Survey of Spatial Logics

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- A spatial logic is a formal language with
  - variables ranging over ‘geometrical entities’
  - non-logical primitives denoting relations and operations defined over those geometrical entities.
- Any spatial logic is thus characterized by by three parameters:
  - a logical syntax:

    propositional logic, FOL, higher-order logic ...
  - a signature of geometrical primitives:

    $\mathrm{conv}(x)$, $c(x)$, $C(x, y)$, $\ldots$, $x + y$, $-x$, $\ldots$
  - a class of interpretations (more on this below).

- To see what is new here, compare the following two axiomatic treatments of geometry:
  - Hilbert's *Grundlagen der Geometrie* (1903):

    > Let a be a line, and A a point not on a. Then, in the plane determined by a and A, there is at most one line which passes through A and does not meet a.

  - Tarski's *What is elementary geometry?* (1958):

$$\forall xyzuv(\delta(x,u,x,v) \wedge \delta(y,u,y,v) \wedge \delta(z,u,z,v) \wedge u \neq v \rightarrow \beta(x,y,v) \vee \beta(y,z,x) \vee \beta(z,x,y))$$

- The new element here is the focus on the formal language.
- Amazingly:

  Theorem 1 (Tarski)**.** *Elementary geometry is decidable.*

- The geometrical primitives in Tarski's logic are points: but there are other possibilities ...
- Consider the spatial logic characterized by the following settings of our three parameters
  - propositional logic;
  - binary predicates for the '$\mathcal{RCC}8$' primitives

$$\begin{array}{lll} \mathsf{DC}(r_1, r_2) & \mathsf{EC}(r_1, r_2) & \mathsf{PO}(r_1, r_2) \\ \mathsf{EQ}(r_1, r_2) & \mathsf{TPP}(r_1, r_2) & \mathsf{NTPP}(r_1, r_2); \end{array}$$

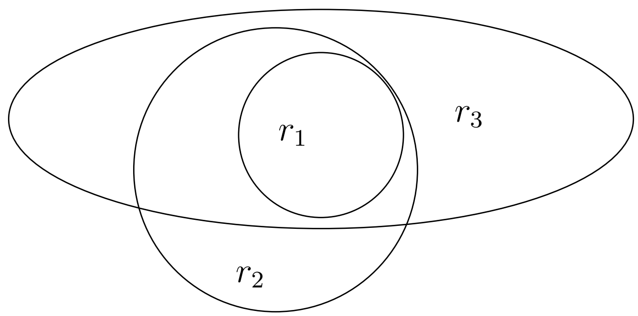
  - the class RegC of regular closed algebras of topological spaces.

(Randall, Cui and Cohn, 1992), (Egenhofer 1991)

- Example of a formula in this logic:

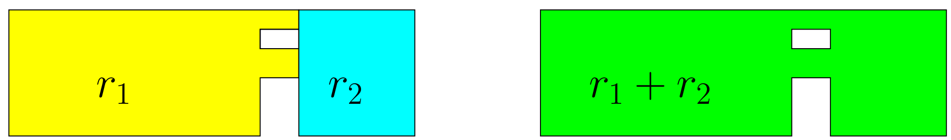
$$\big(\mathsf{TPP}(r_1, r_2) \wedge \mathsf{NTPP}(r_1, r_3)\big) \rightarrow \big(\mathsf{PO}(r_2, r_3) \vee \mathsf{TPP}(r_2, r_3) \vee \mathsf{NTPP}(r_2, r_3)\big).$$

- This formula is valid over RegC:



- Warning: this is a claim about all topological spaces. You cannot rely on diagrams to establish it!

- The language formerly known as $\mathcal{BRCC}8$ (Wolter and Zakharyaschev, 2000) adds Boolean operators to this language, i.e. we have the primitives $\mathbf{0}$, $\mathbf{1}$, $+$, $\cdot$, $-$ in addition to the $\mathcal{RCC}8$-predicates.



