Geometrical Logics

Ian Pratt-Hartmann
School of Computer Science
Manchester University

Handbook of Spatial Logics Launch
University of Groningen, November, 2007.

- Euclid's *Elements* (c. 300 B.C.)

Postulate 1 To draw a straight line from any point to any point.

...

Postulate 5 That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

(Tr. Sir Thomas L. Heath, 1908)

- Hilbert's *Grundlagen der Geometrie* (3rd Ed., 1909)

  I.1 *Two distinct points A and B determine a line.*

  ...

  IV *Let a be a line, and A a point not on a. Then, in the plane determined by a and A, there is at most one line which passes through A and does not meet a.*

  V.2 *The elements (points, lines, planes) of the geometry form a system of objects which is not capable of any extension, subject to maintenance of all the preceding axioms.* That is to say: it is not possible to add to the system of points, lines and planes another system in such a way that, in the combined system, all axioms I–IV and V.1 are satisfied.

- What had happened to make the axioms so complicated?

- Two obvious developments:
  - the use of the algebraic method in geometry following the work of Descartes (1637), and the ensuing arithmetization of geometry;
  - the rise of non-Euclidean geometries due to the work of Lobachevsky (1826) and Bolyai (1829).
- Thus, we have a radical change in the subject matter of geometry: from
  - the study of the consequences of a given set of postulates about geometrical entities and relations

  to

  - the study of the properties of (classes of) geometrical structures, which need not be given axiomatically.

- Hilbert axiomatized the structure formed by $\mathbb{R}^3$, with the standard interpretation of the geometrical primitives.
- The axioms make sense only as an attempt to do just this.
- Note that Hilbert (like Euclid) wrote his axioms out in natural language; his *Foundations of Geometry* is (implicitly) clear about the geometrical primitives of his system, but not about the set-theoretical and logical resources at his disposal.
- This shift from **consequences of axiom-systems** to **facts about (classes of) structures** opens new mathematical possibilities.

- Tarski: *What is elementary geometry?* (1958)

  A1 $\forall xy(\beta(x,y,x) \rightarrow x = y)$

  ...

  A12 $\forall xyzuv(\delta(x,u,x,v) \wedge \delta(y,u,y,v) \wedge \delta(z,u,z,v) \wedge u \neq v \rightarrow$
  $\beta(x,y,v) \vee \beta(y,z,x) \vee \beta(z,x,y))$

  ...

- Here the variables range over points, and only first-order logic is employed.

- Tarski was well-aware of alternatives: in 1929, he had developed a *geometry of solids* employing second-order logic, and with object-variables ranging over *spheres* in $\mathbb{R}^3$.

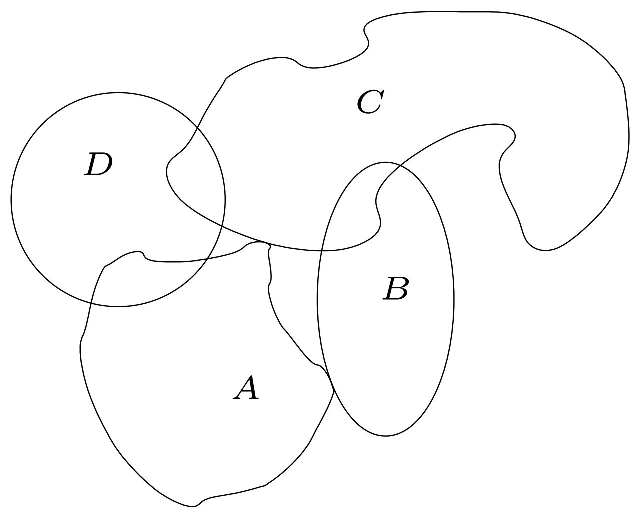
- The shift from the axiomatic to the constructive method enables us to study (classes of) structures from the point of view of different formal languages.
- Let us call a formal language interpreted over a (class of) geometrical structures, broadly construed, a geometrical logic.
- Informally, we can identify three 'dimensions' of variation in geometrical logics:
  1. the (classes of) structures considered;
  2. the primitive geometrical terms interpreted thereover;
  3. the logical (and set-theortical) resources by means of which these terms may be combined.

- Here are some of the questions you can ask about a geometrical logic:
  - Can you axiomatize its validities?
  - Is its satisfiability problem decidable, and if so, with what complexity?
  - What relations can formulas (with free variables) express over the intended interpretations?
  - What are the other models of the theory?
- For a variety of geometrical logics, we have answers to these questions.

- Example 1: The Boolean theory of plane discs over the RCC-8 primitives
  - The formula $\phi(x, y, z, u)$ given by

$$\mathrm{EC}(x, y) \wedge \mathrm{EC}(x, z) \wedge \mathrm{O}(x, u) \wedge \mathrm{O}(y, z) \wedge \mathrm{DC}(y, u) \wedge \mathrm{O}(z, u)$$

    is satisfied by the tuple (A, B, C, D)



  - Schaefer, Sedgwick and Štefankovič (2003) showed that satisfiability in this logic is NP-complete.

