# Mereotopology: a Survey

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- A topological space is a pair $\langle X, \mathcal{O} \rangle$ where $\mathcal{O}$ is a collection of subsets of $X$ s.t. ....
- Let $X$ be a topological space and $p$ a subset of $X$. Then

$$-\left((p^{-0})^{0-}\right) \cup p^{-0} = X.$$

- Changing notation slightly, we obtain the modal logic formula

$$\neg\Box\Diamond\Box\Diamond p \vee \Box\Diamond p$$

which happens to be an S4-theorem.

- More generally, McKinsey and Tarski, 1944, showed:

**Theorem:** Let $\phi$ be a formula of modal logic. TFAE:

1. $\phi$ is an S4-theorem;
2. $\phi$ is valid in the class of topological spaces;
3. $\phi$ is valid in $X$, where $X$ is any dense-in-itself, separable metric space.

- Consider the formal language $\mathcal{T}$:
  - *Terms*: $\tau :: \quad x \mid 0 \mid 1 \mid -\tau \mid \tau_1 \cup \tau_2 \mid \tau_1 \cap \tau_2 \mid \tau^- \mid \tau^0$
  - *Statements*: $\phi :: \quad \tau_1 = \tau_2 \mid \phi_1 \wedge \phi_2 \mid \neg\phi$.
- An *interpretation* for $\mathcal{T}$ is simply a topological space $X$ (belonging to some class), with variables ranging over $\mathbb{P}(X)$.
- Using the obvious semantics for the above primitives, we obtain the notion of a $\mathcal{T}$-validity. For example:

$$\models -\left((p^{-0})^{0-}\right) \cup p^{-0} = 1$$

- The McKinsey-Tarski theorem tells us that the logic of $\mathcal{T}$, over the class of all topological spaces, or indeed over any single dense-in-itself, separable metric space $X$, is (in effect) the logic S4. In particular, the corresponding satisfiability problems are PSPACE-complete.

- If $X$ is a topological space, a subset $u \subseteq X$ is regular open if $u$ is equal to the interior of its closure: $u = (u^{-})^{0}$.
- the set of regular open subsets of $X$ is denoted $\mathrm{RO}(X)$.
- $(\mathrm{RO}(X), \subseteq)$ is always a (complete) Boolean algebra under the interpretation:

$$\begin{aligned} 1 &= X & x \cdot y &= x \cap y \\ 0 &= \emptyset & x + y &= (x \cup y)^{-0} \\ & & -x &= (X \setminus x)^{0}. \end{aligned}$$

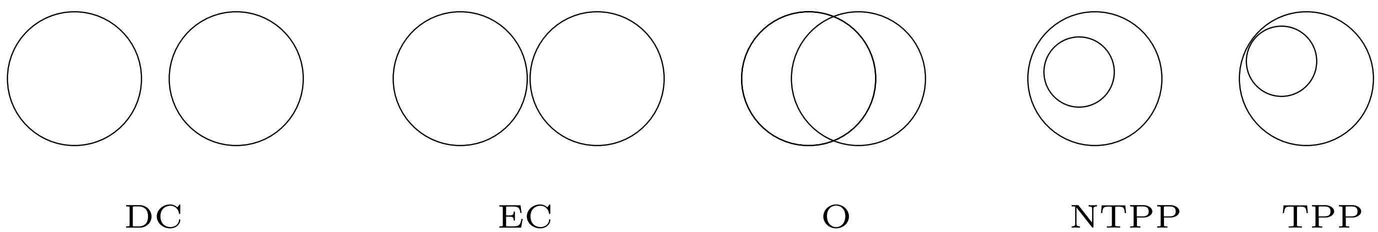
  It is called the regular open algebra of $X$.
- The valid formula in the previous example states (in effect) that the regular open sets are exactly those of the form $p^{-0}$.

- This leads to the following fragment of $\mathcal{T}$:
  - take variables to range only over regular open sets
  - take atomic formulas to be only those of the forms

    $\mathrm{DC}(x, y) \equiv x^- \cap y^- = 0$

    $\mathrm{EC}(x, y) \equiv x \cap y = 0 \wedge x^- \cap y^- \neq 0$

    etc.




