On the Computability of Euclidean Logics

Yavor Nenov

School of Computer Science
University of Manchester, UK

(Joint work with Ian Pratt-Hartmann)

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Pioneering research and skills

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Euclidean Logics

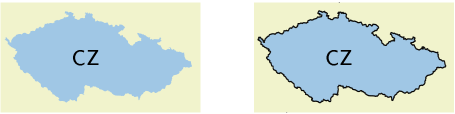
Euclidean Logic: A logical language whose variables are interpreted as **subsets** of $\mathbb{R}^n$, for a fixed $n > 0$, and whose non-logical primitives are interpreted as **geometrical** properties, relations and operations involving those sets.

Euclidean Logics - Regions

What collection of subset of $\mathbb{R}^n$ shall we choose?

- subsets which are likely to be occupied by **physical objects**

Open/Closed Sets



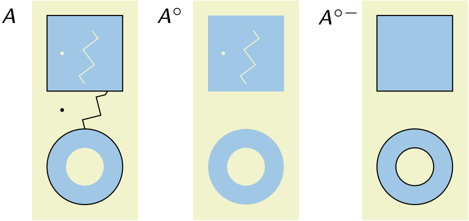
Regular Closed Sets

$A = A^{\circ-}$

$RC(\mathcal{X}) =$

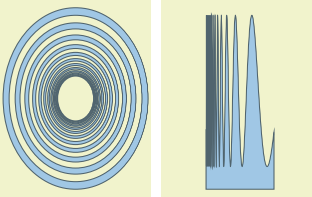
$\{A \subseteq X \mid A = A^{\circ-}\}$

$RC(\mathcal{X})$ — Boolean algebra



Euclidean Logics - Regions

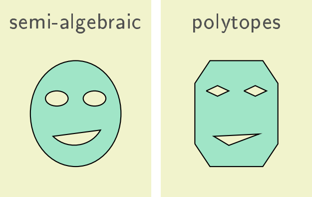
$RC(\mathbb{R}^n)$ still include many pathological sets:



Wild

vs

Tame



We consider different Boolean sub-algebras of $RC(\mathbb{R}^n)$.

$RC(\mathbb{R}^n)$	the set of all regular closed sets
$RCS(\mathbb{R}^n)$	semi-algebraic sets
$RCP(\mathbb{R}^n)$	selmi-linear sets (polytopes)
$RCP_{\mathbb{A}}(\mathbb{R}^n)$	algebraic polytopes
$RCP_{\mathbb{Q}}(\mathbb{R}^n)$	rational polytopes

Let $\Sigma = \{RC(\mathbb{R}^n), RCS(\mathbb{R}^n), RCP(\mathbb{R}^n), RCP_{\mathbb{A}}(\mathbb{R}^n), RCP_{\mathbb{Q}}(\mathbb{R}^n)\}$.

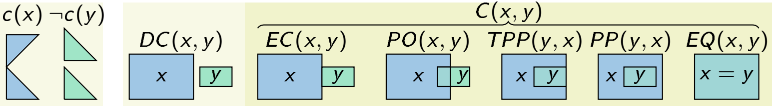
Euclidean Logics - The Languages $\mathcal{L}_C$, $\mathcal{L}_{conv}$ and $\mathcal{L}_{closer}$

Logical Syntax First-order logic

Non-logical Primitives

Boolean: $(\leq, +, -, \cdot, 0, 1)$

Topological: connectedness and **contact**



Euclidean: convexity and **relative closeness**

$\mathcal{L}_C := \langle C \rangle$ $\qquad$ $\mathcal{L}_{conv} := \langle conv, \leq \rangle$ $\qquad$ $\mathcal{L}_{closer} = \langle closer \rangle$

Lemma For $\mathcal{M} \in \Sigma$, $(\mathcal{M}, \mathcal{L}_C) \leq_m^p (\mathcal{M}, \mathcal{L}_{conv}) \leq_m^p (\mathcal{M}, \mathcal{L}_{closer})$.

Theories

<table>
<tr><th colspan="2" rowspan="2"></th><th colspan="3">Languages</th></tr>
<tr><th>$\mathcal{L}_C$</th><th>$\mathcal{L}_{conv}$</th><th>$\mathcal{L}_{closer}$</th></tr>
<tr><td rowspan="10">Domains</td><td>$RC(\mathbb{R})$</td><td colspan="2">Decidable, NONELEMENTARY</td><td>Δ^1_ω-complete</td></tr>
<tr><td>$RCS(\mathbb{R})$</td><td colspan="2" rowspan="4">Decidable, NONELEMENTARY</td><td>Δ^1_ω-complete</td></tr>
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<tr><td>$RCP_{\mathbb{Q}}(\mathbb{R}^n)$,$n > 1$</td><td>$\Delta^0_\omega$-complete</td><td>$\Delta^0_\omega$-complete</td><td>$\Delta^0_\omega$-complete</td></tr>
</table>

Table: A complexity map of the first-order Euclidean spatial logics.

$\mathcal{L}_C$ and $\mathcal{L}_{conv}$ Over $\mathbb{R}$

Lemma For $\mathcal{M} \in \Sigma$ $(\mathcal{M}, \mathcal{L}_C) \equiv_m^p (\mathcal{M}, \mathcal{L}_{conv})$.

Lemma $(RCP_{\mathbb{Q}}(\mathbb{R}), \mathcal{L}_C) \prec (RCP_{\mathbb{A}}(\mathbb{R}), \mathcal{L}_C) \prec (RCP(\mathbb{R}), \mathcal{L}_C)$
Proof. Using Tarski-Vaught Test.

