# On the Maximum Number of Common Cards between Various Classes of Graphs

## by Paul Brown

A thesis presented to Birkbeck College

University of London

in the fulfillment of the thesis requirement

for the degree of

Doctor of Philosophy

in

Mathematics

London 2008



Birkbeck UNIVERSITY OF LONDON

University of London

I hereby declare that the work presented in this thesis is my own:

Paul Brown.

**Abstract**

# On the Maximum Number of Common Cards between Various Classes of Graphs

The *Reconstruction Conjecture* is one of the foremost unsolved problems in graph theory. It conjectures that a graph can be uniquely determined, up to isomorphism, by its collection of unlabelled vertex-deleted subgraphs (called its *deck of cards*). Like many mathematical problems, its appeal lies in the simplicity of its hypothesis and its accessibility to non-experts. However, although many graph theorists have tried to resolve the status of conjecture, it is still an open problem.

Since the conjecture has remained unresolved, attention has focused on related reconstruction problems. One such area is the study of the two *reconstruction numbers* of some particular graph $G$: the *existential* reconstruction number $rn(G)$, defined to be the minimum $k$ such that there exists $k$ cards from which $G$ can be reconstructed, and the *universal* reconstruction number $urn(G)$, defined to be the minimum $k$ such that $G$ can be reconstructed from any $k$ cards.

Most work on reconstruction numbers yet published concerns $rn(G)$. This thesis instead focusses on $urn(G)$ and will be one of the first to contain substantial results on this topic. $urn(G)$ can also be studied in terms of the maximum number of common cards that $G$ can have with any other graph, and that is the approach that we take. We find upper bounds for the maximum number of common cards between pairs of graphs in various classes and, in all cases, we show that these bounds can be attained by infinite families. Moreover, we completely characterise the families of pairs of graphs that attain the bounds. In doing so, we present many families of graph pairs with different values on various parameters that have, by far, the largest number of common cards yet published.

A *sunshine* graph (*caterpillar* graph) is a graph where the removal of all of its leaves reduces the graph to a cycle (path). A pair of graphs have a *common isomorphic component* $C$ if there is a component isomorphic to $C$ in both graphs. A 2UC graph pair is a pair of graphs, in which after the iterative removal of all common isomorphic components, at least one of the resulting graphs is disconnected. For pairs of graphs of order $n$, the major results in this thesis are:

(a) The maximum number of common cards between a connected graph and a disconected graph is $\left\lfloor \frac{n}{2} \right\rfloor + 1$. Moreover, with the exception of six pairs of graphs of order at most 7, any such pair that attains the bound is in one of four families, up to isomorphism.

(b) For $n \geq 62$, the maximum number of common cards between a sunshine graph and a caterpillar graph is $\left\lfloor \frac{2(n+1)}{5} \right\rfloor$. Moreover, in this case there is only one family of such pairs of graphs with $\frac{2(n+1)}{5}$ common cards, up to isomorphism.

(c) For $n \geq 13$, the maximum number of common cards between a 2UC graph pair is $2\left\lfloor \frac{1}{3}(n-1) \right\rfloor$. Moreover, for all values of $n \geq 22$, there is precisely one 2UC graph pair when $n \equiv 1$ or $2 \pmod 3$, and two 2UC graph pairs when $n \equiv 0 \pmod 3$ that attain this bound, up to isomorphism.

(d) For $n \geq 11$, there are families of 2UC graph pairs with the same number of edges having $2\left\lfloor \frac{1}{3}(n-4) \right\rfloor$ common cards and, for certain values of $n \geq 25$, there is an infinite family of 2UC graph pairs with the same degree sequence having $\frac{2}{3}(n+5-2\sqrt{3n+6})$ common cards.

(e) There exist other pairs of graphs in various classes with almost as many common cards as those in (c). In particular, there is a family of pairs of trees with $2\left\lfloor \frac{1}{3}(n-5) \right\rfloor$ common cards.

Additionally we conjecture that, for large enough $n$:

(i) Every simple finite undirected graph is determined, up to isomorphism, by any $2\left\lfloor \frac{1}{3}(n-1)\right\rfloor + 1$ of its vertex-deleted subgraphs.

(ii) There are only eighteen distinct families of pairs of graphs, and at most twelve for any $n$, that have $2\left\lfloor \frac{1}{3}(n-1)\right\rfloor$ common cards.

(iii) Whether a graph is a tree or not can be determined from any $\left\lfloor \frac{n}{2}\right\rfloor + 2$ of its vertex-deleted subgraphs.

**For my parents**

I would like to thank all my friends and colleagues who have supported me throughout. In particular, I would like to thank my two supervisors, Andrew Bowler and Trevor Fenner for all their hard work over the last few years. I would also like to thank my two examiners Graham Brightwell (LSE) and Josef Lauri (University of Malta) for their time and useful comments.

# Contents

# List of Figures

## Notation

$G$ and $H$ are graphs.

$u$ and $v$ are vertices of $G$; $e$ is an edge of $G$.

| | | |
|---|---|---|
| $V(G)$ | vertices of $G$ | 17 |
| $E(G)$ | edges of $G$ | 17 |
| $n$ | order (number of vertices) of $G$ | 18 |
| $m$ | size (number of edges) of $G$ | 18 |
| $d(v)$ | degree of $v$ | 18 |
| $d_i(G)$ | number of vertices of degree $i$ in $G$ | 18 |
| $d_i(v)$ | number of vertices of degree $i$ adjacent to $v$ | 18 |
| $v^*$ | unique leaf adjacent to $v$ (only meaningful if $d_1(v) = 1$) | 18 |
| $G^C$ | complement of $G$ | 18 |
| $H \subseteq G$ | $H$ is a subgraph of $G$ | 19 |
| $G(W)$ | graph induced by $W \subseteq V(G)$ | 19 |
| $G - S$ | subgraph of $G$ obtained by deleting every element of $S \subseteq V(G)$ | 19 |
| $G - T$ | subgraph of $G$ obtained by deleting every edge of $T \subseteq E(G)$ | 19 |
| $G - v$ | the subgraph of $G$ obtained by deleting the vertex $v$ | 19 |
| $G - e$ | the subgraph of $G$ obtained by deleting the edge $e$ | 19 |
| $A + B$ | the disconnected graph consisting | |
| | of precisely two components $A$ and $B$ | 20 |
| $\mathcal{T}$ | a collection of components of a graph | 21 |
| $\kappa(G)$ | connectivity of $G$ | 21 |
| $G \cong H$ | $G$ is isomorphic to $H$ | 21 |
| $A \oplus B$ | a disconnected graph consisting of precisely | |
| | two components, one isomorphic to $A$ and the other to $B$ | 22 |
| $\bigoplus_k \beta_k H_k$ | component structure of $H$ | 22 |
| $h_k$ | order of component isomorphic to $H_k$ | 22 |
| $T$ | an arbitrary tree | 23 |
| $P_n$ | path of order $n$ | 23 |
| $S_p^k$ | $k$-Star with $p$ spokes | 23 |

| | | |
|---|---|---|
| $C_n$ | cycle of order $n$ | 24 |
| $K_n$ | complete graph of order $n$ | 24 |
| $K_{p,\,q}$ | complete bipartite graph of bi-degree $(p,\, q)$ | 25 |
| $S_q[F]$ | graph $F$ with $q$ leaves added to each of its vertices | 26 |
| $\mathcal{D}(G)$ | vertex-deck of $G$ | 28 |
| $\mathcal{E}D(G)$ | edge-deck of $G$ | 32 |
| $s(F,\, G)$ | number of subgraphs of $G$ isomorphic to $F$ | 33 |
| $rn(G)$ | existential or ally reconstruction number of $G$ | 40 |
| $urn(G)$ | universal or adversary reconstruction number of $G$ | 40 |
| $A_H(G)$ | set of active vertices of $G$ with respect to $H$ | 41 |
| $a_H(G)$ | number of active vertices of $G$ with respect to $H$ | 41 |
| $B(G,\, H)$ | bipartite graph with edges joining associated vertices | 41 |
| $b(G,\, H)$ | number of common cards of $G$ and $H$ | 41 |
| $A_{H_j}(G)$ | set of $H_j$-active vertices of $G$ | 48 |
| $a_{H_j}(G)$ | number of $H_j$-active vertices of $G$ | 48 |
| $a_G(H_j)$ | number of active vertices in a component of $H$ isomorphic to $H_j$ | 48 |
| $b(G,\, H_j)$ | number of common cards of $G$ and $H$ restricted to the $H_j$-active vertices of $G$ | 49 |
| $S'_q[K_p]$ | $S_q[K_p]$ with a single leaf removed | 50 |
| $S'_q[K_p]$ | $S_q[K_p]$ with two leaves adjacent to different vertices removed | 50 |
| $X_{uv}$ | component of $G - u$ that contains $v$ | 55 |
| $U$ | arbitrary unicyclic graph | 79 |
| $A^*_T(U)$ | set of active non cut-vertices of $U$ with respect to $T$ | 80 |
| $a^*_T(U)$ | number of active non cut-vertices of $U$ with respect to $T$ | 80 |
| $b^*(U,\, T)$ | number of connected common cards of $U$ and $T$ | 80 |
| $\delta_i(U)$ | number of vertices of degree $i$ on the unique cycle $C$ of $U$ | 80 |
| $S$ | arbitrary sunshine graph | 81 |
| $CT$ | arbitrary caterpillar graph | 82 |
| $c_i(G)$ | number of cut 2-paths of length $i$ in $G$ | 86 |

