Topological logics with connectedness over Euclidean spaces

Roman Kontchakov, Birkbeck, University of London, UK
Yavor Nenov, University of Oxford, UK
Ian Pratt-Hartmann, University of Manchester, UK
Michael Zakharyaschev, Birkbeck, University of London, UK

We consider the quantifier-free languages, $\mathcal{B}c$ and $\mathcal{B}c^\circ$, obtained by augmenting the signature of Boolean algebras with a unary predicate representing, respectively, the property of being connected, and the property of having a connected interior. These languages are interpreted over the regular closed sets of $\mathbb{R}^n$ ($n \geq 2$) and, additionally, over the regular closed semilinear sets of $\mathbb{R}^n$. The resulting logics are examples of formalisms that have recently been proposed in the Artificial Intelligence literature under the rubric Qualitative Spatial Reasoning. We prove that the satisfiability problem for $\mathcal{B}c$ is undecidable over the regular closed semilinear sets in all dimensions greater than 1, and that the satisfiability problem for $\mathcal{B}c$ and $\mathcal{B}c^\circ$ is undecidable over both the regular closed sets and the regular closed semilinear sets in the Euclidean plane. However, we also prove that the satisfiability problem for $\mathcal{B}c^\circ$ is NP-complete over the regular closed sets in all dimensions greater than 2, while the corresponding problem for the regular closed semilinear sets is EXPTIME-complete. Our results show, in particular, that spatial reasoning is much harder over Euclidean spaces than over arbitrary topological spaces.

Categories and Subject Descriptors: I.2.4 [**Knowledge Representation Formalisms and Methods**]: Representation languages; F.4.1 [**Mathematical Logic**]: Computability theory

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1. INTRODUCTION

Let $\mathcal{B}c$ be the quantifier-free fragment of first-order logic in the signature $(+, \cdot, -, 0, 1, c)$, where c is a unary predicate; and let $\mathsf{RCP}(\mathbb{R}^n)$ be the collection of polyhedra in n-dimensional Euclidean space, where a *polyhedron* is understood as any finite union of finite intersections of closed half-spaces. (Thus, our polyhedra need not be connected, need not be bounded, and need not have connected complements.) The collection $\mathsf{RCP}(\mathbb{R}^n)$ forms a Boolean algebra under the subset ordering; we interpret $\mathcal{B}c$ over $\mathsf{RCP}(\mathbb{R}^n)$ by taking $+, \cdot, -, 0, 1$ to have their Boolean algebra meanings, and by taking c to denote the property of *being connected*. Intuitively, we think of elements of $\mathsf{RCP}(\mathbb{R}^n)$ as regions of space, and of formulas of $\mathcal{B}c$ as descriptions of arrangements of these regions. Our primary concern is the *satisfiability problem*: given a $\mathcal{B}c$-formula, is there an assignment of elements of $\mathsf{RCP}(\mathbb{R}^n)$ to its variables making it true?

The motivation for studying this problem comes from the field of *Qualitative Spatial Reasoning* in Artificial Intelligence, the aim of which is to develop formal languages for

representing and processing qualitative spatial information. In this context, $\mathcal{B}c$ constitutes a parsimonious language: it has no quantifiers, and its non-logical primitives express only the basic region-combining operations and the property of connectedness. At the same time, the structures $\mathsf{RCP}(\mathbb{R}^n)$—particularly in the cases $n = 2$ and $n = 3$—constitute its most natural domains of interpretation, given current practice in the fields of Qualitative Spatial Reasoning, Geographic Information Systems and Spatial Databases.

For reasons discussed below, we broaden the subject of enquiry slightly. Let $\mathcal{B}c^\circ$ denote the quantifier-free fragment of first-order logic in the signature $(+, \cdot, -, 0, 1, c^\circ)$, where $+, \cdot, -, 0, 1$ are as before, and c° is a unary predicate interpreted as the property of *having a connected interior*. Further, let $\mathsf{RC}(\mathbb{R}^n)$ denote the collection of regular closed sets in n-dimensional Euclidean space, where a *regular closed* set is defined as the closure of any open set. Again, $\mathsf{RC}(\mathbb{R}^n)$ forms a Boolean algebra under the subset ordering, and has $\mathsf{RCP}(\mathbb{R}^n)$ as a sub-algebra. Intuitively, we think of $\mathsf{RC}(\mathbb{R}^n)$ as a more liberal model of spatial regions than $\mathsf{RCP}(\mathbb{R}^n)$. In the sequel, we consider the satisfiability problem for $\mathcal{B}c$ and $\mathcal{B}c^\circ$ over the structures $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$. The results of this paper are as follows: (*i*) the satisfiability problem for $\mathcal{B}c$ over $\mathsf{RCP}(\mathbb{R}^n)$ is undecidable for all $n \geq 2$; (*ii*) the satisfiability problem for $\mathcal{B}c$ over $\mathsf{RC}(\mathbb{R}^2)$ is undecidable, as are the satisfiability problems for $\mathcal{B}c^\circ$ over both $\mathsf{RC}(\mathbb{R}^2)$ and $\mathsf{RCP}(\mathbb{R}^2)$; (*iii*) the satisfiability problem for $\mathcal{B}c^\circ$ over $\mathsf{RC}(\mathbb{R}^n)$ is NP-complete for all $n \geq 3$, while over $\mathsf{RCP}(\mathbb{R}^n)$ the corresponding problem is ExpTime-complete. (It may be of interest to note that, over $\mathsf{RC}(\mathbb{R})$ and $\mathsf{RCP}(\mathbb{R})$, the satisfiability problem for $\mathcal{B}c$ and $\mathcal{B}c^\circ$ is NP-complete.) The decidability of the satisfiability problems for $\mathcal{B}c$ over $\mathsf{RC}(\mathbb{R}^n)$, for $n \geq 3$, is left open. Results (*ii*) and (*iii*) were announced, without proofs, in [Kontchakov et al. 2010b; 2011].

Mathematically, it is also meaningful to consider the satisfiability of $\mathcal{B}c$- and $\mathcal{B}c^\circ$-formulas over the regular closed subsets of *any* topological space. If T is a topological space, we denote the collection of regular closed subsets of T by $\mathsf{RC}(T)$; again, this collection always forms a Boolean algebra under the subset ordering. The satisfiability problem for $\mathcal{B}c$ over the class of structures of the form $\mathsf{RC}(T)$ is known to be ExpTime-complete, while for $\mathcal{B}c^\circ$, the corresponding problem is NP-complete [Kontchakov et al. 2010a; 2010b]. However, satisfiability over *arbitrary* topological spaces is of at most marginal relevance to Qualitative Spatial Reasoning. Indeed, the results reported here show that, for languages able to express the property of connectedness, reasoning over *Euclidean* spaces is a different kettle of fish altogether. In the remainder of this section, we discuss the significance of these results in the context of recent developments in spatial, algebraic and modal logics.

1.1. Spatial logic

A spatial logic is a formal language interpreted over some class of geometrical structures. Spatial logics, thus understood, have a long history, tracing their origins back both to the axiomatic tradition in geometry [Hilbert 1909; Tarski 1959] and also the region-based theory of space [Whitehead 1929; de Laguna 1922], subsequently developed in [Clarke 1981; 1985; Biacino and Gerla 1991]. They were proposed as a formalism for Qualitative Spatial Reasoning in the seminal paper [Randell et al. 1992]. The basic idea is as follows: numerical coordinate-based descriptions of the objects that surround us are hard to acquire, inherently error-prone, and probably unnecessary for everyday spatial reasoning tasks; therefore—so goes the argument—we should employ a representation language whose variables range over spatial regions (rather than points), and whose non-logical primitives are interpreted as qualitative (rather than quantitative) relations and operations. On this view, formulas are to be understood as expressing descriptions of (putative) configurations of objects in space, with the satisfiability of a formula over the space in question equating to the geometrical realizability of the described arrangement. If we imagine an intelligent agent employing such a language to represent spatial arrangements of objects, then the problem of recognizing

satisfiable formulas amounts to that of eliciting the geometrical knowledge latent in that agent's operating environment and cognitive design.

The best-known, and most intensively studied, qualitative spatial representation language is $\mathcal{RCC}8$ [Egenhofer and Franzosa 1991; Randell et al. 1992; Smith and Park 1992]. This language features predicates for the six topological relations DC (disconnection), EC (external connection), PO (partial overlap), EQ (equality), TPP (tangential proper part) and NTPP (non-tangential proper part) illustrated, for the case of closed discs, in Fig. 1. (The name $\mathcal{RCC}8$ becomes less puzzling when we observe that the relations TPP and NTPP are asymmetric.) Note that $\mathcal{RCC}8$ has no individual constants or function symbols, and no quantifiers. Traditionally, $\mathcal{RCC}8$ is interpreted over the regular closed sets in some topological space.
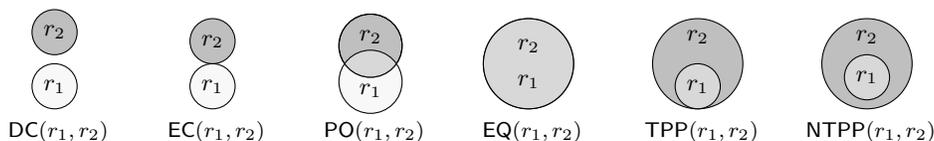

Fig. 1. $\mathcal{RCC}8$-relations over discs in $\mathbb{R}^2$.

The satisfiability problem for $\mathcal{RCC}8$ over the class of structures of the form $\mathsf{RC}(T)$ (for T a topological space) is NP-complete, though tractable fragments have been explored [Renz and Nebel 2001]. In particular, satisfiability of conjunctions of atomic $\mathcal{RCC}8$-formulas is known to be NLOGSPACE-complete [Bennett 1994; Nebel 1995; Renz 1998; Griffiths 2008]. Further, satisfiability of an $\mathcal{RCC}8$-formula over any structure in this class implies satisfiability over $\mathsf{RCP}(\mathbb{R}^n)$, for all $n \geq 1$ [Renz and Nebel 1999]. Thus, the satisfiability problems for $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$, for all $n \geq 1$, coincide and are NP-complete—a fact which testifies to the restricted expressive power of $\mathcal{RCC}8$.

A word of caution is in order at this point. Satisfiability of an $\mathcal{RCC}8$-formula over $\mathsf{RC}(\mathbb{R}^2)$ does not necessarily imply satisfiability by *natural* or *familiar* regions—for example, closed disc-homeomorphs. The $\mathcal{RCC}8$-satisfiability problem for such interpretations requires specialized, and highly non-trivial, techniques. A landmark result [Schaefer et al. 2003] in the area shows, however, that the satisfiability problem for $\mathcal{RCC}8$ interpreted over the closed disc-homeomorphs in $\mathbb{R}^2$ is still in NP. The contribution of present paper, with its emphasis on Euclidean spaces and the property of connectedness, imposes severe limits on what further results of this kind we can hope for.

We mentioned above that, if T is a topological space, the collection $\mathsf{RC}(T)$ always forms a Boolean algebra under the subset ordering. This enables us to extend $\mathcal{RCC}8$ with the function symbols $+$, $\cdot$, $-$ and constants 0, 1, interpreting these in the natural way over any structure $\mathsf{RC}(T)$. Such an extended language was originally introduced in [Wolter and Zakharyaschev 2000] under the name $\mathcal{BRCC}8$ (*Boolean $\mathcal{RCC}8$*). Intuitively, if a_1 and a_2 are regular closed sets, we may think of $a_1 + a_2$ as the agglomeration of a_1 and a_2, $a_1 \cdot a_2$ as the common part of a_1 and a_2, $-a_1$ as the complement of a_1, 0 as the empty region and 1 as the whole space. The satisfiability problem for $\mathcal{BRCC}8$ over the class of structures of the form $\mathsf{RC}(T)$ is still NP-complete; however, restricting attention to *connected* spaces T yields a PSPACE-complete satisfiability problem. Thus, $\mathcal{BRCC}8$, unlike $\mathcal{RCC}8$, has sufficient expressive power to distinguish between satisfiability over arbitrary spaces and satisfiability over connected spaces. But that is about as far as this extra expressive power takes us: satisfiability of a $\mathcal{BRCC}8$-formula over any structure $\mathsf{RC}(T)$, for T connected, implies satisfiability over $\mathsf{RCP}(\mathbb{R}^n)$ for all $n \geq 1$. Hence, the satisfiability problems for $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$, for all $n \geq 1$, coincide, and are PSPACE-complete. Note in particular that $\mathcal{BRCC}8$ does not enable us to say that a given *region* of space is connected.

We end this discussion of $\mathcal{RCC}8$ and $\mathcal{BRCC}8$ with a remark on the absence of quantification from these languages. This restriction is motivated by computability considerations: all reasonable region-based spatial logics with full first-order syntax interpreted over $\mathbb{R}^n$ ($n \geq 2$) have undecidable satisfiability problems, and so are considered unsuitable for Qualitative Spatial Reasoning [Grzegorczyk 1951; Dornheim 1998; Davis 2006; Lutz and Wolter 2006]. To be sure, first-order spatial logics are nevertheless of considerable model-theoretic interest; see [Pratt-Hartmann 2007] for a survey. We note in particular that, if we can quantify over regions, then the $\mathcal{RCC}8$-primitives easily enable us to define, over most interesting classes of interpretations, all of the primitives $+$, $\cdot$, $-$, 0, 1, c and c°. However, as computability considerations are to the fore in this paper, we too confine ourselves to quantifier-free formalisms in the sequel.

1.2. Algebraic and modal logic

The standard view of topology takes a topological space to consist of a set of points on which a collection of open subsets is defined. However, a dual view is possible, in which one begins with a Boolean algebra, and then adds algebraic structure defining distinctively topological relations between its objects. There are two main approaches to developing this second view. On the first, we think of the underlying Boolean algebra as a field of sets, and we augment this Boolean algebra with a pair of unary operators, conceived of as representing the operations of *closure* and *interior*, and assumed to obey the standard Kuratowski axioms [McKinsey and Tarski 1944]. The striking similarity between these axioms and the axioms for the propositional modal logic $\mathcal{S}4$ [Orlov 1928; Gödel 1933] led to the development of topological semantics for modal logics. Under this semantics, the (propositional) variables are taken to range over any collection of subsets of a topological space (not just regular closed sets), and the logical connectives are interpreted by the operations of union, intersection, complement and topological interior (for necessity) and closure (for possibility). The extension of this language with the universal modality, denoted $\mathcal{S}4_u$ [Goranko and Passy 1992], is known to be a super-logic for $\mathcal{RCC}8$ and $\mathcal{BRCC}8$ [Bennett 1994; Renz and Nebel 1997; Cohn et al. 1997; Nutt 1999; Wolter and Zakharyaschev 2000]. The satisfiability problem for $\mathcal{S}4_u$ is the same over every connected, separable, dense-in-itself metric space, and this problem is PSpace-complete [McKinsey and Tarski 1944; Shehtman 1999; Areces et al. 2000]. We remark that, as for $\mathcal{RCC}8$ and $\mathcal{BRCC}8$, $\mathcal{S}4_u$ is unable to express the condition that a region is connected. For a survey of the relationship between spatial and modal logics see [van Benthem and Bezhanishvili 2007; Gabelaia et al. 2005; Kontchakov et al. 2008b] and references therein.

On the second approach, we instead think of the underlying Boolean algebra as an algebra of *regular closed* sets, and we augment this algebra with a binary predicate C, conceived of as representing the relation of *contact*. (Two sets are said to be *in contact* if they have a non-empty intersection). This binary predicate is assumed to satisfy the axioms of *contact algebras*, a category which is known to be dual to the category of dense sub-algebras of regular closed algebras of topological spaces [Düntsch and Winter 2005; Dimov and Vakarelov 2006a; 2006b; Balbiani et al. 2007; Tinchev and Vakarelov 2010; Vakarelov 2007]. The contact relation as a basis for topology actually has a venerable career, having originally been introduced in [Whitehead 1929] under the name 'extensive connection.' More relevantly for the present paper, it is straightforward to show that all the $\mathcal{RCC}8$ relations can be expressed, in purely propositional terms, using this signature [Balbiani et al. 2007; Kontchakov et al. 2008b]. (Thus, for example, $\mathsf{EC}(\tau_1, \tau_2)$ is equivalent to $C(\tau_1, \tau_2) \wedge (\tau_1 \cdot \tau_2 = 0)$.) For this reason, we regard the propositional language over the signature $(+, \cdot, -, 0, 1, C)$, here denoted $\mathcal{C}$, as equivalent to the language $\mathcal{BRCC}8$ mentioned above. The purely Boolean fragment of $\mathcal{C}$ (without the contact predicate C) is denoted by $\mathcal{B}$. This language is in fact equivalent to the extension of the spatial logic $\mathcal{RCC}5$ [Bennett 1994] with $+$, $\cdot$ and $-$.

1.3. Spatial logics with connectedness

Many spatial regions of interest—plots of land in a cadastre, the space occupied by physical objects, paths swept out by moving objects—are either connected or at least contain few connected components [Cohn and Renz 2008]. It seems, therefore, that to be genuinely useful, logics for Qualitative Spatial Reasoning should possess some means of expressing this notion. The simplest way of proceeding is to consider languages featuring a unary predicate denoting this property. Various such languages have been investigated before [Pratt-Hartmann 2002; Kontchakov et al. 2008a; 2010a; Vakarelov 2007; Tinchev and Vakarelov 2010]; the language $\mathcal{B}c$ is chosen for study here because it is so parsimonious.

It is worth bearing in mind, however, that 'connectedness,' in the topologists' sense may not be exactly what we want. For example, a region consisting of two closed discs externally touching is, in this sense, connected, yet, in certain contexts, may be functionally equivalent to a disconnected region. (Imagine having a garden that shape.) In such contexts, it may be more useful to employ the notion of a region's having a connected *interior*, a property we refer to as *interior-connectedness*. Note that every regular closed, interior-connected set is connected; also, in the space $\mathbb{R}$, connectedness and interior-connectedness coincide. So as not to prejudge the issue here, we employ predicates for both notions: c to denote the standard property of connectedness, c° to denote the property of interior-connectedness. Hence, in addition to the 'minimal' language $\mathcal{B}c$, we have its counterpart $\mathcal{B}c^\circ$.

Strikingly, the languages $\mathcal{RCC}8$ and $\mathcal{BRCC}8$, which cannot represent connectedness (or, for that mater, interior-connectedness), are far less sensitive to the underlying geometrical interpretation than the languages $\mathcal{B}c$ and $\mathcal{B}c^\circ$, which can. For example, an $\mathcal{RCC}8$-formula that is satisfiable over the regular closed algebra of *any* topological space is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$, for all $n \geq 1$ [Renz 1998]. Or again, a $\mathcal{BRCC}8$-formula that is satisfiable over the regular closed algebra of any *connected* topological space is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$, for all $n \geq 1$ [Wolter and Zakharyaschev 2000]. Thus, $\mathcal{RCC}8$ and $\mathcal{BRCC}8$ care neither about the dimension of the (Euclidean) space we are reasoning about, nor about the distinction between arbitrary regular closed sets and polyhedra. Not so with the languages $\mathcal{B}c$ or $\mathcal{B}c^\circ$, which are sensitive both to the dimension of space and to the restriction to polyhedra. This sensitivity is easy to demonstrate for $\mathcal{B}c^\circ$, and we briefly do so here, by way of illustration.

Consider first sensitivity to dimension. The $\mathcal{B}c^\circ$-formula

$$\bigwedge_{1 \leq i \leq 3} \big(c^\circ(r_i) \wedge (r_i \neq 0) \big) \quad \wedge \quad \bigwedge_{1 \leq i < j \leq 3} \big(c^\circ(r_i + r_j) \wedge (r_i \cdot r_j = 0) \big) \tag{1}$$

says that r_1, r_2 and r_3 are non-empty regions with connected interiors, such that each forms an interior-connected sum with the other two, without overlapping them. It is obvious that this formula is not satisfiable over $\mathsf{RC}(\mathbb{R})$. For the non-empty, (interior-) connected regular closed sets on the real line are precisely the non-punctual, closed intervals, and it is impossible for three such intervals to touch each other without overlapping. On the other hand, (1) is easily seen to be satisfiable over $\mathsf{RC}(\mathbb{R}^n)$ for all $n \geq 2$. Likewise, the $\mathcal{B}c^\circ$-formula

$$\bigwedge_{1 \leq i \leq 5} \big(c^\circ(r_i) \wedge (r_i \neq 0) \big) \quad \wedge \quad \bigwedge_{1 \leq i < j \leq 5} \big(c^\circ(r_i + r_j) \wedge (r_i \cdot r_j = 0) \big), \tag{2}$$

which makes the analogous claim for regions $r_1, \ldots, r_5$, is not satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, since any satisfying assignment would permit a plane drawing of the graph K_5. On the other hand, (2) is easily seen to be satisfiable over $\mathsf{RC}(\mathbb{R}^n)$ for all $n \geq 3$. Thus, the satisfiability problems for $\mathcal{B}c^\circ$ over $\mathsf{RC}(\mathbb{R})$, $\mathsf{RC}(\mathbb{R}^2)$ and $\mathsf{RC}(\mathbb{R}^3)$ are all different. (We shall see in Sec. 6, however, that the satisfiability problem for $\mathcal{B}c^\circ$ over $\mathsf{RC}(\mathbb{R}^n)$ is the same for all $n \geq 3$.)

Consider next sensitivity to the restriction to *polyhedra*. The $\mathcal{B}c^\circ$-formula

$$\bigwedge_{1 \leq i \leq 3} c^\circ(r_i) \quad \wedge \quad c^\circ(r_1 + r_2 + r_3) \quad \wedge \quad \bigwedge_{i=2,3} \neg c^\circ(r_1 + r_i) \tag{3}$$

is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, as we see from the regular closed sets in Fig. 2, where r_2 and r_3 lie, respectively, above and below the graph of the function $\sin \frac{1}{x}$ on the interval $(0, 1]$. By contrast, formula (3) is unsatisfiable over $\mathsf{RCP}(\mathbb{R}^n)$ for all $n \geq 1$ [Pratt and Lemon 1997, p. 231]. Actually, the result can be sharpened: (3) is unsatisfiable over *any* Boolean
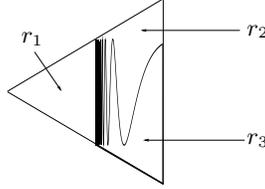

Fig. 2. Three regions in $\mathsf{RC}(\mathbb{R}^2)$ satisfying (3).

sub-algebra of $\mathsf{RC}(\mathbb{R}^n)$ whose regions all satisfy a form of the *curve selection lemma* from real algebraic geometry [Bochnak et al. 1998]. As we might say, in dimensions 2 and above, $\mathcal{B}c^\circ$ is sensitive to the presence of 'non-tame' regions. And since—at least conceivably—non-tame regions may be thought implausible models of the space occupied by any physical objects—it is natural to consider satisfiability of $\mathcal{B}c^\circ$-formulas over $\mathsf{RCP}(\mathbb{R}^n)$ rather than over $\mathsf{RC}(\mathbb{R}^n)$.

The language $\mathcal{B}c$ is similarly sensitive to the dimension of the Euclidean space over which it is interpreted, and also to the restriction to polyhedra. For dimensionality, this sensitivity can be demonstrated by examples similar to (1) and (2); see [Kontchakov et al. 2008b]. For the restriction to polyhedra, this result follows from Sec. 3, where we show that there exists a $\mathcal{B}c$-formula satisfiable in $\mathsf{RC}(\mathbb{R}^n)$ for all $n \geq 2$, but only by tuples of regions having infinitely many connected components!

1.4. Plan of the paper and summary of results

The remainder of this paper is organized as follows. Sec. 2 defines the syntax and semantics of $\mathcal{B}c$ and $\mathcal{B}c^\circ$. To simplify proofs, we also employ the more expressive languages $\mathcal{C}c$ and $\mathcal{C}c^\circ$, obtained by adding the predicates c and c°, respectively, to $\mathcal{C}$ ($= \mathcal{BRCC}8$). In Sec. 3, we prove that there exist $\mathcal{C}c$-, $\mathcal{C}c^\circ$- and $\mathcal{B}c$-formulas satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, for all $n \geq 2$, but only by tuples of regions some of which have infinitely many connected components, and hence which cannot belong to $\mathsf{RCP}(\mathbb{R}^n)$. By further developing the ideas encountered in this proof, we show in Sec. 4 that $\mathcal{C}c$, $\mathcal{C}c^\circ$ and $\mathcal{B}c$ (but not $\mathcal{B}c^\circ$) are r.e.-hard over $\mathsf{RCP}(\mathbb{R}^n)$, for all $n \geq 2$. Using a different approach, we show in Sec. 5 that all four of our logics—$\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$—are r.e.-hard over both $\mathsf{RCP}(\mathbb{R}^2)$ and $\mathsf{RC}(\mathbb{R}^2)$. Finally, we show in Sec. 6 that $\mathcal{B}c^\circ$ is NP-complete over $\mathsf{RC}(\mathbb{R}^n)$, and ExpTime-complete over $\mathsf{RCP}(\mathbb{R}^n)$, for all $n \geq 3$. The decidability of satisfiability for $\mathcal{C}c$, $\mathcal{C}c^\circ$ and $\mathcal{B}c$ over $\mathsf{RC}(\mathbb{R}^n)$, for all $n \geq 3$, is left open.

2. PRELIMINARIES

We begin by formally defining the syntax and semantics of the topological logics considered in this paper. This section also contains the basic technical definitions and results we need in what follows.

2.1. Basic topological notions

A *topological space* is a pair $(T, \mathcal{O})$, where T is a set and $\mathcal{O}$ a collection of subsets of T containing $\emptyset$ and T, and closed under arbitrary unions and finite intersections. The elements of $\mathcal{O}$ are referred to as *open sets*; their complements are *closed* sets. If $\mathcal{O}$ is clear from context, we refer to the topological space $(T, \mathcal{O})$ simply as T. If $X \subseteq T$, the *closure* of X, denoted X^-, is the smallest closed set including X, and the *interior* of X, denoted X°, is the largest open set included in X. These sets always exist. The *boundary* of X, denoted δX, is the set $X^- \setminus X^\circ$. The Euclidean space $\mathbb{R}^n$ is assumed always to have the usual metric topology. We may treat any subset $X \subseteq T$ as a topological space in its own right by defining the *subspace topology* on X to be the collection of sets $\mathcal{O}_X = \{O \cap X \mid O \in \mathcal{O}\}$.

We call X *regular closed* if it is the closure of an open set—equivalently, if $X = (X^\circ)^-$. We denote by $\mathsf{RC}(T)$ the set of regular closed subsets of T. It is a standard result that $\mathsf{RC}(T)$ forms a complete Boolean algebra, with operations $\sum A = (\bigcup A)^-$, $\prod A = (\bigcap A)^{\circ -}$ and $-X = (T \setminus X)^-$ (see, e.g. [Koppelberg 1989]). The partial order induced by this Boolean algebra is simply $(T, \subseteq)$; we often write $X \leq Y$ in preference to $X \subseteq Y$ where X and Y are regular closed. Note that, if $A = \{X_1, X_2\}$, then $\sum A = X_1 + X_2 = X_1 \cup X_2$.