- The following $\mathcal{C}$-formula is valid over RegC:

$$\mathsf{EC}(r_1 + r_2, r_3) \leftrightarrow \big(\mathsf{EC}(r_1, r_3) \vee \mathsf{EC}(r_2, r_3)\big)$$

- Using the function symbols $+$, $\cdot$ and $-$, we can replace the $\mathcal{RCC}8$-predicates with the single binary relation of contact:

$$C(r_1, r_2) \text{ iff } r_1 \cap r_2 = \emptyset.$$

- For this reason, the language is now called, simply, $\mathcal{C}$.

- All the logics we are interested in are (effectively) closed under negation, so we may consider satisfiability rather than validity.
- If $\mathcal{L}$ is a spatial logic and $\mathcal{K}$ a class of interpretations, we denote the satisfiability problem for $\mathcal{L}$-formulas over $\mathcal{K}$ by $\mathit{Sat}(\mathcal{L}, \mathcal{K})$.

Theorem 2 (≈ Renz 1998)**.** *The problem* $\mathit{Sat}(\mathcal{RCC}8, \textsc{RegC})$ *is* NP*-complete. Indeed, for any* $n \geq 0$*,*

$$\mathit{Sat}(\mathcal{RCC}8, \mathsf{RC}(\mathbb{R}^n)) = \mathit{Sat}(\mathcal{RCC}8, \textsc{RegC}).$$

- Actually, by restricting the language somewhat, we get better complexities:
  - if we consider only conjunctions of $\mathcal{RCC}8$-primitives, complexity of satisfiability goes down to NLogSpace
  - Various (larger) tractable fragments have been found (Nebel and Bürckert 1995), (Renz 1999), . . . ,

- For the language $\mathcal{C}$, however, things are more interesting

  Theorem 3 (Wolter and Zakharyaschev, 2000)**.** *The problem $Sat(\mathcal{C}, \text{RegC})$ is* NP*-complete. For any $n \geq 1$, the problem $Sat(\mathcal{C}, \mathsf{RC}(\mathbb{R}^n))$ is* PSPACE*-complete.*

- The critical difference here is that the spaces $\mathbb{R}^n$ are connected. (The PSPACE-hardness result applies when $\mathcal{C}$ is interpreted over the class of regular closed algebras of connected topological spaces.)

- Logics which cannot express the property of connectedness are of limited interest. So let's add it!

- We consider the languages
  - $\mathcal{RCC}8c$: $\mathcal{RCC}8$ plus the unary predicate c;
  - $\mathcal{C}c$: W+Z's language (i.e. C, $+$, $\cdot$, $-$, 0, 1) plus the unary predicate c;
  - $\mathcal{B}c$: like $\mathcal{C}$, but without C.
- Example of an $\mathcal{RCC}8c$-formula in the 15 variables r_i $(1 \leq i \leq 5)$ and $r_{i,j}$ $(1 \leq i < j \leq 5)$:

$$\bigwedge_{1 \leq i < j \leq 5} c(r_{i,j}) \;\wedge \bigwedge_{\{i,j\} \cap \{k,\ell\} = \emptyset} \mathsf{DC}(r_{i,j}, r_{k,\ell}) \;\wedge \bigwedge_{i \in \{j,k\}} \mathsf{TPP}(r_i, r_{j,k}),$$

- Various complexity results are known here

  Theorem 4 (Kontchakov, P-H, W+Z, forthcoming)**.**

  $Sat(\mathcal{RCC}8c, \textsc{RegC})$ *is* NP*-complete;*
  $Sat(\mathcal{B}c, \textsc{RegC})$ *is* $\textsc{ExpTime}$*-complete;*
  $Sat(\mathcal{C}c, \textsc{RegC})$ *is* $\textsc{ExpTime}$*-complete.*