| | | |
|---|---|---|
| $l_i(G)$ | number of leaf 2-paths of length $i$ in $G$ | 86 |
| $\gamma$ | number of cut-vertices of $S$ | 86 |
| $\mathcal{A}_i(S)$ | set of vertices of $S$ of degree two adjacent to $i$ vertices of degree 2 | 86 |
| $\lvert\overline{\mathcal{A}(S)}\rvert$ | $d_2(S) - b^*(S, CT)$ | 86 |
| $\lambda_i(CT)$ | number of leaves in $CT$ adjacent to a vertex of degree $i$ | 90 |
| $\lambda^*(CT)$ | number of leaves in $CT$ adjacent to a vertex of degree 4 or more | 90 |
| $g_i$ | order of component isomorphic to $G_i$ | 110 |
| $A_Z(Y, G)$ | set of $Z$-active vertices of $G$ in a component isomorphic to $Y$ | 111 |
| $a_Z(Y, G)$ | number of $Z$-active vertices of $G$ in a component isomorphic to $Y$ | 111 |
| $A_{H_j}(G_i)$ | set of $H_j$-active vertices of $G$ in a component isomorphic to $G_i$ | 113 |
| $a_{H_j}(G_i)$ | number of $H_j$-active vertices of $G$ in a component isomorphic to $G_i$ | 113 |
| $b(F_k, F_k)$ | number of common cards of $G$ and $H$ restricted to the $F_k$-active vertices of $G$ and $H$ | 114 |
| $b(G_i, H_j)$ | number of common cards of $G$ and $H$ restricted to the $H_j$-active vertices of $G$ and the $G_i$-active vertices of $H$ | 114 |
| $\overline{a_H(G_i)}$ | number of non-active vertices of $G$ in a component isomorphic to $G_i$ | 148 |
| $b_j$ | $b(G_1, H_j)$ | 148 |
| $\overline{b_j(G)}$ | number of $H_j$-active vertices of $G$ not used for common cards | 148 |
| $R(G)$ | number of vertices of $G$ in a component isomorphic to some $G_i$ that is not used for common cards | 161 |
| $VT_{2(p-1)}$ | $2(p-1)$-regular graph of order $2p$ | 176 |

# Chapter 1

# Preliminaries

In this chapter, we recount some basic graph theoretic terminology and results. With the exception of the concept of 2-paths, the disconnected graph and $p$-star notation, and the construction of the graph $S_1[F]$, all the terminology is standard. We generally follow Bondy and Murty [10, 11], although we also refer the reader to Lauri and Scapellato [23] and Wilson [41]. Proofs of the assertions in this chapter can be found in these references.

## 1.1 Introductory Concepts

A *graph* $G$ consists of two disjoint sets: $V(G)$, whose elements are called the *vertices* of $G$, and $E(G)$, whose elements, called the *edges* of $G$, are pairs of distinct elements of $V(G)$. The number of vertices of $G$ is called the *order* of $G$ and the number of edges of $G$ is called the *size* of $G$. $G$ is *finite* if both its vertex and edge sets are finite. If $E(G)$ consists of ordered pairs, then $G$ is called a *directed graph*; otherwise $G$ is called an *undirected* graph.

If $E(G) = \emptyset$, then $G$ is said to be an *empty* graph. Many authors, however, stipulate that $V(G) \neq \emptyset$. In this thesis we follow [11] and allow $V(G)$ to be empty. We call such a graph the *null* graph, and define it to have order and size zero. A graph with only one vertex and no edges is called the *trivial* graph.

Let $G$ be a graph and let $u$ and $v$ be two distinct vertices of $G$. We denote the edge $e$ consisting of the vertices $u$ and $v$ by $uv$. The vertices $u$ and $v$ are said to be *adjacent* to each other, and the edge $e$ *incident* to $u$ and $v$. Two edges are adjacent, if they are incident to the same vertex. If $G$ is undirected then $uv = vu$. The vertices of $V(G)$ that are adjacent to $v$ are called the *neighbours* of $v$.

Note that, we have restricted our definition so that a graph must be *simple*; that is, $G$ does not contain any *loops* (edges that are only incident to the same vertex) or *multiple edges* (when two or more edges are incident to the same pair of vertices). In addition, unless otherwise specified, all graphs in this thesis will be finite and undirected. For the rest of this chapter, $G$ will denote a (simple) finite undirected graph of order $n$ and size $m$.

The *degree* of $v$, $d(v)$, is the number of vertices adjacent to $v$. If $d(v) = 0$, then $v$ is called an *isolated vertex* and if $d(v) = 1$ then $v$ is called a *leaf*. We denote by $d_i(G)$ the number of vertices of degree $i$ in $G$. If we label the vertices of $G$ by $v_1, v_2, \ldots, v_n$, where $d(v_i) \geq d(v_{i+1})$ for all $i$, then the sequence $d(G) = (d(v_1), d(v_2), \ldots d(v_n))$ is called the *degree sequence* of $G$. If every vertex in $G$ has the same degree $d$ then $G$ is said to be a *$d$-regular* graph. An $(n-1)$-regular graph is called a *complete* graph with $n$ vertices.

It is often useful to consider the number of vertices of a particular degree that are adjacent to $v$. We denote by $d_i(v)$, the number of neighbours of $v$ of degree $i$. In particular, we define a *$k$-leaf adjacent vertex of degree $d$* to be a vertex $v$ such that $d_1(v) = k$ and $d(v) = d$. If $d_1(v) = 1$, then we denote the leaf of $G$ adjacent to $v$ by $v^*$. We sometimes use the term *non-leaf* to describe any vertex that is not a leaf.

The *complement* of $G$ is the graph $G^C$ with vertex set $V(G)$, such that two vertices adjacent are in $G^C$ if and only if they are not adjacent in $G$. Clearly, the size of $G^C$ is equal to $\frac{n(n-1)}{2} - m$, and the degree of $v$ in $G^C$ is equal to $n - d(v) - 1$.

Let $M \subseteq E(G)$. Then $M$ is called a *matching* in $G$ if none of the edges of $M$ are adjacent. In other words, $M$ is a matching of $G$ if all of the edges in $M$ are incident to distinct vertices of $G$. $M$ is a *maximum matching* if there is no matching in $G$ of greater size than $M$.

## 1.2 Subgraphs

A graph $H$ is a *subgraph* of $G$, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and for every edge $e$ of $H$, both vertices incident to $e$ are in $V(H)$. We denote this relationship by $H \subseteq G$. If $V(H) \neq V(G)$, then $H$ is called a *proper* subgraph of $G$. If $V(H) = V(G)$, then $H$ is called a *spanning* subgraph of $G$. For any $W \subseteq V(G)$, the subgraph of $G$ *induced* by $W$ is the graph $G(W)$ with vertex set equal to $W$, and whose edge-set consists of all the edges of $G$ that join any two vertices in $W$.