A topological space T is said to be *connected* if it cannot be decomposed into two disjoint, non-empty closed sets; likewise, X is *connected* if it is a connected space under the subspace topology. We call X *interior-connected* if X° is connected. A maximal connected subset of X will be called a *component* of X (some authorities prefer the term *connected component*). The following facts are easily verified: every non-empty connected subset of X is included in a unique component of X; every component of a closed set is closed.

The space T is said to be *locally connected* if every neighbourhood of any point of T includes a connected neighbourhood of that point (a *neighbourhood* of a point p is a set X that includes an open set containing p). In a locally connected space, every component of an open set is open; note however that components of regular closed sets are closed but, in general, not regular closed, even in locally connected spaces. The space T is said to be *unicoherent* if, for any closed, connected subsets X_1, X_2 such that $T = X_1 \cup X_2$, the set $X_1 \cap X_2$ is connected. For all $n \geq 1$, the Euclidean space $\mathbb{R}^n$ is (obviously) locally connected and (much less obviously) unicoherent [Kuratowski 1928]. A simple example of a non-locally connected space is the rational numbers $\mathbb{Q}$ under the usual metric topology. Simple examples of non-unicoherent spaces are the Jordan curve and the torus.

The most important properties of local connectedness and unicoherence, from our point of view, are given by the following lemmas.

LEMMA 2.1. *Let X be a regular closed subset of a topological space T and S a component of $-X$. If $-X$ has finitely many components, then $\delta S \subseteq X$. Alternatively, if T is locally connected, then $\delta S \subseteq X$.*

PROOF. For the first statement, let Z be the union of all components of $-X$ other than S. By definition, $T = X^\circ \cup S \cup Z$. Further, both $S \cap X^\circ$ and $S \cap Z$ are empty, whence $T \setminus S = X^\circ \cup Z$. Since X is regular closed and Z is closed (as the union of finitely many closed sets), $(T \setminus S)^- = X \cup Z$. Finally, since S is closed, and $S \cap Z = \emptyset$, $\delta S = S \cap (T \setminus S)^- \subseteq X$. For the second statement, suppose, to the contrary, that δS contains a point p lying in $(-X)^\circ$. Since S is closed, $p \in S$. By local connectedness, let Y be a connected open set such that $p \in Y \subseteq (-X)^\circ$. Since $p \in S$ and Y is a connected subset of $-X$, we have $p \in Y \subseteq S$. But this contradicts the assumption that $p \in \delta S$. □

LEMMA 2.2. *Let T be a unicoherent space and $X \in \mathsf{RC}(T)$ be connected. Then every component of $-X$ has a connected boundary.*

PROOF. Let S be a connected component of $-X$, and let Z be the union of all components of $-X$ other than S. Thus, $T \setminus S = Z \cup X^\circ$. We write $S^* = (T \setminus S)^-$. Since X is

regular closed, $S^* = Z^- \cup X$. By connectedness of T, X intersects every component of $-X$. It follows that $Z \cup X$, and hence $Z^- \cup X = S^*$ are connected. By definition, S is connected, whence, by unicoherence of T, $\delta S = S \cap S^*$ is connected. $\square$

2.2. Frames

A *frame* is a pair $(T, \mathcal{S})$, where T is a topological space, and $\mathcal{S}$ is a Boolean sub-algebra of $\mathsf{RC}(T)$. Where T is clear from context, we refer to $(T, \mathcal{S})$, simply, as $\mathcal{S}$. Furthermore, where $\mathcal{S}$ is clear from context, we refer to elements of $\mathcal{S}$ as *regions*. We denote the class of frames of the form $(T, \mathsf{RC}(T))$ by RC. Note that not all frames are of this form: in particular, when working in n-dimensional Euclidean spaces, we shall be principally interested in the following proper sub-algebra of $\mathsf{RC}(\mathbb{R}^n)$. Any $(n-1)$-dimensional hyperplane bounds two elements of $\mathsf{RC}(\mathbb{R}^n)$ called *half-spaces*; a *polyhedron* in $\mathbb{R}^n$ is an element of the Boolean sub-algebra of $\mathsf{RC}(\mathbb{R}^n)$ generated by the half-spaces, or, equivalently, a finite union of finite intersections of half-spaces. We denote this collection of sets by $\mathsf{RCP}(\mathbb{R}^n)$, and, in the case $n = 2$, we speak of *polygons*, rather than polyhedra. Note that the polyhedra in $\mathbb{R}^n$ may be alternatively characterized as the regular closed *semilinear* sets. Polyhedra will be regarded as 'well-behaved' or, in topologists' parlance, 'tame.' We call $(T, \mathcal{S})$ *unicoherent* if T is unicoherent, and *finitely decomposible* if, for all $s \in \mathcal{S}$, there exist connected elements $s_1, \ldots, s_k$ of $\mathcal{S}$, such that $s = s_1 + \cdots + s_k$. Evidently, $(\mathbb{R}^n, \mathsf{RCP}(\mathbb{R}^n))$ is finitely decomposible, since any product of half-spaces is connected. Equally obviously:

LEMMA 2.3. *Suppose the frame $(T, \mathcal{S})$ is finitely decomposible, and $s \in \mathcal{S}$. Then every component of s is in $\mathcal{S}$, and s is equal to the sum of those components.*

The following basic concepts will be used repeatedly in the sequel. Let $(T, \mathcal{S})$ be a frame. A tuple $(s_0, \ldots, s_{k-1})$, where $k \geq 1$, will be called a *partition*, provided $s_0 + \cdots + s_{k-1} = 1$ and $s_i \cdot s_j = 0$ for $0 \leq i < j < k$. We do not insist that the s_i are non-empty. We call a partition $(s_0, \ldots, s_{k-1})$ *sub-cyclic* if the s_i are non-empty and $s_i \cap s_j = \emptyset$, for $0 \leq i, j < k$ such that $1 < j - i < k - 1$. The term 'sub-cyclic' refers to an imagined graph with nodes $\{s_0, \ldots, s_{k-1}\}$ and edges $\{(s_i, s_j) \mid i \neq j$ and $s_i \cap s_j \neq \emptyset\}$: this graph is required to be a (not necessarily proper) subgraph of the cyclic graph on $\{s_0, \ldots, s_{k-1}\}$.

Suppose s is a non-empty element of a frame $(T, \mathcal{S})$, and $\vec{s} = (s_0, \ldots, s_{k-1})$ a partition in that frame. We say that $\vec{s}$ is a *colouring* of the components of s if every component of s is included in exactly one of the regions of $\vec{s}$. Colourings will be used repeatedly in the sequel, particularly in situations where we may regard the components of s as positions in a finite sequence; by regarding the set of elements of $\vec{s}$ as an alphabet, colourings define *words* over that alphabet in the obvious way.

2.3. Topological logics

In this paper, the focus of attention is not on frames themselves, but rather, on frames *as they are described in some language*. The languages considered here all employ a countably infinite collection of variables $r_1, r_2, \ldots$. The language $\mathcal{C}$ is defined by the following syntax:

$$\tau \quad ::= \quad r \quad | \quad \tau_1 + \tau_2 \quad | \quad \tau_1 \cdot \tau_2 \quad | \quad -\tau_1 \quad | \quad 0 \quad | \quad 1;$$
$$\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad C(\tau_1, \tau_2) \quad | \quad \varphi_1 \vee \varphi_2 \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \neg\varphi_1.$$

The language $\mathcal{B}$ is defined analogously, but without the predicate C. Thus, $\mathcal{B}$ is the quantifier-free language of the variety of Boolean algebras.

An *interpretation over* a frame $(T, \mathcal{S})$ is a function $\cdot^{\mathfrak{I}}$ mapping variables r to elements $r^{\mathfrak{I}}$ of $\mathcal{S}$. We extend $\cdot^{\mathfrak{I}}$ to terms τ by setting $(\tau_1 + \tau_2)^{\mathfrak{I}} = \tau_1^{\mathfrak{I}} + \tau_2^{\mathfrak{I}}$, $(\tau_1 \cdot \tau_2)^{\mathfrak{I}} = \tau_1^{\mathfrak{I}} \cdot \tau_2^{\mathfrak{I}}$, $(-\tau_1)^{\mathfrak{I}} = -(\tau_1^{\mathfrak{I}})$, $0^{\mathfrak{I}} = \emptyset$ and $1^{\mathfrak{I}} = T$. We write $\mathfrak{I} \models \tau_1 = \tau_2$ if and only if $\tau_1^{\mathfrak{I}} = \tau_2^{\mathfrak{I}}$, and $\mathfrak{I} \models C(\tau_1, \tau_2)$ if and only if $\tau_1^{\mathfrak{I}} \cap \tau_2^{\mathfrak{I}} \neq \emptyset$, extending this relation to non-atomic formulas in the standard way. We read $C(\tau_1, \tau_2)$ as 'τ_1 *contacts* τ_2.' If φ is a formula whose variables,

taken in some order, are $\vec{r} = (r_1, \ldots, r_n)$, and $\mathfrak{I} \models \varphi$, then the tuple $\vec{a} = (r_1^{\mathfrak{I}}, \ldots, r_n^{\mathfrak{I}})$ is said to *satisfy* $\varphi(\vec{r})$; in such a case, we will often say '$\vec{a}$ satisfies $\varphi(\vec{r})$.'

The property that a k-tuple forms a partition is expressible using the $\mathcal{B}$-formula

$$\mathsf{part}(r_0, \ldots, r_{k-1}) \;\; = \;\; \Big(\sum_{0 \le i < k} r_i = 1 \Big) \; \wedge \bigwedge_{0 \le i < j < k} (r_i \cdot r_j = 0).$$

The property that a k-tuple forms a sub-cyclic partition is expressible using the $\mathcal{C}$-formula

$$\mathsf{sc\text{-}part}(r_0, \ldots, r_{k-1}) \;\; = \;\; \mathsf{part}(r_0, \ldots, r_{k-1}) \wedge \bigwedge_{0 \le i < k} (r_i \ne 0) \wedge \bigwedge_{1 < j-i < k-1} \neg C(r_i, r_j).$$

And, assuming that $\mathsf{part}(r_0, \ldots, r_{k-1})$ is satisfied, the $\mathcal{C}$-formula

$$\mathsf{colour}(r; r_0, \ldots, r_{k-1}) \;\; = \;\; \bigwedge_{0 \le i < j < k} \neg C((r \cdot r_i), \, (r \cdot r_j))$$

ensures that the partition $r_0, \ldots, r_{k-1}$ colours the components of r. Conversely, over finitely decomposable frames, any colouring of r by a partition $r_0, \ldots, r_{k-1}$ must satisfy $\mathsf{colour}(r; r_0, \ldots, r_{k-1})$.

Turning to connectedness predicates, we define the languages $\mathcal{B}c$ and $\mathcal{C}c$ to be extensions of $\mathcal{B}$ and $\mathcal{C}$, respectively, with the unary predicate c. We set $\mathfrak{I} \models c(\tau)$ if and only if $\tau^{\mathfrak{I}}$ is connected in the topological space under consideration. Similarly, we define $\mathcal{B}c^{\circ}$ and $\mathcal{C}c^{\circ}$ to be extensions of $\mathcal{B}$ and $\mathcal{C}$ with the predicate c°, setting $\mathfrak{I} \models c^{\circ}(\tau)$ if and only if $(\tau^{\mathfrak{I}})^{\circ}$ is connected. If $\mathcal{K}$ is a class of frames, and $\mathcal{L}$ is one of $\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{B}c^{\circ}$ or $\mathcal{C}c^{\circ}$, then $Sat(\mathcal{L}, \mathcal{K})$ is the set of $\mathcal{L}$-formulas satisfiable over $\mathcal{K}$.

In the case $\mathcal{K} = \mathsf{RC}$, the complexity of $Sat(\mathcal{L}, \mathcal{K})$ is known for all of the languages $\mathcal{L}$ considered above [Kontchakov et al. 2010a; 2010b]. If φ is a formula of any of the languages $\mathcal{B}c$, $\mathcal{C}c$ or $\mathcal{C}c^{\circ}$, and φ is satisfiable over RC, then φ is satisfiable over some frame $\mathsf{RC}(T)$, where $|T|$, the cardinality of T, is bounded by an exponential function of $|\varphi|$; and the problems $Sat(\mathcal{L}, \mathsf{RC})$, for $\mathcal{L} \in \{\mathcal{B}c, \mathcal{C}c, \mathcal{C}c^{\circ}\}$, are all ExpTime-complete. On the other hand, if ψ is a $\mathcal{B}c^{\circ}$-formula satisfiable over RC, then ψ is satisfiable over some frame $\mathsf{RC}(T)$, where $|T|$ is polynomial in $|\psi|$; and the problem $Sat(\mathcal{B}c^{\circ}, \mathsf{RC})$ is NP-complete. Thus, we observe a difference between $\mathcal{B}c$, $\mathcal{C}c$ and $\mathcal{C}c^{\circ}$ on the one hand, and $\mathcal{B}c^{\circ}$ on the other.

However, satisfiability over RC is of little interest from the point of view of spatial logic in AI, where the majority of applications concern the frames over Euclidean space of dimensions 2 or 3. Accordingly, we shall be concerned with $Sat(\mathcal{L}, \mathcal{K})$, where $\mathcal{L}$ is any of $\mathcal{B}c$, $\mathcal{B}c^{\circ}$, $\mathcal{C}c$ or $\mathcal{C}c^{\circ}$, and $\mathcal{K}$ is $\{\mathsf{RC}(\mathbb{R}^n)\}$ or $\{\mathsf{RCP}(\mathbb{R}^n)\}$ for $n \ge 2$. For ease of reading, we write $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n))$ and $Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$ rather than $Sat(\mathcal{L}, \{\mathsf{RC}(\mathbb{R}^n)\})$ and $Sat(\mathcal{L}, \{\mathsf{RCP}(\mathbb{R}^n)\})$.

2.4. Graphs

Unless explicitly indicated to the contrary, all graphs in this paper are taken to be finite, and to have no multiple edges and no self-loops: i.e., if $G = (V, E)$ is a graph, $(v, v') \in E$ implies $v \ne v'$. We also assume that the edges have no direction, i.e., $(v, v') \in E$ if and only if $(v', v) \in E$. A *path* in G is a sequence of distinct vertices $v_0, \ldots, v_{n-1}$ such that (v_i, v_{i+1}) is an edge, for all $0 \le i < n-1$; further, a *cycle* in G is a path $v_0, \ldots, v_{n-1}$, with $n \ge 3$, such that, in addition, (v_{n-1}, v_0) is an edge. Informally, in this case, we speak of the sequence $v_0, \ldots, v_{n-1}, v_0$ as a cycle. A graph is connected if any two nodes are joined by some path; a graph which contains no cycles is *acyclic*; and a connected, acyclic graph is a *tree*. If G is a tree, then any pair of nodes in G is joined by a unique path. Further, if $v_0, \ldots, v_{n-1}$ is a sequence of nodes in a tree such that (v_i, v_{i+1}) is an edge for $0 \le i < n-1$, and $v_i \ne v_{i+2}$ for $0 \le i < n-2$, then this sequence contains no duplicates, and thus is a path.

Let $\mathcal{S}$ be a finitely decomposable frame over some topological space, and $\vec{s}$ a connected partition in $\mathcal{S}$. We can associate a graph with $\vec{s}$, denoted $H(\vec{s})$, as follows: the vertices of

$H(\vec{s})$ are the components of the elements of $\vec{s}$; the edges of $H(\vec{s})$ are the pairs (X, Y) such that $X \neq Y$ and $X \cap Y \neq \emptyset$. We refer to $H(\vec{s})$ as the *component graph* of $\vec{s}$. Note that the number of vertices of $H(\vec{s})$ is in general larger than the number of elements in $\vec{s}$; however, since $\mathcal{S}$ is finitely decomposible, this number is still finite.

We prove a simple but powerful lemma connecting some of the notions encountered above.

LEMMA 2.4. *Let T be a unicoherent topological space, $\mathcal{S}$ a finitely decomposible frame on T, and $\vec{s}$ a sub-cyclic partition in $\mathcal{S}$. Then the component graph, $H(\vec{s})$, is a tree.*

PROOF. Write $\vec{s} = (s_0, \ldots, s_{n-1})$. Since $\mathcal{S}$ is finitely decomposible, and T is connected, $H(\vec{s})$ is obviously finite and connected. We need only show that it contains no cycles. If $n = 1$, then $|H(\vec{s})| = 1$, and this is trivial. We assume, for ease of formulation, that $n \geq 4$, since a similar (and in fact simpler) argument applies if $n = 2$ or $n = 3$.

Suppose (X_0, X_1) is an edge of $H(\vec{s})$. We may assume, without loss of generality, that X_0 is a component of s_0, and X_1 a component of s_1. The sub-cyclicity condition ensures that $s_0 \cap s_i = \emptyset$ for $2 \leq i < n-1$, and $s_1 \cap s_i = \emptyset$ for $3 \leq i < n$. Now let S be the component of $-X_1$ containing X_0: we claim that $\delta S \subseteq s_0$. By the first statement of Lemma 2.1, $\delta S \subseteq X_1 \subseteq s_1$, whence δS contains no point of $s_3 + \cdots + s_{n-1}$. On the other hand, δS is obviously included in $-s_1 = s_0 + s_2 + s_3 + \cdots s_{n-1}$, and hence in $s_0 + s_2$. Since $s_0 \cap s_2 = \emptyset$, and, by Lemma 2.2, δS is connected, we have either $\delta S \subseteq s_0$ or $\delta S \subseteq s_2$. Now, since (X_0, X_1) is an edge of $H(\vec{s})$, and any point of $X_0 \cap X_1$ must lie in both S and $-S$, we have $\delta S \cap X_0 \neq \emptyset$, and, therefore, $\delta S \subseteq s_0$, as claimed.

Now suppose (X_1, X_2) is also an edge of $H(\vec{s})$, with X_0 and X_2 distinct. We claim that X_0 and X_2 lie in different components of $-X_1$ (i.e., $X_2 \nsubseteq S$). For suppose otherwise. Again, since any point of $X_1 \cap X_2$ lies in both S and $-S$, $\delta S \cap X_2 \neq \emptyset$. Furthermore, since $s_1 \cap s_i = \emptyset$ for $3 \leq i < n$, X_2 must be a component of either s_0 or s_2. But if $X_2 \subseteq s_0$, then the connected set $\delta S \subseteq s_0$ has points in common with the components X_0, X_2 of s_0, contradicting the assumption that X_0 and X_2 are distinct. On the other hand, if $X_2 \subseteq s_2$ then $\delta S \subseteq s_0$ contains a point of s_2, which is again impossible.

Finally, suppose that $X_0, X_1, X_2, \ldots, X_m$ is a cycle in $H(\vec{s})$, where $m \geq 3$ and $X_m = X_0$. Then the connected set $X_2 + \cdots + X_{m-1} + X_0$ lies entirely in $-X_1$, contradicting the fact that X_0 and X_2 lie in different components of $-X_1$. □

2.5. Post correspondence problem

In the sequel, we make use of the well-known *Post correspondence problem* (PCP). Fix finite alphabets T and U, where $|\mathsf{T}| \geq 7$ and $|\mathsf{U}| \geq 2$. A *morphism* from T to U is a function $\mathsf{w} \colon \mathsf{T} \to \mathsf{U}^*$ mapping each element of T to a word over U. We extend w to a mapping $\mathsf{w} \colon \mathsf{T}^* \to \mathsf{U}^*$ by defining, for any word $\tau = t_1 \cdots t_k \in \mathsf{T}^*$, $\mathsf{w}(\tau) = \mathsf{w}(t_1) \cdots \mathsf{w}(t_k)$. An *instance* of the PCP is a pair of morphisms $\mathsf{W} = (\mathsf{w}^1, \mathsf{w}^2)$ from T to U. The instance W is *positive* if there exists a non-empty word $\tau \in \mathsf{T}^*$ such that $\mathsf{w}^1(\tau) = \mathsf{w}^2(\tau)$. Intuitively, we are invited to think of each element of T as a 'tile' inscribed with an 'upper' word over U, given by $\mathsf{w}^1(t)$, and a 'lower' word over U, given by $\mathsf{w}^2(t)$; we are asked to determine, for the given collection of tiles, whether there exists a non-empty, finite sequence of these tiles (repeats allowed) such that the concatenation of their upper words equals the concatenation of their lower words. The set of positive PCP instances is known to be r.e.-complete [Post 1946], and remains so even under the restriction that $\mathsf{w}^k(t)$ is non-empty for every $t \in \mathsf{T}$. In fact, nothing hinges on the exact choice of T and U, subject to the restrictions mentioned above. In particular, we may assume T and U are disjoint.

3. FORCING INFINITELY MANY COMPONENTS IN LOCALLY CONNECTED UNICOHERENT SPACES

In this section, we construct $\mathcal{C}c$-, $\mathcal{C}c^\circ$- and $\mathcal{B}c$-formulas φ with the following properties: (i) φ is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$ for all $n \geq 2$; (ii) if T is a locally connected, unicoherent space

and $\vec{r}$ is a tuple from $\mathsf{RC}(T)$ satisfying φ, then $\vec{r}$ includes members with *infinitely many* connected components. Since $\mathsf{RCP}(\mathbb{R}^n)$ is finitely decomposible, these properties entail that $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n)) \neq Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$ for $\mathcal{L}$ any of Cc, Cc° or Bc, and all $n \geq 2$. Furthermore, the techniques developed in this section will be used in Sec. 4 to prove that satisfiability of Cc-, Cc°- and Bc-formulas over $\mathsf{RCP}(\mathbb{R}^n)$, for $n \geq 2$, is undecidable.

We now construct a Cc-formula, φ_∞ with properties (*i*) and (*ii*). The first conjunct of φ_∞ states that r_0, r_1, r_2, r_3 form a sub-cyclic partition:

$$\mathsf{sc\text{-}part}(r_0, r_1, r_2, r_3). \tag{4}$$

We also require non-empty sub-regions r_i' of r_i and a non-empty region t:

$$\bigwedge_{0 \leq i < 4} \left((r_i' \neq 0) \wedge (r_i' \leq r_i) \right) \quad \wedge \quad (t \neq 0). \tag{5}$$

As an aid to intuition, take T to be the Euclidean plane $\mathbb{R}^2$, and consider the configuration depicted in Fig. 3, where components of the r_i are arranged like the layers of an onion (r_i' is included in r_i). The 'innermost' component of r_0 is surrounded by a component of r_1,
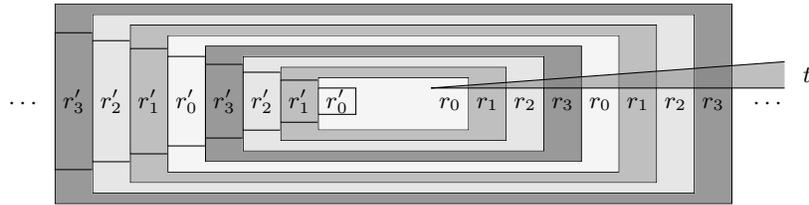
Fig. 3. Regions satisfying φ_∞.

which in turn is surrounded by a component of r_2, and so on. The region t passes through every layer, but avoids the r_i'. To enforce a configuration of this sort, we need the following formulas. Throughout this section, we write $\lfloor i \rfloor$ to denote the value of i modulo 4.

$$\bigwedge_{0 \leq i < 4} c(r_i' + r_{\lfloor i+1 \rfloor} + t), \tag{6}$$

$$\bigwedge_{0 \leq i < 4} \neg C(r_i', t), \tag{7}$$

$$\bigwedge_{0 \leq i < 4} \neg C(r_i', \ r_{\lfloor i+1 \rfloor} \cdot (-r_{\lfloor i+1 \rfloor}')). \tag{8}$$

Observe that (6)–(8) ensure each component of r_i' is in contact with $r_{\lfloor i+1 \rfloor}'$. Denote by φ_∞ the conjunction of (4)–(8).

THEOREM 3.1. *The Cc-formula φ_∞ is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$. On the other hand, if T is a locally connected, unicoherent space, then any tuple from $\mathsf{RC}(T)$ satisfying φ_∞ features sets that have infinitely many components.*

PROOF. Fig. 3 shows how φ_∞ can be satisfied over $\mathsf{RC}(\mathbb{R}^2)$. By cylindrification, it is also satisfiable over any $\mathsf{RC}(\mathbb{R}^n)$, for $n > 2$. This establishes the first statement of the lemma. For the second statement, we suppose that φ_∞ is satisfied by a tuple $r_0, \ldots, r_3, r_0', \ldots, r_3', t$ over $\mathsf{RC}(T)$, where T is a locally connected, unicoherent space. Our strategy is to prove that this tuple looks approximately like the arrangement in Fig. 3. More precisely, we construct a sequence $X_0, X_1, \ldots$ of elements of $\mathsf{RC}(T)$, such that: (*i*) X_i is a component of $r_{\lfloor i \rfloor}$; and (*ii*) each X_i separates any X_j ($j < i$) from any X_k ($k > i$). It follows that each r_i has infinitely

many components. Note however that these regions may in general be badly-behaved. In particular, $r_0, \dots, r_3$ may have other components in addition to the X_i.

We construct the sequence of components X_i of $r_{\lfloor i \rfloor}$ together with open sets Z_i connecting X_i to X_{i+1}; see Fig. 4. By the first conjunct of (5), let X_0 be a component of r_0 containing

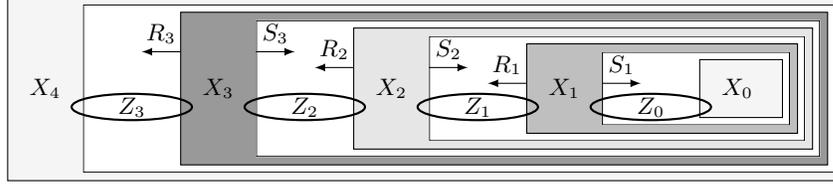Fig. 4. Sequences $X_0, X_1, \dots$ of components X_i of $r_{\lfloor i \rfloor}$ and $Z_0, Z_1, \dots$ of open sets Z_i connecting X_i to X_{i+1} with the 'holes' S_{i+1} and R_{i+1} of X_{i+1} containing X_i and X_{i+2}, respectively.

points in r_0'. Suppose X_i has been constructed. By (6)–(8), X_i is in contact with $r_{\lfloor i+1 \rfloor}'$. By (4) and local connectedness of T, one can find a component X_{i+1} of $r_{\lfloor i+1 \rfloor}$ which has points in $r_{\lfloor i+1 \rfloor}'$, and a connected open set Z_i such that $Z_i \cap X_i$ and $Z_i \cap X_{i+1}$ are non-empty, but $Z_i \cap r_{\lfloor i+2 \rfloor}$ is empty.