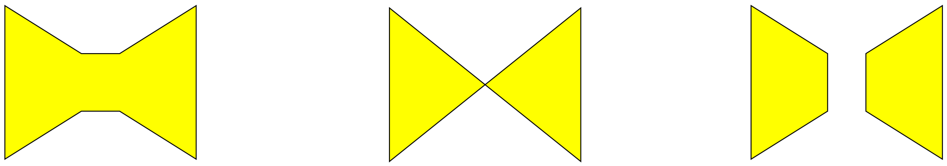
- However, once the property of connectedness is expressible, all the logics become sensitive to the underlying space.

$$Sat(\mathcal{RCC}8c, \textsc{RegC}) \neq Sat(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}^n)) \qquad \text{for } n = 1, 2$$

$$Sat(\mathcal{B}c, \textsc{RegC}) \neq Sat(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^n)) \qquad \text{for } n = 1, 2$$

$$Sat(\mathcal{C}c, \textsc{RegC}) \neq Sat(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^n)) \qquad \text{for } n = 1, 2$$

- We may wish to distinguish between connectedness and interior connectedness:



- We employ the predicate c° where $c^\circ(r)$ has the interpretation "r° is connected".

- This gives us the further languages $\mathcal{RCC}8c^\circ$, $\mathcal{B}c^\circ$, $\mathcal{C}c^\circ$.

- Example of an $\mathcal{RCC}8c^\circ$-formula

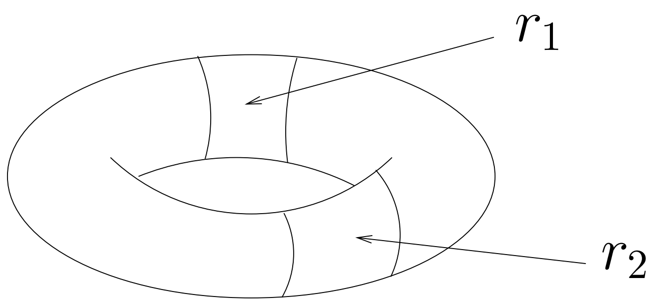
$$c^\circ(-r_1) \wedge c^\circ(-r_2) \wedge \mathsf{DC}(r_1, r_2) \wedge \neg c^\circ(-(r_1 + r_2))$$

- Complexity results for $\mathcal{RCC}8c^\circ$ $\mathcal{B}c^\circ$ and $\mathcal{C}c^\circ$ are the same as those for for $\mathcal{RCC}8c$ $\mathcal{B}c$ and $\mathcal{C}c$, respectively.

- All too simple? The $\mathcal{C}c^\circ$-formula

$$c^\circ(-r_1) \wedge c^\circ(-r_2) \wedge \mathsf{DC}(r_1, r_2) \wedge \neg c^\circ(-(r_1 + r_2))$$

is satisfiable over RegC, thus:



But it is not satisfiable over $\mathsf{RC}(\mathbb{R}^n)$ for *any* n!

- More generally, we have:

$$Sat(\mathcal{RCC}8c^\circ, \text{RegC}) = Sat(\mathcal{RCC}8c^\circ, \mathsf{RC}(\mathbb{R}^n)) \quad \text{for } n \geq 3$$

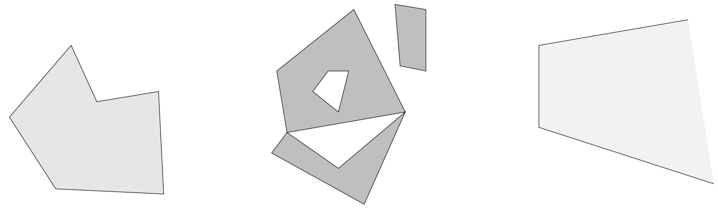
$$Sat(\mathcal{B}c^\circ, \text{RegC}) = Sat(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^n)) \quad \text{for } n \geq 3$$

$$Sat(\mathcal{C}c^\circ, \text{RegC}) \neq Sat(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^n)) \quad \text{for } n \geq 1$$

- Actually, matters are even more delicate than this: $\mathsf{RC}(\mathbb{R}^n)$ contains some very pathological sets:

- This prompts us to consider interpretations of spatial logics over collections of tame regions.