- Call this language $\mathcal{RCC}8$ (Egenhofer and Franzosa, Bennett …).
- For example,

$$\models \mathrm{EC}(x, y) \wedge \mathrm{NTPP}(y, z) \rightarrow (\mathrm{O}(x, z) \vee \mathrm{TPP}(x, z) \vee \mathrm{NTPP}(x, z))$$

- We can extend $\mathcal{R}CC8$ by adding function symbols $+$, $\cdot$ and $-$ denoting the obvious operations in $\mathrm{RO}(X)$.
- Call this language $\mathcal{B}RCC8$ (Wolter and Zakharyaschev, 2000)
- For example, we have the validity

$$\models \mathrm{EC}(x, y+z) \rightarrow (\mathrm{EC}(x, y) \vee \mathrm{EC}(x, z)).$$

**Theorem:** Sat-$\mathcal{R}CC8$ is NP-hard. Sat-$\mathcal{B}RCC8$ is in NP.

- Returning to the language $\mathcal{T}$, one can extend to obtain a language $\mathcal{T}C$ by adding an additional unary predicate $c$:
  - *Terms*: $\ldots$
  - *Statements*: $\phi ::\quad \ldots \mid c(\tau)$

  with the interpretation: $X \models c[s]$ iff $s \subseteq X$ is connected.
- Let $X$ be a topological space and $r$, $s$ subsets of $X$:
  - if $r$ is connected and $r \subseteq s \subseteq r^{-}$, then $s$ is connected;
  - if $r$ and $s$ are connected and $r \cap s \neq \emptyset$, then $r \cup s$ is connected.
- We can express these (textbook) results as the $\mathcal{T}C$-validities:
  - $\models c(x) \wedge -x \cup y = 1 \wedge -y \cup x^{-} = 1 \rightarrow c(y)$
  - $\models c(x) \wedge c(y) \wedge x \cap y \neq 0 \rightarrow c(x \cup y)$.
- The problem Sat-$\mathcal{T}C$ is in NEXPTIME.

- Obvious next step: add quantifiers.
- Consider the language $\mathcal{C}A$ defined as follows:
  - *Terms*: $\tau :: \quad x \mid 0 \mid 1 \mid -\tau \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2$
  - *Statements* $\phi :: \quad \tau_1 = \tau_2 \mid C(\tau_1, \tau_2) \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \exists x \phi$,

  with variables ranging over certain collections (details to follow) of regular open of subsets of topological spaces belonging to some class, and and the predicate $C$ is interpreted as:

  $$X \models C[r, s] \text{ iff } r^- \cap s^- \neq \emptyset.$$
- The predicate $C$ is the contact predicate (and the relation it expresses, the contact relation).
- Whitehead (1919) introduced this relation, calling it "connection".
- Whitehead's motivation was metaphysical/epistemological, rather than computational.

- We now explain the ‘certain collections’ of regular open sets ...

  **Definition:** Let $X$ be a topological space. A mereotopology over $X$ is a Boolean sub-algebra $M$ of $\mathrm{RO}(X)$ such that every neighbourhood in $X$ contains a neighbourhood in $M$: if $q \in o \subseteq X$ with $o$ open, there exists $r \in M$ such that $q \in r \subseteq o$.

- Where $M$ is clear from context, we refer its elements as regions.

- Important: not every regular open subset of the space in question need count as a region.

- We shall always interpret the language $\mathcal{C}A$ over mereotopologies.

- A word on etymology:
  - Mereology (Leśniewski): the logic of the part-whole relationship ($\leq$).
  - Mereotopology is simply the study of topological spaces with regions functioning as the primary objects.
  - I am not sure where the term first appeared in print.
- It is easy to see that, for most interesting classes of mereotopologies, deciding satisfiability of $\mathcal{C}A$-formulas is undecidable. But there is plenty else we can ask about these logics …

**Definition:** A contact algebra is a structure interpreting the signature $(C, \leq, +, \cdot, -, 0, 1)$ satisfying the usual axioms of Boolean algebra together with

(C0) $\forall x \neg C(x, 0)$

(C1) $\forall x (x > 0 \to C(x, x))$

(C2) $\forall x \forall y (C(x, y) \to C(y, x))$

(C3) $\forall x \forall y (C(x, y) \wedge y \leq z \to C(x, z))$

(C4) $\forall x \forall y (C(x, y + z) \to C(x, y) \vee C(x, z))$

- We consider also the following additional axioms:

(Ext) $\forall x \forall y (\forall z (C(x, z) \to C(y, z)) \to x \leq y)$

(Int) $\forall x \forall y (\neg C(x, y) \to \exists z (\neg C(x, -z) \wedge \neg C(y, z)))$

(Con) $\forall x \forall y (x + y = 1 \to C(x, y))$.

