

Boolean Role Inclusions in DL-Lite With and Without Time

Roman Kontchakov¹, Vladislav Ryzhikov¹, Frank Wolter², Michael Zakharyashev¹

¹Department of Computer Science, Birkbeck, University of London, UK

²Department of Computer Science, University of Liverpool, UK

{roman, vlad, michael}@dcs.bbk.ac.uk, wolter@liverpool.ac.uk

Abstract

Traditionally, description logic has focused on representing and reasoning about classes rather than relations (roles), which has been justified by the deterioration of the computational properties if expressive role inclusions are added. The situation is even worse in the temporalised setting, where monodicity is viewed as an almost necessary condition for decidability. We take a fresh look at the description logic *DL-Lite* with expressive role inclusions, both with and without a temporal dimension. While we confirm that full Boolean expressive power on roles leads to FO²-like behaviour in the atemporal case and undecidability in the temporal case, we show that, rather surprisingly, the restriction to Krom and Horn role inclusions leads to much lower complexity in the atemporal case and to decidability (and EXPTIME-completeness) in the temporal case, even if one admits full Booleans on concepts. The latter result is one of very few instances breaking the monodicity barrier in temporal FO. This is also reflected on the data complexity level, where we obtain new rewritability results into FO with relational primitive recursion and FO with unary divisibility predicates.

1 Introduction

Description logics (DLs) have often been described as decidable fragments of first-order logic (FO) that model a domain by introducing complex concept descriptions and subsumptions between them. In fact, the main syntactic difference between DLs and FO is that, in the former, one can construct new, complex, concept descriptions from atomic concepts using concept constructors without the explicit use of individual variables. The subsumption relationship between complex concepts is then expressed using concept inclusions (CIs). Interestingly, corresponding role (binary relation) constructors taking as input atomic roles and describing complex roles have never become mainstream except for role composition, thus admitting role inclusions (RIs) of the form $R_1 \circ \dots \circ R_n \sqsubseteq R$, with appropriate restrictions (Baader et al. 2017). The advantages of even a very limited form of Boolean expressivity on roles is well known (Hustadt and Schmidt 1998; Lutz and Sattler 2000b; Rudolph, Krötzsch, and Hitzler 2008a; Rudolph, Krötzsch, and Hitzler 2008b), so one can only speculate about the reasons for them not becoming more popular. The main issue appears to be that, from a computational perspective, adding Boolean operators on roles leads to expressivity similar to

that of the two-variable fragment FO² of FO (Lutz and Sattler 2000a; Lutz, Sattler, and Wolter 2001), which, while still decidable, is significantly more challenging for automated reasoning than typical DL fragments of FO with some form of the tree model property (Grädel, Kolaitis, and Vardi 1997; Vardi 1996). In temporal DLs, the addition of expressivity for roles is even more problematic: just declaring a role to remain constant in time often leads to undecidability (Lutz, Wolter, and Zakharyashev 2008; Gabbay et al. 2003). Again, the reason is well understood: if one goes beyond the monodic fragment of first-order temporal logic and is thus able to represent how relations change in time, one typically can encode the halting problem for Turing machines by using the relations to represent the tape and time to encode the computation (Gabbay et al. 2003).

Our aim here is to revisit Boolean RIs in the context of (temporal) *DL-Lite* and introduce logics with new expressivity for roles, for which the *knowledge base (KB) satisfiability problem* is decidable in the temporal case and of significantly lower complexity than FO² in the atemporal one.

Recall that in *DL-Lite_R* (Calvanese et al. 2007), also denoted *DL-Lite_{core}^{tl}* in the classification of (Artale et al. 2009), CIs and RIs take the form of binary Horn (aka core) inclusions $\vartheta_1 \sqsubseteq \vartheta_2$ or $\vartheta_1 \sqcap \vartheta_2 \sqsubseteq \perp$, where the ϑ_i are either both concepts (that is, concept names or of the form $\exists R$) or roles. The *DL-Lite* languages we consider extend this schema by allowing CIs and RIs of the form

$$\vartheta_1 \sqcap \dots \sqcap \vartheta_k \sqsubseteq \vartheta_{k+1} \sqcup \dots \sqcup \vartheta_{k+m}, \quad (1)$$

where the ϑ_i are all concepts or, respectively, roles. We classify ontologies by the form of their inclusions. Let $c, r \in \{bool, g\text{-}bool, horn, krom, core\}$. Then *DL-Lite_c^r* is the DL whose ontologies contain CIs and RIs of the form (1) satisfying the following conditions for c and r , respectively:

(horn) $m \leq 1$, **(core)** $k + m \leq 2$ and $m \leq 1$,

(krom) $k + m \leq 2$, **(bool)** any $k \geq 0$ and $m \geq 0$,

(g-bool) any $k \geq 1$ and $m \geq 0$.

It follows that *core* is included in both *krom* and *horn*, which are in *bool* (*g-bool* stands for *guarded bool*). The resulting languages provide a new way of classifying ontologies. While the languages *DL-Lite_c^{bool}* all have essentially the same expressivity as FO² and inherit NEXPTIME-completeness of KB satisfiability, the *DL-Lite_c^{krom}* provide a

way of introducing ‘covering’ RIs $\top \sqsubseteq R_1 \sqcup R_2$ and also the complement of a role via disjointness and covering. Rather surprisingly, these disjunctions come for free as far as the complexity of KB satisfiability is concerned: even combined with Boolean CIs, satisfiability is still in NP, and combined with Krom CIs, it is even in NL. The full table of our complexity results is given below:

RIs \ CIs	(g-)bool	krom	horn	core
bool		NEXPTIME		
g-bool		EXPTIME		
krom	NP	NL	NP	NL
horn	NP	P	P	P
core	NP	NL	P	NL

Our main aim in this paper is to investigate extensions of these *DL-Lite* languages with the standard linear temporal logic (*LTL*) operators \square_F/\square_P (always in the future/past) and \circ_F/\circ_P (in the next/previous moment) interpreted over the timeline $(\mathbb{Z}, <)$. The temporal DLs have an additional parameter $\mathfrak{o} \in \{\square, \circ, \square\circ\}$: $DL-Lite_{\mathfrak{o}/r}^{\mathfrak{o}}$ allows ontologies whose axioms (1) may contain operators from \mathfrak{o} (e.g., $\mathfrak{o} = \square$ permits \square_F/\square_P only) and comply with \mathfrak{c} for CIs and \mathfrak{r} for RIs. A CI or RI is satisfied in a model if it holds globally, at all time points in \mathbb{Z} . Even in the minimal language $DL-Lite_{\text{core/core}}^{\square}$, we can state that a role R is expanding ($R \sqsubseteq \circ_F R$) or constant (by adding $\circ_F R \sqsubseteq R$). Using an auxiliary relation, we can also express $R \sqsubseteq \square_F Q$ in $DL-Lite_{\text{core/core}}^{\square}$. Moving to $DL-Lite_{\text{core/horn}}^{\square\circ}$, we can express that R is convex or has a finite lifespan, and $DL-Lite_{\text{core/krom}}^{\square\circ}$ makes it possible to state that R causes Q to hold eventually; see Section 2 for more details and discussions.

Using temporalised RIs we can thus represent temporal knowledge about relations that goes significantly beyond the expressive power of languages where only concepts and/or axioms are temporalised (Baader, Ghilardi, and Lutz 2012; Lutz, Wolter, and Zakharyashev 2008; Gabbay et al. 2003; Borgwardt, Lippmann, and Thost 2015; Gutiérrez-Basulto, Jung, and Kontchakov 2016). We show that, nevertheless, KB satisfiability is decidable (in fact, EXPSpace-complete) for both $DL-Lite_{\text{bool/krom}}^{\square\circ}$ and $DL-Lite_{\text{bool/horn}}^{\square\circ}$, that is, even with arbitrary Boolean concepts, neither Krom nor Horn RIs lead to undecidability. This is optimal, as we also show that satisfiability of $DL-Lite_{\text{g-bool/g-bool}}^{\square}$ KBs is undecidable.

RIs \ CIs	(g-)bool	horn
(g-)bool	undecidable	
krom	? (EXPSpace for \circ -only RBox)	
horn	EXPSpace	
core	PSpace	

We also investigate whether the satisfiability problem for KBs in our languages can be reduced to the query evaluation problem over the underlying temporal database, which clarifies the data complexity of the former. We show that $DL-Lite_{\text{krom/core}}^{\square\circ}$ ontologies are rewritable to FO($<, \equiv_{\mathbb{N}}$), extending FO($<$) with unary predicates $t \equiv 0 \pmod{n}$, for any $n > 1$, which corresponds to the data complexity in AC⁰. On the other hand, we prove that $DL-Lite_{\text{bool/horn}}^{\square\circ}$ ontologies can only be rewritten to FO(RPR), extending

FO with relational primitive recursion, which entails NC¹-completeness for data complexity. The inevitable fly in the ointment is that there is a $DL-Lite_{\text{g-bool/g-bool}}^{\circ}$ ontology for which consistency with a given input data is undecidable.

2 Preliminaries

We use the standard DL syntax and semantics. Let $a_i, i < \omega$, be *individual names*, A_i *concept names*, and P_i *role names*. We define *roles* S , *basic concepts* B , *temporalised roles* R and *temporalised concepts* C by the following grammar:

$$\begin{aligned} S &::= P_i \mid P_i^-, & B &::= A_i \mid \exists S, \\ R &::= S \mid \square_F R \mid \square_P R \mid \circ_F R \mid \circ_P R, \\ C &::= B \mid \square_F C \mid \square_P C \mid \circ_F C \mid \circ_P C. \end{aligned}$$

A *concept* or *role inclusion* (CI or RI) takes the form (1), where the ϑ_i are all temporalised concepts or, respectively, all temporalised roles. (The empty \square is \top and the empty \sqcup is \perp .) A *TBox* \mathcal{T} and an *RBox* \mathcal{R} are finite sets of CIs and, respectively, RIs; their union $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ is an *ontology*. The atemporal $DL-Lite_{\mathfrak{c}}^{\mathfrak{o}}$ and temporal $DL-Lite_{\mathfrak{c}/r}^{\mathfrak{o}}$ were defined in the introduction. We also set $DL-Lite_{\mathfrak{c}}^{\mathfrak{o}} = DL-Lite_{\mathfrak{c}/\mathfrak{c}}^{\mathfrak{o}}$.

To illustrate, imagine an estate agency describing properties by their proximity to various amenities, using roles wd for ‘walking distance’ and dd for ‘driving distance’. Then we can state in $DL-Lite_{\text{core}}^{\text{krom}}$ that $\top \sqsubseteq wd \sqcup dd$, that these roles are disjoint ($wd \sqcap dd \sqsubseteq \perp$) and symmetric (e.g., $wd \sqsubseteq wd^{-}$), and describe locations using CIs such as $FamilyLocation \sqsubseteq \exists wd.School \sqcap \exists dd.Pub$ (which requires fresh auxiliary role names). In $DL-Lite_{\text{core}}^{\text{bool}}$, we can further say that $Station \sqsubseteq \forall wd.WellConnected$ (see Theorem 1). In $DL-Lite_{\text{core/krom}}^{\circ}$, we can also express that, over the past three years, there has been a pub within walking distance: $SocialLocation \sqsubseteq \exists wd.Pub \sqcap \circ_P \exists wd.Pub \sqcap \circ_P \exists wd.Pub$.

