The Computational Complexity of Topological Logics

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- A spatial logic is a formal language with
  - variables ranging over ‘geometrical entities’
  - non-logical primitives denoting relations and operations defined over those geometrical entities.
- Any spatial logic is thus characterized by by three parameters:
  - a logical syntax:

    propositional logic, FOL, higher-order logic $\ldots$
  - a signature of non-logical (geometrical) primitives:

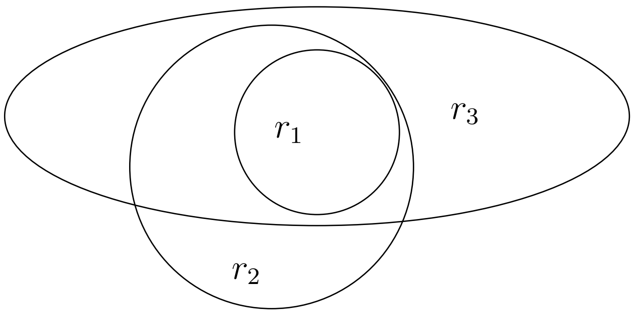
    conv(x), $c(x)$, $C(x, y)$, $\ldots$
  - a class of interpretations (more on this below).
- A topological logic is a spatial logic whose non-logical primitives are all topological in character.

- Probably the best-known topological logic is the '$\mathcal{RCC}8$' language (Randall, Cui and Cohn, 1992), (Egenhofer 1991)

$\mathsf{DC}(r_1, r_2)$	$\mathsf{EC}(r_1, r_2)$
$\mathsf{PO}(r_1, r_2)$	$\mathsf{EQ}(r_1, r_2)$
$\mathsf{TPP}(r_1, r_2)$	$\mathsf{NTPP}(r_1, r_2)$

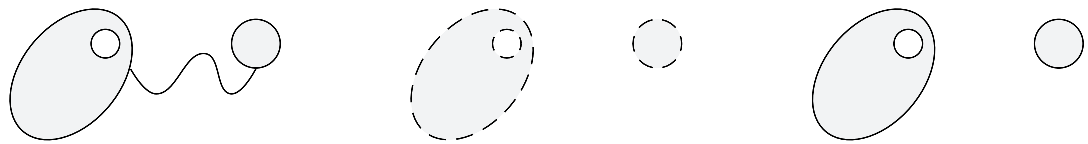
- Example of a formula in this logic:

$$\big(\mathsf{TPP}(r_1, r_2) \wedge \mathsf{NTPP}(r_1, r_3)\big) \rightarrow \big(\mathsf{PO}(r_2, r_3) \vee \mathsf{TPP}(r_2, r_3) \vee \mathsf{NTPP}(r_2, r_3)\big).$$



- If X is a topological space, a frame on X is a pair $(X, \mathbf{R})$, where $\mathbf{R}$ is a (non-empty) collection of subsets of X—called regions.

- For example, we can consider the frame $(X, \mathsf{RC}(X))$ of regular closed sets in X. (A regular closed set is the closure of an open set).



- If X is a topological space, $\mathsf{RC}(X)$ is a Boolean algebra under natural operations:

$$
\begin{aligned}
r_1 + r_2 &= r_1 \cup r_2 \\
r_1 \cdot r_2 &= \mathrm{cl}(\mathrm{int}(r_1 \cap r_2)) \\
-r_1 &= \mathrm{cl}(\mathrm{cmp}(r_1))
\end{aligned}
$$

So frames of the form $\mathsf{RC}(X)$ are natural structures over which to interpret topological logics.

- Denote the class of frames $\{(X, \mathsf{RC}(X)) \mid X \text{ a topological space}\}$ by REGC.
- And given an assignment of variables to regions of a frame in REGC, the $\mathcal{RCC}8$-primitives have natural formal interpretations:

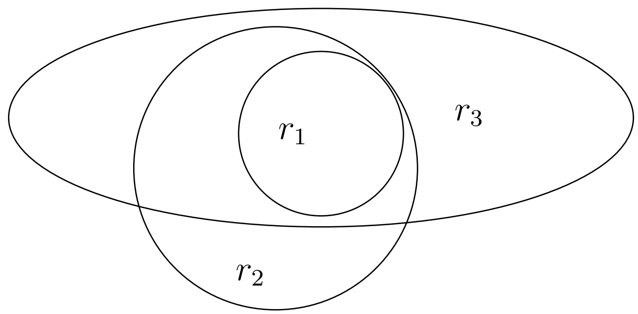
$$
\begin{array}{rcl}
\mathsf{DC}(r_1, r_2) & \text{iff} & r_1 \cap r_2 = \emptyset \\
\mathsf{TPP}(r_1, r_2) & \text{iff} & r_1 \subseteq r_2 \text{ but } r_1 \not\subseteq \text{int}(r_2) \\
\mathsf{NTPP}(r_1, r_2) & \text{iff} & r_1 \subseteq \text{int}(r_2) \\
\dots & \dots & \dots
\end{array}
$$