- A topological space is semi-regular if it has a basis of regular open sets; a topological space is weakly regular if it is semi-regular and, for any non-empty open set $u$, there exists a non-empty open set $v$ with $v^- \subseteq u$.
- $X$ is regular $\Rightarrow$ $X$ is weakly regular $\Rightarrow$ $X$ is semi-regular.

  **Theorem:** Let $X$ be a topological space, and let $M$ be a mereotopology over $X$, regarded as a structure interpreting the signature $(C, \leq, +, \cdot, -, 0, 1)$. Then $M \models$ (C0)–(C4). In addition:

  1. If $X$ is weakly regular, then $M \models$ (Ext).
  2. If $X$ is compact and Hausdorff, then $M \models$ (Int).
  3. If $X$ is connected, then $M \models$ (Con).

  **Proof:** Routine.

**Theorem:** (Dimov and Vakarelov, 2006) Let $\mathfrak{A}$ be a structure interpreting $(C, \leq, +, \cdot, -, 0, 1)$, whose reduct to $(\leq, +, \cdot, -, 0, 1)$ is a Boolean algebra. If $\mathfrak{A} \models$ (C0)–(C4), then $\mathfrak{A}$ is isomorphic to a mereotopology over some topological space $X$. Moreover:

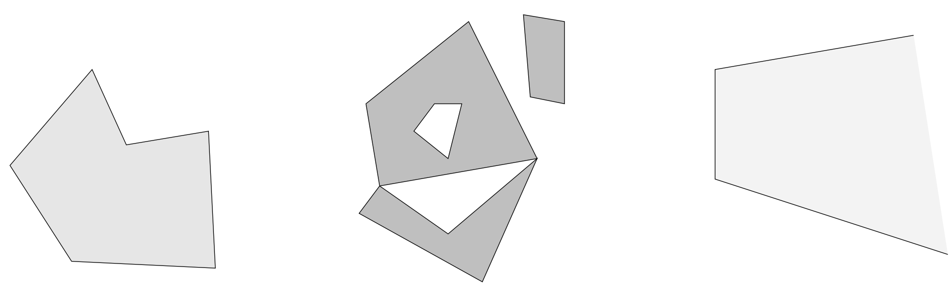
1. if $\mathfrak{A} \models$ (Ext), then $X$ can be chosen to be weakly regular (Düntsch and Winter, 2004);
2. if $\mathfrak{A} \models$ (Int) and (Ext), then $X$ can be chosen to be compact and Hausdorff (Roeper, 1997); and
3. if $M \models$ (Con), then $X$ can be chosen to be connected.

**Proof sketch:** Define the points of $X$ to be ultrafilter-like subsets of $A$; define a mapping $g : A \to \mathbb{P}(X)$ by

$$g(a) = \{x \in X \mid a \in X\};$$

use these sets as the basis of a topology.

- Examples of mereotopologies:
  - $\mathrm{RO}(X)$ for any semi-regular space $X$.
  - $\mathrm{ROS}(\mathbb{R}^n)$: the regular open semi-algebraic sets in $\mathbb{R}^n$;
  - $\mathrm{ROP}(\mathbb{R}^n)$: the regular open polyhedra in $\mathbb{R}^n$;
  - $\mathrm{ROQ}(\mathbb{R}^n)$: the regular open rational polyhedra in $\mathbb{R}^n$.



  - These have their closed-space analogues: $\mathrm{RO}(\mathbb{S}^n)$, $\mathrm{ROS}(\mathbb{S}^n)$, $\mathrm{ROP}(\mathbb{S}^n)$, $\mathrm{ROQ}(\mathbb{S}^n)$.