An *ABox*, \mathcal{A} , is a finite set of atoms of the form $A_i(a, \ell)$ and $P_i(a, b, \ell)$, where a, b are individual names and $\ell \in \mathbb{Z}$. We denote by $\text{ind}(\mathcal{A})$ the set of individual names in \mathcal{A} , by $\min \mathcal{A}$ and $\max \mathcal{A}$ the minimal and maximal integers in \mathcal{A} , and set $\text{tem}(\mathcal{A}) = \{n \in \mathbb{Z} \mid \min \mathcal{A} \leq n \leq \max \mathcal{A}\}$. For simplicity, we assume that $\min \mathcal{A} = 0$. A $DL-Lite_{\mathfrak{c}/r}^{\mathfrak{o}}$ *knowledge base* (KB) is a pair $(\mathcal{O}, \mathcal{A})$, where \mathcal{O} is a $DL-Lite_{\mathfrak{c}/r}^{\mathfrak{o}}$ ontology and \mathcal{A} an ABox. The *size* $|\mathcal{O}|$ of \mathcal{O} is the number of occurrences of symbols in it; the size of a TBox, RBox, ABox and KB is defined in the same way, with *unary* encoding of numbers in ABoxes.

A (*temporal*) *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}(n)})$, where $\Delta^{\mathcal{I}} \neq \emptyset$ and, for each $n \in \mathbb{Z}$,

$$\mathcal{I}(n) = (\Delta^{\mathcal{I}}, a_0^{\mathcal{I}}, \dots, A_0^{\mathcal{I}(n)}, \dots, P_0^{\mathcal{I}(n)}, \dots)$$

is a standard DL interpretation with $a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, $A_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}}$ and $P_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The DL constructs and temporal operators are interpreted in $\mathcal{I}(n)$ as usual:

$$\begin{aligned} (P_i^-)^{\mathcal{I}(n)} &= \{(u, v) \mid (v, u) \in P_i^{\mathcal{I}(n)}\}, \\ (\exists S)^{\mathcal{I}(n)} &= \{u \mid (u, v) \in S^{\mathcal{I}(n)}, \text{ for some } v\}, \\ (\square_F \vartheta)^{\mathcal{I}(n)} &= \bigcap_{k > n} \vartheta^{\mathcal{I}(k)}, & (\square_P \vartheta)^{\mathcal{I}(n)} &= \bigcap_{k < n} \vartheta^{\mathcal{I}(k)}, \\ (\circ_F \vartheta)^{\mathcal{I}(n)} &= \vartheta^{\mathcal{I}(n+1)}, & (\circ_P \vartheta)^{\mathcal{I}(n)} &= \vartheta^{\mathcal{I}(n-1)}. \end{aligned}$$

CIs and RIs are interpreted in \mathcal{I} globally in the sense that inclusion (1) is true in \mathcal{I} if

$$\vartheta_1^{\mathcal{I}(n)} \cap \dots \cap \vartheta_k^{\mathcal{I}(n)} \subseteq \vartheta_{k+1}^{\mathcal{I}(n)} \cup \dots \cup \vartheta_{k+m}^{\mathcal{I}(n)}, \quad \text{for all } n \in \mathbb{Z}.$$

For an inclusion α , we write $\mathcal{I} \models \alpha$ if α is true in \mathcal{I} . We call \mathcal{I} a *model* of $(\mathcal{O}, \mathcal{A})$ and write $\mathcal{I} \models (\mathcal{O}, \mathcal{A})$ if $\mathcal{I} \models \alpha$ for all $\alpha \in \mathcal{O}$, $a^{\mathcal{I}} \in A^{\mathcal{I}(\ell)}$ for $A(a, \ell) \in \mathcal{A}$, and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in P^{\mathcal{I}(\ell)}$ for $P(a, b, \ell) \in \mathcal{A}$. A KB is *satisfiable* if it has a model.

It is to be noted that the *LTL* operators \diamond_F (eventually), \mathcal{U} (until) and their past counterparts can be expressed in *bool* using \circ_F/\circ_F and \square_F/\square_F (Fisher, Dixon, and Peim 2001; Artale et al. 2013). In many cases, one does not need full Booleans: $\diamond_P R \sqsubseteq Q$ is equivalent to $R \sqsubseteq \square_F Q$, which can be expressed in *DL-Lite_{core}^o* as $R \sqsubseteq \circ_F S$, $S \sqsubseteq \circ_F S$, $S \sqsubseteq Q$, where S is fresh. It immediately follows that convexity of R (that is, $\diamond_P R \sqcap \diamond_F R \sqsubseteq R$) can be expressed in *DL-Lite_{horn}^o* and *DL-Lite_{horn}^o*. Then $R \sqsubseteq \diamond_F Q$ can be simulated in *DL-Lite_{krom}^o* with $\top \sqsubseteq \bar{Q} \sqcup Q$ and $R \sqcap \square_F \bar{Q} \sqsubseteq \perp$. That the lifespan of R is bounded can be expressed in *DL-Lite_{core}^o* using $\square_P R \sqsubseteq \perp$ and $\square_F R \sqsubseteq \perp$.

We are interested in the *combined* and *data* complexities of the *satisfiability problem* for KBs: for the former, both the ontology and the ABox of a KB are regarded as input, while for the latter, the ontology is fixed. We assume that $|\text{tem}(\mathcal{A})| \geq |\text{ind}(\mathcal{A})|$ in any input ABox \mathcal{A} (if this is not so, we add the required number of dummies with the missing timestamps to \mathcal{A}). Let $\text{ind}(\mathcal{A}) = \{a_1, \dots, a_m\}$. We encode \mathcal{A} as a structure $\mathfrak{S}_{\mathcal{A}}$ with domain $\text{tem}(\mathcal{A})$ ordered by $<$ such that $\mathfrak{S}_{\mathcal{A}} \models A(k, \ell)$ iff $A(a_k, \ell) \in \mathcal{A}$ and $\mathfrak{S}_{\mathcal{A}} \models P(k, k', \ell)$ iff $P(a_k, a_{k'}, \ell) \in \mathcal{A}$.

We establish our data complexity results by ‘rewriting’ ontologies to FO-sentences ‘accepting’ or ‘rejecting’ the input ABoxes. Let \mathcal{L} be a class of FO-sentences interpreted over $\mathfrak{S}_{\mathcal{A}}$. Say that $\Phi \in \mathcal{L}$ is an \mathcal{L} -rewriting of \mathcal{O} if, for any ABox \mathcal{A} , the KB $(\mathcal{O}, \mathcal{A})$ is satisfiable iff $\mathfrak{S}_{\mathcal{A}} \models \Phi$. Here, we need three classes \mathcal{L} : (i) FO($<$) with binary and ternary predicates of the form $A_i(x, t)$ and $P_i(x, y, t)$ as well as $<$ and $=$; (ii) FO($<, \equiv_{\mathbb{N}}$) with extra unary predicates $t \equiv 0 \pmod{n}$, for any $n > 1$, and (iii) FO(RPR) that extends FO with *relational primitive recursion*, which allows one to construct formulas such as

$$\left[\begin{array}{l} Q_1(z_1, t) \equiv \Theta_1(z_1, t, Q_1(z_1, t-1), \dots, Q_n(z_n, t-1)) \\ \dots \\ Q_n(z_n, t) \equiv \Theta_n(z_n, t, Q_1(z_1, t-1), \dots, Q_n(z_n, t-1)) \end{array} \right] \Psi,$$

where $[\dots]$ defines recursively, via the formulas Θ_i , the interpretations of the predicates Q_i in Ψ . For data complexity, evaluation of FO($<, \equiv_{\mathbb{N}}$)-sentences over $\mathfrak{S}_{\mathcal{A}}$ is known to be in LOGTIME-uniform AC⁰ (Immerman 1999) and evaluation of FO(RPR)-sentences is in NC¹ (Compton and Laflamme 1990).

3 Reasoning with Atemporal DL-Lite

To begin with, we establish the complexity of reasoning with the plain DLs underlying the temporal *DL-Lite_{c/r}^o* introduced above. We denote them by *DL-Lite_c^r*, where as before $c, r \in \{\text{bool}, g\text{-bool}, \text{horn}, \text{krom}, \text{core}\}$. The satisfiability problem for DLs of the form *DL-Lite_c^{core}* was stud-

ied by (Calvanese et al. 2007; Artale et al. 2009): it is NP-complete for *DL-Lite_{bool}^{core}*, P-complete for *DL-Lite_{horn}^{core}*, and NL-complete for *DL-Lite_{krom}^{core}* and *DL-Lite_{core}^{core}* KBs.

We show that *DL-Lite_{bool}^{core}* can be regarded as a notational variant of the extension $\mathcal{ALCT}^{\square, \neg}$ of \mathcal{ALC} with inverse roles and Boolean operators on roles. This logic has, in turn, almost the same expressive power as FO², except that the identity role has to be added. In detail, let $\mathcal{ALCT}^{\square, \neg}$ be the DL with roles S and concepts C defined by

$$\begin{aligned} S, S' &::= \top \mid P_i \mid S \sqcap S' \mid \neg S \mid S^{-}, \\ C, C' &::= \top \mid A_i \mid \exists S.C \mid C \sqcap C' \mid \neg C. \end{aligned}$$

An $\mathcal{ALCT}^{\square, \neg}$ CI takes the form $C \sqsubseteq C'$ (Lutz, Sattler, and Wolter 2001; Lutz and Sattler 2000a; Gargov and Passy 1990). We say that a KB \mathcal{K} is a *model conservative extension* of a KB \mathcal{K}' if $\mathcal{K} \models \mathcal{K}'$, the signature of \mathcal{K} contains the signature of \mathcal{K}' , and every model of \mathcal{K}' can be extended to a model of \mathcal{K} by providing interpretations of the fresh symbols of \mathcal{K} and leaving the domain and the interpretation of the symbols in \mathcal{K}' unchanged.

Theorem 1. (i) For every *DL-Lite_{bool}^{bool}* KB, one can compute in logarithmic space an equivalent $\mathcal{ALCT}^{\square, \neg}$ KB.

(ii) For every $\mathcal{ALCT}^{\square, \neg}$ KB, one can compute in log-space a model conservative extension in *DL-Lite_{bool}^{bool}*.

Proof. (i) Clearly, any CI in *DL-Lite_{bool}^{bool}* is an $\mathcal{ALCT}^{\square, \neg}$ CI ($\exists R = \exists R. \top$). Any RI $S_1 \sqcap \dots \sqcap S_k \sqsubseteq S_{k+1} \sqcup \dots \sqcup S_{k+m}$ in *DL-Lite_{bool}^{bool}* is equivalent to the $\mathcal{ALCT}^{\square, \neg}$ CI $\exists R. \top \sqsubseteq \perp$, where R abbreviates $S_1 \sqcap \dots \sqcap S_k \sqcap \neg S_{k+1} \sqcap \dots \sqcap \neg S_{k+m}$.

(ii) For any $\mathcal{ALCT}^{\square, \neg}$ KB \mathcal{K} , we construct a model conservative extension of \mathcal{K} in $\mathcal{ALCT}^{\square, \neg}$ with CIs in normal form:

$$A \sqsubseteq \forall S.B, \forall S.B \sqsubseteq A, A_1 \sqcap A_2 \sqsubseteq B, A \sqsubseteq \neg B, \neg A \sqsubseteq B,$$

where A, B, A_1, A_2 range over concept names and \top . Next, we replace CIs $A \sqsubseteq \forall S.B$ and $\forall S.B \sqsubseteq A$ by $S \sqsubseteq Q \sqcup R$, $\exists Q \sqsubseteq B$, $\exists R \sqsubseteq \neg A$, and, respectively, $\neg A \sqsubseteq \exists R$, $R \sqsubseteq S$, $\exists R \sqsubseteq \neg B$, with fresh role names Q, R . Finally, RIs with a Boolean S are transformed into normal form (1) to obtain a model conservative extension of \mathcal{K} in *DL-Lite_{bool}^{bool}*. \square

The NEXPTIME-completeness of $\mathcal{ALCT}^{\square, \neg}$ KB satisfiability (Lutz, Sattler, and Wolter 2001) implies that *DL-Lite_{bool}^{bool}* KB satisfiability is also NEXPTIME-complete. To bring down the complexity to EXPTIME, it suffices to avoid unguarded quantification by admitting only RIs with a non-empty left-hand side, as in the *gbool* RIs. Then, for any *DL-Lite_{bool}^{gbool}* KB, it is straightforward to compute in linear time an equivalent KB in the guarded two-variable fragment GF² of FO. Using the fact that KB satisfiability for the latter logic is in EXPTIME (Grädel 1999), we obtain the following:

Theorem 2. KB satisfiability is NEXPTIME-complete for *DL-Lite_{bool}^{bool}* and EXPTIME-complete for *DL-Lite_{bool}^{gbool}*.