To see that the X_i are distinct, let S_{i+1} and R_{i+1} be the components of $-X_{i+1}$ containing X_i and X_{i+2}, respectively. It suffices to show that we have $S_{i+1} \subseteq S_{i+2}^\circ$. Note that the connected set Z_i must intersect δS_{i+1}. By the second statement of Lemma 2.1, $\delta S_{i+1} \subseteq X_{i+1} \subseteq r_{\lfloor i+1 \rfloor}$. Also, $\delta S_{i+1} \subseteq -X_{i+1}$; hence, by (4), $\delta S_{i+1} \subseteq r_{\lfloor i \rfloor} \cup r_{\lfloor i+2 \rfloor}$. By Lemma 2.2, δS_{i+1} is connected, and therefore, by (4), δS_{i+1} is entirely contained either in $r_{\lfloor i \rfloor}$ or in $r_{\lfloor i+2 \rfloor}$. Since $Z_i \cap \delta S_{i+1} \neq \emptyset$ and $Z_i \cap r_{\lfloor i+2 \rfloor} = \emptyset$, we have $\delta S_{i+1} \not\subseteq r_{\lfloor i+2 \rfloor}$, so $\delta S_{i+1} \subseteq r_{\lfloor i \rfloor}$. Similarly, $\delta R_{i+1} \subseteq r_{\lfloor i+2 \rfloor}$. By (4), then, $\delta S_{i+1} \cap \delta R_{i+1} = \emptyset$, and since S_{i+1} and R_{i+1} are components of the same set, and have non-empty boundaries, they are disjoint. Hence, we obtain $S_{i+1} \subseteq (-R_{i+1})^\circ$, and since $X_{i+2} \subseteq R_{i+1}$, also $S_{i+1} \subseteq (-X_{i+2})^\circ$. So, using local connectedness again, S_{i+1} lies in the interior of a component of $-X_{i+2}$, and since $\delta S_{i+1} \subseteq X_{i+1} \subseteq S_{i+2}$, that component must be S_{i+2}. $\square$

Now we show how the $\mathcal{C}c$-formula φ_∞ can be transformed to $\mathcal{C}c^\circ$- and $\mathcal{B}c$-formulas with similar properties. Note first that all occurrences of c in φ_∞ have positive polarity. Let φ_∞° be the result of replacing them with the predicate c°. In Fig. 3, the connected regions mentioned in (6) are in fact interior-connected; hence φ_∞° is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$. Since interior-connectedness implies connectedness, φ_∞° entails φ_∞, and we obtain:

COROLLARY 3.2. *The $\mathcal{C}c^\circ$-formula φ_∞° is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$. On the other hand, if T is a locally connected, unicoherent space, then any tuple from $\mathsf{RC}(T)$ satisfying φ_∞° features sets that have infinitely many components.*

We next consider the language $\mathcal{B}c$. Observe that all occurrences of C in φ_∞ are negative. We eliminate these using the predicate c: we use the fact that, if the sum of two connected regions is not connected, then they are not in contact. If τ_1 and τ_2 are any terms, we employ the abbreviation

$$\mathsf{notC}(\tau_1, \tau_2) \quad = \quad c(\tau_1) \wedge c(\tau_2) \wedge \neg c(\tau_1 + \tau_2).$$

Observe that $\mathsf{notC}(\tau_1, \tau_2)$ is a $\mathcal{B}c$-formula. Furthermore, $\mathsf{notC}(\tau_1, \tau_2)$ implies $\neg C(\tau_1', \tau_2')$ for any $\tau_1' \leq \tau_1$ and $\tau_2' \leq \tau_2$. Now we replace (7) by

$$\mathsf{notC}(r_0' + r_1' + r_2' + r_3', \ t). \tag{7^c}$$

The resulting formula thus implies the original; on the other hand, it is satisfied by the configuration of Fig. 3. Next, we replace each conjunct $\neg C(r'_i, r_{\lfloor i+1 \rfloor} \cdot (-r'_{\lfloor i+1 \rfloor}))$ in (8) by

$$\mathsf{notC}(r'_i + s, \quad r_{\lfloor i+1 \rfloor} \cdot (-r'_{\lfloor i+1 \rfloor}) + t), \tag{8^c}$$

where s is a fresh variable. Again, the resulting formula implies the original, and, furthermore, is evidently satisfied by the configuration of Fig. 5, where s lies inside $\sum_{j=0}^{3} r'_j$, symmetrically to t lying inside $\sum_{j=0}^{3}(r_j \cdot (-r'_j))$. The only remaining occurrences of the
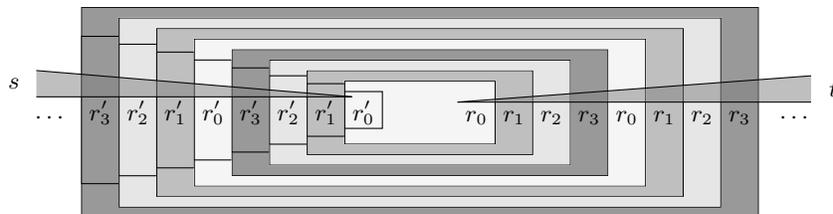
Fig. 5.   Region s lying inside $\sum_{j=0}^{3} r'_j$ and connecting the components of each r'_i.

contact predicate C are in (4). We deal with them by partitioning the regions; instead of each $\neg C(r_i, r_{\lfloor i+2 \rfloor})$ we consider the equivalent conjunction of 4 formulas:

$$\neg C(r'_i, r'_{\lfloor i+2 \rfloor}) \quad \wedge \quad \neg C(r_i \cdot (-r'_i), \quad r_{\lfloor i+2 \rfloor} \cdot (-r'_{\lfloor i+2 \rfloor})) \quad \wedge$$
$$\neg C(r_i \cdot (-r'_i), \quad r'_{\lfloor i+2 \rfloor}) \quad \wedge \quad \neg C(r'_i, \quad r_{\lfloor i+2 \rfloor} \cdot (-r'_{\lfloor i+2 \rfloor})),$$

which are then replaced by

$$\mathsf{notC}(r'_i + s_i, \quad r'_{\lfloor i+2 \rfloor} + s_{\lfloor i+2 \rfloor}) \quad \wedge \quad \mathsf{notC}(r_i \cdot (-r'_i) + t_i, \quad r_{\lfloor i+2 \rfloor} \cdot (-r'_{\lfloor i+2 \rfloor}) + t_{\lfloor i+2 \rfloor}) \quad \wedge$$
$$\mathsf{notC}(r_i \cdot (-r'_i) + t, \quad r'_{\lfloor i+2 \rfloor} + s) \quad \wedge \quad \mathsf{notC}(r'_i + s, \quad r_{\lfloor i+2 \rfloor} \cdot (-r'_{\lfloor i+2 \rfloor}) + t).$$

Again, the resulting formula implies the original. The conjuncts of the second row are satisfied by the configuration of Fig. 5. To see that the conjuncts in the first row are still satisfiable, we select regions $s_0, \ldots, s_3$, with s_i and s_{i+2} disjoint ($i = 0, 1$), such that each s_i ($0 \leq i < 4$) connects together the components of r'_i as shown in Fig. 6. In a symmetric way,
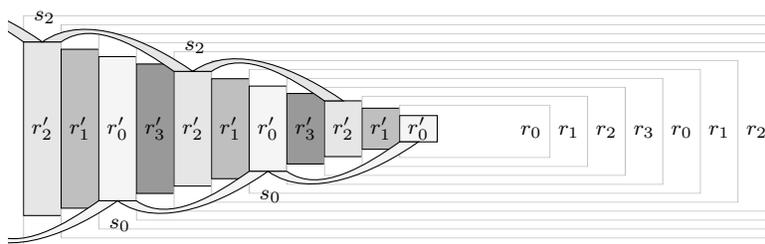
Fig. 6.   Disjoint connected regions s_0 and s_2 containing the components of r'_0 and r'_2, respectively.

select regions $t_0, \ldots, t_3$, with t_i and t_{i+2} disjoint ($i = 0, 1$), such that each t_i ($0 \leq i < 4$) connects together the components of $r_i \cdot (-r'_i)$. Transforming φ_∞ in the way just described, we obtain a $\mathcal{B}c$-formula φ_∞^c with the required properties.

THEOREM 3.3. *The $\mathcal{B}c$-formula φ_∞^c is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$. On the other hand, if T is a locally connected, unicoherent space, then any tuple from $\mathsf{RC}(T)$ satisfying φ_∞^c features sets that have infinitely many components.*

The results of this section make no reference to the language $\mathcal{B}c^\circ$. In fact, an analogue of Theorem 3.3 for $\mathcal{B}c^\circ$ will be proved in the special case $n = 2$, in Sec. 5.1, using a planarity argument. For $n \geq 3$, however, this result fails, as we show in Sec. 6. As we observed above, Theorem 3.3 shows that, for all $n \geq 2$, $Sat(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^n)) \neq Sat(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}^n))$. The reader will recall from Sec. 1.3 that the corresponding inequations for the language $\mathcal{B}c^\circ$ hold anyway, by (3). Finally, we remark on the case of the real line, $\mathbb{R}$, which was considered in [Kontchakov et al. 2010b]. The analogue of Theorem 3.1 for the case $n = 1$ holds (though we need to use a different formula to force an infinitude of components); however, the analogue of Theorem 3.3 for $n = 1$ fails: indeed, we have $Sat(\mathcal{B}c, \mathsf{RC}(\mathbb{R})) = Sat(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}))$.

4. UNDECIDABILITY: THE POLYHEDRAL CASE

We use the techniques of the previous section to prove that the satisfiability problem for any of the languages $\mathcal{B}c$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$ over the frame $\mathsf{RCP}(\mathbb{R}^n)$, $n \geq 2$, is undecidable. Recall that a frame $(T, \mathcal{S})$ is unicoherent if T is unicoherent; and $(T, \mathcal{S})$ is finitely decomposible if, for all $s \in \mathcal{S}$, there exist connected elements $s_1, \ldots, s_k$ of $\mathcal{S}$, such that $s = s_1 + \cdots + s_k$.

THEOREM 4.1. *Let $\mathcal{K}$ be any class of unicoherent, finitely decomposible frames, such that $\mathcal{K}$ contains some frame of the form $(\mathbb{R}^n, \mathcal{S})$, $n \geq 2$, where $\mathsf{RCP}(\mathbb{R}^n) \subseteq \mathcal{S}$. Then the problem $Sat(\mathcal{C}c, \mathcal{K})$ is r.e.-hard.*

PROOF. We proceed via a reduction of the Post correspondence problem (PCP), constructing, for any instance W, a formula φ_W with the property that (i) if W is positive then φ_W is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$, $n \geq 2$, and (ii) if φ_W is satisfiable over a unicoherent, finitely decomposible frame then W is positive.

We begin with a sketch of the proof strategy. Let T and U be the alphabets of W. Our formula φ_W enforces the existence of two finite sequences of regions, $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$, with each sequence forming an 'onion-like' configuration, much as in the proof of Theorem 3.1. Each region A_i is 'coloured' with some letter of U, and each region B_j is likewise coloured with some tile of T, thus yielding words $\upsilon \in \mathsf{U}^*$ and $\tau \in \mathsf{T}^*$. In addition, φ_W establishes a pair of regions, e^1 and e^2, ensuring that every element $B_j \cdot e^k$ is the union of one or more of the elements $A_i \cdot e^k$, $k = 1, 2$. Thus, the word υ is segmented in two different ways—once by e^1 and once by e^2—with each segment corresponding to a position in the word τ. Finally, φ_W ensures that, if B_j is coloured with some tile $t \in \mathsf{T}$, then the regions A_i such that $A_i \cdot e^1 \subseteq B_j \cdot e^1$ are coloured with the letters of U so as to spell out the upper string on tile t; similarly, for e^2, but spelling out the lower string of t. In this way we establish that any satisfying assignment for φ_W in $\mathsf{RCP}(\mathbb{R}^n)$ yields a solution to W.

For example, let $\mathsf{U} = \{0, 1\}$ and T contain the tiles t_1: $\boxed{\begin{smallmatrix} 101 \\ 1 \end{smallmatrix}}$, t_2: $\boxed{\begin{smallmatrix} 01 \\ 0 \end{smallmatrix}}$ and t_3: $\boxed{\begin{smallmatrix} 1 \\ 1011 \end{smallmatrix}}$. Thus, W is a positive PCP-instance since the tile string $t_1 t_2 t_3$ yields, on both top and bottom rows, 101011. Notice how $t_1 t_2 t_3$ segments 101011 as two sequences of substrings with different lengths: 3, 2, 1 on the top, and 1, 1, 4 on the bottom. Fig. 7 shows an arrangement satisfying our formula φ_W, where $A_1, \ldots, A_6$ are indicated by thin outlines, and B_1, B_2, B_3 by thick outlines. (The region $A_7 = B_4$ functions as a full stop, and may be ignored here.) Notice how e^1 and e^2 segment $A_1, \ldots, A_6$ in different ways, with lengths 3, 2, 1 in e^1, and 1, 1, 4 in e^2, just like our solution of W. By colouring the regions $A_1, \ldots, A_6$ with the letters 1, 0, 1, 0, 1, 1, respectively, and the regions B_1, B_2, B_3 with the tiles t_1, t_2 and t_3, respectively, we satisfy the remaining constraints of φ_W.

We now proceed with the detailed proof (in which $\lfloor i \rfloor$ means i modulo 4). Let the PCP-instance $\mathsf{W} = (\mathsf{w}^1, \mathsf{w}^2)$ over alphabets T and U be given, and let $\vec{r} = (r_0, \ldots, r_3)$ and $\vec{s} = (s_0, \ldots, s_3)$ be quadruples of variables. The first conjuncts of φ_W ensure that $\vec{r}$ and $\vec{s}$

are sub-cyclic partitions:

$$\text{sc-part}(r_0, r_1, r_2, r_3), \tag{9}$$

$$\text{sc-part}(s_0, s_1, s_2, s_3). \tag{10}$$

By Lemma 2.4, the component graphs $H(\vec{r})$ and $H(\vec{s})$ are trees. Thus, any two vertices of $H(\vec{r})$ are joined by a unique path, and likewise for $H(\vec{s})$. The vertices of $H(\vec{r})$ will be used to represent letters in some word $\upsilon \in U^*$, and those of $H(\vec{s})$, letters in some word $\tau \in T^*$.

Let e^1 and e^2 be fresh variables. We shall use these to represent the morphisms w^1 and w^2, respectively. The next conjuncts of φ_W ensure that, for all $0 \le i < 4$, the components of both $r_i \cdot e^1$ and $r_i \cdot e^2$ are coloured by the elements $\vec{s}$, as defined in Sec. 2:

$$\bigwedge_{k=1,2} \bigwedge_{0 \le i < 4} \text{colour}(r_i \cdot e^k; \ \ s_0, s_1, s_2, s_3). \tag{11}$$

Fig. 7 shows a configuration conforming to conditions (9)–(11). In this arrangement, $H(\vec{r})$ has vertices $\{A_1, \ldots, A_7\}$, where A_i is a component of $r_{\lfloor i \rfloor}$, and $H(\vec{s})$ has vertices $\{B_1, \ldots, B_4\}$ (indicated by thick boundaries), where B_j is a component of $s_{\lfloor j \rfloor}$. Observe that, for $0 \le i < 4$ and $1 \le k \le 2$, each component of $r_i \cdot e^k$ is included in exactly one of $s_0, \ldots, s_3$, and hence in a single vertex B_j of $H(\vec{s})$; however, outside $e^1 + e^2$, elements of $H(\vec{s})$ may intersect elements of $H(\vec{r})$ without including them. A word of warning: in the configuration of Fig. 7, the various sets $A_i \cdot e^k$ and $B_j \cdot e^k$ are all *connected*; however, the formula φ_W does not enforce this. That is, there is nothing to prevent the sets e^k from chopping elements of $H(\vec{r})$ and $H(\vec{s})$ into several pieces.
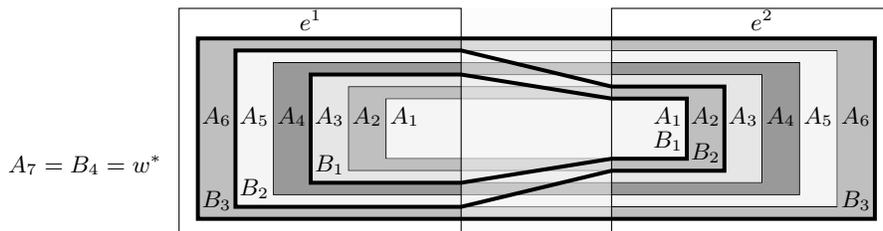
Fig. 7. Arrangement satisfying (9)–(14) such that $H(\vec{r})$ has nodes $A_1, \ldots, A_7$ (different shades) and $H(\vec{s})$ has nodes $B_1, \ldots, B_4$ (surrounded by thick lines).

Let w^* be a fresh variable. The next conjuncts of φ_W ensure that the graphs $H(\vec{r})$ and $H(\vec{s})$ contain a common vertex, w^*:

$$c(w^*) \wedge (w^* \ne 0), \tag{12}$$

$$\text{colour}(w^*; \ \ r_0, r_1 r_2, r_3) \ \ \wedge \ \ \bigwedge_{0 \le i < 4} \text{colour}(r_i; \ w^*, -w^*), \tag{13}$$

$$\text{colour}(w^*; \ \ s_0, s_1, s_2, s_3) \ \ \wedge \ \ \bigwedge_{0 \le i < 4} \text{colour}(s_i; \ w^*, -w^*). \tag{14}$$

To see why, note that the first conjunct of (13) ensures that w^* is included in one of the sets r_i, and hence—since it is connected, by (12)—in one of the vertices of $H(\vec{r})$; on the other hand, the remaining conjuncts of (13) ensure that every vertex of $H(\vec{r})$ is included in either w^* or $-w^*$. Since w^* is non-empty, it must be identical to a single vertex of $H(\vec{r})$. The same conclusion holds for $H(\vec{s})$ using (14). In the arrangement of Fig. 7, we have $w^* = A_7 = B_4$.

We need to impose a little more structure on the graphs $H(\vec{r})$ and $H(\vec{s})$. Let $r'_0, \ldots, r'_3$ and w_1 be fresh variables, and let φ_W contain the conjuncts:

$$c(w_1) \;\wedge\; (w_1 \leq r_1) \;\wedge\; (w_1 \leq s_1) \;\wedge\; (w_1 \cdot w^* = 0), \tag{15}$$

$$\bigwedge_{0 \leq i < 4} (r'_i \leq r_i), \tag{16}$$

$$\bigwedge_{k=1,2} (e^k \cdot r'_1 \cdot w_1 \neq 0). \tag{17}$$

Since w_1 is a non-empty, connected subset of both r_1 and s_1, let A_1 be the component of r_1 including w_1, and let B_1 be the component of s_1 including w_1. It follows that $A_1 \leq B_1$; the final conjunct of (15) ensures that A_1 and B_1 are both distinct from w^*. In the sequel, we shall construct a path in the graph $H(\vec{r})$ from A_1 to w^*, and a path in the graph $H(\vec{s})$ from B_1 to w^*. The proof will hinge on analysing the properties of these paths.

Let t be a fresh variable, and let φ_W contain the conjuncts:

$$\bigwedge_{k=1,2} (e^k \cdot t \neq 0), \tag{18}$$

$$\bigwedge_{k=1,2} \; \bigwedge_{0 \leq i < 4} c(e^k \cdot ((r'_i \cdot (-w^*)) + r_{\lfloor i+1 \rfloor} + t)), \tag{19}$$

$$\bigwedge_{0 \leq i < 4} \neg C(r'_i, \; t), \tag{20}$$

$$\bigwedge_{0 \leq i < 4} \neg C(r'_i, \; r_{\lfloor i+1 \rfloor} \cdot (-r'_{\lfloor i+1 \rfloor})). \tag{21}$$

Fixing the value of k for the moment ($1 \leq k \leq 2$), from (17), select a point q_1^k in the interior of $e^k \cdot r'_1 \cdot w_1$. By (15), $q_1^k \notin w^*$. Let X_1^k be the component of $e^k \cdot r_1$ containing q_1^k, and A_1^k the component of r_1 including X_1^k; note that A_1^k is a vertex of the graph $H(\vec{r})$. Evidently, $A_1^k \neq w^*$, and so $X_1^k \leq (e^k \cdot r_1 \cdot (-w^*))$. Now suppose X_i^k has been defined and contains some point $q_i^k \in e^k \cdot r'_{\lfloor i \rfloor} \cdot (-w^*)$. From (18)–(20), X_i^k contains a point $q_{i+1}^k \in e^k \cdot r_{\lfloor i+1 \rfloor}$, which, by (21), is in fact in $e^k \cdot r'_{\lfloor i+1 \rfloor}$. Let X_{i+1}^k be the component of $e^k \cdot r_{\lfloor i+1 \rfloor}$ containing q_{i+1}^k, and A_{i+1}^k the component of $r_{\lfloor i+1 \rfloor}$ including X_{i+1}^k; again, A_{i+1}^k is a vertex of $H(\vec{r})$. Note that either $q_{i+1}^k \in w^*$ or $q_{i+1}^k \notin w^*$, and in the latter case, $q_{i+1}^k \in e^k \cdot r'_{\lfloor i+1 \rfloor} \cdot (-w^*)$. This process either continues forever, or, at some point, $q_{i+1}^k \in w^*$. But now consider any sequence $A_1^k, A_2^k, \ldots A_\ell^k$ obtained in this way. Evidently, (A_i, A_{i+1}) is an edge of $H(\vec{r})$ for all $1 \leq i < \ell$; moreover, since $A_i \leq r_{\lfloor i \rfloor}$, we see from (9) that $A_i \neq A_{i+2}$ for all $1 \leq i < \ell - 1$, whence, since $H(\vec{r})$ is a tree, $A_1^k, A_2^k, \ldots A_\ell^k$ is a path (i.e., has no repeated nodes). It follows that, for some value of i, denoted by n^k, the condition $q_{i+1}^k \in w^*$ must hold, for otherwise, $H(\vec{r})$ would contain an infinite path, contradicting the assumption that the frame in question is finitely decomposable. Since $q_{n^k+1} \in X_{n^k+1}^k$, we have $A_{n^k+1}^k = w^*$, and hence there is a path $A_1^k, A_2^k, \ldots, A_{n^k+1}^k$ in $H(\vec{r})$ from $A_1^k = A_1$ to $A_{n^k+1}^k = w^*$. Indeed, this must be the *same* path for both $k = 1, 2$, so that we may drop the k-superscripts, and write: $A_1, A_2, \ldots, A_n, A_{n+1}$, where $A_{n+1} = w^*$. (Note that the letter n here is simply a convenient label for the length of this path: it has nothing to do with the dimension of the space.) It is important to remember that the sets X_i^1 and X_i^2, for a fixed value of i, will in general be distinct (Fig. 8).

Let us now turn our attention to the graph $H(\vec{s})$. Fixing the value of k again, consider the sequence $X_1^k, \ldots, X_{n+1}^k$. Since X_i^k is a connected subset of $e^k \cdot r_{\lfloor i \rfloor}$, it follows from (11) that
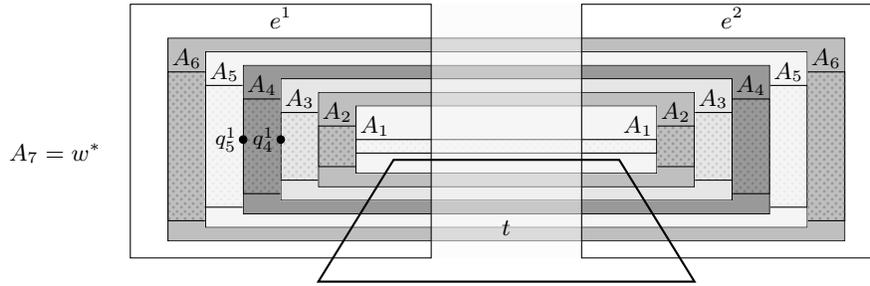
Fig. 8. Regions $A_1, \ldots, A_7$ and e^1, e^2 from Fig. 7; the components of $r'_{\lfloor i \rfloor}$ lying in each A_i are indicated by shading.

each X_i^k is included in some vertex of $H(\vec{s})$, say, $\hat{B}_i^k$. Thus, we have a sequence $\hat{B}_1^k, \ldots, \hat{B}_{n+1}^k$, where $A_1 \leq \hat{B}_1^k$ and $\hat{B}_{n+1}^k = w^*$. Of course, this sequence may contain adjacent duplicates, since there is nothing to stop X_i^k and X_{i+1}^k being included in the same vertex of $H(\vec{s})$. Furthermore, the two sequences (for $k = 1, 2$) may be distinct, since, for fixed i, there is nothing to stop X_i^1 and X_i^2 lying in different vertices of $H(\vec{s})$; see Figs. 7 and 8. But now suppose we remove adjacent duplicates, obtaining sequences: $B_1^k, \ldots, B_{m^k+1}^k$, where $A_1 \leq B_1^k$ and $B_{m^k+1}^k = w^*$ with $m^k \leq n$. Thus, every B_j^k is the result of coalescing a contiguous block of identical vertices $\hat{B}_i^k$. Evidently, (B_j^k, B_{j+1}^k) must be an edge of $H(\vec{s})$, for $1 \leq j \leq m^k$.

Now let φ_W contain the conjuncts:

$$\bigwedge_{k=1,2} \bigwedge_{0 \leq i < 4} \bigwedge_{0 \leq j < 4} \neg C(e^k \cdot r_i \cdot s_j, \ e^k \cdot r_{\lfloor i+1 \rfloor} \cdot s_{\lfloor j-1 \rfloor}). \tag{22}$$

We claim that $B_j^k \leq s_{\lfloor j \rfloor}$ for all $1 \leq j \leq m^k$. The proof is by induction on j. By (15), $X_1^k \leq s_1$, and so $B_1^k \leq s_1$. Suppose, then, $B_j^k \leq s_{\lfloor j \rfloor}$ for some $1 \leq j < m^k$. Let $\hat{B}_i^k$, be the last vertex in the block coalescing to B_j^k, so that $\hat{B}_{i+1}^k$ is the first element of the block coalescing to B_{j+1}^k. Thus, $X_i^k \leq r_{\lfloor i \rfloor}$ and $X_{i+1}^k \leq r_{\lfloor i+1 \rfloor}$. But (10) and (22) then ensure that either $B_{j+1}^k \leq s_{\lfloor j \rfloor}$ or $B_{j+1}^k \leq s_{\lfloor j+1 \rfloor}$; and the former is impossible, since then $\hat{B}_i^k$ and $\hat{B}_{i+1}^k$ would have coalesced to the same block. This proves the claim. By (10) (and the fact that $B_{m^k-1}^k \neq w^*$), we then have $B_j^k \neq B_{j+2}^k$ for $1 \leq j < m^k$. And since $H(\vec{s})$ is a tree, it follows that $B_1^k, \ldots, B_{m^k+1}^k$ is a path through $H(\vec{s})$ with $B_1^1 = B_1^2$ and $B_{m^1+1}^1 = B_{m^2+1}^2 = w^*$. Indeed, this is the *same* path through $H(\vec{s})$ for both values of k, so that we can again drop the superscripts, and just write: $B_1, \ldots, B_m, B_{m+1}$, where $B_{m+1} = w^*$.