- It is interesting to ask what first-order sentences (with various signatures of topological primitives) are true in mereotopologies over certain classes of spaces.
- Consider, for example, the sentence $\psi_{\text{con}}$ given by

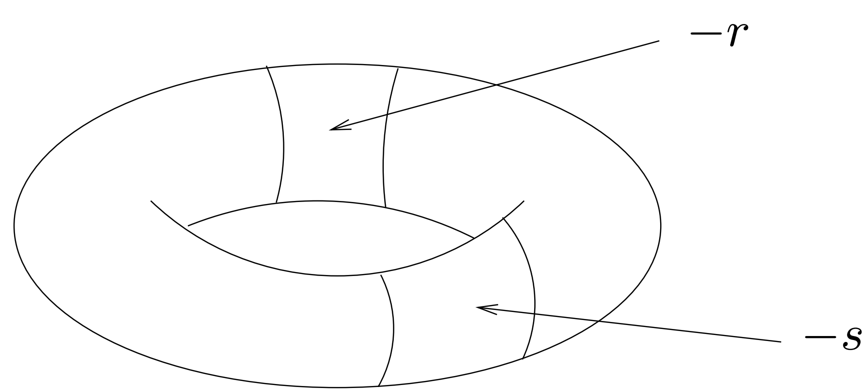
$$\forall x \forall y (c(x) \wedge c(y) \wedge x \cdot y > 0 \rightarrow c(x + y))$$

- If $M$ is any mereotopology, then $M \models \psi_{\text{con}}$.

- Consider the sentence $\psi_{\mathrm{Eucl}}$ given by

$$\forall x \forall y (c(x) \wedge c(y) \rightarrow (c(x \cdot y) \vee C(-x, -y))).$$

- $\psi_{\mathrm{Eucl}}$ is not true in all mereotopologies:

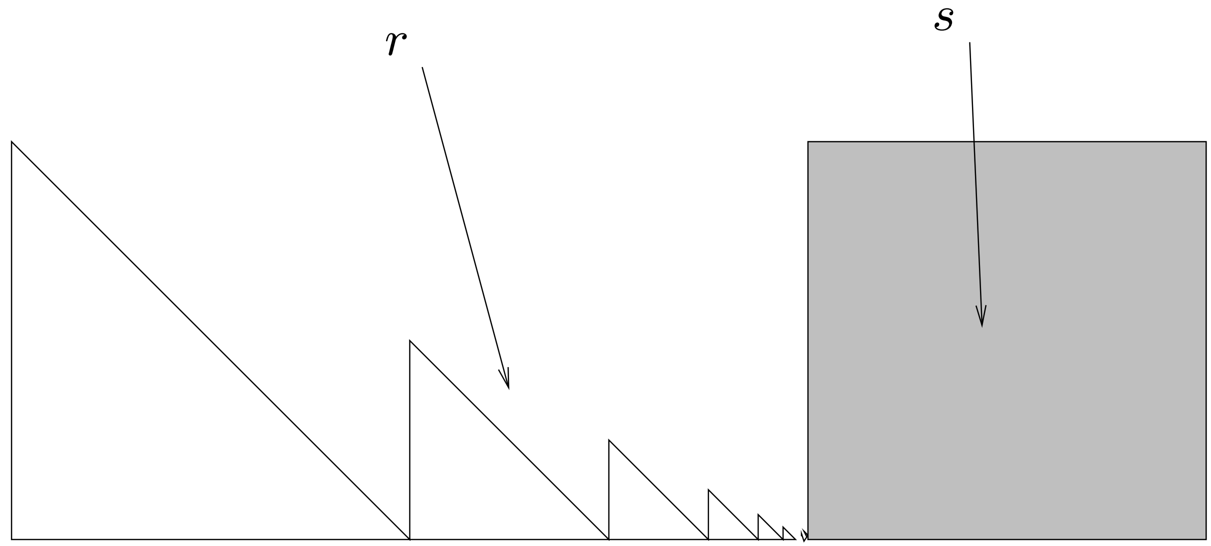



- However, if $M$ is any mereotopology over $\mathbb{R}^n$ ($n \geq 1$), then $M \models \psi_{\mathrm{Eucl}}$.