We now show that the *DL-Lite* logics with Horn and Krom RIs are reducible to propositional logic. For an ontology \mathcal{O} , let $\text{role}^{\pm}(\mathcal{O}) = \{P, P^{-} \mid P \text{ a role in } \mathcal{O}\}$ and let $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. We assume that \mathcal{R} is closed under taking the inverses of roles in RIs. Denote by $\text{sub}_{\mathcal{T}}$ the set

of concepts in \mathcal{T} and their negations. A *concept type* τ for \mathcal{T} is a maximal subset τ of $\text{sub}_{\mathcal{T}}$ that is ‘propositionally consistent’ with \mathcal{T} : if $B_1, \dots, B_k \in \tau$ and \mathcal{T} contains $B_1 \sqcap \dots \sqcap B_k \sqsubseteq B_{k+1} \sqcup \dots \sqcup B_{k+m}$, then one of B_{k+1}, \dots, B_{k+m} is also in τ (note, however, that τ does not have to be consistent with \mathcal{T} as it can contain $\exists P$ even if $\exists P^- \sqsubseteq \perp$ is in \mathcal{T}). Clearly, for an interpretation \mathcal{J} and $u \in \Delta^{\mathcal{J}}$, the set comprising all $B \in \text{sub}_{\mathcal{T}}$ with $u \in B^{\mathcal{J}}$ and all $\neg B \in \text{sub}_{\mathcal{T}}$ with $u \notin B^{\mathcal{J}}$ is a concept type for \mathcal{T} ; it is denoted by $\tau_u^{\mathcal{J}}$ and called the *concept type of u in \mathcal{J}* . Similarly, let $\text{sub}_{\mathcal{R}}$ be the set of roles in \mathcal{R} and their negations. A *role type* ρ for \mathcal{R} is a maximal subset of $\text{sub}_{\mathcal{R}}$ propositionally consistent with \mathcal{R} . For $(u, v) \in \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}}$, the set comprising all $S \in \text{sub}_{\mathcal{R}}$ with $(u, v) \in S^{\mathcal{J}}$ and all $\neg S \in \text{sub}_{\mathcal{R}}$ with $(u, v) \notin S^{\mathcal{J}}$ is a role type for \mathcal{R} ; it is denoted by $\rho_{u,v}^{\mathcal{J}}$ and called the *role type of (u, v) in \mathcal{J}* . For a set of role literals (roles and their negations) Ξ , let $\text{cl}_{\mathcal{R}}(\Xi)$ be the set of all role literals L' such that $\mathcal{R} \models \prod_{L \in \Xi} L \sqsubseteq L'$. The following lemma plays a key role in the reduction.

Lemma 3. *For any satisfiable $DL\text{-Lite}_{bool}^{krom} KB \mathcal{K} = (\mathcal{O}, \mathcal{A})$, $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$, there is a model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of \mathcal{K} such that*

$$\Delta^{\mathcal{I}} = \text{ind}(\mathcal{A}) \cup \{w_S^i \mid S \in \text{role}^{\pm}(\mathcal{O}) \text{ and } 0 \leq i < 3\}$$

and $(u, v) \in S^{\mathcal{I}}$, for every $u \rightarrow_S v$ with $u \in (\exists S)^{\mathcal{I}}$, where

$$\rightarrow_S = \{(a, w_S^0) \mid a \in \text{ind}(\mathcal{A})\} \cup \{(w_R^i, w_S^{i \oplus 1}) \mid w_R^i \in \Delta^{\mathcal{I}}\}$$

and \oplus is addition modulo 3. In particular, $DL\text{-Lite}_{bool}^{krom}$ has the linear model property: $|\Delta^{\mathcal{I}}| = |\text{ind}(\mathcal{A})| + 3|\text{role}^{\pm}(\mathcal{O})|$.

Proof. Given a model $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ of \mathcal{K} , we construct \mathcal{I} as follows. For any $S \in \text{role}^{\pm}(\mathcal{O})$, if $S^{\mathcal{J}} \neq \emptyset$, then we pick $w_S \in (\exists S^-)^{\mathcal{J}}$; otherwise, we pick any $w_S \in \Delta^{\mathcal{J}}$. We assume that the w_S are distinct. Let $\Delta^{\mathcal{I}}$ comprise $\text{ind}(\mathcal{A})$ and three copies w_S^0, w_S^1, w_S^2 of each w_S ; cf. (Börger, Grädel, and Gurevich 1997, Proposition 8.1.4). This also fixes the \rightarrow_S . Define $f: \Delta^{\mathcal{I}} \rightarrow \Delta^{\mathcal{J}}$ by taking $f(a) = a$, for all $a \in \text{ind}(\mathcal{A})$, and $f(w_S^i) = w_S$, for all S and i . We then set $\tau_u = \tau_{f(u)}^{\mathcal{J}}$, for all $u \in \Delta^{\mathcal{I}}$. To define $\rho_{u,v}$ for $u, v \in \Delta^{\mathcal{I}}$, we consider the following three cases.

- If $u, v \in \text{ind}(\mathcal{A})$, then we take $\Xi = \{S \mid S(a, b) \in \mathcal{A}\}$, assuming $P_i^-(a, b) \in \mathcal{A}$ whenever $P_i(b, a) \in \mathcal{A}$.
- If $\exists S \in \tau_u$ and $u \rightarrow_S v$, then we take $\Xi = \{S\}$.
- Otherwise, we take $\Xi = \emptyset$.

We begin with $\rho_{u,v} = \text{cl}_{\mathcal{R}}(\Xi)$ and perform the following procedure for each RI $\top \sqsubseteq S_1 \sqcup S_2$ in \mathcal{R} such that none of S_i and $\neg S_i$ is in $\rho_{u,v}$ yet. As $\mathcal{J} \models \mathcal{R}$, either S_1 or S_2 is in $\rho_{f(u), f(v)}^{\mathcal{J}}$. So $\rho_{u,v}$ is extended with the respective $\text{cl}_{\mathcal{R}}(\{S_i\})$. Since any contradiction derivable from Krom formulas is derivable from two literals, the resulting $\rho_{u,v}$ is consistent with \mathcal{R} and both τ_u - and τ_v -compatible: that is, $\exists R \in \tau_u$ and $\exists R^- \in \tau_v$, for all $R \in \rho_{u,v}$. One can check that the constructed τ_u and $\rho_{u,v}$, for $u, v \in \Delta^{\mathcal{I}}$, are types for \mathcal{T} and \mathcal{R} , respectively, and give rise to a model of \mathcal{K} . \square

The existence of a model \mathcal{I} from Lemma 3 can be encoded by a *propositional* formula $\varphi_{\mathcal{K}}$ whose propositional

variables take the form $B^{\dagger}(u)$ and $P_i^{\dagger}(u, v)$, for $u, v \in \Delta^{\mathcal{I}}$, assuming that $(P_i^-)^{\dagger}(u, v) = P_i^{\dagger}(v, u)$. The formula $\varphi_{\mathcal{K}}$ is a conjunction of the following, for all $u, v \in \Delta^{\mathcal{I}}$:

$$\begin{aligned} B_1^{\dagger}(u) \wedge \dots \wedge B_k^{\dagger}(u) &\rightarrow B_{k+1}^{\dagger}(u) \vee \dots \vee B_{k+m}^{\dagger}(u), \\ &\text{for CI } B_1 \sqcap \dots \sqcap B_k \sqsubseteq B_{k+1} \sqcup \dots \sqcup B_{k+m} \text{ in } \mathcal{T}, \\ S_1^{\dagger}(u, v) &\rightarrow S_2^{\dagger}(u, v), \quad \text{for RI } S_1 \sqsubseteq S_2 \text{ in } \mathcal{R}, \\ \neg S_1^{\dagger}(u, v) &\vee \neg S_2^{\dagger}(u, v), \quad \text{for RI } S_1 \sqcap S_2 \sqsubseteq \perp \text{ in } \mathcal{R}, \\ S_1^{\dagger}(u, v) \vee S_2^{\dagger}(u, v) &\quad \text{for RI } \top \sqsubseteq S_1 \sqcup S_2 \text{ in } \mathcal{R}, \\ A^{\dagger}(a) &\quad \text{for } A(a) \in \mathcal{A}, \text{ and } P^{\dagger}(a, b), \text{ for } P(a, b) \in \mathcal{A}, \\ (\exists S)^{\dagger}(u) &\rightarrow S^{\dagger}(u, v), \quad \text{for each } S \text{ with } u \rightarrow_S v, \\ S^{\dagger}(u, v) &\rightarrow (\exists S)^{\dagger}(u), \quad \text{for each } S. \end{aligned}$$

Clearly, \mathcal{K} is satisfiable iff $\varphi_{\mathcal{K}}$ is satisfiable. Also, if \mathcal{K} is in $DL\text{-Lite}_{krom}^{krom}$, then $\varphi_{\mathcal{K}}$ is a Krom formula constructed by a logspace transducer. Now, since $DL\text{-Lite}_{krom}^{krom}$ can express $DL\text{-Lite}_{bool}^{krom}$ (Krom RIs can simulate Krom CIs, and the latter can express the complement of concepts), we obtain:

Theorem 4. *Satisfiability is NP-complete for $DL\text{-Lite}_{bool}^{krom}$ and $DL\text{-Lite}_{horn}^{krom}$ KBs, and NL-complete for $DL\text{-Lite}_{krom}^{krom}$.*

The next theorem is proved by a similar argument. However, for $DL\text{-Lite}_{krom}^{horn}$ we use a polynomial (rather than logspace) reduction into Krom propositional logic.

Theorem 5. *Satisfiability is NP-complete for $DL\text{-Lite}_{bool}^{horn}$ KBs, and P-complete for $DL\text{-Lite}_{horn}^{horn}$ and $DL\text{-Lite}_{krom}^{horn}$ KBs.*

4 Satisfiability of Temporal KBs

We now consider extensions $DL\text{-Lite}_{c/r}^o$ of $DL\text{-Lite}_c^r$ with temporal operators in $o \in \{\square, \circ, \square\circ\}$ that can be applied to concepts and roles. Our first observation is negative:

Theorem 6. *Satisfiability in $DL\text{-Lite}_{g\text{-bool}}^o$ is undecidable.*

Proof. The proof is by reduction of the undecidable $\mathbb{N} \times \mathbb{N}$ -tiling problem (Berger 1966). Given a set $\mathfrak{T} = \{1, \dots, m\}$ of tile types, with the colours on the four edges of tile type i denoted by $up(i)$, $down(i)$, $left(i)$ and $right(i)$, we define the following $DL\text{-Lite}_{g\text{-bool}}^o$ ontology \mathcal{O} , where R_i is a role name associated with the tile type $i \in \mathfrak{T}$:

$$\begin{aligned} I &\sqsubseteq \bigsqcup_{i \in \mathfrak{T}} \exists R_i, & R_i &\sqsubseteq \bigsqcup_{\text{right}(i)=\text{left}(j)} \circ_F R_j, \\ \exists R_i^- &\sqsubseteq \bigsqcup_{\text{up}(i)=\text{down}(j)} \exists R_j, & \exists R_i \sqcap \exists R_j &\sqsubseteq \perp, \text{ for } i \neq j. \end{aligned}$$

Then $(\mathcal{O}, \{I(a, 0)\})$ is satisfiable iff \mathfrak{T} can tile $\mathbb{N} \times \mathbb{N}$. \square

Fortunately, the temporal $DL\text{-Lite}$ languages with Krom, Horn and core RIs turn out to be less naughty. In the remainder of this section, we develop reductions of these languages to propositional and first-order LTL with one variable.

