$, $a_i \ll a_{i+1}$ and $a_i \ll a$. Take $a_{-1} = 0$ and consider the following definitions:

$$\begin{aligned} b_i = a_{2i} - a_{2i-1}, \quad b = \textstyle\sum_{i\in\omega} b_i, \quad b_{i-} = \sum_{j<i} b_j, \quad b_{i+} = \sum_{j>i} b_j, \\ c_i = a_{2i+1} - a_{2i}, \quad c = \textstyle\sum_{i\in\omega} c_i, \quad c_{i-} = \sum_{j<i} c_j, \quad c_{i+} = \sum_{j>i} c_j. \end{aligned}$$

Since $\mathcal{M}$ is complete, it follows that $b, c, b_{i-}, c_{i-}, b_{i+}, c_{i+} \in \mathcal{M}$. (See Figure 1.)

Claim 1 For $i \in \omega$, $\neg C(b_i, b - b_i)$ and $\neg C(c_i, c - c_i)$

Proof. From $b_{i-} \leq a_{2i-2} \ll a_{2i-1}$ and $b_i \leq -a_{2i-1}$, we get that $\neg C(b_i, b_{i-})$. From $b_{i+} \leq -a_{2i+1} \ll -a_{2i}$ and $b_i \leq a_{2i}$, we get $\neg C(b_i, b_{i+})$. From $b - b_i = b_{i-} + b_{i+}$, we get that $\neg C(b_i, b - b_i)$. Similarly $\neg C(c_i, c - c_i)$.

Claim 2 From $b' \leq b$ and $\mathcal{M} \models \psi_c[b']$, it follows $b' \ll a$. From $c' \leq c$ and $\mathcal{M} \models \psi_c[c']$, it follows $c' \ll a$.

Proof. We will show that there is some $i \in \omega$ such that $b' \leq b_i$. Since $b' \leq b$ and $b = \sum_{i\in\omega} b_i$ there is some $i \in \omega$ such that $b' \cdot b_i \neq 0$. We have that $b' = b' \cdot b = b' \cdot (b + b_i - b_i) = b' \cdot b_i + b' \cdot (b - b_i)$. From Claim 1 it follows that $\neg C(b_i, (b - b_i)$ and thus $\neg C(b' \cdot b_i, b' \cdot (b - b_i))$. Now from $\mathcal{M} \models \psi_c[b']$ and $b' \cdot b_i \neq 0$ it follows that $b' \cdot (b - b_i) = 0$ and thus $b' = b_i \cdot b'$, which is $b' \leq b_i$. Finally, we get that $b' \leq b_i \ll a$. Similarly $c' \leq c$ and $\mathcal{M} \models \psi_c[c']$ imply $c' \ll a$.

Finally, from $a = b + c$ and $C(a, -a)$, it follows that either $C(b, -a)$ or $C(c, -a)$. W.l.o.g., let $C(b, -a)$ be the case. By Claim 2 we get that $\mathcal{M} \models \psi_{\odot}[b, -a]$ and so $\mathcal{M} \models \psi_{cmp}$. □

Lemma 15. $RCP(\mathbb{R}^2) \not\models \psi_{cmp}$.

Proof. It is well known that $RCP(\mathbb{R}^2) \models \psi_{ext} \wedge \psi_{int} \wedge \psi_{conn} \wedge \neg\psi_{triv}$, so it suffices to show that $RCP(\mathbb{R}^2) \not\models (\exists x)(\exists y)(\psi_{\odot}(x, y))$. Let a and b be regular closed polygons, such that $a \cap b \neq \emptyset$. Since a is a polygon, it can be represented as a finite sum of basic polygons, say $a = \sum_{i=1}^{n} a_i$. Since the sum of finitely many regular closed sets is just their union, we get that $a_i C b$ for some $i \leq n$. Since the basic polygons are connected and Lemma 12, we get that $RCP(\mathbb{R}^2) \not\models \psi_{\odot}[a, b]$. So, we get that $RCP(\mathbb{R}^2) \models (\forall x)(\forall y)(\neg\psi_{\odot}(x, y))$ and thus $RCP(\mathbb{R}^2) \models \neg\psi_{cmp}$. □

Theorem 16. *The $\mathcal{L}-$theory of complete mereotopologies is different from the $\mathcal{L}-$theory of:*

$i)$ *the class of all mereotopologies;*

$ii)$ *the class of mereotopologies over weakly regular topological spaces;*

$iii)$ *the class of mereotopologies over compact Hausdorff topological spaces.*

Proof. $RCP(\mathbb{R}^2)$ is a member of each of the above classes. $\square$

Theorem 17. *Let X be a connected, compact, Hausdorff topological space and let the complete mereotopology, $\mathcal{M}$, over X be non-trivial. Then the $\mathcal{L}-$theory of $\mathcal{M}$ is different from the $\mathcal{L}-$theory of every finitely decomposable mereotopology, $\mathcal{M}'$, over X, that is a close base for X.*

Proof. From Theorem 8 and $\mathcal{M}$ being non-trivial, it follows that $\mathcal{M}' \models \psi_{ext} \wedge \psi_{int} \wedge \psi_{conn} \wedge \neg\psi_{triv}$. Since $\mathcal{M}'$ is finitely decomposable, one can show, as in Lemma 15, that $\mathcal{M}' \not\models \psi_{cmp}$. But since $\mathcal{M}$ is a complete mereotopology, we have that $\mathcal{M} \models \psi_{cmp}$. So $\mathcal{M}$ and $\mathcal{M}'$ have different $\mathcal{L}-$theories. $\square$

Corollary 18. The $\mathcal{L}-$theories of $RC(\mathbb{R}^2)$ and $RCP(\mathbb{R}^2)$ are different.

5. Conclusions and Future Work

We showed that the theory of complete mereotopologies is different from the theory of all mereotopologies. As a future step, one can establish an axiomatization for the theory of complete mereotopologies or the theories of specific complete mereotopologies such as the mereotopologies of the regular closed sets in the real line, real plane or higher dimensional topological spaces.

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