- Example 2: The first-order theory of the regular open semialgebraic plane sets $\mathrm{ROS}(\mathbb{R}^2)$ over the primitives c ("is a connected region") and $\leq$ ("is a part of").
  - Some formulas in this theory are:

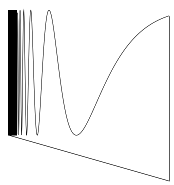
$$\forall x \forall y \forall z \Big(\big(c(x+y) \wedge c(y+z) \wedge y \neq 0 \big) \rightarrow c(x+y+z) \Big);$$

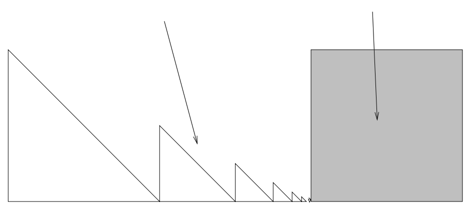
$$\forall x_1 \forall x_2 \forall x_3 \big(c(x_1) \wedge c(x_2) \wedge c(x_3) \wedge c(x_1+x_2+x_3) \rightarrow c(x_1+x_2) \vee c(x_1+c_3) \big).$$

- This theory can in fact be axiomatically characterized

- Example 2 (contd.)
  - The language $\mathcal{L}_{(c,\leq)}$ is not 'topologically lossless' over ROS($\mathbb{R}^2$): an open disc and its complement (which are topologically distinguishable) satisfy exactly the same set of formulas.
  - But it becomes topologically lossless if an extra predicate b ('is bounded') is added: every tuple satisfies a formula of $\mathcal{L}_{(c,\leq,b)}$ which is identifies it up to topological similarity.
  - The first-order $\mathcal{L}_{(c,\leq)}$-theory of ROS($\mathbb{R}^2$) has, up to isomorphism, exactly one countable model in which every element is the sum of finitely many connected elements.

- Example 3: The first-order theory of the regular open plane sets $\mathrm{RO}(\mathbb{R}^2)$ over the signature $(c, \leq)$
  - The set $\mathrm{RO}(\mathbb{R}^2)$ contains some pathological regions



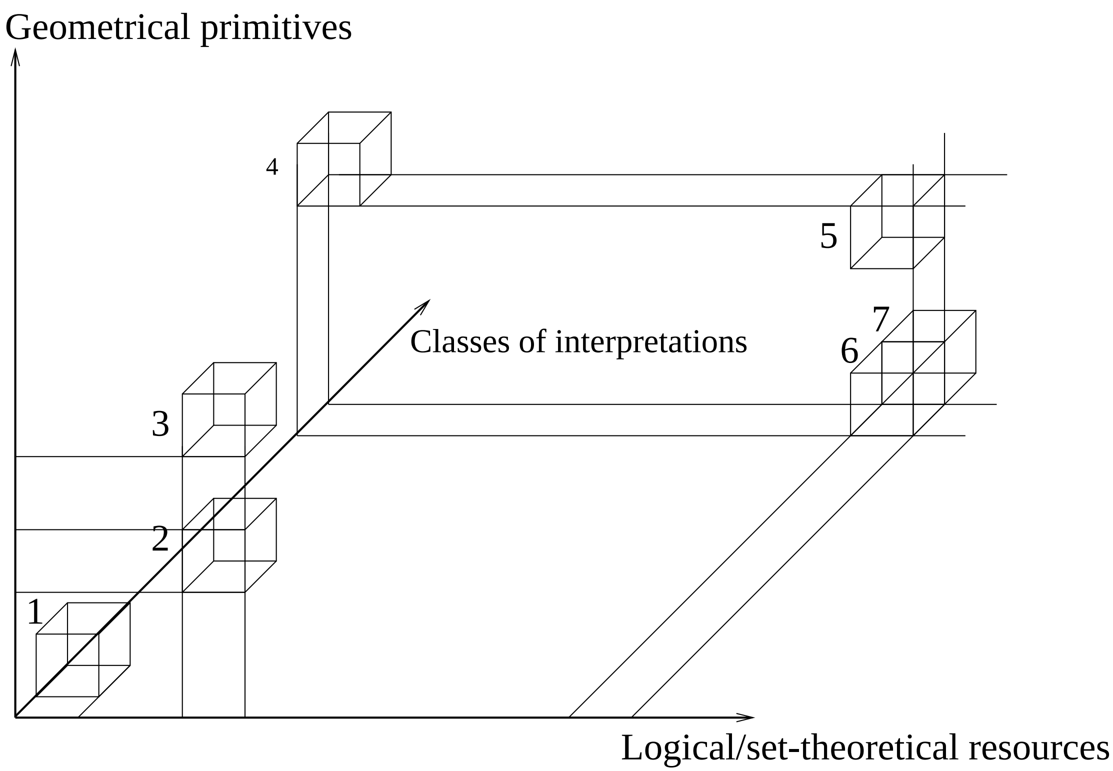


  - Almost nothing is known about this theory, except that it is not identical to the theory of $\mathrm{ROS}(\mathbb{R}^2)$ over the same signature. For example,

$$\forall x_1 \forall x_2 \forall x_3 (c(x_1) \wedge c(x_2) \wedge c(x_3) \wedge c(x_1 + x_2 + x_3) \rightarrow c(x_1 + x_2) \vee c(x_1 + c_3)).$$

    does not belong to it.

- The resulting space of geometrical logics can be visualized thus:



1. RCC-type constraint languages
2: Nutt/Bennett's PSPACE mereotopological language
3: TCC (topological constraints with component counting)
4: Davis et al's constraint language with convexity
5: First order theory of convexity in Euclidean plane
6: First-order polygonal mereotopology
7: First-order polyhedral mereotopology

- Conclusion
  - A geometrical logic is a formal language interpreted over a (class of) geometrical structures, broadly construed.
  - The study of geometrical logics is a way of pursuing the study of geometry—but with particular emphasis on the relationship between sentences of those logics and the structures they describe.
  - This study involves questions which cannot formulated without the model-theoretic view that I have outlined.
  - That view is the product of a long historical development.