Given a $DL\text{-Lite}_{bool/krom}^o$ KB $\mathcal{K} = (\mathcal{T} \cup \mathcal{R}, \mathcal{A})$, we construct a first-order temporal sentence $\Phi_{\mathcal{K}}$ with one free variable x . We assume that \mathcal{K} has no nested temporal operators and that, in RIs of the form $\top \sqsubseteq R_1 \sqcup R_2$ from \mathcal{R} , both R_i

are plain (atemporal) roles; also, \mathcal{R} is closed under taking the inverses of roles in RIs. First, we set $\Phi_{\mathcal{K}} = \perp$ if $(\mathcal{R}, \mathcal{A})$ is unsatisfiable. Otherwise, we treat concept names and basic concepts in \mathcal{K} as unary predicates and define $\Phi_{\mathcal{K}}$ as a conjunction of the following sentences, where $\square = \square_F \square_P$:

$$\begin{aligned} & \square \forall x [C_1(x) \wedge \dots \wedge C_k(x) \rightarrow C_{k+1}(x) \vee \dots \vee C_{k+m}(x)], \\ & \quad \text{for CI } C_1 \sqcap \dots \sqcap C_k \sqsubseteq C_{k+1} \sqcup \dots \sqcup C_{k+m} \text{ in } \mathcal{T}, \\ & \square \forall x [\exists S_1(x) \vee \exists S_2(x)] \text{ and} \\ & \quad \square [\forall x \exists S_1(x) \vee \forall x \exists S_2^-(x)], \text{ for RI } \top \sqsubseteq S_1 \sqcup S_2 \text{ in } \mathcal{R}, \\ & \square_F^\ell A(a), \quad \text{for } A(a, \ell) \in \mathcal{A}, \\ & \square_F^\ell \exists P(a) \text{ and } \square_F^\ell \exists P^-(b), \quad \text{for } P(a, b, \ell) \in \mathcal{A}, \\ & \square [\exists x \exists P(x) \leftrightarrow \exists x \exists P^-(x)], \quad \text{for role name } P \text{ in } \mathcal{T}, \end{aligned}$$

and, for every RI $\circ_1 S_1 \sqsubseteq \circ_2 S_2$ with $\mathcal{R} \models \circ_1 S_1 \sqsubseteq \circ_2 S_2$, where each \circ_i is \circ_F , \circ_P or blank, and $\circ_1 S_1$ can be \top and $\circ_2 S_2$ can be \perp , the sentence

$$\square \forall x [\circ_1 \exists S_1(x) \rightarrow \circ_2 \exists S_2(x)].$$

We observe that $\mathcal{R} \models \circ_1 S_1 \sqsubseteq \circ_2 S_2$ can be checked in P (Artale et al. 2014, Lemma 5.3), and so $\Phi_{\mathcal{K}}$ is constructed in polynomial time.

Lemma 7. *A DL-Lite $_{bool/krom}^{\square}$ KB is satisfiable iff $\Phi_{\mathcal{K}}$ is satisfiable.*

Theorem 8. *The satisfiability problem for DL-Lite $_{bool/krom}^{\square}$ KBs is EXSPACE-complete.*

Proof. The upper bound follows from Lemma 7 since the one-variable fragment of first-order LTL is known to be EXSPACE-complete (Halpern and Vardi 1989; Gabbay et al. 2003); hardness is proved by reduction of the $(2^n - 1)$ corridor tiling problem (Van Emde Boas 1997): given a finite set \mathfrak{T} of tile types $\{1, \dots, m\}$ with four colours $up(i)$, $down(i)$, $left(i)$ and $right(i)$ and a distinguished colour W , decide whether \mathfrak{T} can tile the grid $\mathbb{N} \times \{s \mid 1 \leq s < 2^n\}$ so that (\mathbf{b}_1) tile 0 is placed at $(0, 1)$, (\mathbf{b}_2) every tile i placed at every $(c, 1)$ has $down(i) = W$, and (\mathbf{b}_3) every tile i placed at every $(c, 2^n - 1)$ has $up(i) = W$.

Let $\mathcal{A} = \{A(a, 0)\}$ and \mathcal{O} contain the inclusions

$$A \sqsubseteq \circ_F^{2^n} D, \quad D \sqsubseteq \circ_F^{2^n} D, \quad A \sqsubseteq \prod_{1 \leq s < 2^n} \circ_F^s \exists P, \quad \exists P^- \sqsubseteq \bigsqcup_{i \in \mathfrak{T}} T_i,$$

$$T_i \sqsubseteq \circ_F^{2^n} \bigsqcup_{right(i)=left(j)} T_j, \quad T_i \sqcap \exists S_i^- \sqsubseteq \perp, \quad \top \sqsubseteq S_i \sqcup Q_i, \quad \text{for } i \in \mathfrak{T},$$

$$\exists Q_i \sqcap \circ_F \exists Q_j \sqsubseteq \perp, \quad \text{for } i, j \in \mathfrak{T} \text{ with } up(i) \neq down(j).$$

Observe that $(\mathcal{O}, \mathcal{A})$ is satisfiable iff there is a placement of tiles on the grid: each of the $(2^n - 1)$ P -successors of a created at moments $1, \dots, 2^n$ represents a corridor column. However, the size of the CIs is exponential in n . We now describe how they can be replaced by polynomial-size CIs.

Consider a CI $A \sqsubseteq \circ_F^{2^n} D$. We express it using the following CIs, for k with $0 \leq k < n$ and j with $k < j < n$:

$$\begin{aligned} & A \sqsubseteq \circ_F (\neg B_{n-1} \sqcap \dots \sqcap \neg B_0) \text{ and } B_{n-1} \sqcap \dots \sqcap B_0 \sqsubseteq D, \\ & \neg B_k \sqcap B_{k-1} \sqcap \dots \sqcap B_0 \sqsubseteq \circ_F (B_k \sqcap \neg B_{k-1} \sqcap \dots \sqcap \neg B_0), \\ & \neg B_j \sqcap \neg B_k \sqsubseteq \circ_F \neg B_j, \text{ and } B_j \sqcap \neg B_k \sqsubseteq \circ_F B_j, \end{aligned}$$

which have to be converted into normal form (1). Intuitively, they encode a binary counter from 0 to $2^n - 1$, where $\neg B_i$ and B_i stand for ‘the i th bit of the counter is 0 and, respectively, 1’. The CIs of the form $C_1 \sqsubseteq \circ_F^{2^n} C_2$ are handled similarly. For $A \sqsubseteq \prod_{1 \leq s < 2^n} \circ_F^s \exists P$, we use the $B_k \sqsubseteq \exists P$, for $0 \leq k < n$, instead of $B_{n-1} \sqcap \dots \sqcap B_0 \sqsubseteq \exists P$.

To ensure that (\mathbf{b}_1) – (\mathbf{b}_3) are satisfied, we add to \mathcal{O} the CIs

$$\begin{aligned} & A \sqcap \circ_F \exists Q_i \sqsubseteq \perp, \quad \text{for } i \in \mathfrak{T} \setminus \{0\}, \\ & D \sqcap \circ_F \exists Q_i \sqsubseteq \perp, \quad \text{for } down(i) \neq W, \\ & \circ_F D \sqcap \exists Q_i \sqsubseteq \perp, \quad \text{for } up(i) \neq W. \end{aligned}$$

One can show that $(\mathcal{O}, \mathcal{A})$ is as required. \square

Let $\mathcal{K} = (\mathcal{T} \cup \mathcal{R}, \mathcal{A})$ be a DL-Lite $_{bool/horn}^{\square}$ KB. We assume that \mathcal{R} is closed under taking the inverses of roles in RIs and contains all roles in \mathcal{T} . A *beam* \mathbf{b} for \mathcal{T} is a function from \mathbb{Z} to the set of concept types for \mathcal{T} such that, for all $n \in \mathbb{Z}$,

$$\circ_F C \in \mathbf{b}(n) \text{ iff } C \in \mathbf{b}(n+1), \quad (2)$$

$$\circ_F C \in \mathbf{b}(n) \text{ iff } C \in \mathbf{b}(k), \text{ for all } k > n, \quad (3)$$

and symmetric conditions for the past-time operators. The function $\mathbf{b}_u^{\mathcal{I}}: n \mapsto \{C \in \text{sub}_{\mathcal{T}} \mid u \in C^{\mathcal{I}(n)}\}$ (we specify only the positive component of types) is a beam, for any \mathcal{I} and $u \in \Delta^{\mathcal{I}}$; we will refer to it as *the beam of u in \mathcal{I}* .

A *rod* \mathbf{r} for \mathcal{R} is a function from \mathbb{Z} to the set of role types for \mathcal{R} such that (2)–(3) and their past-time counterparts hold for all $n \in \mathbb{Z}$ with \mathbf{b} replaced by \mathbf{r} and C by temporalised roles S . For any \mathcal{I} and any $u, v \in \Delta^{\mathcal{I}}$, the function $\mathbf{r}_{u,v}^{\mathcal{I}}: n \mapsto \{R \in \text{sub}_{\mathcal{R}} \mid (u, v) \in R^{\mathcal{I}(n)}\}$ is a rod for \mathcal{R} . Fix individual names d, e . Since the RIs in \mathcal{R} are Horn, given any ABox \mathcal{A} with atoms of the form $S(d, e, \ell)$, define the *\mathcal{R} -canonical rod $\mathbf{r}_{\mathcal{A}}$* for \mathcal{A} (consistent with \mathcal{R}): $\mathbf{r}_{\mathcal{A}}: n \mapsto \{R \in \text{sub}_{\mathcal{R}} \mid \mathcal{R}, \mathcal{A} \models R(d, e, n)\}$. In other words, \mathcal{R} -canonical rods are the minimal rods for \mathcal{R} ‘containing’ all atoms of \mathcal{A} : for any R and $n \in \mathbb{Z}$,

$$\begin{aligned} & R \in \mathbf{r}_{\mathcal{A}}(n) \quad \text{iff} \quad R \in \mathbf{r}(n), \text{ for all rods } \mathbf{r} \text{ for } \mathcal{R} \\ & \quad \text{such that } S \in \mathbf{r}(\ell), \text{ for each } S(d, e, \ell) \in \mathcal{A}. \end{aligned}$$

Given a beam \mathbf{b} , a rod \mathbf{r} is called *\mathbf{b} -compatible* if $\exists S \in \mathbf{b}(n)$ whenever $S \in \mathbf{r}(n)$, for $n \in \mathbb{Z}$ and a basic concept $\exists S$. We are now fully equipped to prove the following characterisation of DL-Lite $_{bool/horn}^{\square}$ KBs satisfiability, where beams can be ‘shifted’ in (4) to achieve a finite representation.

Lemma 9. *Let $\mathcal{K} = (\mathcal{T} \cup \mathcal{R}, \mathcal{A})$ be a DL-Lite $_{bool/horn}^{\square}$ KB. Let $\Delta = \text{ind}(\mathcal{A}) \cup \{w_S \mid S \in \text{role}^{\pm}(\mathcal{R})\}$. Then \mathcal{K} is satisfiable iff there are beams \mathbf{b}_w , $w \in \Delta$, for \mathcal{T} such that*

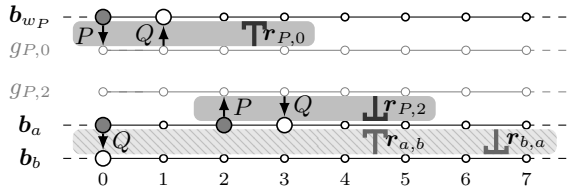
$$\begin{aligned} & A \in \mathbf{b}_a(\ell), \text{ for all } A(a, \ell) \in \mathcal{A}, \\ & \text{if } \exists S \in \mathbf{b}_w(n), \text{ then } \exists S^- \in \mathbf{b}_{w_{S^-}}(k), \text{ for some } k \in \mathbb{Z}, \quad (4) \end{aligned}$$

$$\begin{aligned} & \text{for all } a, b \in \text{ind}(\mathcal{A}), \text{ there is a } \mathbf{b}_a\text{-compatible rod } \mathbf{r} \text{ for } \mathcal{R} \\ & \quad \text{with } S \in \mathbf{r}(\ell), \text{ for all } S(a, b, \ell) \in \mathcal{A}, \quad (5) \end{aligned}$$

$$\begin{aligned} & \exists S \in \mathbf{b}_w(n) \text{ iff there is a } \mathbf{b}_w\text{-compatible rod } \mathbf{r} \text{ for } \mathcal{R} \\ & \quad \text{with } S \in \mathbf{r}(n). \quad (6) \end{aligned}$$

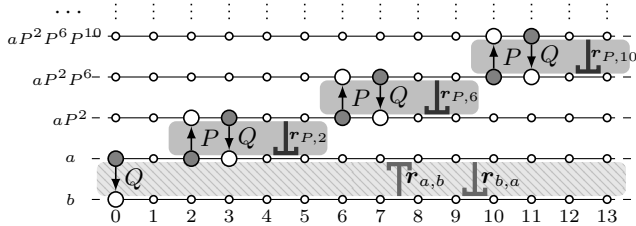
We illustrate the construction by the following example.