- Natural candidates for tame subalgebras of $\mathsf{RC}(\mathbb{R}^n)$:
  - The regular closed polyhedra in $\mathbb{R}^n$, $\mathrm{RCP}(\mathbb{R}^n)$:

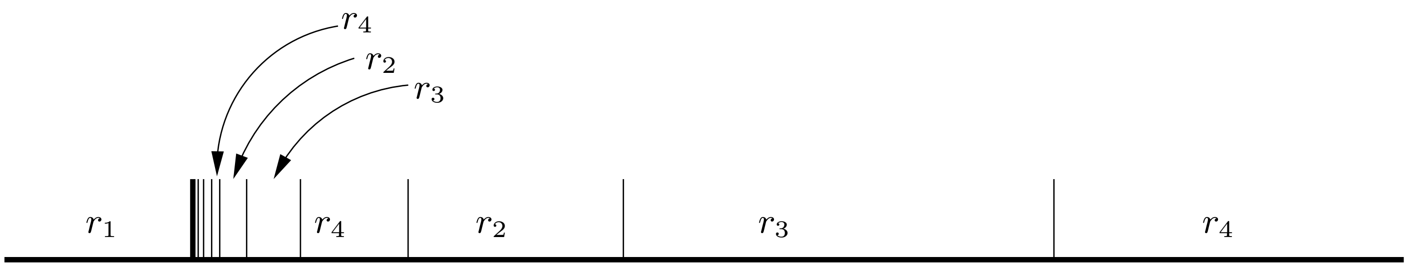


  - The regular closed semi-algebraic subsets of $\mathbb{R}^n$, $\mathrm{RCS}(\mathbb{R}^n)$.

- We consider first logics interpreted over 1-dimensional space.
- Consider the $\mathcal{RCC}8c$-formula

$$c(r_1) \wedge \bigwedge_{1 \leq i < j \leq 4} \mathsf{EC}(r_i, r_j),$$

- This formula is satisfiable over $\mathsf{RC}(\mathbb{R})$:



- But the only satisfying tuples are those in which some of the members have infinitely many components.
- That is, the formula is not satisfiable over $\mathsf{RCP}(\mathbb{R})$.

- Thus, we have shown:

$$\begin{aligned} Sat(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R})) &\neq Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R})) \\ Sat(\mathcal{C}c, \mathsf{RC}(\mathbb{R})) &\neq Sat(\mathcal{C}c, \mathsf{RCP}(\mathbb{R})). \end{aligned}$$

- These problems do, however, have the same complexity:

  Theorem 5. *$Sat(\mathcal{C}c, \mathsf{RC}(\mathbb{R}))$ and $Sat(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}))$ are both* NP*-complete.*

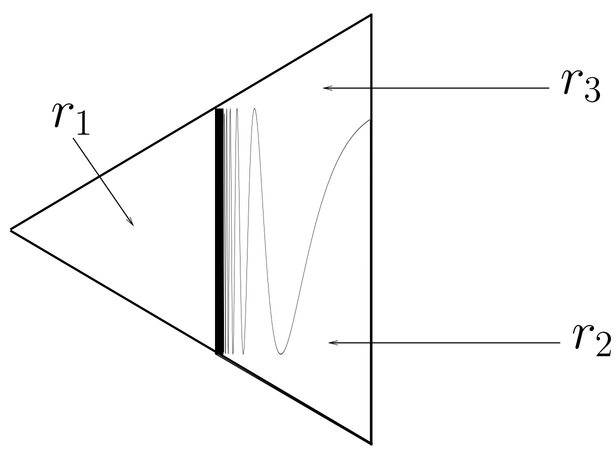
- However, we have:

  Theorem 6. *$Sat(\mathcal{B}c, \mathsf{RC}(\mathbb{R})) = Sat(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}))$ is* NP*-complete.*

- Now let us consider topological logics with connectedness interpreted over 2-dimensional space.
- Consider the $\mathcal{B}c^\circ$-formula:

$$\bigwedge_{1 \leq i \leq 3} c^\circ(r_i) \quad \wedge \quad c^\circ\Big(\sum_{1 \leq i \leq 3} r_i\Big) \quad \wedge \neg(c^\circ(r_1 + r_3) \vee c^\circ(r_1 + r_3)).$$

- This formula is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$:



- However, it is unsatisfiable over $\mathsf{RCP}(\mathbb{R}^2)$.