Suppose that $G_1$ and $G_2$ are subgraphs of $G$. Then $G_1$ and $G_2$ are *vertex-disjoint* if $V(G_1) \cap V(G_2) = \emptyset$, and *edge-disjoint* if $E(G_1) \cap E(G_2) = \emptyset$. Since vertex-disjoint implies edge-disjoint, we use *disjoint* to mean vertex-disjoint. A collection of subgraphs of $G$ is disjoint if these are each pair-wise disjoint.

Let $S \subseteq V(G)$ and let $T \subseteq E(H)$. We define $G - S$, the $S$ *vertex-deleted subgraph of* $G$, to be the subgraph of $G$ induced by $V(G) - S$. Similarly, we define $G - T$, the $T$ *edge-deleted subgraph of* $G$, to be the subgraph of $G$ formed by the deletion of all the elements of $T$ from $E(G)$. Note that, if $S = V(G)$, then $G - S$ is the null graph. Similarly, if $T = E(G)$, then $G - T$ is an empty graph with $n$ vertices.

If $S = \{v\}$, then $G - S$ is called a *vertex-deleted subgraph* of $G$, and is denoted by $G - v$. Similarly, if $T = \{e\}$, then $G - T$ is called an *edge-deleted subgraph* and is denoted by $G - e$. There are $n$ distinct vertex-deleted subgraphs (one for each vertex) and $m$ edge-deleted subgraphs of $G$. These subgraphs form the basis of all *Graph Reconstruction* problems.

## 1.3 Paths and Connectivity

Suppose that $v_1, v_2, \ldots, v_{s+1}$ are distinct vertices of $G$ such that every pair $v_i$ and $v_{i+1}$ are adjacent. Then the sequence $P = v_1, v_2, \ldots, v_{s+1}$ is called a *path* in $G$ of length $s$ from $v_1$ to $v_{s+1}$ (or between $v_1$ and $v_{s+1}$). If every vertex of $P$ except $v_1$ and $v_{s+1}$ is of degree two in $G$ then $P$ is called a *2-path*. More specifically, if $d(v_1) \geq 3$ and $d(v_{s+1}) \geq 3$, then $P$ is called a *cut 2-path* of length $s$, whereas if $d(v_1) \geq 3$ and $d(v_{s+1}) = 1$, then $P$ is called a *leaf 2-path* of length $s$. In these two cases, the vertices $v_1$ and $v_{s+1}$ are called the *end-vertices* and every other vertex is called an *interior* vertex (we discuss 2-paths in greater depth in Chapters 4 and 5). Note that, we assume that any 2-path is of length at least 1. A *cycle* $C$ of length $s \geq 3$ in $G$ is a sequence of adjacent vertices $C = u_1, u_2, \ldots, u_{s+1}$ of $G$, where each $u_1, u_2, \ldots, u_s$ is distinct and $u_1 = u_{s+1}$.

If $u$ and $v$ are vertices of $G$, then $u$ and $v$ are said to be *connected* if there is a path from $u$ to $v$. A path between $u$ and $v$ is a *shortest path* if it has minimum length over all paths in $G$ between $u$ and $v$. If $s$ is the length of a shortest path, we say that the *distance* between $u$ and $v$, $d(u, v)$, is equal to $s$. If there is no path in $G$ between $u$ and $v$, then $d(u, v)$ is undefined.

A natural equivalence relation on the vertices of $G$ is defined to be $u \sim v$ if and only if $u$ and $v$ are connected. If $V_1, V_2, \ldots, V_r$ are the equivalence classes of $\sim$, then the induced-subgraphs $G(V_1), G(V_2), \ldots, G(V_r)$ are called the *connected components* of $G$. If $r = 1$, then $G$ is said to be *connected*; otherwise $G$ is said to be a *disconnected* graph with $r$ components. We express the component structure of $G$ as

$$G = G(V_1) + G(V_2) + \ldots + G(V_r). \tag{1.1}$$

If $D = G(V_i)$ is a component of $G$, then $G - D$ denotes the subgraph of $G$ with the component $D$ removed.

We use the script notation $\mathcal{T}$, $\mathcal{S}$ to denote a subgraph of $G$ consisting of a collection of its components. We usually employ this notation to indicate that our main interest is in the other components of the graph. If $D$ is a component of $\mathcal{T}$, then $\mathcal{T} - D$ denotes the subgraph of $G$ consisting of all the components of $\mathcal{T}$ with $D$ removed. Note that, if $\mathcal{T}$ consists of only the component $D$, $\mathcal{T} - D$ is the null graph.

For a connected graph $G$, if $S \subset V(G)$ such that $G - S$ is a disconnected graph, then $S$ is said to *disconnect* $G$. If $S = \{v\}$, then $v$ is called a *cut-vertex* of $G$. Similarly, if $S = \{u, v\}$ and $u$ and $v$ are not cut-vertices, then $u$ and $v$ are called a *cut-pair* of $G$. The *connectivity* of $G$, denoted $\kappa(G)$, is the size of the smallest such subset that disconnects $G$ (or reduces $G$ to an isolated vertex). If $G$ is disconnected, then $\kappa(G) = 0$; if $G$ contains a cut-vertex, then $\kappa(G) = 1$. $G$ is said to be *separable* if $\kappa(G) \leq 1$; $G$ is $k$-connected if it is of connectivity at least $k$. Any connected graph with three or more vertices contains at least two vertices that are not cut-vertices.

## 1.4 Isomorphism and Isomorphism Classes

Two simple graphs $G$ and $H$ are *isomorphic*, denoted $G \cong H$, if there is a bijection $\phi : V(G) \longrightarrow V(H)$ that preserves adjacency. In other words, $G$ and $H$ are isomorphic if, for all vertices $u$ and $v$ in $G$, $uv$ is an edge in $G$ if and only if $\phi(u)\phi(v)$ is an edge of $H$. If $\phi$ is the identity map, $G$ and $H$ are said to be *identical*.

An isomorphism $\phi$ from $G$ to itself is called an *automorphism* of $G$; in this case, $\phi$ is a permutation of the vertices of $G$ that preserves adjacency. Two vertices $u$ and $v$ are said to be *similar* if there is an automorphism $\phi_v$ of $G$ such $\phi_v(u) = v$. If every pair of vertices of $G$ are similar, then $G$ is said to be *vertex-transitive*.

Isomorphism is an equivalence relation on the set of all graphs. We call the equivalence classes, the *isomorphism classes*. To indicate we are considering some representative of a particular graph isomorphism class, we draw graphs without any labelling of their vertices or edges.

Let $A$ and $B$ be connected graphs. We denote by $A \oplus B$ a disconnected graph with two components, one isomorphic to $A$ and the other to $B$. If $B = A$, then this denotes a graph with precisely two components both isomorphic to $A$. For any non-negative integer $\beta$, we frequently denote a graph with $\beta$ components, all of which are isomorphic to $A$, by $\beta A$; so $2A$ and $A \oplus A$ denote isomorphic graphs. We further extend this notation in the natural way when $A$ or $B$ is disconnected. Note that, when $\beta = 0$, $\beta A$ is the null graph.

Now suppose that $H$ is a disconnected graph whose components are in $r$ distinct isomorphism classes with representatives $H_1,\ H_2,\ \ldots, H_r$. Then we express the component structure of $H$ as

$$H \cong \beta_1 H_1 \oplus \beta_2 H_2 \oplus \ldots \oplus \beta_r H_r, \qquad \text{or just} \qquad H \cong \textstyle\bigoplus_k \beta_k H_k, \tag{1.2}$$

where the coefficients $\beta_1,\ \beta_2,\ \ldots, \beta_r$ are positive integers. We define $h_i = |V(H_i)|$, and order the isomorphism classes so that $h_1 \geq h_2 \geq \ldots \geq h_r$. Finally, we define $\beta_i = 0$ for $i > r$. Note that, if $\mathcal{T} \cong \oplus_{i=1}^{t} \beta_i H_i$ and $\mathcal{S} \cong \oplus_{j=t+1}^{r} \beta_j H_j$, then $H \cong \mathcal{T} \oplus \mathcal{S}$.

## 1.5 Graph Parameters

A graph *parameter* is any function that can be defined on the set of all graphs and is invariant under isomorphism. For example, the order, size, connectedness, connectivity and degree sequence are all graph parameters. One parameter, due to Kelly [20, 21], that has been used extensively in *Graph Reconstruction* is $s(F, G)$, the number of subgraphs of $G$ isomorphic to $F$, for any given graph $F$. Parameters we shall also make use of in Chapters 4 and 5 are the numbers of cut 2-paths and leaf 2-paths of $G$. Whenever we refer to *a graph* that has some particular value(s) for a parameter, we mean a representative of the isomorphism class with that parameter value(s).