Example 1. Let $\mathcal{K} = (\mathcal{O}, \{Q(a, b, 0)\})$, where \mathcal{O} consists of $\exists Q \sqcap \square_P A \sqsubseteq \perp$, $\top \sqsubseteq A \sqcup \exists P$ and $P^- \sqsubseteq \circ_F Q$, obtained by converting $\exists Q \sqsubseteq \diamond_F \exists P$ and $P^- \sqsubseteq \circ_F Q$ into normal form (1). Beams and rods in Lemma 9 are depicted below:



Beams b_a , b_b and b_{w_P} are shown by horizontal lines: the concept type contains $\exists P$ or $\exists Q$ whenever the large node is grey; similarly, the type contains $\exists P^-$ or $\exists Q^-$ whenever the large node is white (the label of the arrow specifies the role); we omit A to avoid clutter. The rods are the arrows between the pairs of horizontal lines. For example, the rod in (5) for a and b is labelled by $r_{a,b}$: it contains only Q at 0 (only the positive components of types are given); the rod in (5) for b and a is labelled by $r_{b,a}$, and in this case, it is the mirror image of $r_{a,b}$. In fact, if we choose \mathcal{R} -canonical rods in (5), then the rod for any b, a will be the mirror image of the rod for a, b . The rod $r_{P,2}$ required by (6) for $\exists P$ on b_a at moment 2 is depicted between b_a and $g_{P,2}$: it contains P at 2 and Q at 3. In fact, it should be clear that, if we choose canonical \mathcal{R} -rods in (6), then they will all be isomorphic copies of at most $|\mathcal{R}|$ -many rods: more precisely, they will be of the form $r_{\{S(d,e,n)\}}$, for a role S from \mathcal{R} .

In the proof of Lemma 9, we show how this collection of beams and \mathcal{R} -canonical rods can be used to obtain a model \mathcal{I} of \mathcal{K} shown below (again, A is omitted):



We now reduce the existence of the required collection of beams to the satisfiability problem for the one-variable first-order *LTL* and thus establish decidability and the upper complexity bound for $DL\text{-Lite}_{bool/horn}^{\square\circ}$, which turns out to be tight.

Theorem 10. *The satisfiability problem for $DL\text{-Lite}_{bool/horn}^{\square\circ}$ KBs is EXPSPACE-complete.*

Proof. We first show the upper bound. Let $\mathcal{K} = (\mathcal{O}, \mathcal{A})$ be a $DL\text{-Lite}_{bool/horn}^{\square\circ}$ KB with $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. We assume that \mathcal{R} is closed under taking the inverses of roles in RIs.

We define a translation $\psi_{\mathcal{K}}$ of \mathcal{K} into first-order *LTL* with a single individual variable x . We treat elements of Δ as constants in the first-order language, basic concepts B as unary predicates and roles P as binary predicates, assuming that $P_i^-(u, x) = P_i(x, u)$, and let $\psi_{\mathcal{K}}$ be a conjunction of the following sentences, for all constants $u \in \Delta$:

$$\square(C_1(u) \wedge \dots \wedge C_k(u) \rightarrow C_{k+1}(u) \vee \dots \vee C_{k+m}(u)), \quad (7)$$

for CI $C_1 \sqcap \dots \sqcap C_k \sqsubseteq C_{k+1} \sqcup \dots \sqcup C_{k+m}$ in \mathcal{T} ,

$$\square \forall x (R_1(u, x) \wedge \dots \wedge R_k(u, x) \rightarrow R(u, x)),$$

$$\text{for RI } R_1 \sqcap \dots \sqcap R_k \sqsubseteq R \text{ in } \mathcal{R},$$

and similarly with \perp for RI $R_1 \sqcap \dots \sqcap R_k \sqsubseteq \perp$ in \mathcal{R} ,

$$\circ_F^\ell A(a), \text{ for } A(a, \ell) \in \mathcal{A}, \quad \circ_F^\ell P(a, b), \text{ for } P(a, b, \ell) \in \mathcal{A},$$

$$\square[(\exists S)(u) \rightarrow \diamond_F \diamond_P (\exists S^-)(w_{S^-})], \text{ for } S \in \text{role}^\pm(\mathcal{O}), \quad (8)$$

$$\square[(\exists S)(u) \leftrightarrow \exists x S(u, x)], \text{ for } S \in \text{role}^\pm(\mathcal{O}). \quad (9)$$

It can be seen that each collection of beams b_u , $u \in \Delta$, for \mathcal{T} gives rise to a model \mathfrak{M} of $\psi_{\mathcal{K}}$: the domain of \mathfrak{M} comprises Δ and the $g_{S,m}$, for a role S and $m \in \mathbb{Z}$. Then, take \mathcal{R} -canonical rods $r_{a,b}$ for $\{S(d, e, \ell) \mid S(a, b, \ell) \in \mathcal{A}\}$, which exist by (5), and \mathcal{R} -canonical rods $r_{S,m}$ for $\{S(d, e, m)\}$ for every S and $m \in \mathbb{Z}$ with $\exists S \in b_u(m)$, for some $u \in \Delta$, which exist by (6), and set, for all $n \in \mathbb{Z}$, basic concepts B , role names P and roles S' ,

$$\mathfrak{M}, n \models B(u) \text{ iff } B \in b_u(n), \text{ for } u \in \Delta,$$

$$\mathfrak{M}, n \models P(a, b) \text{ iff } P \in r_{a,b}(n), \text{ for } a, b \in \text{ind}(\mathcal{A}),$$

$$\mathfrak{M}, n \models S'(u, g_{S,m}) \text{ iff } S' \in r_{S,m}(n), \text{ for } u \in \Delta,$$

$$m \in \mathbb{Z} \text{ and roles } S \text{ with } \exists S \in b_u(m).$$

It is readily checked that \mathfrak{M} is as required (in Example 1, the $g_{S,m}$ are represented explicitly by grey horizontal lines). Conversely, it can be verified that every model \mathfrak{M} of $\psi_{\mathcal{K}}$ gives rise to the required collection of beams for \mathcal{T} .

The lower bound is established by reduction of the non-halting problem for deterministic Turing machines with exponential tape. \square

We now modify the technique developed above to reduce $DL\text{-Lite}_{bool/core}^{\square\circ}$ to *LTL*. The reduction is based on the following observation. Let \mathcal{R} be a $DL\text{-Lite}_{bool/core}^{\square\circ}$ RBox. Consider the \mathcal{R} -canonical rod r for some $\mathcal{A}_R = \{R(d, e, 0)\}$. Then $S \in r(n)$ iff one of the following conditions holds:

- $\mathcal{R}', \mathcal{A}_R \models S(d, e, n)$, where \mathcal{R}' is obtained from \mathcal{R} by removing the RIs with \square ,
- there is $m > n$ with $|m| \leq 2^{|\mathcal{R}|}$ and $\square_P S \in r(m)$,
- there is $m < n$ with $|m| \leq 2^{|\mathcal{R}|}$ and $\square_F S \in r(m)$.

Let $\min_{R,S}$ be the minimal integer with $\square_F S \in r(m)$; if it exists, then $|\min_{R,S}| \leq 2^{|\mathcal{R}|}$. The maximal integer $\max_{R,S}$ with $\square_P S \in r(m)$ has the same bound (if exists). The following example shows that these integers can indeed be exponential in $|\mathcal{R}|$.

Example 2. Let \mathcal{R} be the following $DL\text{-Lite}_{bool/core}^{\square\circ}$ RBox:

$$P \sqsubseteq R_0, \quad R_i \sqsubseteq \circ_F R_{(i+1) \bmod 2}, \quad \text{for } 0 \leq i < 2,$$

$$P \sqsubseteq Q_0, \quad Q_i \sqsubseteq \circ_F Q_{(i+1) \bmod 3}, \quad \text{for } 0 \leq i < 3,$$

$$R_1 \sqsubseteq Q, \quad Q_1 \sqsubseteq Q, \quad Q_2 \sqsubseteq Q, \quad P \sqsubseteq Q, \quad P \sqsubseteq \square_P Q.$$

Clearly, $\mathcal{R} \models P \sqsubseteq \circ_F^6 \square_P Q$. If instead of the 2- and 3-cycles we use p_i -cycles, where p_i is the i th prime number and $1 \leq i \leq n$, then $\mathcal{R} \models P \sqsubseteq \circ_F^{p_1 \times \dots \times p_n} \square_P Q$.

In any case, the existence and binary representations of $\min_{R,S}$ and $\max_{R,S}$ can be computed in PSPACE.

Theorem 11. *For $DL\text{-Lite}_{bool/core}^{\square\circ}$ and $DL\text{-Lite}_{horn/core}^{\square\circ}$ KBs, the satisfiability problem is PSPACE-complete.*

Proof. We encode \mathcal{K} in *LTL* following the proof of Theorem 10 and representing (7)–(8) as *LTL*-formulas with variables of the form $C^\dagger(u)$, $R^\dagger(u, v)$, for $u, v \in \Delta$. Sentences (9), however, require a different treatment. First, take

$$\Box(\Box_1(\exists S_1)^\dagger(u) \rightarrow \Box_2(\exists S_2)^\dagger(u)), \quad (10)$$

for every $\Box_1 S_1 \sqsubseteq \Box_2 S_2$ in \mathcal{R} , where each \Box_i is \Box_F , \Box_P or blank. Then we need CIs of the form $\exists R \sqsubseteq \Box^{\max_{R,S}} \Box_P \exists S$ and $\exists R \sqsubseteq \Box^{\min_{R,S}} \Box_P \exists S$, for all R and S with defined $\max_{R,S}$ and $\min_{R,S}$, which are not entailed by (10). These integers can be represented in binary using n bits, where n is polynomial in $|\mathcal{R}|$. Assuming that $\max_{R,S} \geq 0$, we encode, for example, $\exists R \sqsubseteq \Box^{\max_{R,S}} \Box_P \exists S$ by

$$\Box(\Box_F \diamond_F (\exists R)^\dagger(u) \rightarrow \Box_F (\exists S)^\dagger(u)), \quad (11)$$

$$\Box((\exists R)^\dagger(u) \wedge \neg \diamond_F (\exists R)^\dagger(u) \rightarrow \Box_F^{\max_{R,S}} D_u^{R,S}), \quad (12)$$

$$\Box(D_u^{R,S} \rightarrow \Box_P (\exists S)^\dagger(u)), \quad (13)$$

where (12) is expressed by $O(n^2)$ formulas encoding the binary counter (similar to those in the proof of Theorem 8). To explain the meaning of (11)–(13), consider any $w \in \Delta^{\mathcal{I}}$ in a model \mathcal{I} of \mathcal{K} . If $w \in (\exists R)^{\mathcal{I}(n)}$ for infinitely many $n > 0$, then $w \in (\exists S)^{\mathcal{I}(n)}$ for all n , which is captured by (11). Otherwise, there is n such that $w \in (\exists R)^{\mathcal{I}(n)}$ and $w \notin (\exists R)^{\mathcal{I}(m)}$ for all $m > n$, whence $w \in (\exists S)^{\mathcal{I}(k)}$, for any $k < n + \max_{R,S}$, which is captured by (12) and (13).