Taking stock, we see that, for each $k = 1, 2$, the path $A_1, \ldots, A_n$ may be grouped into m contiguous blocks $E_1^k, \ldots, E_m^k$ by taking the vertex A_i to be in the jth block E_j^k just in case $e^k \cdot X_i^k \leq B_j$. We may depict this grouping as follows:

$$A_1, \ldots, A_n \;=\; \underbrace{A_{1,1}^k, \ldots, A_{1,h_1^k}^k}_{E_1^k}, \ldots, \underbrace{A_{j,1}^k, \ldots, A_{j,h_j^k}^k}_{E_j^k}, \ldots, \underbrace{A_{m,1}^k, \ldots, A_{m,h_m^k}^k}_{E_m^k}.$$

It is important to realize that, although there is only one path $A_1, \ldots, A_{n+1}$ and one path $B_1, \ldots, B_{m+1}$, the two values $k = 1$ and $k = 2$ will in general give rise to different groupings of the vertices of the former into blocks corresponding to the vertices of the latter, as in the example of Fig. 8 (hence, the two sequences of indices $h_1^k, \ldots, h_m^k$).

Recall the PCP-instance $W = (w^1, w^2)$ over the alphabets T and U, which we wish to encode. We regard the elements of these alphabets as fresh variables, and order them in some way to form tuples $\vec{t}$ and $\vec{u}$. We use these variables to colour the vertices of $H(\vec{s})$ and $H(\vec{r})$, respectively, by taking φ_W to contain the conjuncts:

$$\mathsf{part}(\vec{t}) \;\wedge\; \bigwedge_{0 \leq j < 4} \mathsf{colour}(s_j;\ \vec{t}) \qquad \wedge \qquad \mathsf{part}(\vec{u}) \;\wedge\; \bigwedge_{0 \leq i < 4} \mathsf{colour}(r_i;\ \vec{u}). \qquad (23)$$

In this way, the path $A_1, \ldots, A_n$ defines a word $\upsilon \in U^*$, and the path $B_1, \ldots, B_m$ defines a word $\tau \in T^*$. Using the groupings of the sequence $A_1, \ldots, A_n$ obtained above, we shall write conjuncts of φ_W ensuring that $w^k(\tau) = \upsilon$ for $k = 1, 2$. This will mean that, if φ_W has a satisfying assignment over some frame $\mathcal{S} \in \mathcal{K}$, then the PCP-instance W is positive.

For $k = 1, 2$, let $\left\{ p_{h,\ell}^k \mid 1 \leq h \leq |T|, 1 \leq \ell \leq |w^k(t_h)| \right\}$ be a collection of fresh variables, enumerated in some way as $\vec{p}^k$, which we shall use to colour the vertices of $H(\vec{r})$. That is, we add to φ_W the conjuncts:

$$\bigwedge_{k=1,2} \left(\mathsf{part}(\vec{p}^k) \;\wedge\; \bigwedge_{0 \leq i < 4} \mathsf{colour}(r_i;\ \vec{p}^k) \right). \qquad (24)$$

We refer to these variables as *position colours*, because we are to think of $p_{h,\ell}^k$ as denoting the ℓth position in the word $w^k(t_h)$. In particular, any position colour $p_{h,\ell}^k$ is naturally associated to the letter t_h of T. Fixing k for the moment, consider the vertices $A_{j,1}^k, \ldots, A_{j,h_j^k}^k$ grouped into the jth block, E_j^k. What ensures that these vertices belong to one block is the existence of a single B_j such that, if A_i is one of these vertices, then the corresponding set $X_i^k \leq A_i$ is included in B_j. By (23), $X_i^k \leq t_h$, for some $1 \leq h \leq |T|$. It follows that the conjuncts

$$\bigwedge_{k=1,2} \bigwedge_{1 \leq h \leq |T|} \left(t_h \leq \sum_{1 \leq \ell \leq |w^k(t_h)|} p_{h,\ell}^k \right) \qquad (25)$$

ensure that the vertices in this block are assigned 'position' colours associated to the common tile t_h. We proceed to write constraints ensuring that these colours are assigned in exactly the canonical order: $p_{h,1}^k, \ldots p_{h,|w^k(t_h)|}^k$ (from which, incidentally, it follows that $|w^k(t_h)| = h_j^k$). We begin by adding to φ_W the conjuncts:

$$\bigwedge_{k=1,2} \bigwedge_{1 \leq h \leq |T|} \bigwedge_{1 < \ell \leq |w^k(t_h)|} \left((w_1 \cdot p_{h,\ell}^k = 0) \;\wedge\; \bigwedge_{0 \leq i < 4} \neg C(s_i,\ s_{\lfloor i+1 \rfloor} \cdot p_{h,\ell}^k) \right). \qquad (26)$$

These ensure that the first vertex of each block is assigned one of the colours $p_{h,1}^k$, for $1 \leq h \leq |T|$. The rules for colouring successive vertices can now be simply stated. Consider the following binary relation on the variables in $\vec{p}^k$:

$$\Psi^k = \left\{ (p_{h,\ell}^k, p_{h,\ell+1}^k) \mid 1 \leq h \leq |T|, 1 \leq \ell < |w^k(t_h)| \right\} \cup \left\{ (p_{h,|w^k(t_h)|}^k, p_{h',1}^k) \mid 1 \leq h, h' \leq |T| \right\}.$$

This relation captures the rules of possible succession for colouring by the variables $p_{h,\ell}^k$: if A_i is coloured $p_{h,\ell}^k$, where ℓ indicates a non-final position in $w^k(t_h)$, then A_{i+1} must be coloured $p_{h,\ell+1}^k$; and, if A_i is coloured $p_{h,|w^k(t_h)|}^k$, indicating the final position in $w^k(t_h)$, then A_{i+1} must be coloured $p_{h',1}^k$ for some $1 \leq h' \leq |T|$. We therefore add to φ_W the conjuncts

$$\bigwedge_{k=1,2} \bigwedge_{0 \leq i < 4} \bigwedge_{(p_{h,\ell}^k, p_{h',\ell'}^k) \notin \Psi^k} \neg C(p_{h,\ell}^k \cdot r_i,\ p_{h',\ell'}^k \cdot r_{\lfloor i+1 \rfloor}). \qquad (27)$$

We also ensure that each block spells out only one word. That is, we ensure that no vertex of the sequence $A_1, \ldots, A_n$ can be coloured with the starting position in a word if the previous

vertex belongs to the same block:

$$\bigwedge_{k=1,2} \bigwedge_{0 \leq i < 4} \bigwedge_{0 \leq j < 4} \bigwedge_{1 \leq h \leq |\mathsf{T}|} \neg C(r_i \cdot s_j, \ r_{\lfloor i+1 \rfloor} \cdot s_j \cdot p_{h,1}^k). \tag{28}$$

Lastly, we ensure that the final vertex of the final block corresponds to the final position in a word. In other words, we ensure that the vertex A_n (which contacts $A_{n+1} = w^*$) is coloured $p_{h,|\mathsf{w}^k(t_h)|}^k$, for some $1 \leq h \leq |\mathsf{T}|$:

$$\bigwedge_{k=1,2} \bigwedge_{1 \leq h \leq |\mathsf{T}|} \bigwedge_{1 \leq \ell < |\mathsf{w}^k(t_h)|} \neg C(p_{h,\ell}^k, \ w^*). \tag{29}$$

At this stage, we have ensured that, for $k = 1$ and $k = 2$, vertices of each block $A_{j,1}^k, \ldots, A_{j,h_j^k}^k$, $1 \leq j \leq m$, are coloured $p_{h,1}^k, \ldots, p_{h,|\mathsf{w}^k(t_h)|}^k$, where t_h is the jth letter of the word τ. This easily enables us to enforce the sought-after conditions $\mathsf{w}^k(\tau) = v$ for $k = 1, 2$. Denoting by $u_{h,\ell}^k$ the variable in $\vec{u}$ (i.e. that letter of the alphabet U) that is the ℓth letter in the word $\mathsf{w}^k(t_h)$, we add to φ_{W} the conjuncts:

$$\bigwedge_{k=1,2} \bigwedge_{1 \leq h \leq |\mathsf{T}|} \bigwedge_{1 \leq \ell \leq |\mathsf{w}^k(t_h)|} \left(p_{h,\ell}^k \leq u_{h,\ell}^k \right). \tag{30}$$

That $\mathsf{w}^k(\tau) = v$ for $k = 1, 2$ then follows from the fact that each vertex A_i is assigned a unique colour from $\vec{u}$. Thus, if φ_{W} is satisfiable over $\mathcal{K}$ then W is positive.

Conversely, if W is positive, it is obvious that φ_{W} may be satisfied over $\mathsf{RCP}(\mathbb{R}^n)$, $n \geq 2$, by suitably extending a configuration similar to that shown in Fig. 7. $\square$

COROLLARY 4.2. *Let $\mathcal{K}$ be any class of unicoherent, finitely decomposable frames, such that $\mathcal{K}$ contains some frame of the form $(\mathbb{R}^n, \mathcal{S})$, $n \geq 2$, where $\mathsf{RCP}(\mathbb{R}^n) \subseteq \mathcal{S}$. Then the problem $Sat(\mathcal{C}c^\circ, \mathcal{K})$ is r.e.-hard.*

PROOF. We start with the formula φ_{W} of Theorem 4.1, and replace all occurrences of c by c°. Denote the resulting $\mathcal{C}c^\circ$-formula by $\varphi_{\mathsf{W}}^\circ$. Since all atoms of the form $c(\tau)$ in φ_{W} occur with positive polarity, $\varphi_{\mathsf{W}}^\circ$ entails φ_{W}. On the other hand, by inspection of Fig. 7, we see that if W is positive, then $\varphi_{\mathsf{W}}^\circ$ will be satisfiable in $\mathsf{RCP}(\mathbb{R}^2)$, and hence in $\mathsf{RCP}(\mathbb{R}^n)$ for all $n \geq 2$. This proves the corollary. $\square$

COROLLARY 4.3. *Let $\mathcal{K}$ be any class of unicoherent, finitely decomposable frames, such that $\mathcal{K}$ contains some frame of the form $(\mathbb{R}^n, \mathcal{S})$, $n \geq 2$, where $\mathsf{RCP}(\mathbb{R}^n) \subseteq \mathcal{S}$. Then the problem $Sat(\mathcal{B}c, \mathcal{K})$ is r.e.-hard.*

PROOF. Consider again the formula φ_{W} of Theorem 4.1. Since all occurrences of the predicate C in φ_{W} have negative polarity, we can replace them with $\mathcal{B}c$-formulas as we did in the proof of Theorem 3.3. The resulting formula φ_{W}^c implies φ_{W}, and is satisfiable in $\mathsf{RCP}(\mathbb{R}^n)$, for all $n \geq 2$, whenever W is positive. $\square$

5. UNDECIDABILITY: THE PLANE CASE

In Sec. 3, we established that, if $\mathcal{L}$ is any of the languages $\mathcal{B}c$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$, there exists an $\mathcal{L}$-formula that is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but only by regions having infinitely many components. Nothing was mentioned in this regard about the language $\mathcal{B}c^\circ$. In Sec. 4, we established the undecidability of $Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$, $n \geq 2$, where $\mathcal{L}$ is any of the languages $\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{C}c^\circ$. Nothing was mentioned in this regard about the problems $Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^n))$, $n \geq 2$, or indeed about the problems $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n))$ where $\mathcal{L}$ is any of $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$, $\mathcal{C}c^\circ$. In this section, we complete the picture in the case $n = 2$. Specifically, we establish the existence of a $\mathcal{B}c^\circ$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, but only by regions having infinitely

many components; and we establish the undecidability of the problems $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ and $Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$, where $\mathcal{L}$ is any of $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$.

We employ the standard terminology of Jordan arcs and curves: a *non-degenerate Jordan arc* is a continuous, 1–1 function α from the unit interval to $\mathbb{R}^2$; a *degenerate Jordan arc* is a constant function from the unit interval to $\mathbb{R}^2$; a *Jordan arc* is a degenerate Jordan arc or a non-degenerate Jordan arc. A *Jordan curve* is a continuous, 1–1 function from the unit circle to $\mathbb{R}^2$. Where no confusion results, we identify Jordan arcs and curves with their loci (ranges). If α_1 and α_2 are Jordan arcs which intersect in the unique point $\alpha_1(1) = \alpha_2(0)$, then we write $\alpha_1\alpha_2$ to denote, ambiguously, any Jordan arc α with locus $\alpha_1 \cup \alpha_2$ such that $\alpha(0) = \alpha_1(0)$ and $\alpha(1) = \alpha_2(1)$. We employ the following notation: if α is a Jordan arc, α^{-1} denotes a Jordan arc with the same locus but opposite direction, e.g., $\alpha^{-1}(t) = \alpha(1 - t)$, for all $0 \le t \le 1$. If, in addition, $p_1 = \alpha(t_1)$, $p_2 = \alpha(t_2)$ are points on α with $t_2 \ge t_1$, $\alpha[p_1, p_2]$ denotes a Jordan arc whose locus is the segment of α between p_1 and p_2, and which has the same direction as α: $\alpha[p_1, p_2](t) = \alpha(t_1 + t(t_2 - t_1))$. An *end-cut* to p in a set X is a non-degenerate Jordan arc $\alpha \subseteq X^\circ \cup \{p\}$ such that $\alpha(1) = p$. A *cross-cut* in X is a non-degenerate Jordan arc α in X intersecting the boundary δX of X only at its endpoints $\alpha(0)$ and $\alpha(1)$.

5.1. Forcing infinitely many components with $\mathcal{B}c^\circ$

We begin by showing that there exists a $\mathcal{B}c^\circ$-formula that is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, but only by regions having infinitely many components. Many of the techniques we employ will prove useful in Sec. 5.2. Our basic tools are two formulas that enable us to construct Jordan arcs and curves containing points in specified regions. But before presenting these formulas, we need to establish the following property of regular closed sets:

LEMMA 5.1. *Let T be any topological space, and a, b_1 and b_2 elements of $\mathsf{RC}(T)$ such that $b_1 \cdot b_2 = 0$. Then $(a + b_1)^\circ \cap (a + b_2)^\circ = a^\circ$.*

PROOF. Note that, for any $s \in \mathsf{RC}(T)$, $-s = T \setminus s^\circ$. Since $b_1 \cdot b_2 = 0$, we have $-a = \sum_{i=1,2}(-(a + b_i))$, which is then equal to $\bigcup_{i=1,2}(T \setminus (a + b_i)^\circ) = T \setminus (\bigcap_{i=1,2}(a + b_i)^\circ)$. □

Consider now the following $\mathcal{B}c^\circ$-formula:

$$\mathsf{frame}^\circ(r_0, \ldots, r_{n-1}) \quad = \bigwedge_{0 \le i < n} \big((r_i \neq 0) \wedge c^\circ(r_i + r_{\lfloor i+1 \rfloor})\big) \ \wedge \bigwedge_{0 \le i < j < n} (r_i \cdot r_j = 0),$$

where $\lfloor k \rfloor$ denotes the value of k modulo n. This formula allows us to construct Jordan curves that contain points of all regions $r_0, \ldots, r_{n-1}$:

LEMMA 5.2. *Fix $n \ge 3$, and let $(a_0, \ldots, a_{n-1})$ be a tuple of elements of $\mathsf{RC}(\mathbb{R}^2)$ satisfying $\mathsf{frame}^\circ(r_0, \ldots, r_{n-1})$. Then there exist Jordan arcs $\alpha_0, \ldots, \alpha_{n-1}$ and points $p_0, \ldots, p_{n-1}$ such that: for all $0 \le i < n$, α_i is a Jordan arc from p_i to $p_{\lfloor i+1 \rfloor}$, with $\alpha_i \subseteq (a_i + a_{\lfloor i+1 \rfloor})^\circ$; $\alpha_0 \cdots \alpha_{n-1}$ is a Jordan curve lying in $(a_0 + \cdots + a_{n-1})^\circ$; and $p_i \in a_i^\circ$, for all $0 \le i < n$.*

PROOF. For every $0 \le i < n$, select points p_i' in the interior of a_i and connect each p_i' to $p_{\lfloor i+1 \rfloor}'$ with an arc $\alpha_i'' \subseteq (a_i + a_{\lfloor i+1 \rfloor})^\circ$. Let p_1 be the first point on α_0'' that is on α_1'', let α_0' be the initial segment of α_0'' ending at p_1, and let α_1' be the final segment of α_1'' starting at p_1. Note that $\alpha_0' \cap \alpha_1' = \{p_1\}$. For $2 \le i < n - 1$, let p_i be the first point on α_{i-1}' that is on α_i'', let α_{i-1} be the initial segment of α_{i-1}' ending at p_i, and let α_i' be the final segment of α_i'' starting at p_i. Note that $\alpha_{i-1} \cap \alpha_i' = \{p_i\}$. Finally, let p_0 be the first point on α_{n-1}' that is on α_0', let α_{n-1} be the initial segment of α_{n-1}' ending at p_0, and let α_0 be the final segment of α_0' starting at p_0. Note that $\alpha_{n-1} \cap \alpha_0 = \{p_0\}$. By construction, for every $0 \le i < n$, α_i connects points p_i and $p_{\lfloor i+1 \rfloor}$, and $\alpha_{\lfloor i-1 \rfloor} \cap \alpha_i = \{p_i\}$, whence, by Lemma 5.1, $p_i \in a_i^\circ$. □

Consider now the $\mathcal{C}c^\circ$-formula, for $n > 1$,

$$\mathsf{stack}^\circ(r_1,\ldots,r_n) \;=\; \bigwedge_{1 \le i \le n} c^\circ(r_i + \cdots + r_n) \;\wedge\; \bigwedge_{1 \le i < j \le n} (r_i \cdot r_j = 0) \;\wedge\; \bigwedge_{\substack{1 \le i < j \le n \\ j-i>1}} \neg C(r_i, r_j),$$

which will allow us to construct arcs containing points of all the regions $r_1,\ldots,r_n$:

LEMMA 5.3. *Let $(a_1,\ldots,a_n)$ be a tuple of elements of $\mathsf{RC}(\mathbb{R}^2)$ satisfying $\mathsf{stack}^\circ(r_1,\ldots,r_n)$. Then every point $p_1 \in a_1^\circ$ can be connected to every point $p_n \in a_n^\circ$ by a Jordan arc $\alpha = \alpha_1 \cdots \alpha_{n-1}$ such that, for all $1 \le i < n$, α_i is a non-degenerate Jordan arc in $(a_i + a_{i+1})^\circ$, starting at a point $p_i \in a_i^\circ$.*

PROOF. Since $a_1 + \cdots + a_n$ is interior-connected, let $\alpha_1' \subseteq (a_1 + \cdots + a_n)^\circ$ be a Jordan arc connecting p_1 to p_n (see Fig. 9). Since $\neg C(a_1, (a_3 + \cdots + a_n))$, α_1' must contain a point $p_1' \in a_2^\circ$ such that $\alpha_1'[p_1, p_1'] \subseteq (a_1 + a_2)^\circ$. For convenience, let $p_0 = p_1$, let α_0 be the degenerate Jordan arc located at p_1, and let a_0 be the empty region.
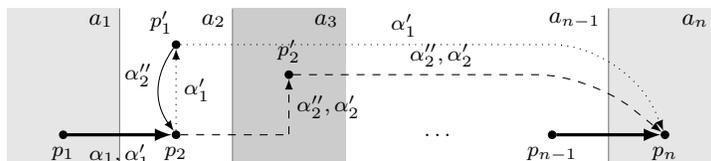
Fig. 9.   The constraint $\mathsf{stack}^\circ(a_1,\ldots,a_n)$ ensures the existence of a Jordan arc $\alpha = \alpha_1 \cdots \alpha_{n-1}$.

We inductively define, for all $1 \le i < n$, arcs α_{i-1}, α_i' and points p_i, p_i' with the following properties: $\alpha_{i-1} \subseteq (a_{i-1} + a_i)^\circ$ and runs from p_{i-1} to p_i; $\alpha_i' \subseteq (a_i + \cdots + a_n)^\circ$; $\alpha_0 \cdots \alpha_{i-1}\alpha_i'$ is a Jordan arc from p_0 to p_n; and $p_i' \in \alpha_i' \cap a_{i+1}^\circ$ with $\alpha_i'[p_i, p_i'] \subseteq (a_i + a_{i+1})^\circ$. Suppose that, for some $1 \le i \le n - 2$, the requisite entities have already been defined (notice that this is already the case for $i = 1$). Since $a_{i+1} + \cdots + a_n$ is interior-connected, let $\alpha_{i+1}'' \subseteq (a_{i+1} + \cdots + a_n)^\circ$ be a Jordan arc connecting p_i' to p_n. Since we certainly have $(a_1 + \cdots + a_i) \cdot (a_{i+1} + \cdots + a_n) = 0$, α_{i+1}'' can intersect $\alpha_0 \cdots \alpha_{i-1}\alpha_i'$ only in its final segment α_i'. Let p_{i+1} be the first point of α_i' lying on α_{i+1}''; let α_i be the initial segment of α_i' ending at p_{i+1}; and let α_{i+1}' be the final segment of α_{i+1}'' starting at p_{i+1}. By construction, then, $\alpha_0 \cdots \alpha_i \alpha_{i+1}'$ is a Jordan arc from p_0 to p_n, and $\alpha_i \subseteq (a_i + a_{i+1})^\circ$. Moreover, since $\neg C(a_{i+1}, (a_{i+3} + \cdots + a_n))$, α_{i+1}' must contain a point $p_{i+1}' \in a_{i+2}^\circ$ such that $\alpha_{i+1}'[p_{i+1}, p_{i+1}'] \subseteq (a_{i+1} + a_{i+2})^\circ$. Continuing up to the value $i = n - 1$, we have defined $\alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}'$ and $p_1, \ldots, p_{n-1}$. It remains only to define α_{n-1}; for this we simply set $\alpha_{n-1} = \alpha_{n-1}'$.

For all $1 < i < n$, we have $p_i \in \alpha_{i-1} \cap \alpha_i$, whence, by Lemma 5.1, $p_i \in a_i^\circ$. It also follows that the α_i are non-degenerate. □

It should be noted that $\mathsf{stack}^\circ(r_1,\ldots,r_n)$ is a not a $\mathcal{B}c^\circ$-formula, as it contains (negative) occurrences of the contact predicate C. It turns out, however, that we can eliminate them. To this end, consider the $\mathcal{B}c^\circ$-formula

$$\mathsf{K5m}(r_1,\ldots,r_5) \;=\; \bigwedge_{1 \le i \le 5} \big(c^\circ(r_i) \wedge (r_i \neq 0)\big) \;\wedge\; \bigwedge_{1 \le i < j \le 5} (r_i \cdot r_j = 0) \;\wedge$$

$$\bigwedge_{3 \le j \le 5} c^\circ(r_1 + r_j) \;\wedge\; \bigwedge_{2 \le i < j \le 5} c^\circ(r_i + r_j).$$

This formula is similar to formula (2) encoding the non-planar graph K_5 (hence the name); however, there is no requirement that $r_1 + r_2$ is interior-connected.

LEMMA 5.4. (*i*) *For each tuple* $(a_1, \ldots, a_5)$ *from* $\mathsf{RC}(\mathbb{R}^2)$ *satisfying* $\mathsf{K5m}(r_1, \ldots, r_5)$, *we have* $\neg C(a_1, a_2)$. (*ii*) *If regions* b_1 *and* b_2 *can be separated by a Jordan curve then there exist polygons* $(a_1, \ldots, a_5)$ *satisfying* $\mathsf{K5m}(r_1, \ldots, r_5)$ *such that* $b_1 \leq a_1$ *and* $b_2 \leq a_2$.

PROOF. (*i*) For all i $(1 \leq i \leq 5)$, pick a point $p_i \in a_i^\circ$. Then, for all j $(3 \leq j \leq 5)$ let $\gamma_{1,j}$ be an arc from p_1 to p_j lying in $(a_1 + a_j)^\circ$, and, for all i, j $(2 \leq i < j \leq 5)$, let $\gamma_{i,j}$ be an arc from p_i to p_j lying in $(a_i + a_j)^\circ$. It is routine to show that the various $\gamma_{i,j}$ can be chosen so that they intersect only at their endpoints. Thus, $\Gamma = \gamma_{3,4}\gamma_{4,5}\gamma_{3,5}^{-1}$ forms a Jordan curve in $(a_3 + a_4 + a_5)^\circ$, and $\gamma_{2,3}$, $\gamma_{2,4}$ and $\gamma_{2,5}$ join Γ to the point p_2 lying in one of its residual domains. (Fig. 10 illustrates the situation where p_2 lies in the bounded residual domain.) Since a_1 and a_2 are (interior-) connected and cannot intersect Γ, they each lie in one of its residual domains. It suffices to show that a_1 and a_2 lie in *different* residual domains. To see this, observe that $\gamma_{2,3}$, $\gamma_{2,4}$ and $\gamma_{2,5}$ divide the residual domain of Γ containing p_2 into three regions, bounded by arcs lying in $(a_2 + a_3 + a_4)^\circ$, $(a_2 + a_4 + a_5)^\circ$ and $(a_2 + a_5 + a_3)^\circ$, respectively. But if a_1 and a_2 lie on the same side of Γ, then p_1 lies in one of these regions, contradicting the existence of arcs $\gamma_{1,j} \subseteq (a_1 + a_j)^\circ$ connecting p_1 to p_j, for $j = 3, 4, 5$.
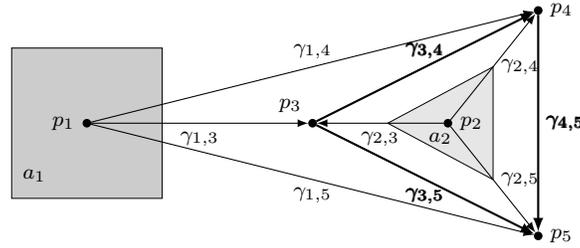
Fig. 10. The Jordan curve $\Gamma = \gamma_{3,4}\gamma_{4,5}\gamma_{3,5}^{-1}$ (thick lines) separating a_1 from a_2.

(*ii*) Let Γ be a Jordan curve separating b_1 and b_2. We may assume that Γ is piecewise-linear. Now thicken Γ to form an annular element of $\mathsf{RCP}(\mathbb{R}^2)$, still disjoint from b_1 and b_2, and divide it into the three interior-connected and non-overlapping polygons a_3, a_4, a_5. Choose a_1 and a_2 to be the components of the complement of $a_3 + a_4 + a_5$ containing b_1 and b_2, respectively. □

We remark that Lemma 5.4 (*ii*) guarantees that $a_1, \ldots, a_5$ are *polygons*. This fact will be important in Sec. 5.2, where we prove the undecidability of $Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$; for the main result of this section, however, we require only that $a_1, \ldots, a_5$ are regular closed sets in $\mathbb{R}^2$:

THEOREM 5.5. *There is a* $\mathcal{B}c^\circ$-*formula satisfiable over* $\mathsf{RC}(\mathbb{R}^2)$, *but only by tuples featuring sets with infinitely many components.*

PROOF. We first write a $\mathcal{C}c^\circ$-formula, φ_∞^*, with the required properties, and then show that all occurrences of C in it can be eliminated. Note that φ_∞^* is not the same as φ_∞° constructed for the proof of Corollary 3.2. If k is an integer, $\lfloor k \rceil$ indicates k modulo 2.