- Thus, we have shown:

$$\begin{aligned} Sat(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^2)) &\neq Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R})) \\ Sat(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^2)) &\neq Sat(\mathcal{C}c^\circ, \mathsf{RCP}(\mathbb{R}^2)). \end{aligned}$$

- However, we have:

$$Sat(\mathcal{RCC}8c^\circ, \mathsf{RC}(\mathbb{R}^2)) = Sat(\mathcal{RCC}8c^\circ, \mathsf{RCP}(\mathbb{R}))$$

- The situation with ordinary connectedness in two-dimensions turns out to be similar (but much harder to analyse):

$$\begin{aligned} Sat(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}^2)) &= Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R})) \\ Sat(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^2)) &\neq Sat(\mathcal{B}c, \mathsf{RCP}(\mathbb{R})) \\ Sat(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^2)) &\neq Sat(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}^2)) \end{aligned}$$

- Much less in known about complexity here:

  Theorem 7. *$Sat(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}^2))$, $Sat(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}^2))$ $Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ and $Sat(\mathcal{C}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ are all* ExpTime*-hard.*

  Theorem 8. *$Sat(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^2))$, $Sat(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^2))$ and $Sat(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^2))$ are all* ExpTime*-hard.*

  Theorem 9 (≈ Schaefer, Sedgwick and Štefankovič)**.** *$Sat(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}^2))$ and $Sat(\mathcal{RCC}8c^\circ, \mathsf{RC}(\mathbb{R}^2))$ are both* NP*-complete.*

- There is nothing sacrosanct about the syntax of propositional logic, or topological primitives.
- Let σ be a signature of any geometrical primitives (e.g. $\sigma = (C)$, or $\sigma = (c^0, +, \cdot, -, 0, 1)$, or $\sigma = (\text{conv}, \leq)$.
- Let $\mathcal{K}$ be a class of interpretations.
- Denote by $\text{Th}_\sigma(\mathcal{K})$ the first-order theory of $\mathcal{K}$ over σ.
- First-order spatial logics are generally undecidable:

  Theorem 10 (Dornheim). *$Th_C(\mathsf{RCP}(\mathbb{R}^2))$ is undecidable.*
- Nevertheless, we can ask about matters such as

  Theorem 11 (Davis). *$Th_{\text{conv},\leq}(\mathsf{RCP}(\mathbb{R}^2))$ is Δ^1_ω-complete.*
- Nevertheless, we can ask about matters such as
  - axiomatization
  - expressive power
  - alternative models.

- Cheerful facts about axiomatizations
  - Many elegant axiomatic systems for $\mathrm{Th}_C(\mathcal{K})$, where $\mathcal{K}$ is the class of dense sub-algebras of regular closed subsets of topological spaces of some kind (Roeper, Düntsch and Winter, Dimov and Vakarelov, de Vries).
- Cheerful facts about expressive power
  - First-order languages over (C) are topologically complete for $\mathsf{RC}(\mathbb{R}^2)$ (Vianu, Suciu and Papadimitriou)
  - First-order languages over (C, conv) are affine complete for $\mathsf{RC}(\mathbb{R}^2)$ (Davis, Gotts and Cohn).

Conclusions

- Technical content:
  - What a spatial logic is
  - The three parameters determining any spatial logic
  - Some of the questions that we can ask about spatial logics
- The authoritative reference for many of these results is

  Aiello, P-H and van Benthem (eds.), *Handbook of Spatial Logics*, Springer, 2007.
- The view to go away with:

  Spatial logic is geometry seen through the lens of a formal language.