The *LTL* translation $\Psi_{\mathcal{K}}$ of \mathcal{K} is a conjunction of (7)–(8), (10) and (11)–(13) for all R and S with defined $\max_{R,S}$, and their counterparts for $\exists R \sqsubseteq \Box^{\min_{R,S}} \Box_P \exists S$. One can show that \mathcal{K} is satisfiable iff $\Psi_{\mathcal{K}}$ is satisfiable. The PSPACE lower bound follows from the fact that every *LTL*-formula is equisatisfiable with some *LTL*_{core}[□] KB. \square

5 FO(RPR)-Rewritability of *DL-Lite*_{bool/horn}[□]

We next investigate the data complexity of the satisfiability problem for temporal *DL-Lite* KBs. Again, our first result is negative; it is proved using Theorem 6 and a representation of the universal Turing machine by a set of tiles.

Theorem 12. *There is a *DL-Lite*_{g-bool}[□] ontology \mathcal{O} for which the satisfiability of $(\mathcal{O}, \mathcal{A})$, for a given \mathcal{A} , is undecidable.*

We obtain our positive results using FO-rewritability. Let $\mathcal{L} \in \{\text{FO}(<), \text{FO}(<, \equiv_{\mathbb{N}}), \text{FO}(\text{RPR})\}$. Our first aim is to show that \mathcal{L} -rewritability of *DL-Lite*_{bool/horn}[□] ontologies can be reduced to \mathcal{L} -rewritability of *ontology-mediated atomic queries* (or OMAQs) with *LTL* ontologies.

In general, by an OMAQ q we mean a pair of the form (\mathcal{O}, A) or (\mathcal{O}, P) , where \mathcal{O} is an ontology, A a concept and P a role name. A *certain answer* to (\mathcal{O}, A) over an ABox \mathcal{A} is any $(a, \ell) \in \text{ind}(\mathcal{A}) \times \text{tem}(\mathcal{A})$ such that $a^{\mathcal{I}} \in A^{\mathcal{I}(\ell)}$ for every model \mathcal{I} of $(\mathcal{O}, \mathcal{A})$; a *certain answer* to (\mathcal{O}, P) over \mathcal{A} is any $(a, b, \ell) \in \text{ind}(\mathcal{A})^2 \times \text{tem}(\mathcal{A})$ with $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in P^{\mathcal{I}(\ell)}$ for every $\mathcal{I} \models (\mathcal{O}, \mathcal{A})$. An \mathcal{L} -rewriting of (\mathcal{O}, A) is an \mathcal{L} -formula $\Phi(x, t)$ such that (a, ℓ) is a certain answer to (\mathcal{O}, A) over any ABox \mathcal{A} iff $\mathfrak{S}_{\mathcal{A}} \models \Phi(a, \ell)$; an \mathcal{L} -rewriting of (\mathcal{O}, P) is defined similarly.

First, we show how to reduce the satisfiability problem for *DL-Lite*_{bool/horn}[□] ontologies \mathcal{O} to answering OMAQs

$(\mathcal{O}', A_{\perp})$ with a \perp -free ontology \mathcal{O}' and a concept name A_{\perp} . More precisely, for any ABox \mathcal{A} , the KB $(\mathcal{O}, \mathcal{A})$ is satisfiable iff $(\mathcal{O}', A_{\perp})$ has no certain answers over \mathcal{A} .

Let $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. We define $\mathcal{O}' = \mathcal{T}' \cup \mathcal{R}'$ as follows. The RBox \mathcal{R}' is obtained by replacing every occurrence of \perp in \mathcal{R} with a fresh role name P_{\perp} and adding the RI $P \sqsubseteq P_{\perp}$, for any P inconsistent with \mathcal{O} in the sense that $(\mathcal{O}, \{P(a, b, 0)\})$ has no models. The TBox \mathcal{T}' results from replacing every \perp in \mathcal{T} with a fresh concept A_{\perp} and adding the CIs $\exists P_{\perp} \sqsubseteq A_{\perp}$, $\exists P_{\perp}^- \sqsubseteq A_{\perp}$ together with $A_{\perp} \sqsubseteq \Box_F A_{\perp}$ and $A_{\perp} \sqsubseteq \Box_P A_{\perp}$ saying that A_{\perp} is global: if $u \in A_{\perp}^{\mathcal{I}(n)}$ for some $n \in \mathbb{Z}$, then $u \in A_{\perp}^{\mathcal{I}(n)}$ for all $n \in \mathbb{Z}$. By Theorem 10, \mathcal{O}' can be constructed in exponential space.

Theorem 13. *If $\Phi_{\perp}(x, t)$ is an \mathcal{L} -rewriting of the OMAQ $(\mathcal{O}', A_{\perp})$, then $\exists x, t \Phi_{\perp}(x, t)$ is an \mathcal{L} -rewriting of \mathcal{O} .*

Next, we show that \mathcal{L} -rewritability of a \perp -free OMAQ with an *DL-Lite*_{bool/horn}[□] ontology is reducible to \mathcal{L} -rewritability of a role-free OMAQ. Ontologies without roles are clearly a notational variant of *LTL* ontologies; hence, in this case we prefer to write '*LTL*_{bool}[□] ontologies'. We explain the reduction by instructive examples. The first two examples illustrate the interaction between the DL and temporal dimensions in *DL-Lite*_{bool/horn}[□] that we need to take into account when constructing the *LTL* OMAQs to which the rewritability of \perp -free *DL-Lite*_{bool/horn}[□] OMAQs is reduced.

Example 3. Suppose $\mathcal{T} = \{B \sqsubseteq \exists P, \exists Q \sqsubseteq A\}$ and $\mathcal{R} = \{P \sqsubseteq \Box_F Q\}$. An obvious idea of constructing a rewriting for the OMAQ $q = (\mathcal{T} \cup \mathcal{R}, A)$ would be to find first a rewriting of the *LTL* OMAQ $(\mathcal{T}^\dagger, A^\dagger)$ obtained from (\mathcal{T}, A) by replacing the basic concepts $\exists P$ and $\exists Q$ with surrogate concept names $(\exists P)^\dagger = E_P$ and $(\exists Q)^\dagger = E_Q$, respectively. This would give us the FO-query $A(x, t) \vee E_Q(x, t)$. By restoring the intended meaning of E_Q , we would then obtain $A(x, t) \vee \exists y Q(x, y, t)$. The second step would be to rewrite, using the RBox \mathcal{R} , the atom $Q(x, y, t)$ into $Q(x, y, t) \vee P(x, y, t - 1)$. However, the resulting formula

$$A(x, t) \vee \exists y (Q(x, y, t) \vee P(x, y, t - 1))$$

is not a rewriting of q : it does not return the certain answer $(a, 1)$ over $\mathcal{A} = \{B(a, 0), C(a, 1)\}$ because so far we have not taken into account the CI $\exists P \sqsubseteq \Box_F \exists Q$, which is a consequence of \mathcal{R} . If we now add the 'connecting axiom' $(\exists P)^\dagger \sqsubseteq \Box_F (\exists Q)^\dagger$ to \mathcal{T}^\dagger , then in the first step we obtain $A(x, t) \vee E_Q(x, t) \vee E_P(x, t - 1) \vee B(x, t - 1)$, which gives us the correct FO($<$)-rewriting of q :

$$A(x, t) \vee \exists y (Q(x, y, t) \vee P(x, y, t - 1)) \vee \exists y P(x, y, t - 1) \vee B(x, t - 1).$$

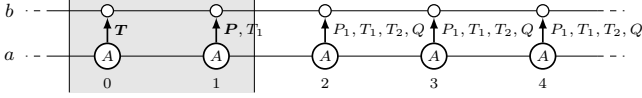
Example 4. Let $q = (\mathcal{T} \cup \mathcal{R}, A)$ with $\mathcal{T} = \{\exists Q \sqsubseteq \Box_P A\}$, $\mathcal{R} = \{P \sqsubseteq \Box_F P_1, T \sqsubseteq \Box_F T_1, T_1 \sqsubseteq \Box_F T_2, P_1 \sqcap T_2 \sqsubseteq Q\}$. The two-step construction outlined in Example 3 would first give us the formula

$$\Phi(x, t) = A(x, t) \vee \exists t' ((t < t') \wedge \exists y Q(x, y, t'))$$

as a rewriting of (\mathcal{T}, A) . It is readily checked that the following formula $\Psi(x, y, t')$ is a rewriting of (\mathcal{R}, Q) :

$$Q(x, y, t') \vee ([P_1(x, y, t') \vee \exists t'' ((t'' < t') \wedge P(x, y, t''))] \wedge [T_2(x, y, t') \vee \exists t'' ((t'' < t') \wedge (T_1(x, y, t'') \vee \exists t''' ((t''' < t'') \wedge T(x, y, t''')))]).$$

However, the result of replacing $Q(x, y, t')$ in $\Phi(x, t)$ with $\Psi(x, y, t')$ is not an FO($<$)-rewriting of (\mathcal{O}, A) : when evaluated over $\mathcal{A} = \{T(a, b, 0), P(a, b, 1)\}$, it does not return the certain answers $(a, 0)$ and $(a, 1)$; see the picture below:



(Note that these answers would be found had we evaluated the obtained ‘rewriting’ over \mathbb{Z} rather than $\{0, 1\}$.) This time, we miss the CI $\exists(\square_F P_1 \sqcap \square_F T_2) \sqsubseteq \square_F \exists Q$, which follows from \mathcal{R} and \mathcal{T} . To fix the problem, we can take a fresh role name G_ρ , for $\rho = \{\square_F P_1, \square_F T_2\}$, and add the ‘connecting axiom’ $\exists G_\rho \sqsubseteq \square_F \exists Q$ to \mathcal{T} . Then we rewrite the extended TBox and A into $\Phi'(x, t)$ defined as follows:

$$A(x, t) \vee \exists t' ((t < t') \wedge \exists y Q(x, y, t')) \vee \exists t' \exists y G_\rho(x, y, t'),$$

where we replace $Q(x, y, t')$ by $\Psi(x, y, t')$ and restore the meaning of $G_\rho(x, y, t')$ by rewriting $(\mathcal{R}, \square_F P_1 \sqcap \square_F T_2)$ to $P(x, y, t') \wedge (T_1(x, y, t') \vee \exists t'' ((t'' < t') \wedge T(x, y, t'')))$ and substituting it for $G_\rho(x, y, t')$ in $\Phi'(x, t)$.

We now formally define the connecting axioms for \mathcal{O} , assuming that \mathcal{R} contains all role names in \mathcal{T} . Let ρ be a set of (temporalised) roles from \mathcal{R} consistent with \mathcal{R} . Denote by \mathbf{r}_ρ the \mathcal{R} -canonical rod for $\{S(d, e, 0) \mid S \in \rho\}$. By the well-known properties of *LTL*, there are positive integers $s^\rho \leq |\mathcal{R}|$ and $p^\rho \leq 2^{2|\mathcal{R}|}$ with

$$\begin{aligned} \mathbf{r}_\rho(n) &= \mathbf{r}_\rho(n - p^\rho), & \text{for } n \leq -s^\rho, \\ \mathbf{r}_\rho(n) &= \mathbf{r}_\rho(n + p^\rho), & \text{for } n \geq s^\rho. \end{aligned}$$

Then we take a fresh role name G_ρ and fresh concept names D_ρ^n , for $-s^\rho - p^\rho < n < s^\rho + p^\rho$, and construct the CIs

$$\begin{aligned} \exists G_\rho \sqsubseteq D_\rho^0, \quad D_\rho^n \sqsubseteq \square_F D_\rho^{n+1}, \quad \text{for } 0 \leq n < s^\rho + p^\rho - 1, \\ D_\rho^{s^\rho + p^\rho - 1} \sqsubseteq \square_F D_\rho^{s^\rho}, \\ D_\rho^n \sqsubseteq \exists S, \quad \text{for } S \in \mathbf{r}_\rho(n), \quad 0 \leq n < s^\rho + p^\rho, \end{aligned}$$

and symmetrical ones for $-s^\rho - p^\rho \leq n \leq 0$. Let **(con)** be the set of such CIs for all possible ρ . Set $\mathcal{T}_\mathcal{R} = \mathcal{T} \cup \mathbf{(con)}$.