We begin with a sketch of the proof strategy. Our formula φ_∞^* will be satisfied by the arrangement shown in Fig. 11, with regions $s_0, \ldots, s_3$, a and b forming a rectangular frame, and the $a_{i,j}$, $b_{i,j}$ $(0 \leq i < 2, 1 \leq j \leq 3)$ having infinitely many components arranged in a repeating pattern, as shown. Our task is to show that any satisfying tuple in $\mathsf{RC}(\mathbb{R}^2)$ looks approximately like this. More precisely, we construct a Jordan curve lying in $(s_0 + \cdots + s_3 + a + b)^\circ$, together with infinite sequences of Jordan arcs $\{\alpha_k\}_{k \geq 1}$, $\{\beta_k\}_{k \geq 1}$
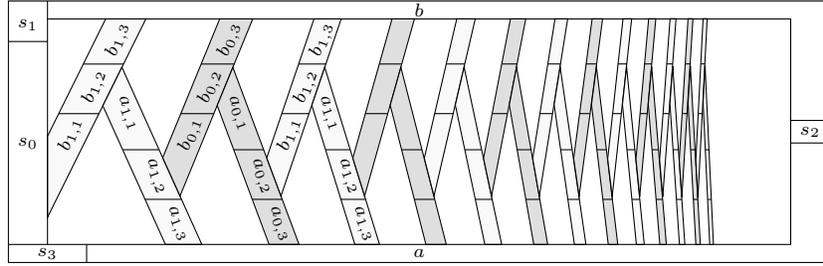
Fig. 11.   A tuple satisfying φ_∞^*: the pattern of components of the $a_{i,j}$ and $b_{i,j}$ repeats forever.

arranged as in Fig. 12. We show that

$$\alpha_k \subseteq (b_{\lfloor k \rfloor,2} + a_{\lfloor k \rfloor,1} + a_{\lfloor k \rfloor,2} + a_{\lfloor k \rfloor,3} + a)^\circ, \quad \text{for all } k \geq 1,$$
$$\beta_k \subseteq (a_{\lfloor k+1 \rfloor,2} + b_{\lfloor k \rfloor,1} + b_{\lfloor k \rfloor,2} + b_{\lfloor k \rfloor,3} + b)^\circ, \quad \text{for all } k \geq 2.$$

(The arc β_1 is exceptional, because it contains points in s_0°.) Finally, we write constraints
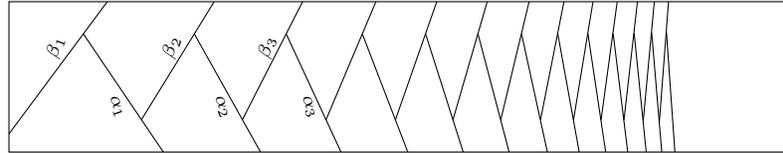
Fig. 12.   Arcs α_k and β_k lying in any tuple satisfying (31)–(34).

forcing all regions involved to have pairwise disjoint interiors. Given the existence of the arcs in Fig. 12, this ensures that the $a_{i,j}$ and $b_{i,j}$ have infinitely many components. Of course, this argument depends heavily on the assumption that we are working in $\mathbb{R}^2$.

We now proceed to the details, beginning with the conjuncts of φ_∞^*. The constraints

$$\mathsf{frame}^\circ(s_0, s_1, b, s_2, a, s_3), \tag{31}$$
$$\mathsf{stack}^\circ(s_0, b_{1,1}, b_{1,2}, b_{1,3}, b), \tag{32}$$
$$\bigwedge\nolimits_{i=0,1} \mathsf{stack}^\circ(b_{i,2}, a_{i,1}, a_{i,2}, a_{i,3}, a), \tag{33}$$
$$\bigwedge\nolimits_{i=0,1} \mathsf{stack}^\circ(a_{\lfloor i-1 \rfloor,2}, b_{i,1}, b_{i,2}, b_{i,3}, b) \tag{34}$$

are evidently satisfied by the arrangement of Fig. 11. Let φ_∞^* be the conjunction of (31)–(34) as well as formulas

$$(r \cdot r' = 0), \quad \text{for distinct variables } r \text{ and } r'. \tag{35}$$

Note that the regions $a_{i,j}$ and $b_{i,j}$ have infinitely many components. We will show that this is true for every tuple satisfying φ_∞^*.

By (31) and Lemma 5.2, there is a Jordan curve $\sigma\lambda_0\mu_0^{-1}$ whose segments σ, λ_0 and μ_0^{-1} are Jordan arcs lying in the respective sets $(s_3 + s_0 + s_1)^\circ$, $(s_2 + a + s_3)^\circ$ and $(s_1 + b + s_2)^\circ$; see Fig. 13. Note that all points in s_0 that are on $\sigma\lambda_0\mu_0^{-1}$ are on σ. Let o' be the common point of μ_0 and λ_0 and $\tilde{q}_{1,1} \in \sigma \cap s_0^\circ$.

A word is required concerning the generality of this and other diagrams in this section. The reader is to imagine the figure drawn on a *spherical* canvas, of which the sheet of paper or computer screen in front of him is simply a small part. This sphere represents the plane with a 'point' at infinity, under the usual stereographic projection. We do not say where this point at infinity is, other than that it never lies on a drawn arc. In this way, a diagram in

Fig. 13. The arcs σ, μ_0 and λ_0.

which the spherical canvas is divided into n cells represents n different configurations in the plane—one for each of the cells in which the point at infinity may be located. For example, Fig. 13 represents two topologically distinct configurations in $\mathbb{R}^2$, and, as such, depicts the arcs σ, λ_0 and μ_0 and points $\tilde{q}_{1,1}$, o' in full generality. All diagrams in this proof are to be interpreted in this way. We stress that our 'spherical diagrams' are simply a convenient device for using one drawing to represent several possible configurations in the Euclidean plane: in particular, we are interested only in the satisfiability of $\mathcal{B}c^\circ$-formulas over $\mathsf{RC}(\mathbb{R}^2)$, not over the regular closed algebra of any other space!

Let $\tilde{q}_{1,3} \in \mu_0 \cap b^\circ$. By (32) and Lemma 5.3, we can connect $\tilde{q}_{1,1}$ to $\tilde{q}_{1,3}$ by a Jordan arc $\tilde{\beta}_{1,1}\tilde{\beta}_{1,2}\tilde{\beta}_{1,3}$ whose segments lie in the respective sets $(s_0 + b_{1,1})^\circ$, $(b_{1,1} + b_{1,2} + b_{1,3})^\circ$ and $(b_{1,3} + b)^\circ$; see Fig. 14. Let $q_{1,1}$ be the last point on $\tilde{\beta}_{1,1}$ that is on σ and let $\beta_{1,1}$ be the final segment of $\tilde{\beta}_{1,1}$ starting at $q_{1,1}$; by Lemma 5.1, $q_{1,1} \in s_0^\circ$. Similarly, let $q_{1,3}$ be the first point on $\tilde{\beta}_{1,3}$ that is on 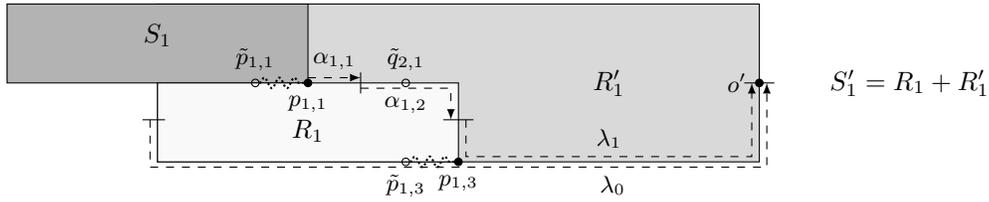μ_0 and let $\beta_{1,3}$ be the initial segment of $\tilde{\beta}_{1,3}$ ending at $q_{1,3}$; by Lemma 5.1, $q_{1,3} \in b^\circ$. Hence, the arc $\beta_1 = \beta_{1,1}\beta_{1,2}\beta_{1,3}$ lies in exactly one of the regions bounded by $\sigma\lambda_0\mu_0^{-1}$: for reasons that will emerge in the course of the proof, we denote that region by R_0'. Now, β_1 divides R_0' into two sub-regions: we denote the sub-region whose boundary is disjoint from a by S_1, and the other sub-region by S_1'. Let $\mu_1 = \beta_{1,3}\mu_0[q_{1,3}, o'] \subseteq (b_{1,3} + s_1 + b + s_2)^\circ$. The arc $\beta_{1,2}$ contains a point $\tilde{p}_{1,1} \in b_{1,2}^\circ$; moreover, all points of β_1 in $b_{1,2}^\circ$ lie on $\beta_{1,2}$.

Fig. 14. Construction of $\beta_1 = \beta_{1,1}\beta_{1,2}\beta_{1,3}$ joining $q_{1,1}$ to $q_{1,3}$.

We will now construct a cross-cut $\alpha_{1,1}\alpha_{1,2}\alpha_{1,3}$ in S_1'. Let $\tilde{p}_{1,3}$ be a point in $\lambda_0 \cap a^\circ$. By (33) and Lemma 5.3, we can connect $\tilde{p}_{1,1}$ to $\tilde{p}_{1,3}$ by a Jordan arc $\tilde{\alpha}_{1,1}\alpha_{1,2}\tilde{\alpha}_{1,3}$ whose segments lie in the respective sets $(b_{1,2} + a_{1,1})^\circ$, $(a_{1,1} + a_{1,2} + a_{1,3})^\circ$ and $(a_{1,3} + a)^\circ$; see Fig. 15. Let $p_{1,1}$ be the last point on $\tilde{\alpha}_{1,1}$ that is on $\beta_{1,2}$ and let $\alpha_{1,1}$ be the final segment of $\tilde{\alpha}_{1,1}$ starting at $p_{1,1}$; by Lemma 5.1, $p_{1,1} \in b_{1,2}^\circ$. Similarly, let $p_{1,3}$ be the first point on $\tilde{\alpha}_{1,3}$ that is on λ_0 and let $\alpha_{1,3}$ be the initial segment of $\tilde{\alpha}_{1,3}$ ending at $p_{1,3}$; by Lemma 5.1, $p_{1,3} \in a^\circ$. Since $\alpha_1 = \alpha_{1,1}\alpha_{1,2}\alpha_{1,3}$ does not intersect the boundaries of S_1 and S_1' except at its endpoints, it is a cross-cut in one of these regions. Moreover, that region has to be S_1' since the boundary of S_1 is disjoint from a. So, α_1 divides S_1' into two sub-regions: we denote the sub-region whose boundary is disjoint from b by R_1, and the other sub-region by R_1'. Let $\lambda_1 = \alpha_{1,3}\lambda_0[p_{1,3}, o'] \subseteq (a_{1,3} + s_3 + a + s_2)^\circ$. The arc $\alpha_{1,2}$ contains a point $\tilde{q}_{2,1} \in a_{1,2}^\circ$; moreover, all points of α_1 in $a_{1,2}^\circ$ lie on $\alpha_{1,2}$.

Fig. 15. Construction of $\alpha_1 = \alpha_{1,1}\alpha_{1,2}\alpha_{1,3}$ joining $p_{1,1}$ to $p_{1,3}$.

We can now forget about the region S_1, and start constructing a cross-cut $\beta_{2,1}\beta_{2,2}\beta_{2,3}$ in R_1'. As before, let $\tilde{q}_{2,3} \in \mu_1 \cap b^\circ$. Then there is a Jordan arc $\tilde{\beta}_{2,1}\beta_{2,2}\tilde{\beta}_{2,3}$ connecting $\tilde{q}_{2,1}$ to $\tilde{q}_{2,3}$ such that its segments are contained in the respective sets $(a_{1,2} + b_{0,1})^\circ$, $(b_{0,1} + b_{0,2} + b_{0,3})^\circ$ and $(b_{0,3} + b)^\circ$. As before, we choose $\beta_{2,1} \subseteq \tilde{\beta}_{2,1}$ and $\beta_{2,3} \subseteq \tilde{\beta}_{2,3}$ so that the Jordan arc $\beta_2 = \beta_{2,1}\beta_{2,2}\beta_{2,3}$ (apart from its endpoints) is disjoint from the boundaries of R_1 and R_1'. Hence β_2 has to be a cross-cut in R_1 or R_1', and since the boundary of R_1 is disjoint from b it has to be a cross-cut in R_1'. So, β_2 separates R_1' into two regions S_2 and S_2' so that the boundary of S_2 is disjoint from a. Let $\mu_2 = \beta_{2,3}\mu_1[q_{2,3}, o'] \subseteq (b_{0,3} + b_{1,3} + s_1 + b + s_2)^\circ$. Now, we can ignore the region R_1, and reasoning as before we can construct a cross-cut $\alpha_2 = \alpha_{2,1}\alpha_{2,2}\alpha_{2,3}$ in S_2' dividing it into two sub-regions R_2 and R_2'.

Evidently, this process continues forever: R_{i-1}' is divided into S_i and S_i' and S_i' is divided into R_i and R_i'. Now, the boundary of S_i contains the arc $\beta_{i,2}$, whence the interior of S_i contains points of $b_{\lfloor i \rfloor,2}$. On the other hand, S_i certainly lies outside S_{i+1}'; moreover, $\delta S_{i+1}'$ is a subset of $\alpha_{i,2} \cup \beta_{i+1,1} \cup \beta_{i+1,2} \cup \mu_{i+1} \cup \lambda_i$, whence

$$\delta S_{i+1}' \subseteq (a_{\lfloor i \rfloor,1} + a_{\lfloor i \rfloor,2} + a_{\lfloor i \rfloor,3})^\circ \cup (a_{\lfloor i \rfloor,2} + b_{\lfloor i+1 \rfloor,1})^\circ \cup (b_{\lfloor i+1 \rfloor,1} + b_{\lfloor i+1 \rfloor,2} + b_{\lfloor i+1 \rfloor,3})^\circ \cup$$
$$(b_{0,3} + b_{1,3} + b + s_1 + s_2)^\circ \cup (a_{0,3} + a_{1,3} + a + s_2 + s_3)^\circ.$$

Hence $\delta S_{i+1}'$ contains no points of $b_{\lfloor i \rfloor,2}$. Yet S_{i+1}' includes all the regions S_{i+2k} for $k \geq 1$, each of which contains points of $b_{\lfloor i \rfloor,2}$. It follows that $b_{\lfloor i \rfloor,2}$ has infinitely many components.
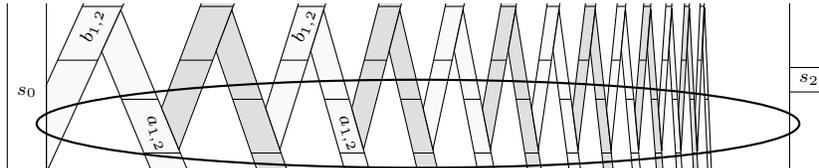

Fig. 16. Separating $a_{1,2}$ from $b_{1,2}$ by a Jordan curve.

So far we know that the $\mathcal{C}c^\circ$-formula φ_∞^* forces infinitely many components. Now we replace every conjunct in φ_∞^* of the form $\neg C(s_1, s_2)$ by $\mathsf{K5m}(\vec{r}) \wedge (s_1 \leq r_1) \wedge (s_2 \leq r_2)$, where $\vec{r}$ is a vector of fresh variables. By Lemma 5.4 (i), the resulting formula entails φ_∞^*. Conversely, to show that the formula is satisfiable, we apply Lemma 5.4 (ii): it suffices to separate every pair of disjoint regions in Fig. 11 by a Jordan curve. Such a Jordan curve is shown in Fig. 16 for $b_{1,2}$ and $a_{1,2}$. Other pairs of disjoint regions are treated analogously. □

5.2. Undecidability in the plane

We now return to the question of decidability. We know from Sec. 4 that $Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ is undecidable, where $\mathcal{L}$ is any of the languages $\mathcal{B}c$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$. We proceed to establish the undecidability of the problems $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$, where $\mathcal{L}$ is any of the languages $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$, and also of the problem $Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$. Most of the techniques required

have been rehearsed in the proof of Theorem 5.5. However, we face a new difficulty. In the language $\mathcal{B}c°$, we can say that the *interior* of a region (rather than merely the region itself) is connected. Since, for open sets, connectedness implies arc-connectedness, we were able, in the proof of Theorem 5.5, to write formulas enforcing various arrangements of Jordan arcs in the plane. When dealing with $\mathcal{B}c$ and $\mathcal{C}c$, however, we can speak merely of the connectedness of a region (rather than of its interior), which, for elements of $\mathsf{RC}(\mathbb{R}^2)$ does not imply arc-connectedness; this complicates the business of enforcing the requisite arrangements of Jordan arcs.

To overcome this difficulty, we employ the technique of 'wrapping' a region inside two bigger ones. If a and b are regions such that $\neg C(a, -b)$, we write $a \ll b$ (pronounced: *a is right inside b*). Let us say that a *3-region* is a triple $\mathfrak{a} = (a, \dot{a}, \ddot{a})$ of elements of $\mathsf{RC}(\mathbb{R}^2)$ such that $0 \neq \ddot{a} \ll \dot{a} \ll a$. It helps to think of $\mathfrak{a} = (a, \dot{a}, \ddot{a})$ as consisting of a kernel, $\ddot{a}$, encased in two protective layers: an inner shell, $\dot{a}$, and an outer shell, a. As a simple example, consider the sequence of 3-regions $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ depicted in Fig. 17, where the kernels form a sequence of externally touching polygons. When describing arrangements of 3-regions, we use the
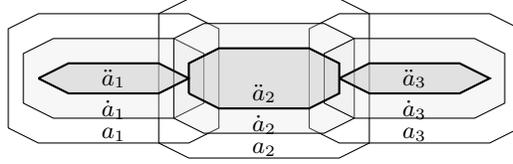

Fig. 17. A chain of 3-regions satisfying $\mathsf{stack}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3)$.

variable $\mathfrak{r}$ for the triple of variables $(r, \dot{r}, \ddot{r})$, taking the following conjuncts to be implicit:

$$(\ddot{r} \neq 0) \quad \wedge \quad (\ddot{r} \ll \dot{r}) \quad \wedge \quad (\dot{r} \ll r).$$

In the sequel, when depicting arrangements of 3-regions, we standardly draw only the kernels of these 3-regions, leaving the reader to imagine the encasing layers of shell. (This is simply to reduce diagrammatic clutter.)

For $n \geq 2$, define the $\mathcal{C}c$-formula

$$\mathsf{stack}(\mathfrak{r}_1, \ldots, \mathfrak{r}_n) \quad = \quad \bigwedge_{1 \leq i < n} c(\dot{r}_i + \ddot{r}_{i+1} + \cdots + \ddot{r}_n) \quad \wedge \quad c(\dot{r}_n) \quad \wedge \quad \bigwedge_{\substack{1 \leq i,j \leq n \\ j - i > 1}} \neg C(r_i, r_j).$$

(Observe that the term $c(\dot{r}_i + \ddot{r}_{i+1} + \cdots + \ddot{r}_n)$ features the *inner shell* of $\mathfrak{r}_1$, and the *kernels* of $\mathfrak{r}_2, \ldots, \mathfrak{r}_n$.) Thus, the triple of 3-regions $(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3)$ in Fig. 17 satisfies $\mathsf{stack}(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3)$. This formula allows us to construct sequences of arcs with useful properties.

LEMMA 5.6. *Fix $n \geq 2$, and let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be a tuple of 3-regions satisfying* $\mathsf{stack}(\mathfrak{r}_1, \ldots, \mathfrak{r}_n)$. *Then, for every point $p_0 \in \dot{a}_1$ and every point $p_n \in \ddot{a}_n$, there exist points $p_1, \ldots, p_{n-1}$ and Jordan arcs $\alpha_1, \ldots, \alpha_n$ such that: (i) $\alpha = \alpha_1 \cdots \alpha_n$ is a Jordan arc from p_0 to p_n; (ii) $p_i \in \dot{a}_{i+1} \cap \alpha_i$, for all $1 \leq i < n$; and (iii) $\alpha_i \subseteq a_i$, for all $1 \leq i \leq n$.*

PROOF. Let $v_0 = p_0$. Since $v_0 \in a_1°$, $p_n \in \dot{a}_n°$ and $\dot{a}_1 + \ddot{a}_2 + \cdots + \ddot{a}_n$ is connected, we see that v_0 and p_0 lie in the same component of $(a_1 + \dot{a}_2 + \cdots + \dot{a}_n)°$. So let β_1 be a Jordan arc connecting v_0 to p_n in $(a_1 + \dot{a}_2 + \cdots + \dot{a}_n)°$. Since a_1 is disjoint from all the a_i except a_2, let p_1 be the first point of β_1 lying in $\dot{a}_2$, so $\beta_1[v_0, p_1] \subseteq a_1° \cup \{p_1\}$, i.e., the arc $\beta_1[v_0, p_1]$ is either included in $a_1°$, or is an end-cut of $a_1°$. (We do not rule out $v_0 = p_1$.) Similarly, let β_2' be a Jordan arc connecting p_1 to p_n in $(a_2 + \dot{a}_3 + \cdots + \dot{a}_n)°$, and let q_1 be the last point of β_2' lying on $\beta_1[v_0, p_1]$. If $q_1 = p_1$, then set $v_1 = p_1$, $\alpha_1 = \beta_1[v_0, p_1]$, and $\beta_2 = \beta_2'$, so that the endpoints of β_2 are v_1 and p_n. Otherwise, we have $q_1 \in a_1°$. We can now construct an

arc $\gamma_1 \subseteq a_1^\circ \cup \{p_1\}$ from p_1 to a point v_1 on $\beta_2'[q_1, p_n]$, such that γ_1 intersects $\beta_1[v_0, p_1]$ and $\beta_2'[q_1, p_n]$ only at its endpoints, p_1 and v_1; see Fig. 18.
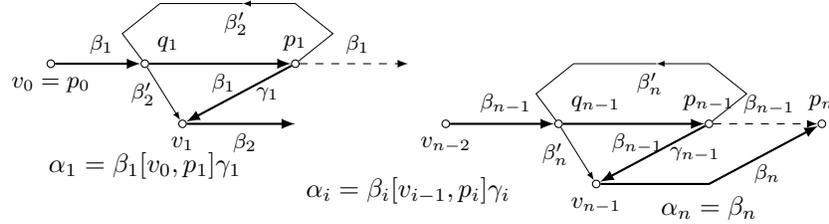

Fig. 18. Proof of Lemma 5.6.

Let $\alpha_1 = \beta_1[v_0, p_1]\gamma_1$, and let $\beta_2 = \beta_2'[v_1, p_n]$. Since β_2 contains a point $p_2 \in \dot{a}_3$, we may iterate this procedure, obtaining $\alpha_2, \alpha_3, \ldots \alpha_{n-1}, \beta_n$. We remark that α_i and α_{i+1} have a single point of contact by construction, while α_i and α_j ($i < j - 1$) are disjoint by the constraint $\neg C(a_i, a_j)$. Finally, we let $\alpha_n = \beta_n$; see Fig. 18. □

In fact, we can add a 'switch' to the formula $\mathsf{stack}(\mathfrak{r}_1, \ldots, \mathfrak{r}_n)$, in the following sense. Recall from Sec. 2 that if $a, a_0, \ldots, a_{n-1}$ are regions satisfying $\mathsf{colour}(r; r_0, \ldots, r_{n-1})$, then every connected subset of a—and in particular, any component of a—is included in exactly one of the $a_0, \ldots, a_{n-1}$. Let z be a variable, and consider what happens when we replace $\dot{r}_1$ in $\mathsf{stack}(\mathfrak{r}_1, \ldots, \mathfrak{r}_n)$ by $(-z) \cdot \dot{r}_1$, and add the conjunct $\mathsf{colour}(\dot{r}_1; z, -z)$. The result is

$$\mathsf{stack}_z(\mathfrak{r}_1, \ldots, \mathfrak{r}_n) = \mathsf{colour}(\dot{r}_1; z, -z) \wedge c(((-z) \cdot \dot{r}_1) + \ddot{r}_2 + \cdots + \ddot{r}_n) \wedge$$
$$\bigwedge_{1 < i < n} c(\dot{r}_i + \ddot{r}_{i+1} + \cdots + \ddot{r}_n) \wedge c(\dot{r}_n) \wedge \bigwedge_{\substack{1 \leq i,j \leq n \\ j-i>1}} \neg C(r_i, r_j).$$

Now let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be 3-regions and d a region satisfying $\mathsf{stack}_z(\mathfrak{r}_1, \ldots, \mathfrak{r}_n)$. The first conjunct of the formula ensures that any component of $\dot{a}_1$ is either included in d or included in $-d$. The remaining conjuncts then have the same effect as $\mathsf{stack}(\mathfrak{r}_1, \ldots, \mathfrak{r}_n)$—but only for those components of $\dot{a}_1$ included in $-d$. That is, if $p \in (-d) \cdot \dot{a}_1$, we can find an arc $\alpha_1 \cdots \alpha_n$ starting at p, with the properties of Lemma 5.6. However, if $p \in d \cdot \dot{a}_1$, no such arc need exist. Thus, the variable z functions so as to 'de-activate' $\mathsf{stack}_z(\mathfrak{r}_1, \ldots, \mathfrak{r}_n)$ when we are dealing with a component of $\dot{r}_1$ included in z.