Lemma$(RCP(\mathbb{R}), \mathcal{L}_C) = (RCS(\mathbb{R}), \mathcal{L}_C) \leq_m^p (RC(\mathbb{R}), \mathcal{L}_C)$

$\mathcal{L}_C$ and $\mathcal{L}_{conv}$ Over $\mathbb{R}$— Upper Bound

Result: The first-order theory of $(RC(\mathbb{R}), \mathcal{L}_C)$ is decidable.

Theorem [Rabin69] The monadic second-order theory (MSO) of $(\mathbb{Q}, <)$ is decidable.

Lemma There exists an interpretation of $(RC(\mathbb{R}), \mathcal{L}_C)$ in the MSO of $(\mathbb{Q}, <)$.

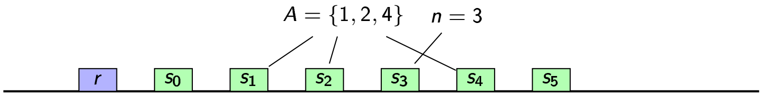
Proof. We identify every regular closed subset A of $\mathbb{R}$ with $A \cap \mathbb{Q}$.

$\mathcal{L}_C$ and $\mathcal{L}_{conv}$ Over $\mathbb{R}$— Lower Bound

Result: The first-order theory of $(RCP(\mathbb{R}), \mathcal{L}_C)$ is non-elementary.

Theorem [Meyer75] The weak monadic second-order theory of $(\mathbb{N}, S)$ (WS1S) is non-elementary.

Lemma WS1S in many-one reducible to $(RCP(\mathbb{R}), \mathcal{L}_C)$.



$\mathcal{L}_C$ over $\mathbb{R}^n (n > 1)$

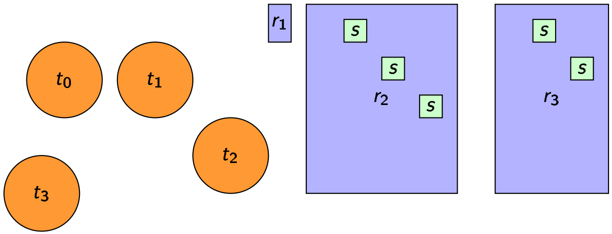
Result: For $\mathcal{M} \in \Sigma$,

- $(\mathcal{M}, \mathcal{L}_C)$ can encode first-order arithmetic (Δ^0_ω-hard).
- $(RC(\mathbb{R}^n), \mathcal{L}_C)$ can encode second-order arithmetic (Δ^1_ω-hard).

Idea.

• Identify a natural number n with the class of regions having exactly n connected components. (Grzegorczyk 1951)

• Identify a set of natural numbers $A \subseteq \mathbb{N}$ with a pair of regions r, s as shown for the set $A = \{0, 2, 3\}$:



$\mathcal{L}_{conv}$ and $\mathcal{L}_{closer}$ over $\mathbb{R}^n$

Theorem [Davis'06] Let $\mathcal{L}$ be either $\mathcal{L}_{conv}$ or $\mathcal{L}_{closer}$ and $n > 1$.

$(RC(\mathbb{R}), \mathcal{L}_{closer}), (RCS(\mathbb{R}), \mathcal{L}_{closer}), (RCP(\mathbb{R}), \mathcal{L}_{closer})$ Δ^1_ω-hard.

$(RCP_{\mathbb{A}}(\mathbb{R}), \mathcal{L}_{closer}), (RCP_{\mathbb{Q}}(\mathbb{R}), \mathcal{L}_{closer})$ Δ^0_ω-hard.

$(RC(\mathbb{R}^n), \mathcal{L}), (RCS(\mathbb{R}^n), \mathcal{L}), (RCP(\mathbb{R}^n), \mathcal{L})$ Δ^1_ω-hard.

$(RCP_{\mathbb{A}}(\mathbb{R}^n), \mathcal{L}), (RCP_{\mathbb{Q}}(\mathbb{R}^n), \mathcal{L})$ Δ^0_ω-hard.

Summary

		Languages		
		$\mathcal{L}_C$	$\mathcal{L}_{conv}$	$\mathcal{L}_{closer}$
Domains	$RC(\mathbb{R})$	Decidable, NONELEMENTARY		Δ^1_ω-complete
	$RCS(\mathbb{R})$	Decidable, NONELEMENTARY		Δ^1_ω-complete
	$RCP(\mathbb{R})$			Δ^1_ω-complete
	$RCP_{\mathbb{A}}(\mathbb{R})$			Δ^0_ω-complete
	$RCP_{\mathbb{Q}}(\mathbb{R})$			Δ^0_ω-complete
	$RC(\mathbb{R}^n),\ n > 1$	Δ^1_ω-complete	Δ^1_ω-complete	Δ^1_ω-complete
	$RCS(\mathbb{R}^n),\ n > 1$	Δ^0_ω-hard	Δ^1_ω-complete	Δ^1_ω-complete
	$RCP(\mathbb{R}^n),\ n > 1$	Δ^0_ω-hard	Δ^1_ω-complete	Δ^1_ω-complete
	$RCP_{\mathbb{A}}(\mathbb{R}^n),\ n > 1$	Δ^0_ω-complete	Δ^0_ω-complete	Δ^0_ω-complete
	$RCP_{\mathbb{Q}}(\mathbb{R}^n), n > 1$	Δ^0_ω-complete	Δ^0_ω-complete	Δ^0_ω-complete

Table: A complexity map of the first-order Euclidean spatial logics.

THANK YOU!