Example 5. In Example 3, for $\rho = \{P, \square_F Q\}$, we have $s^\rho = 2, p^\rho = 1$, and so $\mathcal{T}_\mathcal{R}$ contains the CIs

$$\begin{aligned} \exists P \sqsubseteq D_\rho^0, \quad D_\rho^0 \sqsubseteq \square_F D_\rho^1, \quad D_\rho^1 \sqsubseteq \square_F D_\rho^2, \\ D_\rho^2 \sqsubseteq \square_F D_\rho^2, \quad D_\rho^0 \sqsubseteq \exists P, \quad D_\rho^1 \sqsubseteq \exists Q, \end{aligned}$$

which imply $\exists P \sqsubseteq \square_F \exists Q$. In the context of Example 4, for $\rho = \{\square_F P_1, \square_F T_2\}$, we have $s^\rho = 1, p^\rho = 1$, and so $\mathcal{T}_\mathcal{R}$ contains the following CIs:

$$\begin{aligned} \exists G_\rho \sqsubseteq D_\rho^0, \quad D_\rho^0 \sqsubseteq \square_F D_\rho^1, \quad D_\rho^1 \sqsubseteq \square_F D_\rho^1, \\ D_\rho^1 \sqsubseteq \exists P_1, \quad D_\rho^1 \sqsubseteq \exists T_2, \quad D_\rho^1 \sqsubseteq \exists Q. \end{aligned}$$

We denote by $\mathcal{T}_\mathcal{R}^\dagger$ the $LTL_{bool}^{\square, \circ}$ TBox obtained from $\mathcal{T}_\mathcal{R}$ by replacing every basic concept B with its surrogate B^\dagger .

Theorem 14. A *DL-Lite* $_{bool/horn}^{\square, \circ}$ OMAQ (\mathcal{O}, A) with a \perp -free $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ is \mathcal{L} -rewritable whenever

- the $LTL_{bool}^{\square, \circ}$ OMAQ $(\mathcal{T}_\mathcal{R}^\dagger, A)$ is \mathcal{L} -rewritable and
- the $LTL_{horn}^{\square, \circ}$ OMAQ (\mathcal{R}, R) is \mathcal{L} -rewritable, for every temporalised role in \mathcal{R} .

As a first consequence of Theorems 13 and 14, we obtain:

Theorem 15. Every *DL-Lite* $_{bool/horn}^{\square, \circ}$ ontology is FO(RPR)-rewritable.

Note that, as follows from (Artale et al. 2015, Theorem 9), satisfiability of $LTL_{horn}^{\square, \circ}$ KBs is NC¹-hard for data complexity, and so satisfiability of *DL-Lite* $_{bool/horn}^{\square, \circ}$ ontologies is NC¹-complete.

6 FO($<, \equiv_{\mathbb{N}}$)-Rewritability of *DL-Lite* $_{krom/core}^{\square, \circ}$

If $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ is a *DL-Lite* $_{krom/core}^{\square, \circ}$ ontology, then the TBox $\mathcal{T}_\mathcal{R}$ constructed above is in *DL-Lite* $_{krom/core}^{\square, \circ}$, and so, by Theorem 14, we can show \mathcal{L} -rewritability of \mathcal{O} by establishing \mathcal{L} -rewritability of every $LTL_{krom}^{\square, \circ}$ OMAQ. It is known from (Artale et al. 2020) that $LTL_{krom}^{\square, \circ}$ OMAQs are FO($<, \equiv_{\mathbb{N}}$)-rewritable. Here we establish FO($<, \equiv_{\mathbb{N}}$)-rewritability of all $LTL_{krom}^{\square, \circ}$ OMAQs. The proof utilises the monotonicity of the \square operators, similarly to the proof of (Artale et al. 2020, Theorem 11). However, the latter relies on partially-ordered NFAs accepting the models of (\mathcal{O}, A) , which do not work in the presence of \circ . Our key observation here is that every model of (\mathcal{O}, A) has at most $O(|\mathcal{O}|)$ timestamps such that the same \square -concepts hold between any two nearest of them. The placement of these timestamps and their concept types can be described by an FO($<$)-formula. However, to check whether these types are compatible (i.e., satisfiable in some model), we require FO($<, \equiv_{\mathbb{N}}$)-formulas similar to those in the proof of (Artale et al. 2020, Theorem 14).

Theorem 16. $LTL_{krom}^{\square, \circ}$ OMAQs are FO($<, \equiv_{\mathbb{N}}$)-rewritable.

Proof. Let $\mathbf{q} = (\mathcal{O}, A)$ be an $LTL_{krom}^{\square, \circ}$ OMAQ. We can assume that A occurs in \mathcal{O} , which has no nested occurrences of temporal operators and contains CIs $\circ B \equiv A \circ B$, for every $\circ B$ in \mathcal{O} with $\circ \in \{\square_F, \square_P\}$. Define an NFA $\mathfrak{A}_\mathcal{O}$ that recognises ABoxes \mathcal{A} consistent with \mathcal{O} , represented as words $X_{\min \mathcal{A}}, \dots, X_{\max \mathcal{A}}$, where

$$X_i = \{B \mid B(i) \in A \text{ and } B \text{ occurs in } \mathcal{O}\}, \quad i \in \text{tem}(\mathcal{A}).$$

The set \mathfrak{T} of states in $\mathfrak{A}_\mathcal{O}$ comprises maximal sets τ of concepts of \mathcal{O} consistent with \mathcal{O} ; we refer to such τ as *types for* \mathcal{O} . Now, for any $\tau, \tau' \in \mathfrak{T}$ and an alphabet symbol X , the NFA $\mathfrak{A}_\mathcal{O}$ has a transition $\tau \rightarrow_X \tau'$ just in case the following conditions hold: (i) $X \subseteq \tau'$, (ii) $\square_F C \in \tau$ iff $C \in \tau'$, (iii) $\square_P C \in \tau$ iff $C, \square_P C \in \tau'$, and their past counterparts. As $\tau \rightarrow_X \tau'$ implies $\tau \rightarrow_\emptyset \tau'$, for any X , we omit \emptyset from \rightarrow_\emptyset . Since all τ in \mathfrak{T} are consistent with \mathcal{O} , every state in $\mathfrak{A}_\mathcal{O}$ has a \rightarrow -predecessor and a \rightarrow -successor. Thus, for any ABox \mathcal{A} represented as X_0, X_1, \dots, X_m , a timestamp ℓ ($0 \leq \ell \leq m$) is not a certain answer to \mathbf{q} over \mathcal{A} iff there is a path

$$\pi = \tau_{-1} \rightarrow_{X_0} \tau_0 \rightarrow_{X_1} \tau_1 \rightarrow_{X_2} \dots \rightarrow_{X_m} \tau_m,$$

in $\mathfrak{A}_{\mathcal{O}}$ with $A \notin \tau_{\ell}$. This criterion can be encoded by an infinite FO-expression $\Psi(t)$ of the form

$$\neg \left[\bigvee_{\substack{\tau_0 \rightarrow \dots \rightarrow \tau_m \\ \text{is a path in } \mathfrak{A}_{\mathcal{O}}}} \left(\bigwedge_{0 \leq i \leq m} \text{type}_{\tau_i}(i) \wedge \bigvee_{0 \leq i \leq m \text{ with } A \notin \tau_i} (t = i) \right) \right],$$

where the disjunction is over all (possibly infinitely many) paths and $\text{type}_{\tau}(t)$ is a conjunction of all $\neg B(t)$ with $B \notin \tau$, for concept names B in \mathcal{O} : the first conjunct ensures, by contraposition, that any B from X_i also belongs to τ_i , while the second conjunct guarantees that $A \notin \tau_{\ell}$ in case $\ell = t$.

We write $\tau \rightarrow^{\square} \tau'$ if τ and τ' satisfy (iii), but not necessarily (ii). One can show that any path $\tau_0 \rightarrow \dots \rightarrow \tau_m$ in $\mathfrak{A}_{\mathcal{O}}$ contains a subsequence

$$\tau_{s_0} \rightarrow^{\square} \tau_{s_1} \rightarrow^{\square} \dots \rightarrow^{\square} \tau_{s_{d-1}} \rightarrow^{\square} \tau_{s_d}$$

such that $0 = s_0 < s_1 < \dots < s_{d-1} < s_d = m$ for $d \leq 2|\mathcal{O}| + 1$ and, for all $i < d$, either $\square C, C \in \tau_{s_i}, \tau_j$ or $\square C \notin \tau_{s_i}, \tau_j$, for all $\square C$ in \mathcal{O} , $\square \in \{\square_P, \square_F\}$, and all $j \in (s_i, s_{i+1})$.

To deal with the \circ -operators, we consider the LTL_{krom}° ontology $\tilde{\mathcal{O}}$ obtained from \mathcal{O} by first extending it with the CIs $\square_F C \sqsubseteq \square_P \square_F C$ and $\square_F C \sqsubseteq \square_F C$ for all $\square_F C$ in \mathcal{O} and their past counterparts, which are obvious $LTL_{krom}^{\square \circ}$ tautologies, and then replacing every $\square_P C$ and $\square_F C$ with its *surrogate*, a fresh concept name. Let $G_{\tilde{\mathcal{O}}}$ be the infinite directed graph whose vertices are pairs (L, n) , for a simple literal L (a concept name or its negation) in $\tilde{\mathcal{O}}$ and $n \in \mathbb{Z}$. It contains an edge from (L, n) to $(L', n+k)$, for $k \in \{-1, 0, 1\}$, iff $\tilde{\mathcal{O}} \models L \sqsubseteq \circ^k L'$, where \circ^k denotes \circ_F^k if $k \geq 0$ and \circ_P^{-k} if $k < 0$. We write $(L_1, n_1) \rightsquigarrow (L_2, n_2)$ if $G_{\tilde{\mathcal{O}}}$ has a path from (L_1, n_1) to (L_2, n_2) , which means that $\tilde{\mathcal{O}} \models \circ^{n_1} L_1 \sqsubseteq \circ^{n_2} L_2$. We slightly abuse notation and write, for example, $L \in \tau$ for a type τ in case L is the surrogate for $\square_P C$ and τ contains $\square_P C$.

Lemma 17. *For any ABox \mathcal{A} , a timestamp $\ell \in \text{tem}(\mathcal{A})$ is not a certain answer to q over \mathcal{A} iff there are $d \leq 2|\mathcal{O}| + 2$, a sequence $\tau_0 \rightarrow^{\square} \dots \rightarrow^{\square} \tau_d$ of types for \mathcal{O} and a sequence $\min \mathcal{A} = s_0 < \dots < s_d = \max \mathcal{A}$ such that*

- $B \in \tau_i$, for all $B(s_i) \in \mathcal{A}$;
- $(B, n) \not\rightsquigarrow (\neg B', n')$, for $s_i < n, n' < s_{i+1}$ with $B(n), B'(n') \in \mathcal{A}$;
- $(L, s_i) \not\rightsquigarrow (\neg B', n')$, for $L \in \tau_i$ and $s_i < n' < s_{i+1}$ with $B'(n') \in \mathcal{A}$;
- $(B, n) \not\rightsquigarrow (\neg L', s_{i+1})$, for $s_i < n < s_{i+1}$ with $B(n) \in \mathcal{A}$ and $L' \in \tau_{i+1}$;
- $(L, s_i) \not\rightsquigarrow (\neg L', s_{i+1})$, for $L \in \tau_i$ and $L' \in \tau_{i+1}$;
- $\ell = s_i$, for some i ($0 \leq i \leq d$) such that $A \notin \tau_i$.