As a further application of Lemma 5.6, consider the formula

$$\mathsf{frame}(\mathfrak{r}_0, \ldots, \mathfrak{r}_n) = \mathsf{stack}(\mathfrak{r}_0, \ldots, \mathfrak{r}_{n-1}) \wedge \neg C(r_n, r_1 + \cdots + r_{n-2}) \wedge$$
$$c(\dot{r}_n) \wedge (\dot{r}_0 \cdot \dot{r}_n \neq 0) \wedge (\ddot{r}_{n-1} \cdot \dot{r}_n \neq 0).$$

This formula allows us to construct Jordan curves in the plane, in the following sense:

LEMMA 5.7. *Fix $n \geq 3$, and let $\mathfrak{a}_0, \ldots, \mathfrak{a}_n$ be a tuple of 3-regions satisfying* $\mathsf{frame}(\mathfrak{r}_0, \ldots, \mathfrak{r}_n)$. *Then there exist Jordan arcs $\gamma_0, \ldots, \gamma_n$ such that $\gamma_0 \cdots \gamma_n$ is a Jordan curve and $\gamma_i \subseteq a_i$, for all $0 \leq i \leq n$.*

PROOF. By Lemma 5.6, let $\alpha_0, \gamma_1, \ldots, \gamma_{n-2}, \alpha_{n-1}$ be Jordan arcs in the respective regions $a_0, \ldots, a_{n-1}$ such that $\alpha_0 \cdots \alpha_{n-1}$ is a Jordan arc connecting a point $\tilde{p} \in \dot{a}_0 \cdot \dot{a}_n$ to a point $\tilde{q} \in \ddot{a}_{n-1} \cdot \dot{a}_n$; see Fig. 19. Because $\dot{a}_n$ is a connected subset of the interior of a_n, let $\alpha_n \subseteq a_n^\circ$ be an arc connecting $\tilde{p}$ and $\tilde{q}$. Note that α_n does not intersect α_i, for $1 \leq i \leq n-2$. Let p be the last point on α_0 that is on α_n (possibly $\tilde{p}$), and q be the first point on α_{n-1} that is on α_n (possibly $\tilde{q}$). Let γ_0 be the final segment of α_0 starting at p and let γ_{n-1} be the
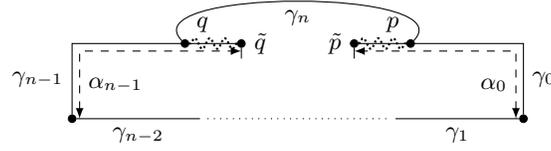
Fig. 19. Establishing a Jordan curve.

initial segment of α_{n-1} ending at q. Finally, let $\gamma_n = \alpha_n[p, q]$ or $\gamma_n = \alpha_n[q, p]$, depending on whether p or q is encountered first on α_n. Then the arcs γ_i, $0 \leq i \leq n$, are as required. □

We are now ready to prove the main result of this section. Again, recall that if $a, a_0, \ldots, a_{n-1}$ are regions satisfying $\mathsf{colour}(r; r_0, \ldots, r_{n-1})$, then every connected subset of a—and in particular, any Jordan arc $\alpha \subseteq a$—is included in exactly one of the a_i, for $0 \leq i < n$. In this case, it is sometimes helpful to think of α as being 'labelled' by a letter of the alphabet $a_0, \ldots, a_{n-1}$.

THEOREM 5.8. *The problems $Sat(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^2))$ and $Sat(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}^2))$ are r.e.-hard.*

PROOF. Again, we proceed via a reduction of the Post correspondence problem (PCP), constructing, for any instance W, a formula ψ_W with the property that the following are equivalent: (*i*) W is positive; (*ii*) ψ_W is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$; (*iii*) ψ_W is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$. This establishes the theorem. In this proof, $\lfloor k \rfloor$ indicates the value of k modulo 3.

The proof proceeds in six stages. In Stage 1, we introduce constraints satisfied by the arrangement in Fig. 20. We show that any tuple of 3-regions in $\mathsf{RC}(\mathbb{R}^2)$ satisfying these constraints contains the system of arcs shown in Fig. 21 (with each arc included in some specified region). These arcs form a large rectangle divided into an 'upper window' and a 'lower window'. In Stage 2, we introduce constraints satisfied by the *finitely* repeating arrangement shown in Fig. 22. We then show, using an argument similar to that of Theorem 5.5, that any tuple of 3-regions in $\mathsf{RC}(\mathbb{R}^2)$ satisfying these constraints contains the finite sequence of triples of arcs $(\zeta_i, \eta_i, \kappa_i)$ shown in Fig. 27 (again, with each arc included in some specified region) such that the η_i all lie in the lower window. Stage 3 simply duplicates Stage 2 to obtain the finite sequence of triples of arcs $(\zeta_i', \eta_i', \kappa_i')$, with the η_i' all lying in the upper window. In Stage 4, we introduce constraints that force these two sequences to have the same length, yielding the arrangement of Fig. 28. At this point, the main geometrical work has been accomplished. In Stage 5, we introduce constraints relating specifically to our PCP-instance W, over the alphabets T and U. These constraints allow us to 'colour' the various arcs η_i with the elements of U, thus spelling out a word of U^*, and similarly for the η_i'; indeed we show that these two words must be identical. In addition, our constraints allow us to group the arcs η_i into contiguous blocks, with each block coloured with some element of T, thus obtaining a word τ of T^*, and similarly for the arcs η_i', obtaining a word τ' of T^*. Using additional constraints, we can ensure that, within any block of the arcs η_i coloured with $t \in \mathsf{T}$, the letters from U spell out the lower string on the tile t; similarly, any block of the η_i' coloured with $t \in \mathsf{T}$ spells out the upper string on the tile t. Finally, in Stage 6, we line up the η_i-blocks and the η_i'-blocks, using the same technique as employed in Stage 4. We show that the words τ and τ' formed by the lower and upper block-sequences must be identical. It follows that, if any tuple of regions in $\mathsf{RC}(\mathbb{R}^2)$ satisfies ψ_W, then W must have a solution. Conversely, if W has a solution, we can construct a tuple from $\mathsf{RCP}(\mathbb{R}^2)$ satisfying ψ_W, in the style of Fig. 22. This completes the reduction.

Stage 1. In the first stage, we define an assemblage of arcs that will serve as scaffolding for the ensuing construction. Consider the arrangement of polygonal 3-regions depicted in Fig. 20, assigned to the 3-region variables $\mathfrak{s}_0, \ldots, \mathfrak{s}_9, \mathfrak{s}_8', \ldots, \mathfrak{s}_1', \mathfrak{d}_0, \ldots, \mathfrak{d}_6$ as indicated. (Note that we have here followed the convention of depicting only the kernels of 3-regions.)
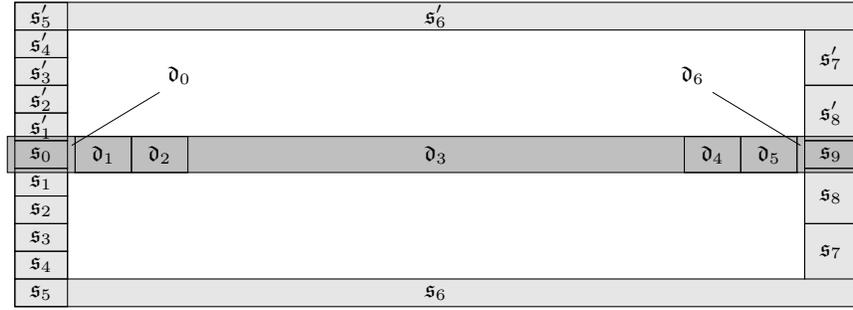
Fig. 20. A tuple of 3-regions satisfying (36)–(38) (showing kernels only).

It is easy to verify that this arrangement can be made to satisfy the following constraints:

$$\mathsf{frame}(\mathfrak{s}_0, \mathfrak{s}_1, \ldots, \mathfrak{s}_8, \mathfrak{s}_9, \mathfrak{s}_8', \ldots, \mathfrak{s}_1'), \tag{36}$$

$$(s_0 \leq \dot{d}_0) \wedge (s_9 \leq \ddot{d}_6), \tag{37}$$

$$\mathsf{stack}(\mathfrak{d}_0, \ldots, \mathfrak{d}_6). \tag{38}$$

And obviously, the arrangement can be made to satisfy any constraint

$$\neg C(r, r'), \tag{39}$$

for which the corresponding 3-regions $\mathfrak{r}$ and $\mathfrak{r}'$ are drawn as not being in contact. (Remember, r is the outer shell of the 3-region $\mathfrak{r}$, and similarly for r'; so we must take these shells to hug the kernels depicted in Fig. 20 quite closely.) Thus, for example, (39) includes $\neg C(s_0, d_1)$, but not $\neg C(s_0, d_0)$ or $\neg C(d_0, d_1)$.

Now suppose $\mathfrak{s}_0, \ldots, \mathfrak{s}_9, \mathfrak{s}_8', \ldots, \mathfrak{s}_1', \mathfrak{d}_0, \ldots, \mathfrak{d}_6$ is *any* collection of 3-regions (not necessarily polygonal) satisfying (36)–(39). By Lemma 5.7 and (36), let $\gamma_0, \ldots, \gamma_9, \gamma_8', \ldots, \gamma_1'$ be Jordan arcs included in the respective regions $s_0, \ldots, s_9, s_8', \ldots, s_1'$, such that $\Gamma = \gamma_0 \cdots \gamma_9 \cdot \gamma_8' \cdots \gamma_1'$ is a Jordan curve (note that γ_i' and γ_i have opposite directions). We select points $\tilde{o}$ on γ_0 and $\tilde{o}'$ on γ_9; see Fig. 21. By (37), $\tilde{o} \in \dot{d}_0$ and $\tilde{o}' \in \ddot{d}_6$. By Lemma 5.6 and (38), let $\tilde{\chi}_1, \chi_2, \tilde{\chi}_3$ be Jordan arcs in the respective regions $(d_0 + d_1), (d_2 + d_3 + d_4), (d_5 + d_6)$ such that $\tilde{\chi}_1 \chi_2 \tilde{\chi}_3$ is a Jordan arc from $\tilde{o}$ to $\tilde{o}'$. Let o be the last point of $\tilde{\chi}_1$ lying on Γ, and let χ_1 be the final segment of $\tilde{\chi}_1$, starting at o. Let o' be the first point of $\tilde{\chi}_3$ lying on Γ, and let χ_3 be the initial segment of $\tilde{\chi}_3$, ending at o'. By (39), we see that the arc $\chi_1 \chi_2 \chi_3$ intersects Γ only in its endpoints, and is thus a chord of Γ, as shown in Fig. 21.

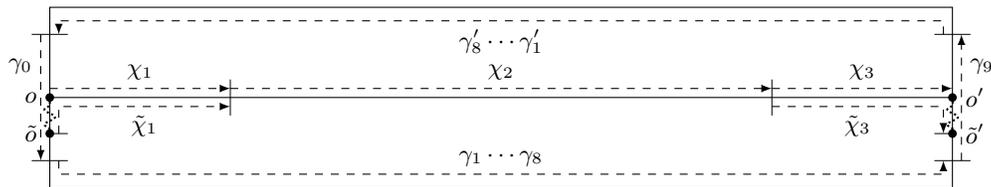Fig. 21. The arcs $\gamma_0, \ldots, \gamma_9, \gamma_8', \ldots, \gamma_1'$ and χ_1, χ_2, χ_3.

As before, we treat these diagrams as being drawn on a spherical canvas. For ease of reference, we refer to the two rectangles in Fig. 21 as the 'upper window' and 'lower window,' it being understood that these are simply handy labels: in particular, either (but not both) of these 'windows' may be unbounded.

Stage 2. In this stage, we construct a sequence of triples $(\zeta_i, \eta_i, \kappa_i)$ of arcs of indeterminate length $n \geq 1$, such that the members of ζ_i all lie in the lower window. (Recall that $\lfloor k \rfloor$ denotes k modulo 3). Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{z}$, $\mathfrak{a}_{i,j}$ and $\mathfrak{b}_{i,j}$ $(0 \leq i < 3, 1 \leq j \leq 6)$ be 3-region variables, and consider the formulas

$$(s_6 \leq \ddot{a}) \wedge (s_6' \leq \ddot{b}) \wedge (s_3 \leq \dot{a}_{0,3}), \tag{40}$$

$$\bigwedge_{i=0,1,2} \mathsf{stack}_{\dot{z}}(\mathfrak{a}_{\lfloor i-1 \rfloor,3}, \mathfrak{b}_{i,1}, \ldots, \mathfrak{b}_{i,6}, \mathfrak{b}), \tag{41}$$

$$\bigwedge_{i=0,1,2} \mathsf{stack}(\mathfrak{b}_{i,3}, \mathfrak{a}_{i,1}, \ldots, \mathfrak{a}_{i,6}, \mathfrak{a}). \tag{42}$$

(Observe that the stack-formula in (41) has a switch, the inner shell $\dot{z}$ of $\mathfrak{z}$.) The arrangement of polygonal 3-regions depicted in Fig. 22 (with $\mathfrak{z}$ assigned appropriately) is one such satisfying assignment. We stipulate that (39) applies now to all regions depicted in either
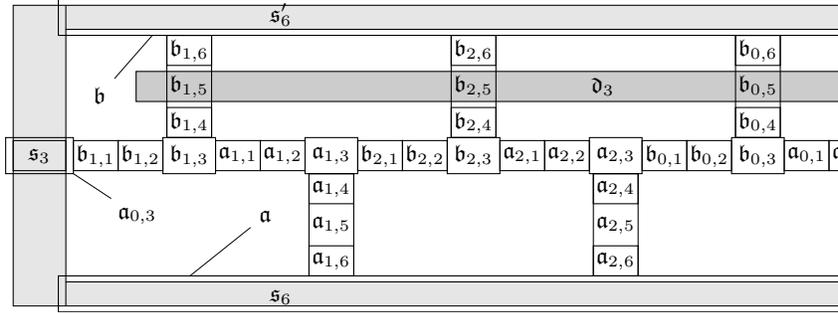

Fig. 22. A tuple of 3-regions satisfying (40)–(42) (showing kernels only).

Fig. 20 or Fig. 22, and we further stipulate

$$z \cdot (s_0 + \cdots + s_9 + s_1' + \cdots + s_8' + d_1 + \cdots + d_4) = 0. \tag{43}$$

Note that d_5 does not appear in this constraint; thus, z° may intersect the arc χ_3. Again, these additional constraints are evidently satisfiable.

Now suppose we are given *any* collection of regions (not necessarily polygonal) satisfying (36)–(43). And let the arcs $\gamma_0, \ldots, \gamma_9, \gamma_8', \ldots, \gamma_1'$ and χ_1, χ_2, χ_3 be as defined above. It will be convenient in this stage to rename γ_6 and γ_6' as λ_0 and μ_0, respectively. Thus, λ_0 forms the bottom edge of the lower window, and μ_0 the top edge of the upper window. Likewise, we rename γ_3 as α_0, forming part of the left-hand side of the lower window. Let $\tilde{q}_{1,1}$ be any point of α_0, p^* any point of λ_0, and q^* any point of μ_0; see Fig. 23. By (40), then, $\tilde{q}_{1,1} \in \dot{a}_{0,3}$, $p^* \in \ddot{a}$, and $q^* \in \ddot{b}$. Certainly, the constraint (43) ensures that $\tilde{q}_{1,1} \in (-\dot{z})$. By Lemma 5.6 and (41), we may draw an arc $\tilde{\beta}_1$ from $\tilde{q}_{1,1}$ to q^*, with successive segments $\tilde{\beta}_{1,1}, \beta_{1,2}, \ldots, \beta_{1,5}, \tilde{\beta}_{1,6}$ lying in the respective regions $a_{0,3} + b_{1,1}, b_{1,2}, \ldots, b_{1,5}, b_{1,6} + b$; further, we can guarantee that $\beta_{1,2}$ contains a point $\tilde{p}_{1,1} \in \dot{b}_{1,3}$. Denote the last point of $\beta_{1,5}$ by $q_{1,2}$. Also, let $q_{1,1}$ be the last point of $\tilde{\beta}_1$ lying on α_0, and $q_{1,3}$ the first point of $\tilde{\beta}_1$ lying on μ_0. Finally, let β_1 be the segment of $\tilde{\beta}_1$ between $q_{1,1}$ and $q_{1,2}$; and let μ_1 be the segment of $\tilde{\beta}_1$ from $q_{1,2}$ to $q_{1,3}$ followed by the final segment of μ_0 from $q_{1,3}$; see Fig. 23. By repeatedly using the constraints in (39), it is easy to see that β_1 and the initial segment of μ_1 up to $q_{1,3}$ together form a chord of Γ. Adding the constraint

$$c(b_{1,5} + d_3), \tag{44}$$

and taking into account (39) ensures that this chord divides the residual domain of Γ containing χ_2 into the regular closed sets S_1 and S_1', as shown in Fig. 23. The wiggly lines indicate that we do not care about the exact positions of $\tilde{q}_{1,1}$ or q^*; otherwise, Fig. 23 is again completely general. Note that μ_1 lies entirely in $b_{1,6} + b$, and hence certainly in
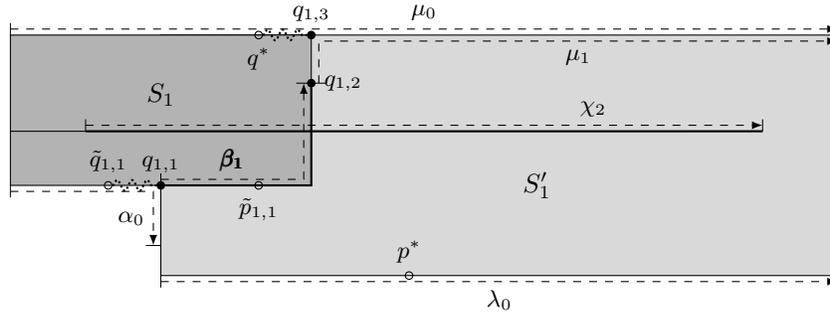
$$b^* = b_{0,6} + b_{1,6} + b_{2,6} + b. \tag{45}$$

Fig. 23. The arc β_1.

Recall that $\tilde{p}_{1,1} \in \dot{b}_{1,3}$ and $p^* \in \ddot{a}$. By Lemma 5.6 and (42), we may draw an arc $\tilde{\alpha}_1$ from $\tilde{p}_{1,1}$ to p^*, with successive segments $\tilde{\alpha}_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,5}, \tilde{\alpha}_{1,6}$ lying in the respective regions $b_{1,3} + a_{1,1}, a_{1,2}, \ldots, a_{1,5}, a_{1,6} + a$; further, we can guarantee that the segment $\alpha_{1,2}$ contains a point $\tilde{q}_{2,1} \in \dot{a}_{1,3}$. (Thus: $\alpha_{1,2}$ lies in $a_{1,2}$, but nevertheless contains at least one point lying in $\dot{a}_{1,3}$.) Denote the last point of $\alpha_{1,5}$ by $p_{1,2}$. Also, let $p_{1,1} \in b_{1,3}$ be the last point of $\tilde{\alpha}_1$ lying on β_1, and $p_{1,3}$ the first point of $\tilde{\alpha}_1$ lying on λ_0. By (39), these points must be arranged as shown in Fig. 24. In particular, the segment of $\tilde{\alpha}_1$ between $p_{1,1}$ and $p_{1,3}$ is a chord in S_1' and divides it into regions R_1 and R_1'. Let α_1 be the segment of $\tilde{\alpha}_1$ between $p_{1,1}$ and $p_{1,2}$. Noting that (39) entails

$$\neg C(a_{1,1} + \cdots + a_{1,6}, \quad s_0 + s_9 + d_0 + \cdots + d_5),$$

we can be sure that α_1 lies entirely in the 'lower' window, whence β_1 crosses the central chord, χ_2 at least once. Let o_1 be the first such point (measured along χ_2 from left to right). Finally, let λ_1 be the segment of $\tilde{\alpha}_1$ between $p_{1,2}$ and $p_{1,3}$, followed by the final segment of λ_0 from $p_{1,3}$. Note that λ_1 lies entirely in $a_{1,6} + a$, and hence certainly in the region

$$a^* = a_{0,6} + a_{1,6} + a_{2,6} + a. \tag{46}$$

The region S_1 may now be forgotten.

By construction, the point $\tilde{q}_{2,1}$ lies in some component of $\dot{a}_{1,3}$, and, from the presence of the 'switching' variable $\dot{z}$ in (42), that component is either included in $\dot{z}$ or included in $-\dot{z}$. Suppose the latter. Then we can repeat the above construction to obtain an arc $\tilde{\beta}_2$ from $\tilde{q}_{2,1}$ to q^*, with successive segments $\tilde{\beta}_{2,1}, \beta_{2,2}, \ldots, \beta_{2,5}, \tilde{\beta}_{2,6}$ lying in the respective regions $a_{1,3} + b_{2,1}, b_{2,2}, \ldots, b_{2,5}, b_{2,6} + b$; further, we can guarantee that $\beta_{2,2}$ contains a point $\tilde{p}_{2,1} \in \dot{b}_{2,3}$. Denote the last point of $\beta_{2,5}$ by $q_{2,2}$. Also, let $q_{2,1}$ be the last point of $\tilde{\beta}_2$ lying on α_1, and $q_{2,3}$ the first point of $\tilde{\beta}_2$ lying on μ_1. Again, we let β_2 be the segment of $\tilde{\beta}_2$ between $q_{2,1}$ and $q_{2,2}$; and we let μ_2 be the segment of $\tilde{\beta}_2$ from $q_{2,1}$ to $q_{2,3}$, followed by the final segment of μ_1 from $q_{2,3}$. Note that μ_2 lies in the set b^*. It is easy to see that the segment of $\tilde{\beta}_2$ from $q_{2,1}$ to $q_{2,3}$ is a cross-cut in R_1' dividing it into regions S_2 and S_2', as shown in Fig. 25. Indeed, $\beta_2 = \tilde{\beta}_2[q_{2,1}, q_{2,2}]$ cannot enter the interior of the region R_1, for, by construction, it can have only one point of contact with α_1, and the constraints (39)
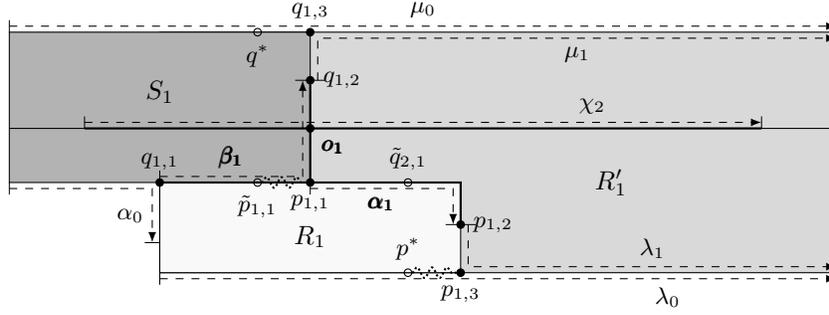
Fig. 24. The arcs β_1 and α_1.

ensure that it cannot intersect any other part of δR_1. Since $q^* \in \ddot{a}$ is guaranteed to lie outside R_1, we evidently have that $\beta_2 \subseteq -R_1$. By the constraints (39), β_2 lies in the interior of R_1' except for its first point, which lies on the boundary of R_1'; hence the reversal of β_2 is an end-cut in R_1'. Similarly, $\tilde{\beta}_2[q_{2,2}, q_{2,3}]$ is an end-cut in R_1' as well, and thus $\tilde{\beta}_2[q_{2,1}, q_{2,3}]$ is a cross-cut in R_1'. This observation having been made, R_1 may now be forgotten.
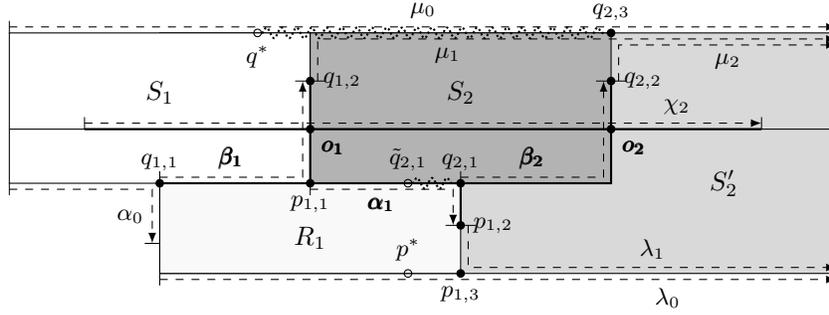
Fig. 25. The arc β_2.

Symmetrically, we construct the arc $\tilde{\alpha}_2$ in $b_{2,3} + a_{2,1} + \cdots + a_{2,6} + a$, and points $p_{2,1}$, $p_{2,2}$, $p_{2,3}$, together with the arcs α_2 and λ_2. Again, we know from (39) that α_2 lies entirely in the 'lower' window, whence β_2 must cross the central chord, χ_2, at least once. Let o_2 be the first such point (measured along χ_2 from left to right); see Fig. 25.
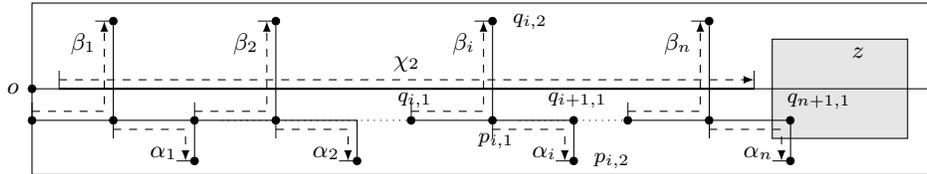
Fig. 26. The sequence of pairs of arcs (β_i, α_i).

This process continues, generating arcs $\beta_i \subseteq a_{\lfloor i-1 \rfloor,3} + b_{\lfloor i \rfloor,1} + \cdots + b_{\lfloor i \rfloor,5}$ and $\alpha_i \subseteq b_{\lfloor i \rfloor,3} + a_{\lfloor i \rfloor,1} + \cdots + a_{\lfloor i \rfloor,5}$, as long as α_i contains a point $\tilde{q}_{i+1,1} \in (-\dot{z})$. That we eventually reach a value $i = n$ for which no such point exists follows from (39). For the conjuncts $\neg C(b_{i,j}, d_k)$, for $j \neq 5$, together entail $o_i \in b_{\lfloor i \rfloor,5}$, for every i such that β_i is defined; and

these points cycle on χ_2 through the regions $b_{0,5}$, $b_{1,5}$ and $b_{2,5}$. If there were infinitely many β_i, the o_i would have an accumulation point, lying in all three regions, contradicting, say, $\neg C(b_{0,5}, b_{1,5})$. The resulting sequence of arcs and points is shown in Fig. 26. It follows that the final arc α_n contains a point $q_{n+1,1} \in \dot{z}$. We finish this stage in the construction by 're-packaging' the pairs of arcs (β_i, α_i). Specifically, for all i $(1 \le i \le n)$, let

$$\zeta_i = \beta_i[q_{i,1}, p_{i,1}]\alpha_i[p_{i,1}, q_{i+1,1}], \quad \eta_i = \beta_i[p_{i,1}, q_{i,2}], \quad \kappa_i = \alpha_i[q_{i+1,1}, p_{i,2}]$$

(see Fig. 27).

Fig. 27. The sequence of triples of arcs $(\zeta_i, \eta_i, \kappa_i)$ formed by 're-packaging' (β_i, α_i) from Fig. 26.

Defining, for $0 \le i < 3$,

$$r_i = a_{\lfloor i-1 \rfloor, 3} + b_{i,1} + \cdots + b_{i,4} + a_{i,1} + \cdots + a_{i,4}, \tag{47}$$

$$b_i = b_{i,2} + \cdots + b_{i,5}, \tag{48}$$

$$a_i = a_{i,2} + \cdots + a_{i,5}, \tag{49}$$

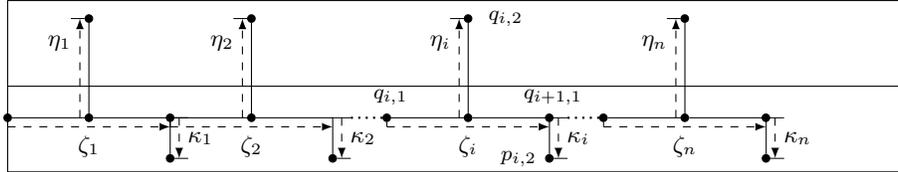the constraints (39) guarantee that $\zeta_i \subseteq r_{\lfloor i \rfloor}$, $\eta_i \subseteq b_{\lfloor i \rfloor}$ and $\kappa_i \subseteq a_{\lfloor i \rfloor}$, for $1 \le i \le n$. Observe that the arcs ζ_i are located entirely in the 'lower window,' and that each arc η_i connects ζ_i to some point $q_{i,2}$, which in turn is connected to $q^* \in \lambda_0$ by an arc in b^*.