We informally define a *family* of graphs to be a (usually infinite) set of graphs that have a similar structure. We further define a *class* of graphs to be a family of graphs that is closed under isomorphism. We introduce some common classes of graphs in the final two sections of this chapter. We extend the notion of a family of graphs to families of graph pairs in the natural way. Note that, when we refer to a family of graphs as *unique*, we mean that family is unique up to isomorphism.

## 1.6 Trees

A graph is *acyclic* if it contains no cycles. If such a graph is connected, it is called a *tree*; if it is disconnected, it is called a *forest*. Obviously, all of the components of a forest must be trees. To distinguish them from other graphs, we denote an arbitrary tree by $T$. A *spanning tree* of $G$ is a spanning subgraph of $G$ that is a tree. Every connected graph contains a spanning tree.

Let $T$ be a tree of order $n \geq 3$. Then $|E(T)| = n - 1$ and every non-leaf of $T$ is a cut-vertex; so $\kappa(T) = 1$. Moreover, if $v$ is a vertex of $T$, then $T - v$ consists of precisely $d(v)$ components. Any two distinct vertices of $T$ are connected by a unique path. In addition, since any non-trivial connected graph of order $n$ contains at least two vertices that are not cut-vertices, $T$ contains at least two leaves.

The simplest type of tree is $P_n$, the *path* of order $n$. This graph consists of $n$ vertices $v_1, v_2, \ldots, v_n$ such that for $2 \leq i \leq n-1$, each $v_i$ is only adjacent to $v_{i-1}$ and $v_{i+1}$, and additionally both $v_1$ and $v_n$ are leaves. So, $d_1(P_n) = 2$ and $d_2(P_n) = n - 2$. Another common type of tree is $S_p^k$, the *$k$-star with $p$ spokes*. This tree consists of $p$ copies of $P_k$ with one leaf of each path adjacent to an additional "central" vertex (and is of order $pk + 1$).

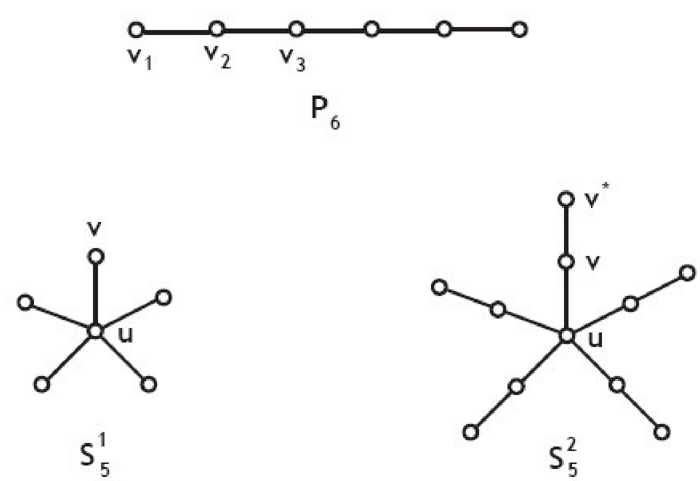


Figure 1.1: $P_6$, $S_5^1$ and $S_5^2$.

$S_{n-1}^1$ is more commonly called the 1-*star* of order $n$ (and often denoted by $K_{1,n-1}$). Since such a graph consists of a central vertex and $n-1$ leaves, $d_1(S_{n-1}^1) = n-1$, $d_{n-1}(S_{n-1}^1) = 1$, and $d_i(S_{n-1}^1) = 0$ for all other $i$. Similarly, $S_{\frac{n-1}{2}}^2$ is called the 2-*star* of order $n$ ($n$ must clearly be odd). In this case, $d_1(S_{\frac{n-1}{2}}^2) = d_2(S_{\frac{n-1}{2}}^2) = \frac{n-1}{2}$, $d_{\frac{n-1}{2}}(S_{\frac{n-1}{2}}^2) = 1$, and $d_i(S_{\frac{n-1}{2}}^1) = 0$ for all other $i$. Figure 1.1 shows the path and 1-star of order 6, and the 2-star of order 11.

## 1.7 Other Common Graphs

The *cycle* of order $n$, denoted by $C_n$, is the 2-regular connected graph. The size of $C_n$ is equal to $n$. For each vertex $v$ of $C_n$, $C_n - v \cong P_{n-1}$. Since $P_{n-1}$ is a tree, it follows that $\kappa(C_n) = 2$.

The *complete graph* of order $n$, denoted by $K_n$, is the $(n-1)$-regular connected graph. Since every vertex $v$ of $K_n$ is adjacent to every other, the size of $K_n$ is equal to $\frac{n(n-1)}{2}$ and, in addition, $K_n - v \cong K_{n-1}$, for each vertex $v$. Moreover, it is easy to see that, for any subset $S \subset V(K_n)$ of cardinality $p \leq n$, $K_n - S \cong K_{n-p}$; thus $\kappa(K_n) = n-1$.

A *bipartite graph* $G$ of order $n$ is a graph in which the vertex-set of $G$ can be partitioned into two disjoint subsets $X$ and $Y$ such that each edge has one incident vertex in $X$ and one incident vertex in $Y$. The partition $(X : Y)$ is called a *bipartition* of $G$.

The *complete bipartite graph* of degrees $(p,\, q)$ is the bipartite graph with bipartition $(X : Y)$ where $|X| = p$, $|Y| = q$, and such that each vertex of $X$ is adjacent to each vertex of $Y$. We denote the complete bipartite graph of degrees $(p,\, q)$ by $K_{p,\,q}$. Clearly, $d_p(K_{p,\,q}) = q$, $d_q(K_{p,\,q}) = p$, and $d_i(K_{p,\,q}) = 0$ for all other $i$. Furthermore, for each vertex $v$ of $X$ and $w$ of $Y$, $K_{p,\,q} - v \cong K_{p-1,\,q}$ and $K_{p,\,q} - w \cong K_{p,\,q-1}$. It is thus easy to see that $\kappa(K_{p,\,q}) = \min(p,\, q)$. The graphs $C_6$, $K_6$ and $K_{2,\,3}$ are shown in Figure 1.2.

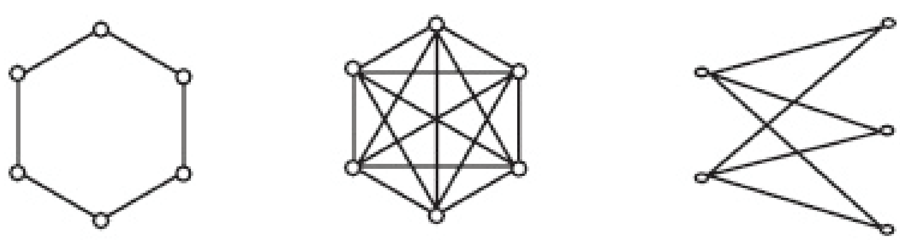

Figure 1.2: $C_6$, $K_6$ and $K_{2,\,3}$.

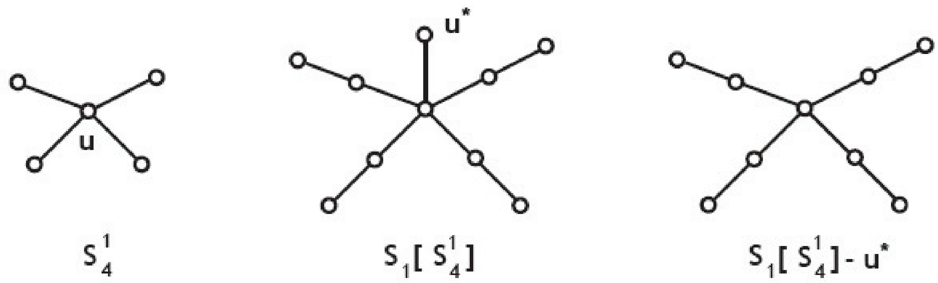


Figure 1.3: $S_4^1$, $S_1[S_4^1]$ and $S_1[S_4^1] - u^*$.