- This gives us notions of satisfiability and validity for formulas, with respect to either frames or, more generally, classes of frames.
- We denote the satisfiability problem for $\mathcal{RCC}8$-formulas over a frame-class $\mathcal{K}$ by *Sat*$(\mathcal{RCC}8, \mathcal{K})$.

- For example,

$$\neg\big(\mathsf{TPP}(r_1, r_2) \wedge \mathsf{NTPP}(r_1, r_3)\big) \rightarrow \big(\mathsf{PO}(r_2, r_3) \vee \mathsf{TPP}(r_2, r_3) \vee \mathsf{NTPP}(r_2, r_3)\big).$$

is not satisfiable over RegC.



- We can also interpret $\mathcal{RCC}8$-formulas over smaller frame-classes: e.g.

$$\mathsf{RC}(\mathbb{R}), \quad \mathsf{RC}(\mathbb{R}^2), \quad \mathsf{RC}(\mathbb{R}), \quad \{\mathsf{RC}(\mathbb{R}^n) \mid n \geq 1\}, \ldots$$

However, this makes no difference to the satisfiability/validity problem: $\mathit{Sat}(\mathcal{RCC}8, \textsc{RegC}) = \mathit{Sat}(\mathcal{RCC}8, \mathsf{RC}(\mathbb{R}^n))$ for all $n \geq 1$.

- Some simple facts:

  Theorem 1 (≈ Renz 1998). *The problem Sat*$(\mathcal{RCC}8, \textsc{RegC})$ *is* NP-*complete. Indeed, for any* $n \geq 0$,

  $$\mathit{Sat}(\mathcal{RCC}8, \mathsf{RC}(\mathbb{R}^n)) = \mathit{Sat}(\mathcal{RCC}8, \textsc{RegC}).$$

- Actually, by restricting the language somewhat, we get better complexities:
  - if we consider only conjunctions of $\mathcal{RCC}8$-primitives, complexity of satisfiability goes down to NLogSpace
  - Various (larger) tractable fragments have been found (Nebel and Bürckert 1995), (Renz 1999), ...,

- Warning:

  Regions need not be connected.

- Now suppose we add $+$, $\cdot$, $-$, 0 and 1 to $\mathcal{RCC}8$, yielding the language $\mathcal{BRCC}8$ (Wolter and Zakharyaschev, 2000), thus:

$$\mathsf{EC}(r_1 + r_2, r_3) \rightarrow \big(\mathsf{EC}(r_1, r_3) \vee \mathsf{EC}(r_2, r_3)\big).$$

- But now, we can replace the $\mathcal{RCC}8$-predicates with the binary relations of equality ($=$) and contact:

$$C(r_1, r_2) \text{ iff } r_1 \cap r_2 = \emptyset.$$

thus:

$$\begin{array}{rcl} \mathsf{DC}(r_1, r_2) & \equiv & \neg C(r_1, r_2) \\ \mathsf{TPP}(r_1, r_2) & \equiv & r_1 \cdot -r_2 = 0 \wedge C(r_1, -r_2) \\ \mathsf{NTPP}(r_1, r_2) & \equiv & r_1 \neq 0 \wedge \neg C(r_1, -r_2) \\ \ldots & \ldots & \ldots \end{array}$$

- For this reason, the language is now called, simply, $\mathcal{C}$.

- Some more simple facts:

  Theorem 2 (Wolter and Zakharyaschev, 2000)**.** *The problem* *Sat*$(\mathcal{C}, \textsc{RegC})$ *is* NP*-complete. For any* $n \geq 1$*, the problem* *Sat*$(\mathcal{C}, \mathsf{RC}(\mathbb{R}^n))$ *is* PSPACE*-complete.*

- The critical difference here is that the spaces $\mathbb{R}^n$ are connected. (The PSPACE-hardness result generally applies when $\mathcal{C}$ is interpreted over the class of regular closed algebras of connected topological spaces.)

- Logics which cannot express the property of connectedness are of limited interest. So let's add it!