We can now define an $\text{FO}(<, \equiv_{\mathbb{N}})$ -rewriting $Q(t)$ of q by encoding the conditions of Lemma 17 as follows:

$$Q(t) = \neg \left[\bigvee_{d \leq 2|\mathcal{O}| + 2} \bigvee_{\tau_0 \rightarrow^{\square} \dots \rightarrow^{\square} \tau_d} \exists t_0, \dots, t_d \left(\text{path}_{\tau_0 \rightarrow^{\square} \dots \rightarrow^{\square} \tau_d}(t_0, \dots, t_d) \wedge \bigvee_{0 \leq i \leq d \text{ with } A \notin \tau_i} (t = t_i) \right) \right],$$

where $\text{path}_{\tau_0 \rightarrow^{\square} \dots \rightarrow^{\square} \tau_d}(t_0, \dots, t_d)$ is the formula

$$\begin{aligned} & (t_0 = \min) \wedge (t_d = \max) \wedge \bigwedge_{0 \leq i < d} (t_i < t_{i+1}) \wedge \bigwedge_{0 \leq i \leq d} \text{type}_{\tau_i}(t_i) \\ & \wedge \bigwedge_{0 \leq i < d} \left[\bigwedge_{L \in \tau_i, L' \in \tau_{i+1}} \neg \text{ent}_{L, \neg L'}(t_i, t_{i+1}) \right. \\ & \quad \wedge \bigwedge_{L \in \tau_i} \forall t' \in (t_i, t_{i+1}) (B'(t') \rightarrow \neg \text{ent}_{L, \neg B'}(t_i, t')) \\ & \quad \wedge \bigwedge_{L' \in \tau_{i+1}} \forall t \in (t_i, t_{i+1}) (B(t) \rightarrow \neg \text{ent}_{B, \neg L'}(t, t_{i+1})) \\ & \quad \left. \wedge \bigwedge_{B, B' \text{ in } \tilde{\mathcal{O}}} \forall t, t' \in (t_i, t_{i+1}) (B(t) \wedge B'(t') \rightarrow \neg \text{ent}_{B, \neg B'}(t, t')) \right] \end{aligned}$$

and where ent_{L_1, L_2} is such that $\mathfrak{S}_{\mathcal{A}} \models \text{ent}_{L_1, L_2}(n_1, n_2)$ iff $\tilde{\mathcal{O}} \models \circ^{n_1} L_1 \sqsubseteq \circ^{n_2} L_2$, for any $n_1, n_2 \in \text{tem}(\tilde{\mathcal{A}})$; see (Artale et al. 2020, Theorem 14). Note that the outermost disjunction in $Q(t)$ can be empty, in particular when \mathcal{O} is inconsistent, in which case the rewriting $Q(t)$ is simply \top . \square

As a consequence of Theorems 13, 14 and 16, we obtain:

Theorem 18. *$DL\text{-Lite}_{krom/core}^{\square \circ}$ ontologies are $\text{FO}(<, \equiv_{\mathbb{N}})$ -rewritable.*

7 Conclusions

We extended the *DL-Lite* family of description logics by languages with Krom, Horn and arbitrary Boolean role inclusions and identified their computational complexity. We observed, in particular, that Boolean RIs make *DL-Lite* as expressive as FO^2 , while covering Krom RIs $\top \sqsubseteq R_1 \sqcup R_2$ come for free as far as satisfiability is concerned.

We used those languages as a basis for defining a new type of temporal DLs. So far the main approach to designing well-behaved fragments of first-order temporal logic has been the monotonicity principle, which disallows temporal operators before a formula with two or more free variables. The main contribution of this paper is to show that by restricting the use of classical connectives one can obtain natural and decidable fragments whose expressivity for binary relations is not captured by the monotonicity principle.

Interesting directions of future work include establishing the tight combined complexity of $DL\text{-Lite}_{horn/krom}^{\square \circ}$ and the data complexity of *DL-Lite* with Krom RIs. We also plan to investigate the problem of answering queries mediated by ontologies in our temporal languages. Answering unions of conjunctive queries (UCQs) is undecidable with $DL\text{-Lite}_{krom}^{\square \circ}$ ontologies (Rosati 2007) and $2\text{EXP}\text{TIME}$ -complete for $DL\text{-Lite}_{bool}^{g\text{-}bool}$ (Bárány, Gottlob, and Otto 2014; Bourhis et al. 2016; Hernich 2020). UCQs with $DL\text{-Lite}_{horn}^{\square \circ}$ ontologies are $\text{FO}(<)$ -rewritable; with $DL\text{-Lite}_{bool}^{g\text{-}bool}$ ontologies they are CONP -complete for data complexity. Temporal instance queries are $\text{FO}(<)$ -rewritable for $DL\text{-Lite}_{core}^{\square \circ}$ and $\text{FO}(<, \equiv_{\mathbb{N}})$ -rewritable for $DL\text{-Lite}_{core}^{\circ}$ (Artale et al. 2015).

Acknowledgements

This work was supported by the EPSRC U.K. grants EP/S032207 and EP/S032282.

References

- Artale, A.; Calvanese, D.; Kontchakov, R.; and Zakharyashev, M. 2009. The DL-Lite family and relations. *Journal of Artificial Intelligence Research (JAIR)* 36:1–69.
- Artale, A.; Kontchakov, R.; Ryzhikov, V.; and Zakharyashev, M. 2013. The complexity of clausal fragments of LTL. In *Proc. of the 19th Int. Conf. on Logic for Programming, Artificial Intelligence and Reasoning (LPAR)*, volume 8312 of *LNCS*, 35–52. Springer.
- Artale, A.; Kontchakov, R.; Ryzhikov, V.; and Zakharyashev, M. 2014. A cookbook for temporal conceptual data modelling with description logics. *ACM Trans. Comput. Logic* 15(3).
- Artale, A.; Kontchakov, R.; Kovtunova, A.; Ryzhikov, V.; Wolter, F.; and Zakharyashev, M. 2015. First-order rewritability of temporal ontology-mediated queries. In *Proc. of the 24th Int. Joint Conf. on Artificial Intelligence, IJCAI 2015*, 2706–2712. AAAI Press.
- Artale, A.; Kontchakov, R.; Kovtunova, A.; Ryzhikov, V.; Wolter, F.; and Zakharyashev, M. 2020. First-order rewritability of ontology-mediated queries in linear temporal logic. *CoRR* abs/2004.07221.
- Baader, F.; Horrocks, I.; Lutz, C.; and Sattler, U. 2017. *An Introduction to Description Logic*. Cambridge University Press.
- Baader, F.; Ghilardi, S.; and Lutz, C. 2012. LTL over description logic axioms. *ACM Trans. Comput. Logic* 13(3):21.
- Bárány, V.; Gottlob, G.; and Otto, M. 2014. Querying the guarded fragment. *Logical Methods in Computer Science* 10(2).
- Berger, R. 1966. *The Undecidability of the Domino Problem*. American Mathematical Society.
- Börger, E.; Grädel, E.; and Gurevich, Y. 1997. *The Classical Decision Problem*. Perspectives in Mathematical Logic. Springer.
- Borgwardt, S.; Lippmann, M.; and Thost, V. 2015. Temporalizing rewritable query languages over knowledge bases. *Journal of Web Semantics* 33:50–70.
- Bourhis, P.; Manna, M.; Morak, M.; and Pieris, A. 2016. Guarded-based disjunctive tuple-generating dependencies. *ACM Trans. Database Syst.* 41(4):27:1–27:45.
- Calvanese, D.; De Giacomo, G.; Lembo, D.; Lenzerini, M.; and Rosati, R. 2007. Tractable reasoning and efficient query answering in description logics: the *DL-Lite* family. *Journal of Automated Reasoning* 39(3):385–429.
- Compton, K. J., and Laflamme, C. 1990. An algebra and a logic for NC^1 . *Inf. Comput.* 87(1/2):240–262.
- Fisher, M.; Dixon, C.; and Peim, M. 2001. Clausal temporal resolution. *ACM Trans. Comput. Logic* 2(1):12–56.
- Gabbay, D.; Kurucz, A.; Wolter, F.; and Zakharyashev, M. 2003. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic*. Elsevier.
- Gargov, G., and Passy, S. 1990. A note on Boolean modal logic. In *Mathematical Logic*. Springer, Boston, MA.
- Grädel, E.; Kolaitis, P. G.; and Vardi, M. Y. 1997. On the decision problem for two-variable first-order logic. *Bulletin of Symbolic Logic* 3(1):53–69.
- Grädel, E. 1999. On the restraining power of guards. *J. Symb. Log.* 64(4):1719–1742.
- Gutiérrez-Basulto, V.; Jung, J. C.; and Kontchakov, R. 2016. Temporalized EL ontologies for accessing temporal data: Complexity of atomic queries. In *Proc. of the 25th Int. Joint Conf. on Artificial Intelligence, IJCAI 2016*, 1102–1108. IJCAI/AAAI Press.
- Halpern, J. Y., and Vardi, M. Y. 1989. The complexity of reasoning about knowledge and time. I. Lower bounds. *J. Comput. Syst. Sci.* 38(1):195–237.
- Hernich, A.; Lutz, C.; Papacchini, F.; and Wolter, F. 2020. Dichotomies in Ontology-Mediated Querying with the Guarded Fragment. *ACM Trans. Comput. Logic* 20:1–47.
- Hustadt, U., and Schmidt, R. A. 1998. Issues of decidability for description logics in the framework of resolution. In *Automated Deduction in Classical and Non-Classical Logics, Selected Papers*, volume 1761 of *LNCS*, 191–205. Springer.
- Immerman, N. 1999. *Descriptive Complexity*. Springer.
- Lutz, C., and Sattler, U. 2000a. The complexity of reasoning with Boolean modal logics. In *Advances in Modal Logic 3*, 329–348. World Scientific.
- Lutz, C., and Sattler, U. 2000b. Mary likes all cats. In *Proc. of the 2000 Int. Workshop on Description Logics DL 2000, CEUR Workshop Proceedings*, 213–226. CEUR-WS.
- Lutz, C.; Sattler, U.; and Wolter, F. 2001. Modal logic and the two-variable fragment. In *Proc. of the 15th Int. Workshop on Computer Science Logic, CSL 2001*, volume 2142 of *LNCS*, 247–261. Springer.
- Lutz, C.; Wolter, F.; and Zakharyashev, M. 2008. Temporal description logics: A survey. In *Proc. of the 15th Int. Symposium on Temporal Representation and Reasoning, TIME 2008*, 3–14.
- Rosati, R. 2007. The limits of querying ontologies. In *Proc. of the 11th Int. Conf. on Database Theory, ICDT 2007*, volume 4353 of *LNCS*, 164–178. Springer.
- Rudolph, S.; Krötzsch, M.; and Hitzler, P. 2008a. All elephants are bigger than all mice. In *Proc. of the 21st Int. Workshop on Description Logics DL 2008*, volume 353 of *CEUR Workshop Proceedings*. CEUR-WS.
- Rudolph, S.; Krötzsch, M.; and Hitzler, P. 2008b. Cheap Boolean role constructors for description logics. In *Proc. of the 11th European Conf. on Logics in Artificial Intelligence, JELIA 2008*, volume 5293 of *Lecture Notes in Computer Science*, 362–374. Springer.
- Van Emde Boas, P. 1997. The convenience of tilings. In *In Complexity, Logic, and Recursion Theory*, 331–363. Marcel Dekker Inc.
- Vardi, M. Y. 1996. Why is modal logic so robustly decidable? In *Proc. of the 1996 DIMACS Workshop on Descriptive Complexity and Finite Models*, volume 31 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, 149–183. DIMACS/AMS.