Finally in this chapter, we introduce a new construction that enables the formation of one graph from another. Let $F$ be a connected graph. We denote by $S_q[F]$, the graph that consists of $F$ with $q$ leaves added to each of its vertices. So if $S_p^1$ is the 1-star of order $p+1$, $S_1[S_p^1]$ is $S_p^2$ with an additional leaf $u^*$ adjacent to its "central" vertex. Thus, if $u$ is the central vertex of $S_p^1$, then $S_1[S_p^1] - u^*$ is the 2-star of order $2p+1$. These constructions are illustrated in Figure 1.3.

# Chapter 2

# Graph Reconstruction

The *Reconstruction Conjecture* is one of the foremost unsolved problems in Graph Theory. It conjectures that a graph can be uniquely determined, up to isomorphism, by its collection of unlabelled vertex-deleted subgraphs. Like many mathematical problems, its appeal lies in the simplicity of its hypothesis, and its accessibility to non-experts. However, although many graph theorists have tried to resolve the status of conjecture, it is still an open problem.

In this chapter we explain the basic concepts, definitions and results in the area of Graph Reconstruction. We initially follow the approach and terminology of Bondy and Hemminger [9], Bondy [6] and Lauri [23], and more information on the material in Sections 2.1 to 2.5 can be found there. From Section 2.6, we introduce a slightly different approach to graph reconstruction - *active vertices*, *common cards* and *reconstruction numbers* - and give references where necessary. Unless otherwise specified, $G$ is a simple finite undirected graph of order $n$.

## 2.1 Vertex Deck

Let $v$ be a vertex of $G$. The vertex-deleted subgraph of $G$, $G - v$, is the subgraph of $G$ obtained by deleting the vertex $v$ and all edges incident to $v$ (see Section 1.2). There are $n$ such subgraphs of $G$, one for each vertex. Following Harary [18], we call such subgraphs *cards* of $G$, and the collection of all $n$ cards of $G$, the *(vertex-)deck* of $G$, denoted by $\mathcal{D}(G)$.

All graphs in $\mathcal{D}(G)$ are unlabelled; that is, we do not differentiate between cards in the same isomorphism class. If $G$ has precisely $k$ vertices, $v_1, v_2, \ldots, v_k$, such that $G - v_1 \cong G - v_2 \cong \ldots \cong G - v_k$, then a representative of the isomorphism class $G - v_1$ occurs in the vertex-deck $k$ times, once for each of these vertices. Thus $\mathcal{D}(G)$ is a multi-set, rather than a set, of representatives of isomorphism classes of graphs.

Suppose that $\mathcal{D}(G)$ contains $r$ distinct isomorphism classes and $\alpha_i$ copies of each isomorphism class $G_i$. Then we express $\mathcal{D}(G)$ as

$$\mathcal{D}(G) = \{(G_i; \alpha_i) \mid 1 \leq i \leq r\}. \tag{2.1}$$

If $\alpha_i = 1$, we write $G_i$ instead of $(G_i; 1)$ in $\mathcal{D}(G)$.

Let $P_n$ be the path of order $n$ with vertices $v_1, v_2, \ldots, v_n$ as in Section 1.6. Then, if we let $P_0$ denote the null graph, it is easy to see that $P_n - v_i \cong P_{n-i} \oplus P_{i-1}$. Figure 2.1 shows the non-isomorphic cards in $\mathcal{D}(P_6)$.

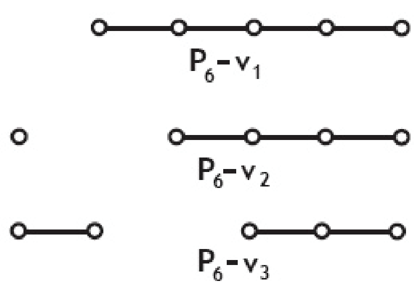


Figure 2.1: The three non-isomorphic cards of $P_6$.

Now consider the 1-star of order $n$, $S^1_{n-1}$. If $u$ is the central vertex and $v$ is any leaf of $S^1_{n-1}$, then $S^1_{n-1} - u \cong (n-1)\,K_1$ and $S^1_{n-1} - v \cong S^1_{n-2}$. Figure 2.2 shows the non-isomorphic cards in $\mathcal{D}(S^1_5)$.

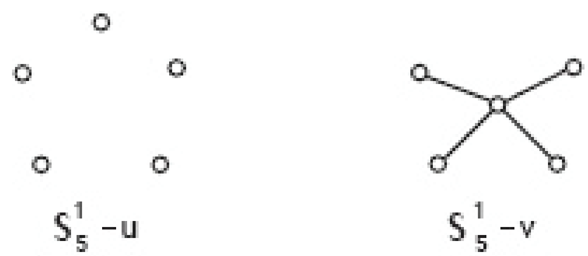


Figure 2.2: The two non-isomorphic cards of $S^1_5$.

Similarly, if $u$ is the central vertex and $v$ is any other non-leaf of the 2-star of order $n$, $S^2_{\frac{n-1}{2}}$, then $S^2_{\frac{n-1}{2}} - u \cong \frac{n-1}{2}\,P_2$, $S^2_{\frac{n-1}{2}} - v \cong S^2_{\frac{n-3}{2}} \oplus K_1$, and
$S^2_{\frac{n-1}{2}} - v^* \cong S_1[S^1_{\frac{n-3}{2}}]$. Figure 2.3 shows the non-isomorphic cards in $\mathcal{D}(S^2_5)$.

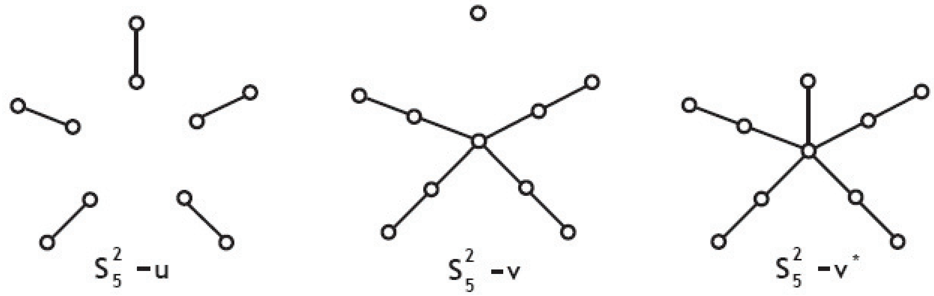


Figure 2.3: The three non-isomorphic cards of $S^2_5$.

These observations, together with those in Section 1.7, allow us to write down the decks of some of the graphs in Chapter 1:

(a) $\mathcal{D}(P_n) = \Big\{(P_{n-1};2),\,(P_{n-2}\oplus P_1;2),\ldots,\,(P_{\frac{n}{2}}\oplus P_{\frac{n-2}{2}};2)\Big\}$, for even $n$.
(b) $\mathcal{D}(P_n) = \Big\{(P_{n-1};2),\,(P_{n-2}\oplus P_1;2),\ldots,\,2P_{\frac{n-1}{2}}\Big\}$, for odd $n$.
(c) $\mathcal{D}(S^1_{n-1}) = \big\{(S^1_{n-2};n-1),\,(n-1)K_1\big\}$.
(d) $\mathcal{D}(S^2_{\frac{n-1}{2}}) = \Big\{(S^2_{\frac{n-3}{2}}\oplus K_1;\frac{n-1}{2}),\,S_1[(S^1_{\frac{n-3}{2}}];\frac{n-1}{2}),\,\frac{n-1}{2}K_2\Big\}$.
(e) $\mathcal{D}(C_n) = \{(P_{n-1};n)\}$.

(f) $\mathcal{D}(K_n) = \{(K_{n-1}; n)\}$.

(g) $\mathcal{D}(K_{p,\,q}) = \{(K_{p-1,\,q}; p)\,(K_{p,\,q-1}; q)\}$.