Stage 3. We now repeat Stage 2 symmetrically, with the 'upper' and 'lower' windows exchanged. Let $\mathfrak{a}'_{i,j}$, $\mathfrak{b}'_{i,j}$ be 3-region variables (with indices in the same ranges as for $\mathfrak{a}_{i,j}$, $\mathfrak{b}_{i,j}$). Let $\mathfrak{a}' = \mathfrak{b}$, $\mathfrak{b}' = \mathfrak{a}$. The formulas

$$(s'_3 \le \dot{a}'_{0,3}), \tag{40'}$$

$$\bigwedge_{i=0,1,2} \mathsf{stack}_{\dot{z}}(\mathfrak{a}'_{\lfloor i-1 \rfloor, 3}, \mathfrak{b}'_{i,1}, \ldots, \mathfrak{b}'_{i,6}, \mathfrak{b}'), \tag{41'}$$

$$\bigwedge_{i=0,1,2} \mathsf{stack}(\mathfrak{b}'_{i,3}, \mathfrak{a}'_{i,1}, \ldots, \mathfrak{a}'_{i,6}, \mathfrak{a}'), \tag{42'}$$

$$c(b'_{1,5} + d_3) \tag{44'}$$

then establish sequences of n' triples of arcs $(\zeta'_i, \eta'_i, \kappa'_i)$ satisfying $\zeta'_i \subseteq r'_{\lfloor i \rfloor}$, $\eta'_i \subseteq b'_{\lfloor i \rfloor}$ and $\kappa'_i \subseteq a'_{\lfloor i \rfloor}$, for $1 \le i \le n'$, where the r'_i, b'_i and a'_i are defined as in (47)–(49) but with the primed variables. The arcs ζ'_i are located entirely in the 'upper window', and each arc η'_i connects ζ'_i to a point $q'_{i,2}$, which in turn is connected to a point $q^{*\prime}$ by an arc in the region $b^{*\prime} = b'_{0,6} + b'_{1,6} + b'_{2,6} + b'$.

Stage 4. Our next task is to write constraints to ensure that $n = n'$, and that, furthermore, each η_i (also each η'_i) connects ζ_i to ζ'_i, for $1 \le i \le n$. From (43), the only arc depicted in Fig. 21 that z° may intersect is χ_3. Recalling that ζ_n and $\zeta'_{n'}$ contain points $q_{n+1,1}$ and $q'_{n'+1,1}$, respectively, both lying in $\dot{z} \subseteq z^\circ$, the constraint

$$c(\dot{z}) \tag{50}$$

ensures that $q_{n+1,1}$ and $q'_{n'+1,1}$ may be joined by an arc, say ζ^*, lying in z°, and also lying entirely in the upper and lower windows, crossing the chord $\chi_1\chi_2\chi_3$ only in χ_3. Without

loss of generality, we may assume that ζ^* contacts each of ζ_n and $\zeta'_{n'}$ in just one point. Bearing in mind that the constraints (39) force $\eta_n \subseteq b_0 + b_1 + b_2$ and $\eta'_{n'} \subseteq b'_0 + b'_1 + b'_2$ to cross the chord $\chi_1 \chi_2 \chi_3$ in its central section, χ_2, and from (43), we see that the following constraint ensures that ζ^* is as shown in Fig. 28:

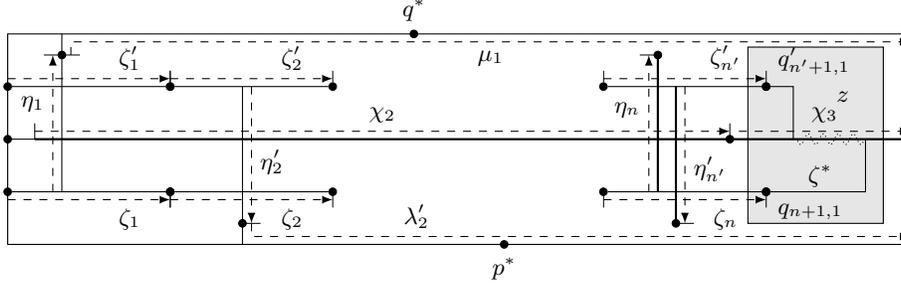$$z \cdot (b^* + b_0 + b_1 + b_2 + b^{*\prime} + b'_0 + b'_1 + b'_2) = 0. \tag{51}$$

Fig. 28. The arc ζ^*.

Now consider the arc η_1. Recalling that η_1 crosses χ_2 and connects ζ_1 to some point $q_{1,2}$, which in turn is connected to the point q^* by an arc in b^*, we see by inspection of Fig. 28 that (51) together with

$$\bigwedge_{i=0,1,2} \neg C(r'_i, b^*) \tag{52}$$

forces η_1 to cross one of the arcs $\zeta'_{j'} \subseteq r'_{\lfloor j' \rfloor}$, for $1 \leq j' \leq n'$; and the constraints

$$\bigwedge_{i=0,1,2} \neg C(r'_i, \quad b_{\lfloor i-1 \rfloor} + b_{\lfloor i+1 \rfloor}) \tag{53}$$

ensure that $j' \equiv 1$ modulo 3. Now suppose $j' \geq 4$. We write the constraints

$$\bigwedge_{i=0,1,2} \neg C(b'_i, \quad r_{\lfloor i-1 \rfloor} + r_{\lfloor i+1 \rfloor}), \tag{53'}$$

$$\bigwedge_{i=0,1,2} \neg C(b'_i, \quad b_{\lfloor i-1 \rfloor} + b_{\lfloor i+1 \rfloor}). \tag{54}$$

The arc η'_2 must connect ζ'_2 to the point $q'_{2,2}$, which in turn is connected to the point p^* on the bottom edge of the lower window by an arc in $b^{*\prime}$, which is now impossible without $\eta'_2 \subseteq b'_2$ crossing either $\zeta_1 \subseteq r_1$ or $\eta_1 \subseteq b_1$—both forbidden by (53′)–(54). Thus, η_1 intersects ζ'_j if and only if $j = 1$. Symmetrically, η'_1 intersects ζ_j if and only if $j = 1$. And the reasoning can now be repeated for $\eta_2, \eta'_2, \eta_3, \eta'_3, \ldots$, leading to the 1–1 correspondence depicted in Fig. 29. In particular, we are guaranteed that $n = n'$.

Stage 5. Recall the given PCP-instance, $\mathsf{W} = (\mathsf{w}, \mathsf{w}')$ over alphabets T and U. In the sequel, we use the standard imagery of tiles, where each tile $t \in \mathsf{T}$ has an 'upper string', $\mathsf{w}'(t) \in \mathsf{U}^*$ and a 'lower string', $\mathsf{w}(t) \in \mathsf{U}^*$. Thus, the problem is to determine whether there is some non-empty sequence of tiles such that the concatenated upper and lower strings both spell out the same word of U^*. We shall label the arcs $\zeta_1, \ldots, \zeta_n$ so as to define a word $\tau \in \mathsf{T}^*$ (with $|\tau| = m \leq n$); likewise we shall label the arcs $\zeta'_1, \ldots, \zeta'_n$ so as to define another word $\tau' \in \mathsf{T}^*$ (with $|\tau'| = m' \leq n$). Then the arcs $\eta_1, \ldots, \eta_n$ will be labelled with the regions in
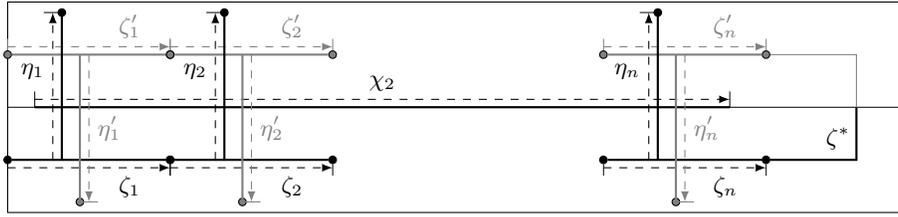
Fig. 29. The 1–1 correspondence between the ζ_i and the ζ_i' established by the η_i and the η_i'.

$\vec{u}$, so to define a word $v \in \mathsf{U}^*$, with $|v| = n$. We shall then add conjuncts to ψ_W ensuring $\mathsf{w}(\tau) = \mathsf{w}'(\tau') = v$ and $\tau = \tau'$, which will guarantee that W is positive.

For $1 \le h \le |\mathsf{T}|$, $1 \le \ell \le |\mathsf{w}(t_h)|$ and $0 \le i < 3$, let $p_{h,\ell}$ be a fresh variable, and let these variables be ordered in some way as the tuple $\vec{p}$. As in the proof of Theorem 4.1, we think of $p_{h,\ell}$ as standing for the ℓth position in the string $\mathsf{w}(t_h)$, where $t_h \in \mathsf{T}$. We use $\vec{p}$ to label the components of r_i, $0 \le i < 3$, but since the r_i are not pairwise disjoint, we require a copy of $\vec{p}$ for each i. Hence, for $1 \le h \le |\mathsf{T}|$, $1 \le \ell \le |\mathsf{w}(t_h)|$ and $0 \le i < 3$, let $p_{h,\ell}^i$ be a fresh variable, and let $\vec{p}_i$ be an ordering of the variables with superscript i. Consider the constraints

$$\bigwedge_{i=0,1,2} \left((r_i = \sum_{h=1}^{|\mathsf{T}|} \sum_{\ell=1}^{|\mathsf{w}(t_h)|} p_{h,\ell}^i) \wedge \mathsf{colour}(r_i; \vec{p}_i) \right) \wedge \bigwedge_{1 \le h \le |\mathsf{T}|} \bigwedge_{1 \le \ell \le |\mathsf{w}(t_h)|} \left(p_{h,\ell} = \sum_{i=0,1,2} p_{h,\ell}^i \right). \tag{55}$$

The first conjunct ensures that each arc $\beta_{i,3} \subseteq b_{\lfloor i \rfloor,3}$ ($1 \le i \le n$) is included in exactly one of the regions $\vec{p}_{\lfloor i \rfloor}$ and is disjoint from the rest of the regions in $\vec{p}_{\lfloor i \rfloor}$ and all the regions in $\vec{p}_{\lfloor i-1 \rfloor}$ and $\vec{p}_{\lfloor i+1 \rfloor}$; the second conjunct then ensures that ζ_i is contained in exactly one of the regions $\vec{p}$, and that $\beta_{i,3}$ is disjoint from the rest of the regions in $\vec{p}$. Note that the $\vec{p}$ do not actually form a partition, because they cannot be made disjoint; nevertheless, we can think of the $\vec{p}$ as 'labels' for arcs ζ_i. The regions in $\vec{p}_0$, $\vec{p}_1$ and $\vec{p}_2$ can now be forgotten.

Next, we organize the arcs ζ_i into (contiguous) blocks, $E_1, \ldots, E_m$ such that, in the jth block, E_j, the sequence of labels reads $p_{h,1}, \ldots, p_{h,|\mathsf{w}(t_h)|}$, for some fixed $1 \le h \le |\mathsf{T}|$. This amounts to insisting that: (i) the very first arc, ζ_1, must be labelled with $p_{h,1}$ for some h; (ii) if ζ_i ($i < n$) is labelled with $p_{h,\ell}$, where $\ell < |\mathsf{w}(t_h)|$, then the next arc, namely ζ_{i+1}, must be labelled with the next position in $\mathsf{w}(t_h)$, namely $p_{h,\ell+1}$; (iii) if ζ_i ($i < n$) is labelled with the final position of $\mathsf{w}(t_h)$, then the next arc must be labelled with the initial position of some possibly different word $\mathsf{w}(t_{h'})$; and (iv) ζ_n must be labelled with the final position of some word $\mathsf{w}(t_h)$. To do this we simply write:

$$\bigwedge_{1 \le h \le |\mathsf{T}|} \bigwedge_{1 < \ell \le |\mathsf{w}(t_h)|} \neg C(p_{h,\ell}, \ s_3), \tag{56}$$

$$\bigwedge_{i=0,1,2} \bigwedge_{h=1}^{|\mathsf{T}|} \bigwedge_{\ell=1}^{|\mathsf{w}(t_h)|-1} \neg C(r_i \cdot p_{h,\ell}, \ r_{\lfloor i+1 \rfloor} \cdot (-r_i) \cdot (\sum_{\ell' \ne \ell+1} p_{h,\ell'} + \sum_{h' \ne h} \sum_{\ell'=1}^{|\mathsf{w}(t_{h'})|} p_{h',\ell'})), \tag{57}$$

$$\bigwedge_{i=0,1,2} \bigwedge_{1 \le h,h' \le |\mathsf{T}|} \bigwedge_{1 < \ell \le |\mathsf{w}(t_h)|} \neg C(r_i \cdot p_{h,|\mathsf{w}(t_h)|}, \ r_{\lfloor i+1 \rfloor} \cdot (-r_i) \cdot p_{h',\ell}), \tag{58}$$

$$\bigwedge_{1 \le h \le |\mathsf{T}|} \bigwedge_{1 \le \ell < |\mathsf{w}(t_h)|} \neg C(p_{h,\ell}, \ z). \tag{59}$$

Supposing the arcs of jth block E_j to have labels reading $p_{h,1}, \ldots, p_{h,|\mathsf{w}(t_h)|}$ (for some fixed h), then, we write h_j to denote the common subscript h. The sequence of indices $h_1, \ldots, h_m$ corresponding to the successive blocks thus defines a word $\tau = t_{h_1} \cdots t_{h_m} \in \mathsf{T}^*$.

Using corresponding formulas, we label the arcs ζ_i' ($1 \leq i \leq n$) with the tuple $\vec{p}'$ of variables $p_{h,\ell}'$, for $1 \leq h \leq |\mathsf{T}|$ and $1 \leq \ell \leq |\mathsf{w}'(t_h)|$, so that, in any satisfying assignment over $\mathsf{RC}(\mathbb{R}^2)$, every arc ζ_i' is labeled with exactly one of the regions $\vec{p}'$ and $\beta_{i,3}' \subseteq b_{i,3}$ is disjoint from the rest of the regions in $\vec{p}'$. Further, we can ensure that these labels are organized into (say) m' contiguous blocks, $E_1', \ldots, E_{m'}'$ such that in the jth block, E_j', the sequence of labels reads $p_{h,1}', \ldots, p_{h,|\mathsf{w}'(t_h)|}'$, for some fixed h. Again, writing h_j' for the common value of h, the sequence of indices $h_1', \ldots, h_{m'}'$ corresponding to the successive blocks defines a word $\tau' = t_{h_1'} \cdots t_{h_{m'}'} \in \mathsf{T}^*$.

Now, the constraints

$$\mathsf{part}(\vec{u}) \quad \wedge \quad \bigwedge_{i=0,1,2} \mathsf{colour}(b_i; \; \vec{u}) \tag{60}$$

ensure that, in any satisfying assignment over $\mathsf{RC}(\mathbb{R}^2)$, every arc $\eta_i \subseteq b_{\lfloor i \rfloor}$, for $1 \leq i \leq n$, is included in ('labelled with') exactly one of the regions in $\vec{u}$, so that the sequence of arcs $\eta_1, \ldots, \eta_n$ defines a word $\upsilon \in \mathsf{U}^*$, with $|\upsilon| = n$.

Securing $\mathsf{w}(\tau) = \mathsf{w}'(\tau') = \upsilon$ is easy. The constraints

$$\bigwedge_{1 \leq h \leq |\mathsf{T}|} \bigwedge_{1 \leq \ell \leq |\mathsf{w}(t_h)|} \bigwedge_{\substack{u_i \text{ is not the } \ell\text{th} \\ \text{letter of } \mathsf{w}(t_h)}} \neg C(u_i, \; p_{h,\ell}) \tag{61}$$

ensure that, since η_i intersects ζ_i, for all $1 \leq i \leq n$, the word $\upsilon \in \mathsf{U}^*$ defined by the arcs η_i must be identical to the word $\mathsf{w}(t_{h_1}) \cdots \mathsf{w}(t_{h_m})$. But this is just to say that $\upsilon = \mathsf{w}(\tau)$. The equation $\upsilon = \mathsf{w}'(\tau')$ is obtained similarly.

Stage 6. In the foregoing stages, we assembled conjuncts of ψ_{W} in such a way that, given any satisfying assignment for ψ_{W}, we can construct sequences of labelled arcs defining words $\upsilon \in \mathsf{U}^*$ and $\tau, \tau' \in \mathsf{T}^*$ with $\mathsf{w}(\tau) = \mathsf{w}'(\tau') = \upsilon$, as described above. In this stage, we add more conjuncts to ψ_{W} to enforce the equation $\tau = \tau'$. This shows that, if ψ_{W} is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, then W is positive.

In particular, it remains to show that $m = m'$ and that $h_j = h_j'$, for all $1 \leq j \leq m$. To do so, we re-use the techniques encountered in Stage 4. We first introduce a new pair of variables, f_0, f_1, which we refer to as 'block colours,' and with which we label the arcs ζ_i. Again, since the regions r_i overlap, we additionally require regions f_0^i and f_1^i, for $0 \leq i < 3$. Consider the constraints:

$$\bigwedge_{i=0,1,2} \big((r_i = f_0^i + f_1^i) \; \wedge \; \mathsf{colour}(r_i; \; f_0^i, f_1^i) \big) \quad \wedge \quad \bigwedge_{k=0,1} \big(f_k = \sum_{i=0,1,2} f_k^i \big). \tag{62}$$

It is readily checked that each $\zeta_i \subseteq r_{\lfloor i \rfloor}$ is included in exactly one of the regions f_0 or f_1, and that $\beta_{i,3}$ is disjoint from the other. (Again, however, f_0, f_1 do not form a partition, because they must overlap.) We force all arcs in each block E_j to have a uniform block colour, and we force the block colours to alternate by writing:

$$\bigwedge_{k=0,1} \bigwedge_{1 \leq h \leq |\mathsf{T}|} \bigwedge_{1 \leq \ell < |\mathsf{w}(t_h)|} \neg C(f_k \cdot p_{h,\ell}, \; f_{1-k} \cdot p_{h,\ell+1}), \tag{63}$$

$$\bigwedge_{i=0,1,2} \bigwedge_{k=0,1} \bigwedge_{1 \leq h,h' \leq |\mathsf{T}|} \neg C(f_k \cdot p_{h,|\mathsf{w}(t_h)|} \cdot r_i, \; f_k \cdot p_{h',1} \cdot r_{\lfloor i+1 \rfloor} \cdot (-r_i)). \tag{64}$$

Thus, we may speak unambiguously of the colour (f_0 or f_1) of a block: if E_1 is coloured f_0, then E_2 will be coloured f_1, E_3 coloured f_0, and so on. Using variables f_0' and f_1', we similarly establish a block structure $E_1', \ldots, E_{m'}'$ on the arcs ζ_i'.

Fig. 30. Arc $\theta_j \tilde{\theta}_j$ intersecting $\zeta_1' \cdots \zeta_n'$.

Now we match up the blocks in a 1–1 fashion, just as we matched up the individual arcs in Stage 4. Let $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g}_0'$ and $\mathfrak{g}_1'$ be new 3-region variables. Recall that every arc ζ_i contains some point of $b_{\lfloor i \rfloor, 3}$ (for instance: $p_{i,1}$) and every such point is unambiguously labeled by a region in $\vec{p}$ and a region in (f_0, f_1). We wish to connect any such arc that starts a block E_j (i.e., any ζ_i labelled by $p_{h,1}$ for some h) to the top edge of the upper window, with the connecting arc depending on the block colour. We can do this using the constraints:

$$\bigwedge_{k=0,1} \left((f_k \cdot (b_{0,3} + b_{1,3} + b_{2,3}) \leq \dot{g}_k) \ \wedge \ \mathsf{stack}(\mathfrak{g}_k, \mathfrak{b}) \right). \tag{65}$$

Specifically, the first (actually: every) arc ζ_i in each block E_j, for $1 \leq j \leq m$, is connected by an arc $\theta_j \tilde{\theta}_j$ to some point on the upper edge of the upper window, where $\theta_j \subseteq g_k$ and $\tilde{\theta}_j \subseteq b$. Using corresponding formulas, we ensure that the first arc in each block E_j', for $1 \leq j \leq m'$, is connected by an arc $\theta_j' \tilde{\theta}_j'$ to some point on the bottom edge of the lower window, where $\theta_j' \subseteq g_k'$ and $\tilde{\theta}_j' \subseteq b'$.

Recall from Stage 3 that $q_{n+1,1}$ is connected by an arc $\kappa_n \subseteq a_0 + a_1 + a_2$ to $p_{n,2}$, which is in turn connected to the lower edge of the lower window by an arc in lying in a^*. And recall from Stage 4 that $q_{n+1,1}$ is connected by $\zeta^* \subseteq z^\circ$ to $q_{n+1,1}'$. Thus, we see from Fig. 30 that the non-contact constraint

$$\neg C(g_0 + g_1, \ s_4' + \cdots + s_1' + s_0 + \cdots + s_5 + a^* + a_0 + a_1 + a_2 + z) \tag{66}$$

ensures that each $\theta_j \tilde{\theta}_j$ ($1 \leq j \leq m$) intersects one of the ζ_i' ($1 \leq i \leq n$). Indeed, since $\tilde{\theta}_j \subseteq b$ cannot intersect any ζ_i', we know that all such points of intersection lie on θ_j. Using a corresponding formula, we ensure that each θ_j' ($1 \leq j \leq m'$) intersects one of the ζ_i ($1 \leq i \leq n$).

We now write the constraints

$$\bigwedge_{k=0,1} \left(\neg C(g_k, f_{1-k}') \wedge \neg C(g_k', f_{1-k}) \right). \tag{67}$$

Thus, any θ_j included in g_k must join some arc ζ_i in a block with colour f_k to some arc $\zeta_{i'}'$ in a block with colour f_k'; and symmetrically for the θ_j'. Adding

$$\neg C(g_0 + g_0', \ \ g_1 + g_1') \tag{68}$$

then ensures, via reasoning similar to that employed in Stage 4, that θ_1 connects the block E_1 to the block E_1', θ_2 connects E_2 to E_2', and so on; and similarly for the θ_j' (as shown,

schematically, in Fig. 31). Thus, we have a 1–1 correspondence between the two sets of blocks, whence $m = m'$.
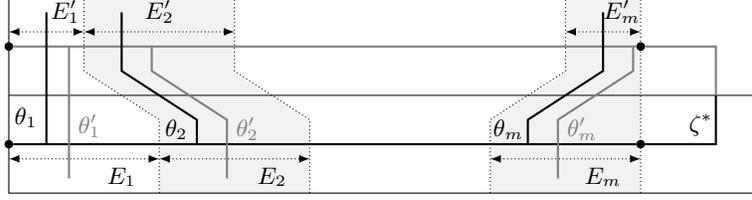

Fig. 31. The 1–1 correspondence between the E_j and the E'_j established by the θ_j and the θ'_j.

Finally, we regard elements of the alphabet T as fresh variables and order them to form the tuple $\vec{t}$. These variables are used for labelling the components of g_0 and of g_1, and hence the arcs $\theta_1, \ldots, \theta_m$:

$$\mathsf{part}(\vec{t}) \quad \wedge \quad \mathsf{colour}(g_0;\ \vec{t}) \quad \wedge \quad \mathsf{colour}(g_1;\ \vec{t}). \tag{69}$$

(Note that this time we can take the regions $\vec{t}$ to form a partition.) Adding the constraints

$$\bigwedge_{k=0,1} \bigwedge_{1 \leq h \leq |\mathsf{T}|} \neg C\big(\sum_{h' \neq h} (g_k \cdot t_{h'}),\ \sum_{1 \leq \ell \leq |\mathsf{w}(t_h)|} p_{h,\ell} + \sum_{1 \leq \ell \leq |\mathsf{w}'(t_h)|} p'_{h,\ell}\big) \tag{70}$$

instantly ensures that the sequences of tile indices $h_1, \ldots, h_m$ and $h'_1, \ldots, h'_m$ are identical. In other words, $\tau = \tau'$.

This completes the argument that, if ψ_W has a satisfying assignment over $\mathsf{RC}(\mathbb{R}^2)$, then W is a positive instance of the PCP. By extending the arrangement of Fig. 22 in the obvious way, we see that, if W is a positive instance of the PCP, then ψ_W has a satisfying assignment over $\mathsf{RCP}(\mathbb{R}^2)$, and hence (trivially) a satisfying assignment over $\mathsf{RC}(\mathbb{R}^2)$. □

The case $\mathcal{C}c^\circ$ is dealt with as for Corollary 3.2: we replace all occurrences of c in ψ_W with c°. Denoting the resulting $\mathcal{C}c^\circ$-formula by ψ_W°, we see that the following are equivalent: (i) W is positive; (ii) ψ_W° is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$; ($iii$) ψ_W° is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$. Thus,

COROLLARY 5.9. *The problems $Sat(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^2))$ and $Sat(\mathcal{C}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ are r.e.-hard.*

Employing the techniques of the proof of Theorem 3.3, we show that

THEOREM 5.10. *The problems $Sat(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^2))$ and $Sat(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}^2))$ are r.e.-hard.*

PROOF. Again, observe that all conjuncts of ψ_W featuring the predicate C are *negative* (remember that there are additional such literals implicit in the use of 3-region variables, e.g., $\dot{r} \ll r$; but let us ignore these for the moment.) Recall the formula $\mathsf{notC}(r, s)$ from the proof of Theorem 3.3 and consider the effect of replacing any literal $\neg C(r, s)$ in ψ_W by the corresponding instance of $\mathsf{notC}(r + r', s + s')$, where r' and s' are fresh variables; denote the resulting formula by ψ. It is easy to see that ψ entails ψ_W; hence if ψ is satisfiable, then W is a positive instance of the PCP.

We next show that, if W is a positive instance of the PCP, then ψ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. For consider a tuple from $\mathsf{RCP}(\mathbb{R}^2)$ satisfying ψ_W, and based on the arrangement of Fig. 22. Note that if $\mathfrak{r}$ and $\mathfrak{s}$ are 3-regions whose outer shells, r and s, are not in contact (e.g., $\mathfrak{a}_{0,1}$ and $\mathfrak{a}_{0,3}$), then r and s have (i) finitely many components, and (ii) connected complements. Hence, it is easy to find polygons r' and s' satisfying $\mathsf{notC}(r + r', s + s')$. Fig. 32 represents the situation schematically. We may therefore assume that all such literals $\neg C(r, s)$ have been eliminated from ψ_W.

Fig. 32. Disjoint connected regions $r + r'$ and $s + s'$ for regions r and s with finitely many components and connected complements.