- Consider the sentence $\psi_{\text{inf}}$ given by

$$\forall x \forall y (C(x, y) \rightarrow \exists z (c(z) \wedge z \leq x \wedge C(y, z)))$$

- $\psi_{\text{inf}}$ is not true in $\mathrm{RO}(\mathbb{R}^2)$:

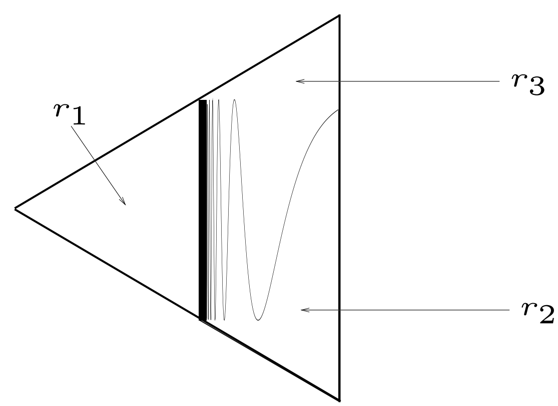



- However, $\psi_{\text{inf}}$ is true in $\mathrm{ROQ}(\mathbb{R}^2)$, $\mathrm{ROP}(\mathbb{R}^2)$, $\mathrm{ROS}(\mathbb{R}^2)$.

- Consider the sentence $\psi_{\text{wiggly}}$ given by

$$\forall x_1 \forall x_2 \forall x_3 (c(x_1) \wedge c(x_2) \wedge c(x_3) \wedge c(x_1 + x_2 + x_3) \rightarrow (c(x_1 + x_2) \vee c(x_1 + x_3))).$$

- $\psi_{\text{wiggly}}$ is not true in $\mathrm{RO}(\mathbb{R}^2)$:



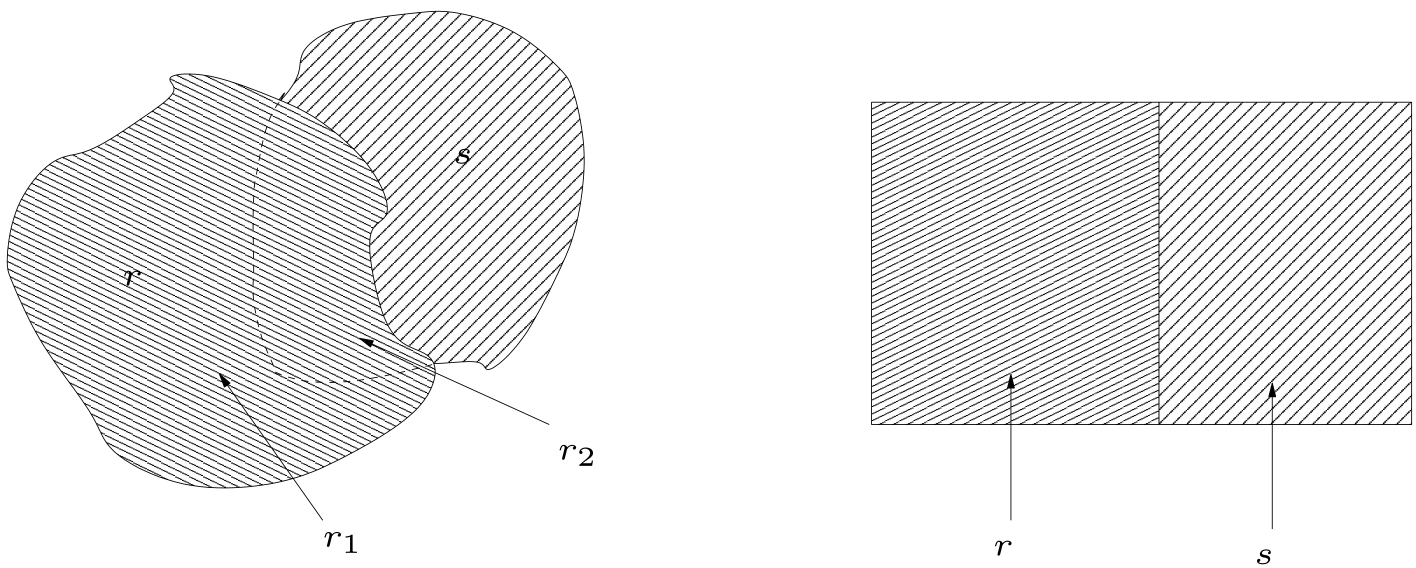

- However, $\psi_{\text{wiggly}}$ is true in $\mathrm{ROQ}(\mathbb{R}^2)$, $\mathrm{ROP}(\mathbb{R}^2)$, $\mathrm{ROS}(\mathbb{R}^2)$.