- We employ a unary predicate c with the semantics

$c(r)$ iff r is connected

- We consider the languages
  - $\mathcal{RCC}8c$: $\mathcal{RCC}8$ plus the unary predicate c;
  - $\mathcal{C}c$: W+Z's language (i.e. C, $+$, $\cdot$, $-$, 0, 1) plus the unary predicate c;
  - $\mathcal{B}c$: like $\mathcal{C}$, but without C.
- Example of an $\mathcal{RCC}8c$-formula in the 3 variables r_1, r_2, r_3:

$$\bigwedge_{1 \leq i \leq 3} c(r_i) \wedge \bigwedge_{1 \leq i < j \leq 3} \mathsf{EC}(r_i, r_j).$$

- Adding the predicate c makes the logic much more sensitive to the underlying space.
- Example:

$$\bigwedge_{1 \le i \le 3} c(r_i) \wedge \bigwedge_{1 \le i < j \le 3} \mathsf{EC}(r_i, r_j)$$

  is not satisfiable in $\mathsf{RC}(\mathbb{R})$ (because any realizing assignment would make r_1, r_2 and r_3 intervals); but it is satisfiable in $\mathsf{RC}(\mathbb{R}^n)$ for $n \ge 2$.

- Example:

$$\bigwedge_{1 \le i < j \le 5} c(r_{i,j}) \quad \wedge \bigwedge_{\{i,j\} \cap \{k,\ell\} = \emptyset} \mathsf{DC}(r_{i,j}, r_{k,\ell}) \quad \wedge \bigwedge_{i \in \{j,k\}} \mathsf{TPP}(r_i, r_{j,k}).$$

  is not satisfiable in $\mathsf{RC}(\mathbb{R}^2)$ (because any realizing assignment would induce a plane embedding of K_5); but it is satisfiable in $\mathsf{RC}(\mathbb{R}^n)$ for $n \ge 3$.

- Various complexity results are known here

Theorem 3 (Kontchakov, P-H, W+Z, forthcoming)**.**

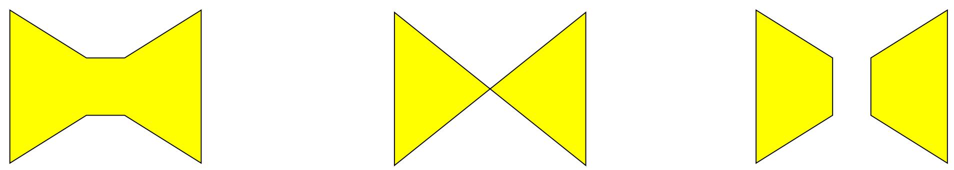
Sat$(\mathcal{RCC}8c, \textsc{RegC})$ *is* NP-*complete* (*trivial*)*;*
Sat$(\mathcal{C}c, \textsc{RegC})$ *is* EXPTIME-*complete;*
Sat$(\mathcal{B}c, \textsc{RegC})$ *is* EXPTIME-*complete.*

Theorem 4.

Sat$(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}^n))$ *is* NP-*complete* ($n \geq 1$)**;*
Sat$(\mathcal{B}c, \mathsf{RC}(\mathbb{R}))$ *is* NP-*complete ;*
Sat$(\mathcal{C}c, \mathsf{RC}(\mathbb{R}))$ *is* PSPACE-*complete;*
Sat$(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^n))$ *is* EXPTIME-*hard* ($n \geq 2$)*;*
Sat$(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^n))$ *is* EXPTIME-*hard* ($n \geq 2$)*.*

* Membership of *Sat*$(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}^2))$ in NPis highly non-trivial (Shaefer, Sedgwick and Štefankovič).

- We may wish to distinguish between connectedness and interior connectedness:



- We employ a unary predicate c° with the semantics

$$c^\circ(r) \text{ iff } \mathrm{int}(r) \text{ is connected}$$

- This gives us the further languages $\mathcal{RCC}8c^\circ$, $\mathcal{B}c^\circ$, $\mathcal{C}c^\circ$.