Let $F$ be a disconnected graph and suppose that $F = D_1 + D_2 + \ldots D_r$ and that $u$ is a vertex in $D_i$. Then, since all edges incident to $u$ are in $E(D_i)$,

$$F - u = (D_i - u) + \sum_{k \neq i} D_k, \tag{2.2}$$

noting that, if $D_i \cong K_1$, then $D_i - u$ is the null graph, and therefore does not correspond to a component of $F - u$. Thus, for example, it follows from (f) that

$$\mathcal{D}(K_a \oplus K_b \oplus K_c) = \{(K_{a-1} \oplus K_b \oplus K_c; a),\,(K_a \oplus K_{b-1} \oplus K_c; b),\,(K_a \oplus K_b \oplus K_{c-1}; c)\}\,. \tag{2.3}$$

## 2.2 Vertex Reconstruction

Let $H$ be a graph of order $n$. Suppose that $\mathcal{D}(G) = \mathcal{D}(H)$, that is, there is some labelling of the vertices of $G$ by $v_1, v_2, \ldots, v_n$ and those of $H$ by $w_1, w_2, \ldots, w_n$ such that $G - v_i \cong H - w_i$ for $1 \leq i \leq n$. Then $H$ is called a *vertex-reconstruction* of $G$. If $H \cong G$, then clearly $\mathcal{D}(G) = \mathcal{D}(H)$. $G$ is said to be *vertex-reconstructible* if every vertex-reconstruction of $G$ is isomorphic to $G$.

Not all graphs are vertex-reconstructible. For example, if $G = K_2$ and $H = 2K_1$, then $\mathcal{D}(G) = \mathcal{D}(H) = \{(K_1; 2)\}$. However, these two graphs are the only known examples of simple finite undirected graphs that are not vertex-reconstructible. The Reconstruction Conjecture states that these are the *only* such simple finite undirected graphs.

**Conjecture 2.2.1 (Vertex-Reconstruction Conjecture)** (Kelly [20], Ulam [40]) All finite simple undirected graphs with at least three vertices are reconstructible.

□

Bondy [6] equivalently defines a vertex-reconstruction $H$ of $G$ to be a graph with $V(H) = V(G)$ and $\mathcal{D}(H) = \mathcal{D}(G)$. He further associates every card in the common deck with a unique element of $V(G)$. For our purposes it is not necessary to place this stipulation on our graphs. However, we shall assume that there is an indexing of the graphs in $\mathcal{D}(G)$ (not uniquely) which induces a labelling of the vertices of $G$ in the natural way.

According to Bondy and Hemminger [9], this conjecture was first "discovered" in the early 1940s by Kelly and Ulam, with the first published record of the problem appearing in Kelly's PhD thesis [20]. No counterexample has ever been found and, moreover, the conjecture has been shown by exhaustive computer search to be true for all graphs of order up to and including 11, by McKay [29], and independently by Baldwin [3], McMullen [30] and Rivshin [38]. It has also been shown to be true for all trees, all regular graphs and all disconnected graphs (see Section 2.5). In addition, Müller [32], Myrvold [33] and Bollobás [4] (independently) proved that the conjecture is true with high probability (that is, the probability of the existence of a non-reconstructible graph of order $n$, approaches zero as $n$ approaches infinity).

## 2.3 Other Reconstruction Areas

This thesis is only concerned with vertex-reconstruction of finite graphs. Other reconstruction topics are mentioned here for interest only. Subsequent to this section, all terms relating to reconstruction refer to vertex-reconstruction.

Analogous to the vertex-deck of $G$ is the *edge-deck* of $G$, $\mathcal{ED}(G)$. This is defined to be the collection of edge-deleted subgraphs $G - e$, for all edges $e$ of $G$. Note that, as for the vertex-deck, $\mathcal{ED}(G)$ is also a multi-set of isomorphism classes of graphs. Any graph $H$ such that $\mathcal{ED}(G) = \mathcal{ED}(H)$ is called an *edge-reconstruction* of $G$ and if every edge-reconstruction of $G$ is isomorphic to $G$, then $G$ is said to be *edge-reconstructible*. The pair $G = 2K_2$ and $H = P_3 \oplus K_1$ clearly have identical edge-decks. In addition, for any $k \geq 1$, the pair of graphs

$$G = K_3 \oplus kK_1 \qquad \text{and} \qquad H = S_3^1 \oplus (k-1)K_1$$

have identical edge-decks. The *Edge-Reconstruction Conjecture*, first proposed by Harary in 1964 [18], essentially states that the above graphs are the only finite simple undirected graphs that are not edge-reconstructible.

**Conjecture 2.3.1 (Edge-Reconstruction Conjecture)** (Harary[18]) All finite simple undirected graphs with at least four edges are edge-reconstructible. □

There has been more progress towards proving the Edge-Reconstruction Conjecture than its vertex equivalent. In addition, it has been proved by Greenwell [15] that any graph without isolated vertices that is vertex-reconstructible, is also edge-reconstructible.

Manvel [26] has proposed extending the vertex-reconstruction conjecture to the $k$-vertex deck of $G$, that is the multi-set of all $\binom{n}{k}$ subgraphs $G - S$, where $S \subset V(G)$ is of cardinality $k$.

**Conjecture 2.3.2 (k-Vertex-Reconstruction Conjecture)** (Manvel [26]) Given any positive integer $k$, there exists an integer $f(k)$ such that all finite simple undirected graphs of order at least $f(k)$ are $k$-vertex-reconstructible. □

The idea of generalising the Reconstruction Conjecture to the $k$-vertex deck was first mentioned by Kelly [20], who observed there are some graphs of small order that are not determined, up to isomorphism, by their 2-vertex deleted subgraphs.

Despite his examples, the conjecture is widely believed to be true for larger graphs, since there are no known counter-examples of large order, for any $k$. In addition, for the 2-vertex deck, Giles [16] has proved that the conjecture is true for all trees and Manvel [25] has proved that it is true for all disconnected graphs with no isolated vertices.

The reconstruction of directed graphs has been studied intensively by Stockmeyer [37]. He has shown that digraphs are not in general reconstructible. In addition, infinite graphs are also not in general reconstructible. For example, if $T_\infty$ denotes a regular tree of infinite degree, then for example, the two graphs $G = T_\infty$ and $H = 2T_\infty$ are reconstructions of one another (see Bondy [6]).

For the rest of this thesis, every graph is finite simple and undirected. Furthermore, since the Reconstruction Conjecture is not true for graphs of order 2, we shall also assume that **the order of $G$ is at least** 3.

## 2.4 Reconstructing Graph Parameters

Let $\rho$ be a graph parameter. Then $\rho$ is said to be *reconstructible* if $\rho(G)$ takes the same value on every reconstruction of $G$; that is, if $\mathcal{D}(G) = \mathcal{D}(H)$ then $\rho(G) = \rho(H)$. For example, the order of $G$ is reconstructible since it corresponds to the number of cards in $\mathcal{D}(G)$ (and is one more than the order of any card of $G$). Since many classes of graphs are defined by the value they take on one or more parameters, the reconstruction of parameters is fundamental to the reconstruction of classes of graphs.

One of the most widely known reconstructible parameters is $s(F, G)$ (see Section 1.5).

**Lemma 2.4.1 (Kelly's Lemma)** (Kelly [20]) Let $G$ and $F$ be graphs of orders $n$ and $f$, respectively, where $f < n$. Then $s(F, G)$, the number of subgraphs of $G$ isomorphic to $F$, is reconstructible.

*Proof* Each subgraph of $G$ that is isomorphic to $F$ occurs in precisely $n - f$ of the cards of $\mathcal{D}(G)$; so

$$(n - f)s(F, G) = \sum_{v \in V(G)} s(F, G - v). \tag{2.4}$$

Clearly the right hand side of the equation is reconstructible. Therefore, so is the left hand side. □

Note that if $G$ and $F$ are of the same order, (2.4) would not enable the calculation of $s(F, G)$. Indeed, if there were a way to extend Lemma 2.4.1 to all subgraphs of $G$, then the Reconstruction Conjecture could be easily shown to be true. Lemma 2.4.1 has some important consequences.

**Corollary 2.4.2** The size of $G$ is reconstructible. In addition, for any card $G - v$, the degree of $v$ can be determined from $\mathcal{D}(G)$.

*Proof* Since an edge of $G$ is a subgraph isomorphic to $K_2$, $|E(G)|$ is reconstructible by Lemma 2.4.1. The second assertion follows since $d(v) = |E(G)| - |E(G - v)|$. □

**Lemma 2.4.3** Let $G$ and $F$ be graphs, with $F$ of smaller order than $G$. For any vertex $v$ of $G$, let $S_v(F, G)$ be the number of subgraphs of $G$ containing $v$ that are isomorphic to $F$. Then $\{S_v(F, G) \,|\, v \in G\}$ is reconstructible.