- We can characterize mereotopologies over large classes of topological spaces abstractly; but what about familiar mereotopologies, such as $\mathrm{ROQ}(\mathbb{R}^n)$, $\mathrm{ROP}(\mathbb{R}^n)$ and $\mathrm{ROS}(\mathbb{R}^n)$?
- We proceed to give a partial answer to this question where $n = 2$.
- Here it turns out to be more convenient to employ the signature $(c, \leq, +, \cdot, -, 0, 1)$ (rather than $(C, \leq, +, \cdot, -, 0, 1)$).
- In fact, for this signature, we have $\mathrm{RO}(\mathbb{R}^n) \simeq \mathrm{RO}(\mathbb{S}^n)$, and similarly, $\mathrm{ROP}(\mathbb{R}^n) \simeq \mathrm{ROP}(\mathbb{S}^n)$ etc.

- Notice that ROQ($\mathbb{R}^n$), ROP($\mathbb{R}^n$) and ROS($\mathbb{R}^n$) are all *tame*, in the following sense:
  - They are all finitely decomposable: each region is the sum of finitely many connected regions (Cell Decomposition Theorem).
  - They exhibit curve-selection: if $r \in M$ and $q \in \mathcal{F}(r)$ there exists a Jordan arc have end $q$ as one of its endpoints, lying in $r \cup \{q\}$ (Curve Selection Lemma).
- They are also all splittable: they make true the following *splitting axiom*:

$$\forall x \forall y \big( x,\ y \text{ and } -(x+y) \text{ are non-empty and connected} \rightarrow \exists u \exists v \big( u_1 \oplus u_2 = x \wedge c(u_1 + y) \wedge \neg c(u_1 + -(x+y)) \wedge c(u_2 + -(x+y)) \wedge \neg c(u_2 + y) \big).$$

- We can illustrate the splitting axiom diagrammatically:




a) b)

- Consider the following axioms

1. the usual axioms of Boolean algebra, and the axiom $0 \neq 1$;
2. the axiom $\forall x \forall y (c(x) \wedge c(y) \wedge x \cdot y \neq 0 \rightarrow c(x+y))$.
3. where $n > 2$, the axioms

$$\forall x_1 \ldots \forall x_n \Big( c(x_1 + \cdots + x_n) \wedge \bigwedge_{1 \leq i \leq n} c(x_i) \rightarrow \bigvee_{2 \leq i \leq n} c(x_1 + x_i) \Big).$$

4. two planarity axioms, e.g.

$$\neg \exists x_1 \ldots \exists x_5 ( \bigwedge_{1 \leq i \leq 5} (c(x_i) \wedge x_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq 5} (c(x_i + x_j) \wedge x_i \cdot x_j = 0) );$$

5. the axioms $c(0)$ and $c(1)$;
6. the splitting axiom;
7. another dreadful axiom to do with splitting up regions.

- If $n \geq 1$, we let $\psi_{\mathrm{c}}^{n}(x)$ stand for the formula

$$\exists z_1 \ldots \exists z_n \Big( \bigwedge_{1 \leq i \leq n} c(z_i) \wedge (x = z_1 + \cdots + z_n) \Big)$$

stating that $x$ can be formed by adding together $n$ connected regions.

- Thus, for any finitely decomposable mereotopology, the following infinitary rule of inference is valid:

$$\frac{\{\forall x(\psi_{\mathrm{c}}^{n}(x) \rightarrow \phi(x)) \mid n \geq 1\}}{\forall x \phi(x)}.$$

- This rule simply says that, if a property holds of all $n$-components regions, for all $n$, then it holds for all regions.

- We have the following:

  **Theorem:** Let $M$ be a finite decomposable mereotopology over $\mathbb{R}^2$ having curve-selection, and satisfying the splitting axiom. Then $M$ satisfies all the above axioms, and makes the infinitary rule of inference valid.

  **Proof:** Routine.

- More interestingly, we have a converse. Let $T_{c,\leq}$ denote the set of sentences which are consequences of the above axioms and the infinitary proof rule.