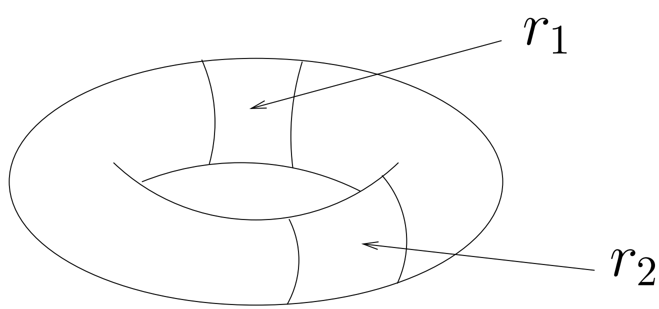
- Example of an $\mathcal{C}c^\circ$-formula

$$c^\circ(-r_1) \wedge c^\circ(-r_2) \wedge \mathsf{DC}(r_1, r_2) \wedge \neg c^\circ(-(r_1 + r_2))$$

- The $\mathcal{C}c^{\circ}$-formula

$$c^{\circ}(-r_1) \wedge c^{\circ}(-r_2) \wedge \mathsf{DC}(r_1, r_2) \wedge \neg c^{\circ}(-(r_1 + r_2))$$

is satisfiable over REGC, thus:

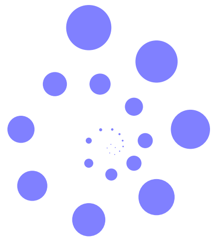


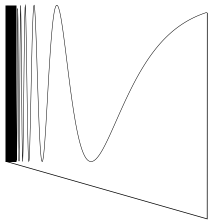
But it is not satisfiable over $\mathsf{RC}(\mathbb{R}^n)$ for *any* n!

Theorem 5.

Sat$(\mathcal{RCC}8c^{\circ}, \mathsf{RC}(\mathbb{R}^n))$ *is* NP-*complete* ($n \geq 2$)*;*
Sat$(\mathcal{C}c^{\circ}, \mathsf{RC}(\mathbb{R}^n))$ *is* EXPTIME-*hard* ($n \geq 2$)*;**
Sat$(\mathcal{B}c^{\circ}, \mathsf{RC}(\mathbb{R}^n))$ *is* NP-*complete* ($n \geq 3$).

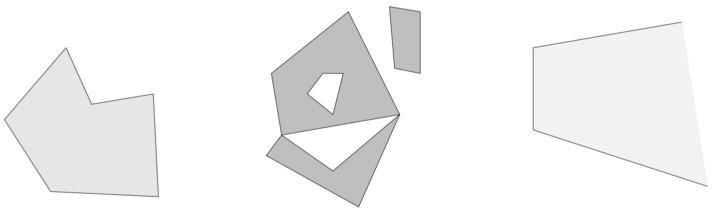
- Actually, matters are even more delicate than this: $\mathsf{RC}(\mathbb{R}^n)$ contains some very pathological sets:





- This prompts us to consider interpretations of spatial logics over collections of tame regions.

- Natural candidates for tame subalgebras of $\mathsf{RC}(\mathbb{R}^n)$:
  - The regular closed polyhedra in $\mathbb{R}^n$, RCP($\mathbb{R}^n$):

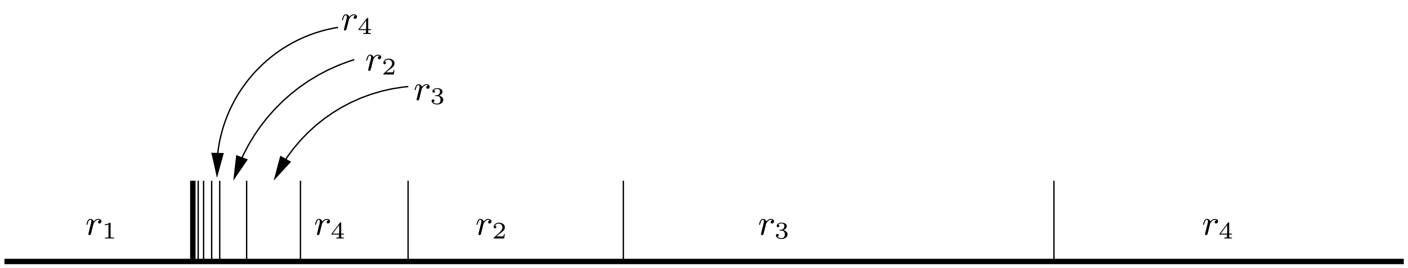


  - The regular closed semi-algebraic subsets of $\mathbb{R}^n$, RCS($\mathbb{R}^n$).

- We consider first logics interpreted over 1-dimensional space.
- Consider the $\mathcal{RCC}8c$-formula

$$c(r_1) \wedge \bigwedge_{1 \le i < j \le 4} \mathsf{EC}(r_i, r_j).$$

- This formula is satisfiable over $\mathsf{RC}(\mathbb{R})$:



- But the only satisfying tuples are those in which some of the members have infinitely many components.
- That is, the formula is not satisfiable over $\mathsf{RCP}(\mathbb{R})$.