*Proof* Any subgraph of $G$ that does not contain $v$ is in the card $G - v$. Therefore, the number of subgraphs of $G$ containing $v$ that are isomorphic to $F$ is equal to $s(F, G) - s(F, G - v)$. The result then follows by Lemma 2.4.1. □

**Corollary 2.4.4** The degree sequence of $G$ is reconstructible.

*Proof* This follows directly by Corollary 2.4.2, or alternatively by setting $F = K_2$ in Lemma 2.4.3. □

**Corollary 2.4.5** For any vertex $v$ of $G$, let $N_v(G) = \{d(u) \,|\, uv \in E(G)\}$. Then $\{N_v(G) \,|\, v \in G\}$ is reconstructible.

*Proof* For any card $G-v$, the degree of $v$ can be determined from $\mathcal{D}(G)$ by Corollary 2.4.2. In addition, the degree sequence of $G$ is reconstructible by Corollary 2.4.4. Let $\mathbf{d}$ be the non-increasing degree sequence of $G$ and $\mathbf{d}'$ be the non-increasing degree sequence of $G-v$, but with the degree of the vertex $v$ inserted in its correct position. The non-zero entries of the vector $\mathbf{d}$ - $\mathbf{d}'$ occur in positions corresponding to the neighbours of $v$. The values of $\mathbf{d}$ corresponding to these positions are then the degrees of the neighbours of $v$. □

Whilst we are discussing degree sequences, we prove the following relation between the degree sequences of a graph and any of its cards. We shall make use of this result in later chapters. We recall from Section 1.1 that if $v$ is a vertex of $G$, then $d_i(v)$ is the number of neighbours of $v$ of degree $i$.

**Lemma 2.4.6** Let $G$ be a graph and $v$ a vertex of $G$ where $d(v) = k$. Then

(a) $d_k(G-v) = d_k(G) + d_{k+1}(v) - d_k(v) - 1$;

(b) $d_i(G-v) = d_i(G) + d_{i+1}(v) - d_i(v)$, for $i \neq k$.

*Proof* The removal of $v$ from $G$ reduces the degree of every vertex adjacent to $v$ by one. Since the removal of $v$ additionally reduces the total number of vertices of $G$ of degree $d(v)$ by one, the result follows. □

We now show that the connectivity of a graph is reconstructible. We begin with the case when $\kappa(G) = 0$.

**Lemma 2.4.7** The connectedness of $G$ is reconstructible.

*Proof* Suppose that $G$ is disconnected and that $v$ is a vertex of $G$. Then $G-v$ is connected if and only if $G$ has precisely two components and, moreover,
$G = \{v\} + (G-v)$; so $\mathcal{D}(G)$ only contains at most one card that is connected.

Suppose, on the other hand, that $G$ is a connected graph. Then $G$ contains at least two vertices that are not cut-vertices; so $\mathcal{D}(G)$ contains at least two cards that are connected. Since the order of $G$ is at least 3, this implies that the connectedness of $G$ can be determined from $\mathcal{D}(G)$. $\square$

**Corollary 2.4.8** $\kappa(G)$, the connectivity of $G$, is reconstructible.

*Proof* If $\kappa(G) = 0$, then $G$ is disconnected and the result will follow from Lemma 2.4.7. We therefore assume that $G$ is connected. In this case, it is easy to see that $\kappa(G) = 1 + \min_{v \in V(G)} \kappa(G - v)$. So $\kappa(G)$ can be determined from $\mathcal{D}(G)$. $\square$

Tutte [39] proved how to reconstruct the number of spanning subtrees of certain types. Kocay later [22] refined the proof using *covers*. Although it is of no importance for any of the main results in this thesis, we state Kocay's Lemma here, since its use is widespread in some aspects of reconstruction.

Suppose that $\mathcal{F} = (F_1, F_2, \ldots, F_k)$ is a sequence of (not necessarily distinct) graphs. A cover of $G$ by $\mathcal{F}$ is a sequence $(G_1, G_2, \ldots, G_k)$ such that $G_i \cong F_i$, $1 \leq i \leq k$, $\bigcup_{i=1}^{k} V(G_i) = V(G)$ and $\bigcup_{i=1}^{k} E(G_i) = E(G)$. The number of covers of $G$ by $\mathcal{F}$ is denoted by $c(\mathcal{F}, G)$.

**Lemma 2.4.9 (Kocay's Lemma)** (Kocay [22]) Let $G$ be a graph of order $n$ and let $\mathcal{F} = (F_1, F_2, \ldots, F_k)$ be a sequence of graphs such that the order of each $F_i$ is less than $n$. Then the parameter

$$\sum_{X} c(\mathcal{F}, X) s(X, G) \tag{2.5}$$

is reconstructible, where the sum in (2.5) extends over all isomorphism types $X$ with $|V(X)| = |V(G)|$. $\square$

Lemma 2.4.9 has been used to prove the following result.

**Lemma 2.4.10** (Tutte [39]) Let $G$ be a graph of order $n$ and let $\mathcal{F} = (F_1,\ F_2, \ldots, F_k)$ be a sequence of graphs such that the order of each $F_i$ is less than $n$. Then the following parameters are reconstructible:

(a) the number of disconnected spanning subgraphs of $G$ with $k$ components isomorphic to $F_1,\ F_2, \ldots, F_k$;

(b) the number of (connected) separable spanning subgraphs of $G$ with $k$ blocks isomorphic to $F_1,\ F_2, \ldots, F_k$;

(c) the number of non-separable spanning subgraphs of $G$ with a specified number of edges;

(d) the number of Hamiltonian cycles of $G$. □

In addition, Tutte proved that the *Tutte polynomial* is reconstructible (this can also be proved using Lemma 2.4.10). From this it follows that the many other algebraic invariants of a graph can be reconstructed (including the *chromatic polynomial*, the *dichromatic polynomial* and the *characteristic polynomial*). See [6] or [23] for more details.

## 2.5 Reconstructing Classes of Graphs

Let $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ is said to be reconstructible if every graph in $\mathcal{C}$ is reconstructible. The most widely used approach to proving that a class is reconstructible is to show that the following two conditions are met:

(a) $\mathcal{C}$ is *recognisable*, that is, for each $G$ in $\mathcal{C}$, every reconstruction of $G$ is a member of $\mathcal{C}$;

(b) $\mathcal{C}$ is *weakly reconstructible*, that is, for each $G$ in $\mathcal{C}$, every reconstruction of $G$ that is in $\mathcal{C}$ is isomorphic to $G$.

Clearly, if both (a) and (b) hold then $\mathcal{C}$ is reconstructible. We demonstrate this approach to showing the reconstructibility of classes by proving that regular graphs and disconnected graphs are reconstructible.

**Theorem 2.5.1** For all integers $r > 0$, the class of $r$-regular graphs is reconstructible.

*Proof* The degree sequence of $G$ is reconstructible by Corollary 2.4.4. Therefore the class of $r$-regular graphs is recognisable.

Suppose that $G$ is a $r$-regular graph and let $v$ be a vertex of $G$. The only way to reconstruct a regular graph of degree $r$ from $G - v$ is to make $v$ incident to all the vertices of $G - v$ that have degree $r - 1$. Clearly, this uniquely reconstructs $G$. Therefore, the class of $r$-regular graphs is weakly reconstructible. This completes the proof. $\square$

We next show that disconnected graphs are reconstructible. There have been many proofs of this. The one we present, due to Manvel [28], is probably the shortest.

**Theorem 2.5.2** The class of disconnected graphs is reconstructible.

*Proof* We note first that the empty graph is immediately reconstructible from Corollary 2.4.4. We therefore only need consider disconnected graphs with at least one edge.

The connectedness of a graph is reconstructible by Lemma 2.4.7, so the class of disconnected graphs is recognisable.

Suppose that $G$ is a disconnected graph and let $C$ be a component of maximum order amongst all the components of the graphs in $\mathcal{D}(G)$. Clearly, $C$ must be a component of $G$. Since $C$ is connected, there is at least one vertex of $C$ that is not a cut-vertex. Let $w$ be one such vertex.