  **Theorem:** $T_{c,\leq}$ is the complete theory of any finitely decomposable mereotopology over $\mathbb{R}^2$ having curve-selection and satisfying the splitting axiom.

  **Proof:** Use the omitting types theorem to get a finitely decomposable model of $T_{c,\leq}$; embed it in $\mathrm{ROP}(\mathbb{R}^2)$, and show that the embedding is elementary.

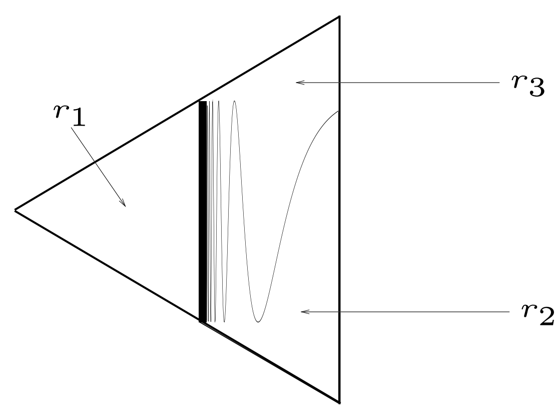
- The above theorem entails that all finitely decomposable, splittable mereotopologies over $\mathbb{R}^2$ having curve-selection, considered as $\{c, \leq\}$-structures, are elementarily equivalent.
- Actually, over the closed plane, more is true:

  **Theorem:** All splittable, finitely decomposable mereotopologies over $\mathbb{S}^2$ with curve-selection have the same $L_\Sigma$-theory for any topological signature $\Sigma$.

- The theory $T_{c,\leq}$ is well-behaved. It is atomic, with $\mathrm{ROQ}(\mathbb{R}^2)$ a prime model.
- In addition, we have:

  **Theorem:** All countable finitely decomposable models of the theory $T_{c,\leq}$ are isomorphic.
- Thus, we can get reasonably close to characterizing the tame-region-based topology of the Euclidean plane axiomatically.

- We conclude with an open problem regarding contact algebras.
- All of the results so far concern *mereotopologies* over various topological spaces $X$—that is, certain sorts of dense subalgebras of $\mathrm{RO}(X)$.
- But what about the contact structure of the whole algebra $\mathrm{RO}(X)$? Can we characterize that?
- The example $\psi_{\mathrm{wiggly}}$, which is true in $\mathrm{ROS}(\mathbb{R}^2)$, but not true in $\mathrm{RO}(\mathbb{R}^2)$, suggests that this problem may not be so simple:



**Theorem** Suppose $\mathfrak{A} \models \Phi_{\text{CA}} \cup \{\phi_{\text{int}}, \phi_{\text{ext}}\}$, and $\mathfrak{A}$ is a (non-trivial) complete Boolean algebra. Then

$$\mathfrak{A} \models \exists x \exists y (C(x,y) \wedge \forall z (z \leq x \wedge$$
$$\forall z_1 \forall z_2 (z_1 > 0 \wedge z_2 > 0 \wedge z = z_1 + z_2 \rightarrow C(z_1, z_2)) \rightarrow \neg C(z,y)))$$

- But this sentence is false in any *finitely decomposable* mereotopology over a topological space.
- Let $\mathcal{X}$ be the class of all topological spaces, and set

$$\mathcal{R} = \{\text{RO}(X) \mid X \in \mathcal{X}\}$$
$$\mathcal{M} = \{M \mid M \text{ a mereotopology over } X \text{ for some } X \in \mathcal{X}\}.$$

Then $\text{Th}(\mathcal{M}) \neq \text{Th}(\mathcal{R})$.

**Open problem:** What is the elementary theory (over a suitable signature) of classes $\{\text{RO}(X) \mid X \in \mathcal{X}\}$, where $\mathcal{X}$ is some salient class of topological spaces?

- Summary
  - Two important ideas:
    - formal language interpreted over classes of geometrical structures (spatial logic),
    - study of topology from a region-based viewpoint (Whitehead's vision).
  - These ideas led us to the notion of a mereotopology.
  - We can prove representation theorems for the first-order theories of various classes of mereotopologies.
  - We can prove an almost-first order representation theorem for the rational polygonal mereotopology over the Euclidean plane.
- See Aiello, Pratt-Hartmann and van Benthem: *Handbook of Spatial Logic* (Springer, 2007) for details . . .