We are not quite done, however. We must show that we can replace the *implicit* non-contact constraints $(\ddot{r} \ll \dot{r})$ and $(\dot{r} \ll r)$ that come with the use of each 3-region variable $\mathfrak{r}$ by suitable $\mathcal{B}c$-formulas. Since the two conjuncts are identical in form, we only show how to deal with $(\dot{r} \ll r)$, which, we recall, is an alternative notation for $\neg C(\dot{r}, -r)$. Since the complement of $-r$ is in general not connected, a direct use of $\mathsf{notC}(\dot{r} + r', (-r) + s')$ will result in an unsatisfiable formula. Instead, we represent $-r$ as the sum of two regions s_1 and s_2 with connected complements, and then proceed as before. In particular, we replace $(\dot{r} \ll r)$ by $((-r) = (s_1 + s_2)) \wedge \mathsf{notC}(\dot{r} + r_1, s_1) \wedge \mathsf{notC}(\dot{r} + r_2, s_2)$. For $i = 1, 2$, $\dot{r} + r_i$ is a connected region that is disjoint from s_i. So, $\dot{r}$ is disjoint from s_1 and s_2, and hence disjoint from their sum, $-r$. Fig. 33 shows regions r_1, s_1 satisfying the above formula; the other pair, r_2, s_2 is the mirror image.
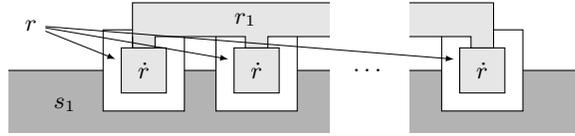
Fig. 33. Disjoint connected regions $\dot{r} + r_1$ and $s_1 \leq (-r)$ for $\dot{r}$ right inside r.

Let $\psi_{\mathsf{W}}^{\mathsf{c}}$ be the result of replacing in ψ_{W} all the (explicit or implicit) conjuncts containing the predicate C, as just described. We have thus shown that if $\psi_{\mathsf{W}}^{\mathsf{c}}$ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$ then W is positive, and, conversely, if W is positive then $\psi_{\mathsf{W}}^{\mathsf{c}}$ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. $\square$

THEOREM 5.11. *The problems $Sat(\mathcal{B}c^{\circ}, \mathsf{RC}(\mathbb{R}^2))$ and $Sat(\mathcal{B}c^{\circ}, \mathsf{RCP}(\mathbb{R}^2))$ are r.e.-hard.*

PROOF. We begin with the $\mathcal{C}c^{\circ}$-formula $\psi_{\mathsf{W}}^{\circ}$ constructed in the proof of the r.e.-hardness result in Corollary 5.9. We proceed by eliminating occurrences of C. However, we cannot directly use Lemma 5.4 as in the proof of Theorem 5.5, because the regions in question may not necessarily be bounded. For instance, consider the formula $(\dot{s}_0 \ll s_0)$, which is an alternative notation for $\neg C(\dot{s}_0, -s_0)$: although the region $\dot{s}_0$ in Fig. 20 is evidently bounded, $-s_0$ is not. We proceed as follows. Say that a region r is *quasi-bounded* if either r itself or its complement, $-r$, is bounded. Since all the polygons in the tuple satisfying $\psi_{\mathsf{W}}^{\circ}$ are quasi-bounded, we can eliminate all occurrences of C from $\psi_{\mathsf{W}}^{\circ}$ using the following fact [Newman 1964, p. 137]:

LEMMA 5.12. *Let F, G be disjoint, closed subsets of $\mathbb{R}^2$ such that both $\mathbb{R}^2 \setminus F$ and $\mathbb{R}^2 \setminus G$ are connected. Then $\mathbb{R}^2 \setminus (F \cup G)$ is connected.*

So, suppose we have a conjunct $\neg C(r, s)$ in $\psi_{\mathsf{W}}^{\circ}$. We consider the following formula:

$$\chi(r, s, \vec{v}) = (r = r_1 + r_2) \wedge (s = s_1 + s_2) \wedge \bigwedge_{1 \leq i,j \leq 2} \left(\mathsf{K5m}(\vec{v}_{ij}) \wedge (r_i \leq v_{ij}^1) \wedge (s_j \leq v_{ij}^2) \right),$$

where $\vec{v}$ is a vector of variables containing r_1, r_2, s_1, s_2 and $v_{ij}^1, \ldots, v_{ij}^5$, for $1 \leq i, j \leq 2$, and $\mathsf{K5m}(v^1, \ldots, v^5)$ is the formula defined before Lemma 5.4. By Lemma 5.4 (i), $\chi(r, s, \vec{v})$ entails $\neg C(r, s)$ over $\mathsf{RC}(\mathbb{R}^2)$. We also show that, conversely, if a and b are disjoint quasi-bounded polygons then there exists a tuple of polygons $\vec{e}$ such that $(a, b, \vec{e})$ satisfies $\chi(r, s, \vec{v})$. Indeed, it is routine to show that, for each quasi-bounded region a, there exist a pair of polygons a_1 and a_2 such that $a = a_1 + a_2$ and both $\mathbb{R}^2 \setminus a_1$ and $\mathbb{R}^2 \setminus a_2$ are connected. Let

b_1 and b_2 be chosen analogously for b. Then, for all $1 \leq i, j \leq 2$, we have $a_i \cap b_j = \emptyset$ and, by Lemma 5.12, $\mathbb{R}^2 \setminus (a_i + b_j)$ is connected. Thus, there exists a piecewise-linear Jordan curve in $\mathbb{R}^2 \setminus (a_i + b_j)$ separating a_i and b_j. By Lemma 5.4 (ii), let $\vec{e}_{ij}$ be a tuple of polygons satisfying $\mathsf{K5m}(\vec{v}_{ij})$ and such that $a_i \leq e_{ij}^1$ and $b_j \leq e_{ij}^2$. It should be clear that the tuple of a_1, a_2, b_1, b_2 and the $\vec{e}_{ij}$, for $1 \leq i, j \leq 2$, is as required.

By replacing all occurrences of C in ψ_W° as described above, we obtain a $\mathcal{B}c^\circ$-formula, say ψ_W^*, such that, if ψ_W^* is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, then W is a positive instance of the PCP; and, conversely, if W is a positive instance of the PCP, then ψ_W^* is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. □

6. THE LANGUAGE $\mathcal{B}c^\circ$ IN DIMENSIONS GREATER THAN 2

In this section, we consider the complexity of satisfying $\mathcal{B}c^\circ$-formulas by polyhedra and regular closed sets in three-dimensional Euclidean space. We proceed by analysing the connections between geometrical and graph-theoretic interpretations of $\mathcal{B}c^\circ$.

A topological space T in which the intersection of *any* family of open sets is open is called an *Aleksandrov space*. Every quasi-order (W, R), that is, a transitive and reflexive relation R on W, can be regarded as an Aleksandrov space by taking $X \subseteq W$ to be open just in case $x \in X$ and xRy imply $y \in X$. (Hence, X is closed just in case $x \in X$ and yRx implies $y \in X$.) It can be shown [Bourbaki 1966] that every Aleksandrov space is the homeomorphic image of one constructed in this way. In the sequel, we shall silently treat any quasi-order (W, R) as a topological space. As we shall see, every $\mathcal{B}c$- or $\mathcal{B}c^\circ$-formula that is satisfiable in RC can also be satisfied in an Aleksandrov space of rather primitive structure.

By a *quasi-saw* we mean a partial order (W, R) of depth 1, in which every point of depth 0 has at least one R-predecessor of depth 1; see Fig. 34. We denote the set of points of depth i by W_i, $i = 0, 1$. Every regular closed set X in such a quasi-saw is uniquely defined by its points of depth 0: a point $z \in W_1$ is in X if and only if there is $x \in W_0 \cap X$ such that zRx. Note also that a set X in (the Aleksandrov space of) a quasi-saw (W, R) is connected if and only if X is connected in the undirected graph with the vertices W and the edges given by R. For example, the set $\{x_3, x_4, z_2, z_3, z_4\}$ in Fig. 34 (shaded) is regular closed and connected, while its interior $\{x_3, x_4, z_3\}$ is not connected.
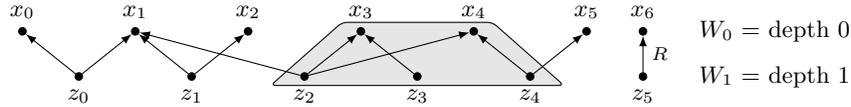

Fig. 34. A quasi-saw.

A *quasi-saw model* is a model based on a quasi-saw (with variables interpreted by regular closed sets). The proof of the following lemma follows from [Kontchakov et al. 2010a, Lemmas 4.1 and 4.2] (see also [Wolter and Zakharyaschev 2000]). But the critical observation can, in essence, be found already in [McKinsey and Tarski 1944] and [Kripke 1963]: for every formula φ and every topological model $\mathfrak{I}$, there exist a finite Aleksandrov model $\mathfrak{A}$ and a continuous function $f \colon \mathfrak{I} \to \mathfrak{A}$ such that $\tau^\mathfrak{A} = f(\tau^\mathfrak{I})$ for every term τ in φ.

LEMMA 6.1. *Let φ be a $\mathcal{B}c$- or $\mathcal{B}c^\circ$-formula. If φ is satisfiable over RC then it can be satisfied in a finite quasi-saw model.*

We begin by briefly discussing the results of [Kontchakov et al. 2010b] for the *polyhedral* case. Denote by ConRC the class of all frames over connected topological spaces with regular closed regions. For a $\mathcal{B}c^\circ$-formula φ, let $\varphi^\bullet$ be the result of replacing every occurrence of c° in φ with c. Evidently, the mapping $\varphi \mapsto \varphi^\bullet$ is a bijection from $\mathcal{B}c^\circ$ to $\mathcal{B}c$.

THEOREM 6.2. *For all $n \geq 3$, the mapping $\varphi \mapsto \varphi^{\bullet}$ constitutes a reduction of $Sat(\mathcal{B}c^{\circ}, \mathsf{RCP}(\mathbb{R}^n))$ to $Sat(\mathcal{B}c, \mathsf{ConRC})$. Hence, the problems $Sat(\mathcal{B}c^{\circ}, \mathsf{RCP}(\mathbb{R}^n))$ coincide, and are all* EXPTIME-*complete.*

PROOF. A *connected partition* in $\mathsf{RCP}(\mathbb{R}^n)$ is a tuple $X_1, \ldots, X_k$ of non-empty polyhedra having connected and pairwise disjoint interiors, which sum to the entire space $\mathbb{R}^n$. The *neighbourhood graph* (V, E) of this partition has vertices $V = \{X_1, \ldots, X_k\}$ and edges

$$E = \{(X_i, X_j) \mid i \neq j \text{ and } (X_i + X_j)^{\circ} \text{ is connected}\};$$

see Fig. 35. Clearly, every connected partition in $\mathsf{RCP}(\mathbb{R}^n)$ has a connected neighbourhood
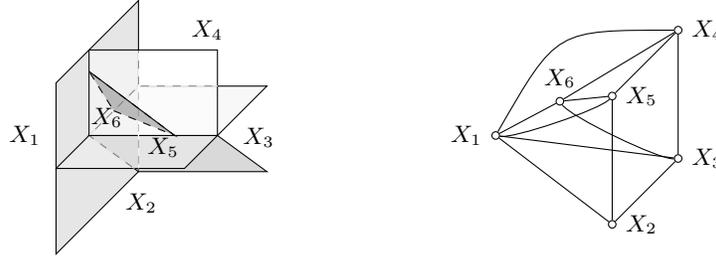
Fig. 35. A connected partition in $\mathsf{RCP}(\mathbb{R}^3)$ and its neighbourhood graph.

graph; and conversely, one can show that *every* connected graph is the neighbourhood graph of some connected partition in $\mathsf{RCP}(\mathbb{R}^n)$. Furthermore, every neighbourhood graph (V, E) gives rise to a quasi-saw $(W_0 \cup W_1, R)$, where $W_0 = V$, $W_1 = \{z_{x,y} \mid (x, y) \in E\}$, and R is the reflexive closure of $\{(z_{x,y}, x), (z_{x,y}, y) \mid (x, y) \in E\}$. Note that in this quasi-saw every point of depth 1 has precisely two R-successors. Such quasi-saws are called *2-quasi-saws*. Conversely, every connected 2-quasi-saw $(W_0 \cup W_1, R)$ can be represented as the neighbourhood graph (W_0, E) of some connected partition, where

$$E = \{(x, y) \mid x \neq y \text{ and there is } z \in W_1 \text{ with } zRx \text{ and } zRy\}.$$

From this, we see that a $\mathcal{B}c^{\circ}$-formula φ is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$ if and only if φ is satisfiable over a connected 2-quasi-saw. But, over 2-quasi-saws, connectedness coincides with interior-connectedness. Thus, φ is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$ if and only if $\varphi^{\bullet}$ is satisfiable over a connected 2-quasi-saw. The problem $Sat(\mathcal{B}c, \mathsf{ConRC})$ is known to be EXPTIME-complete [Kontchakov et al. 2010a]. □

Similarly, one can show that a $\mathcal{B}c^{\circ}$-formula φ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$ if and only if there is a connected partition of $\mathsf{RCP}(\mathbb{R}^2)$ such that its neighbourhood graph is *planar* and φ is satisfiable over the corresponding 2-quasi-saw [Kontchakov et al. 2010b, Lemma 4]. However, as we show in Theorem 5.5, a 'planar' 2-quasi-saw model of a $\mathcal{B}c^{\circ}$-formula may be infinite, and the satisfiability problem is in fact undecidable (cf. Theorem 5.11).

Having shown that the problem $Sat(\mathcal{B}c^{\circ}, \mathsf{RCP}(\mathbb{R}^3))$ is EXPTIME-complete, we now turn our attention to the satisfiability of $\mathcal{B}c^{\circ}$-formulas over the complete Boolean algebra $\mathsf{RC}(\mathbb{R}^3)$, where the picture changes drastically: for instance, the $\mathcal{B}c^{\circ}$-formula (3) is not satisfiable over 2-quasi-saws, but has a quasi-saw model as in Fig. 36. In fact, it is known [Kontchakov et al. 2010b] that every $\mathcal{B}c^{\circ}$-formula φ satisfiable over ConRC can be satisfied in a connected quasi-saw model of size bounded by a polynomial function of $|\varphi|$, and thus the problem $Sat(\mathcal{B}c^{\circ}, \mathsf{ConRC})$ is NP-complete. The following theorem says, in essence, that such polynomial models also give rise to 'small' models over regular closed subsets of $\mathbb{R}^n$, for $n \geq 3$:

THEOREM 6.3. *The problems $Sat(\mathcal{B}c^{\circ}, \mathsf{RC}(\mathbb{R}^n))$, for all $n \geq 3$, coincide with $Sat(\mathcal{B}c^{\circ}, \mathsf{ConRC})$ and are all* NP-*complete.*
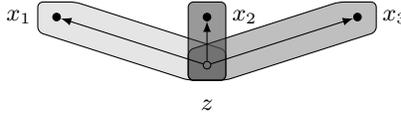
Fig. 36. A quasi-saw model $\mathfrak{A}$ of (3): $r_i^{\mathfrak{A}} = \{x_i, z\}$.

PROOF. We show that, if a $\mathcal{B}c^\circ$-formula is satisfiable over ConRC, then it can be satisfied over $\mathsf{RC}(\mathbb{R}^n)$, for any $n \geq 3$. (The converse implication is trivial.) Indeed, we need only establish the special case $n = 3$, since the others follow by cylindrification. The second statement of the theorem follows from the complexity of $Sat(\mathcal{B}c^\circ, \mathsf{ConRC})$.

So, suppose φ is satisfied in a model $\mathfrak{A}$ over a connected quasi-saw (W, R) of size bounded by a polynomial function of $|\varphi|$. Let W_i be the set of points of depth $i = 0, 1$. Without loss of generality we may assume that there is a point $z_0 \in W_1$ with $z_0 R x$ for all $x \in W_0$, since adding such a point cannot change the truth-values of subformulas of φ of the form $(\tau_1 = \tau_2)$ or $c^\circ(\tau)$. (The only danger here is that adding z to W_1 will change a formula of the form $c^\circ(r)$ from false to true. Now, since z is related by R to *every* point of W_0, the only region that it can cause to become connected is the entire space; but the quasi-saw (W, R) is connected by hypothesis.) We show now how $\mathfrak{A}$ can be embedded into a model over $\mathsf{RC}(\mathbb{R}^3)$. In the remainder of this proof, we repeatedly rely on the fact that, if r and s are interior-connected, regular closed subsets of some topological space, with $r \cdot s \neq 0$, then $r + s$ is also interior-connected.

By an *open ball* we mean a subset of the form $\{x \in \mathbb{R}^3 \mid \|x - y\| < a\}$, where $y \in \mathbb{R}^3$ and a is a positive real. We select open balls D_z for $z \in W_1 \setminus \{z_0\}$ such that their closures are pairwise non-intersecting, and define $D_{z_0} = \mathbb{R}^3 \setminus \bigcup_{z \in W_1 \setminus \{z_0\}} D_z^-$. Thus, each D_z is connected and open, and the open set $D = \bigcup_{z \in W_1} D_z$ is dense. Then we take pairwise disjoint sets B_x^1 for $x \in W_0$, each homeomorphic to a *closed* ball, and arranged so that, for all $x \in W_0$ and $z \in W_1$, $D_z \not\subseteq B_x^1$ and $B_x^1 \cap D_z \neq \emptyset$ if and only if zRx.

We describe a construction in which the regular closed sets B_x^1 are expanded to sets B_x so as to exhaust the entire space, $\mathbb{R}^3$. First, let $q_1, q_2, \dots$ be an enumeration of all the points in D with *rational* coordinates. Consider any piecewise-linear Jordan arc α such that the endpoints of each linear segment of α have rational coordinates: call such an α *rational piecewise-linear*; and let $\alpha_1, \alpha_2, \dots$ be an enumeration of all the rational piecewise-linear arcs with both endpoints in the open set D. We define, for all $k \geq 1$, a collection $\{B_x^k \mid x \in W_0\}$ of interior-connected, pairwise disjoint, regular closed sets in $\mathbb{R}^3$. The case $k = 1$ has already been dealt with. Suppose, then, for $k \geq 1$, the sets B_x^k have been defined; we construct the sets B_x^{k+1} in two steps:

1. If $q_k \in B_x^k$ for some $x \in W_0$, let $\hat{B}_{x'}^k = B_{x'}^k$, for every $x' \in W_0$. Otherwise, $q_k \in D_z$ for some $z \in W_1$. Pick some $x \in W_0$ with zRx and let $C \subseteq D_z$ be a regular closed interior-connected set containing q_k and a point in $(B_x^1)^\circ \cap D_z$ in its interior (e.g., a closed ball centred on q_k and a regular closed 'rod' connecting it to B_x^1, as depicted in Fig. 37). Let $\hat{B}_x^k = B_x^k + C$, and let $\hat{B}_{x'}^k = B_{x'}^k$ for all other $x' \in W_0$. The sets $\hat{B}_{x'}^k$, for $x' \in W_0$, are interior-connected, and, since we are working in $\mathbb{R}^3$, C can obviously be chosen so that the $\hat{B}_{x'}^k$ are pairwise disjoint. (Note that this would not in general be true in $\mathbb{R}$ or $\mathbb{R}^2$.)

2. Let $\hat{B}^k = \bigcup_{x \in W_0} \hat{B}_x^k$. For each $z \in W_1$ such that $\alpha_k \cap D_z$ is not contained in $\hat{B}^k$ and for each $x \in W_0$ such that zRx, choose a distinct point $p_{z,x} \in \alpha_k \cap D_z$, not lying in $\hat{B}^k$. If $p_{z,x}$ is defined, let $C_{z,x} \subseteq D_z$ be a regular closed interior-connected set containing $p_{z,x}$ and a point in $(B_x^k)^\circ \cap D_z$ in its interior, see Fig. 37; otherwise, let $C_{z,x} = \emptyset$. Set $B_x^{k+1} = \hat{B}_x^k + \sum_{z \in W_1} C_{z,x}$, for all $x \in W_0$. The sets B_x^{k+1} are interior-connected; moreover,
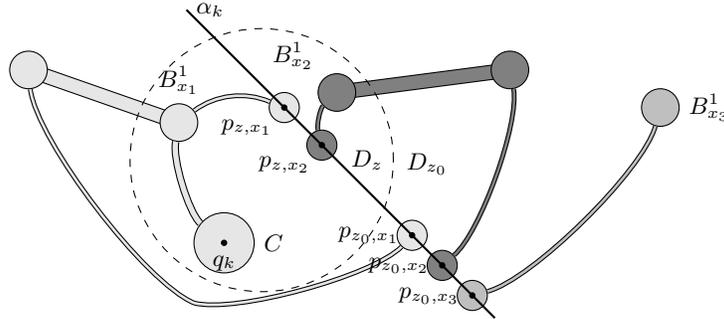
Fig. 37. Filling D_z and D_{z_0} with the sets B_{x_i}, for zRx_1, zRx_2 and z_0Rx_i, $i = 1, 2, 3$.

since we are working in $\mathbb{R}^3$, the $C_{z,x}$ can obviously be chosen such that the B_x^{k+1} are also pairwise disjoint. (Again, this would not in general be true in $\mathbb{R}$ or $\mathbb{R}^2$.)

Since $\mathsf{RC}(\mathbb{R}^3)$ is a complete Boolean algebra, define $B_x = \sum_{k=1}^{\infty} B_x^k$, for each $x \in W_0$. We show that the sets B_x are interior-connected and form a partition (i.e., their pairwise products are empty, and they sum to $\mathbb{R}^3$). Indeed, for distinct $x, y \in W_0$, we certainly have, for all $k, \ell \geq 1$, $B_x^k \cdot B_y^\ell = 0$, whence, by the distributivity law, $B_x \cdot B_y = 0$. And since, for all $k \geq 1$, B_x^k is interior-connected and includes the non-empty, interior-connected set B_x^1, the set $\bigcup_{k=1}^{\infty} \left(B_x^k\right)^\circ$ is connected. But B_x° lies in between $\bigcup_{k=1}^{\infty} \left(B_x^k\right)^\circ$ and its closure, and hence is also connected. Finally, by Step 1 of the above construction, every rational point of the set D lies in some B_x, so that $\sum_{x \in W_0} B_x \supseteq D$, whence $\sum_{x \in W_0} B_x = 1$. This completes the definition of the partition $\{B_x \mid x \in W_0\}$.

Now define a function $f : \mathsf{RC}(W, R) \to \mathsf{RC}(\mathbb{R}^3)$ by taking $f(X) = \sum_{x \in X \cap W_0} B_x$. Let $X \in \mathsf{RC}(W, R)$, and let z be a point of W_1. We claim that $z \in X^\circ$ implies $D_z \subseteq f(X)$; further, if zRx, then $D_z + B_x$ is interior-connected. Indeed, if $z \in X^\circ$ with zRx, then $x \in X$. And since, by Step 1, every rational point of D_z lies in B_x for some such x, it follows that $D_z \subseteq \sum\{B_x \mid x \in X \cap W_0\} = f(X)$. The second statement follows easily from the choice of the sets B_x^1 and the fact that the sets B_x are interior-connected.

We can now show that f is a Boolean algebra homomorphism, and that $X \in \mathsf{RC}(W, R)$ is interior-connected if and only if $f(X)$ is interior-connected. Trivially, $f(X + Y) = f(X) + f(Y)$; and since the B_x form a partition, $f(-X) = \sum_{x \in W_0 \setminus X} B_x = -f(X)$. Now suppose $X \in \mathsf{RC}(W, R)$ is interior-connected, and let p, q be points in $f(X)^\circ$. Then there exist points p', q' in the same components of $f(X)^\circ$ as p, q, respectively, such that, for some $k \geq 1$ and $x, y \in X \cap W_0$, we have $p' \in B_x^k$ and $q' \in B_y^k$. Since X is interior-connected, we can find sequences of points $x_0, \ldots, x_m$ in $X \cap W_0$, and $z_1, \ldots, z_m \in X^\circ \cap W_1$ such that $x = x_0$, $y = x_m$ and z_iRx_{i-1} and z_iRx_i, for all $1 \leq i \leq m$. But we have shown above that the sets D_{z_i} are subsets of $f(X)$, and that the sets $D_{z_i} + B_{x_i}$ and $D_{z_i} + B_{x_{i-1}}$ are interior-connected. Hence, p' and q' lie in the same component of $f(X)^\circ$, whence p and q do as well. That is: $f(X)$ is interior-connected, as required. Finally, suppose $X \in \mathsf{RC}(W, R)$ is not interior-connected, so that we may find elements $x, y \in W_0$ lying in different components of X°. We show that $f(X)$ is not interior-connected. For suppose otherwise. Then there exists a rational piecewise-linear arc α with endpoints in the sets $D \cap \left(B_x^1\right)^\circ$ and $D \cap \left(B_y^1\right)^\circ$, and lying entirely in $f(X)^\circ$. But α occurs as some α_k in our enumeration. It follows that there will be a first point q' of α_k lying in a set $\hat{B}_{y'}^k$ such that x and y' lie in different components of X°; and there will be a last point p' of α_k, occurring strictly before q' and lying in a set $\hat{B}_{x'}^k$. Obviously, x' and y' lie in different components of X°. Let α' be the interior segment

of α_k between p' and q' (i.e. without the end-points); thus α' does not intersect $\hat{B}^k$. By construction of the sets D_z ($z \in W_1$), either α' lies entirely in D_z for some $z \in W_1 \setminus \{z_0\}$, or α' intersects D_{z_0}. In the former case, since zRx' and zRy', but x' and y' lie in different components of X°, it follows that $z \notin X^\circ$. Thus, there exists $x'' \in W_0$ such that zRx'' and $x'' \notin X$. But then Step 2 in the above construction ensures that $\alpha' \subseteq \alpha_k$ contains points of $B_{x''}^{k+1} \subseteq B_{x''}$, contradicting the supposition that $\alpha_k \subseteq f(X)^\circ$. On the other hand, if $\alpha' \cap D_{z_0} \neq \emptyset$, then, since X is not interior-connected, it follows that $X \neq 1$, and so there certainly exists $x'' \in W_0$ such that z_0Rx'' and $x'' \notin X$. By the same reasoning as before, α' contains points of $B_{x''}^{k+1} \subseteq B_{x''}$, contradicting the supposition that $\alpha_k \subseteq f(X)^\circ$.

Now simply define an interpretation $\mathfrak{I}$ over $\mathsf{RC}(\mathbb{R}^3)$ by setting $r^{\mathfrak{I}} = f(r^{\mathfrak{A}})$. It immediately follows from the previous paragraph that φ is true in $\mathfrak{I}$. $\square$

This resolves the decidability and complexity of the problems $Sat(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^n))$ and $Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^n))$ for $n \geq 3$. Recall from Sec. 4 that $Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$ is undecidable for $n \geq 2$, where $\mathcal{L}$ is any of $\mathcal{B}c$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$; and recall from Sec. 5 that $Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ is undecidable, and that $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ is also undecidable, where $\mathcal{L}$ is any of $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$. At the time of writing, it is not known whether any of $Sat(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^n))$, $Sat(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^n))$ or $Sat(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^n))$, for $n \geq 3$, is decidable. The best currently available lower bound can be found in [Kontchakov et al. 2010a], where all three problems are shown to be ExpTime-hard.

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