- Thus, we have shown:

$$
\begin{aligned}
Sat(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R})) &\neq Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R})) \\
Sat(\mathcal{C}c, \mathsf{RC}(\mathbb{R})) &\neq Sat(\mathcal{C}c, \mathsf{RCP}(\mathbb{R})).
\end{aligned}
$$

- These problems do, however, have the same complexity:

  Theorem 6.

  $Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}))$ *is* NP-*complete;*
  $Sat(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}))$ *is* PSPACE-*complete.*

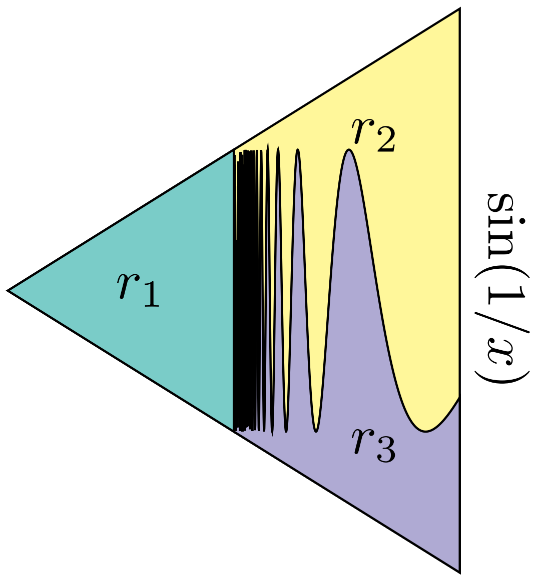
- On the other hand:

  Theorem 7. $Sat(\mathcal{B}c, \mathsf{RCP}(\mathbb{R})) = Sat(\mathcal{B}c, \mathsf{RC}(\mathbb{R}))$*, and hence is* NP-*complete.*

- In two dimensions, we get a different pattern of sensitivity to tameness:
- For example, the $\mathcal{B}c^{\circ}$-formula

$$\bigwedge_{1\leq i\leq 3} c^{\circ}(r_i) \;\wedge\; c^{\circ}\Big(\sum_{1\leq i\leq 3} r_i\Big) \;\wedge \neg(c^{\circ}(r_1+r_2) \vee c^{\circ}(r_1+r_3))$$

is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, thus,



but is unsatisfiable over $\mathsf{RCP}(\mathbb{R}^2)$.

- Thus, we have shown:

$$\mathit{Sat}(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^2)) \neq \mathit{Sat}(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$$
$$\mathit{Sat}(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^2)) \neq \mathit{Sat}(\mathcal{C}c^\circ, \mathsf{RCP}(\mathbb{R}^2)).$$

- Similarly (via a more elaborate construction):

$$\mathit{Sat}(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^2)) \neq \mathit{Sat}(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}^2))$$
$$\mathit{Sat}(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^2)) \neq \mathit{Sat}(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}^2)).$$

Theorem 8.

$\mathit{Sat}(\mathcal{RCC}8c\{\circ\}, \mathsf{RCP}(\mathbb{R}^2)) = \mathit{Sat}(\mathcal{RCC}8c\{\circ\}, \mathsf{RC}(\mathbb{R}^2)).$

Theorem 9.

$\mathit{Sat}(\mathcal{B}c, \mathsf{RCP})(\mathbb{R}^n))$ *is* EXPTIME-*hard* ($n \geq 2$)*;*
$\mathit{Sat}(\mathcal{C}c, \mathsf{RCP})(\mathbb{R}^n))$ *is* EXPTIME-*hard* ($n \geq 2$)*;*
$\mathit{Sat}(\mathcal{C}c^\circ, \mathsf{RCP}(\mathbb{R}^n))$ *is* EXPTIME-*hard* ($n \geq 2$)*;*
$\mathit{Sat}(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ *is* EXPTIME-*hard;*
$\mathit{Sat}(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^n))$ *is* EXPTIME-*complete* ($n \geq 3$).

Conclusions

- We have explained what a topological logic (more generally, a spatial logic) is.
- We have reviewed some well-known results on $\mathcal{RCC}8$, and considered the effect of adding connectedness predicates.
- We showed that even the simplest logics with connectedness:
  - are sensitive to the underlying space;
  - exhibit complex patterns of sensitivity to tameness in Euclidean spaces of different dimension;
  - reveal a complicated (and still, to some extent uncharted) complexity-theoretic landscape.
- The authoritative reference for more results on spatial logics

  Aiello, P-H and van Benthem (eds.), *Handbook of Spatial Logics*, Springer, 2007.