Let $S \subseteq \mathcal{D}(G)$ be the set of cards of $G$ that contain the least number of components isomorphic to $C$, and let $G - v$ be a card in $S$ that has the maximum number of components isomorphic to $C - w$. Then $G$ is uniquely reconstructible from $G - v$ by replacing a component of $G - v$ that is isomorphic to $C - w$ with $C$. So the class of disconnected graphs is weakly reconstructible, which completes the proof. $\square$

The class of trees was first proved to be reconstructible by Kelly [21]. Bondy [5] has shown that any tree $T$ is reconstructible from a subdeck $S \subseteq \mathcal{D}(T)$, where each card in $S$ is formed by the deletion of a peripheral vertex (an end-leaf of a longest path in $T$). In addition, Myrvold [36] has shown that, for $n \geq 5$, you only need three well-chosen cards in its deck to reconstruct a tree.

A connected graph $G$ is a tree if and only if $|E(G)| = n - 1$. So the recognisability of trees is immediate by Corollary 2.4.2 and Lemma 2.4.7. Weak reconstructibility of trees is more difficult to show, however. Most proofs of this consider various sub-classes of trees and use the fact that the centre (bi-centres) of a tree can be determined from its deck. The various branches of the tree are then reconstructed and "glued" back onto its (bi-)centre(s). The simplest proof is probably that by Bondy [9], although even this proof uses case-by-case analyses.

Unicyclic graphs (connected graphs that contain precisely one cycle) can be easily shown to be recognisable using Lemma 2.4.1 and Lemma 2.4.7. Cacti (connected graphs such that no two cycles have an edge in common) are also recognisable since such graphs contain no subgraph homeomorphic to a complete graph with an edge deleted. In both cases, however, weak reconstructibility is more difficult to show, and the proofs are again completed via an examination of various subclasses. For detailed proofs of the reconstructibility of these classes, see Manvel [27] or Bowler [12].

Bondy [7] proved that connected separable graphs with no leaves are reconstructible. Yongzhi [42] made use of this to prove the following very surprising result.

**Theorem 2.5.3** (Yongzhi [42]) Every connected graph is reconstructible if and only if every 2-connected graph is reconstructible. □

Unfortunately, however, no progress has been made in proving that every 2-connected graph is reconstructible.

## 2.6 Subdeck Reconstruction

$G$ is *reconstructible from a subdeck* $\mathcal{A} \subseteq \mathcal{D}(G)$ if it is uniquely determined from $\mathcal{A}$, up to isomorphism; that is, every graph that has $\mathcal{A}$ as a subdeck of its deck is isomorphic to $G$. *Subdeck Reconstruction* is concerned with the following two questions:

(a) What is the minimum $k$ such that $G$ is reconstructible from *some* subdeck of size $k$?

(b) What is the minimum $k$ such that $G$ is reconstructible from *any* subdeck of size $k$?

The minimum such $k$ in (a) is called the *existential* or *ally reconstruction number* of $G$, denoted by $rn(G)$, and the minimum such $k$ in (b) is called the *universal* or *adversary reconstruction number* of $G$, denoted by $urn(G)$. The existential reconstruction number is often simply called the *reconstruction number* of $G$.

The terms "ally" and "adversary" were introduced by Myrvold [33]. She conceived of a two-player game in which player $A$ holds the whole deck and gives $B$ cards one at a time. Player $B$ must then determine the graph from the cards that $A$ gives him. If $A$ is helping $B$ then she gives him cards from which he can identify the graph most easily; in this case, the number of cards given is the ally reconstruction number. On the other hand, if $A$ is obstructing $B$ then she gives him cards that make it most difficult to identify the graph; in this case, the number of cards given is the adversary reconstruction number.

One might also ask whether a graph parameter can be reconstructed from a subset of the deck. Myrvold [35] has proved that the number of edges and hence the degree sequence can be reconstructed from any subdeck of cardinality $n - 1$. It is also relatively straightforward to show that the connectivity of a graph can be reconstructed from a subdeck of that size.

We now introduce some new terminology to make the subject of sub-deck reconstruction more easily accessible. Let $G$ and $H$ be two graphs and suppose that $v$ and $w$ are vertices of $G$ and $H$, respectively, such that $G-v \cong H-w$. Then $v$ is said to be an *active* vertex of $G$ with respect to $H$, and $w$ is said to be a vertex *associated* with $v$. Clearly this relationship is symmetric, that is $w$ is an active vertex of $H$ and $v$ is associated with $w$. We denote the set of active vertices of $G$ with respect to $H$ by $A_H(G)$ and its cardinality by $a_H(G)$. Similarly, we denote the set of active vertices of $H$ with respect to $G$ by $A_G(H)$ and its cardinality by $a_G(H)$.

Any active vertex in $G$ must have an associated vertex in $H$ (and conversely). However, for many pairs of graphs, an active vertex of $G$ (or $H$) may have many associated vertices. In addition, $a_H(G)$ and $a_G(H)$ may not be equal. For example, if $G = K_3$ and $H = P_3$, then both leaves of $H$ are associated with every vertex of $G$, and $a_H(G) > a_G(H)$. So knowing $a_H(G)$ is not sufficient to determine the largest common subdeck of $G$ and $H$. Assuming (without loss of generality) that $V(G)$ and $V(H)$ are disjoint, we therefore define a bipartite graph $B(G, H)$ whose vertices consist of the vertices of $G$ and $H$. Two vertices are adjacent in $B(G, H)$ if and only if they are associated active vertices. That is:

$$\begin{aligned} V(B(G,\, H)) &= V(G) \cup V(H), \\ E(B(G,\, H)) &= \{vw \mid v \in V(G),\, w \in V(H),\, G - v \cong H - w\}\,. \end{aligned} \tag{2.6}$$

The *number of common cards* of $G$ and $H$ (or *between* $G$ and $H$) is defined to be the size of a maximum matching in $B(G, H)$. We denote this number by $b(G, H)$. Clearly, if $b(G, H) < n$ for all graphs $H$ that are not isomorphic to $G$, then $G$ is *reconstructible*. In addition,

$$urn(G) = \max_{H \not\cong G} b(G,\, H) + 1, \tag{2.7}$$

and we define $urn(G) = n + 1$ if $G$ is not reconstructible.

The Reconstruction Conjecture can now be restated using this terminology.

**Conjecture 2.6.1 (Reconstruction Conjecture)** Suppose that $G$ and $H$ are two finite simple undirected graphs, both of order $n \geq 3$. Then $b(G, H) < n$, unless $G$ and $H$ are isomorphic. $\square$

Myrvold [35] (amongst others) defines the number of common cards slightly differently from this. She makes the following (equivalent) definition: suppose that $v_1, v_2, \ldots, v_k$ and $w_1, w_2, \ldots, w_k$ are distinct vertices of $G$ and $H$, respectively, such that $G - v_i \cong H - w_i$ for all $i$. Then $G - v_i$ and $H - w_i$ are *common* cards of $G$ and $H$ and we say that $G$ and $H$ have (at least) $k$ cards in common. $b(G, H)$ is defined to be the maximum number of cards that $G$ and $H$ can have in common. One can also (equivalently) define it as the cardinality of the multi-set intersection of $\mathcal{D}(G)$ and $\mathcal{D}(H)$. However, although these definitions are perhaps more intuitive, for our purposes it is more convenient to use the previous definition.

## 2.7 Results on Reconstruction Numbers

Suppose that $G - u$ and $G - v$ are cards of $G$. We construct a new graph $H$ as follows: if $e = uv$ is an edge of $G$, then we define $H = G - e$; otherwise we define $H = G + e$. Then $G \ncong H$ but both $G - u$ and $G - v$ are cards of $H$. Therefore $G - u$ and $G - v$ cannot alone distinguish between $G$ and $H$. Thus, for all graphs $G$, $rn(G) > 2$. However, a far more important result concerning (ally) reconstruction numbers has been proved to be true.

Suppose that, for some parameter $\rho$, the proportion of graphs $G$ of order $n$ such that $\rho(G) \neq k$ approaches zero as $n$ approaches infinity. Then $\rho$ is said to take the value $k$ on all graphs *with high probability*, or on *almost all* graphs. Myrvold [33] (using a result by Müller [32]) and Bollobás [4] have independently proved the following result.

**Theorem 2.7.1** (Myrvold [33], Müller [32], Bollobás [4]) Every graph has reconstruction number 3 with high probability; that is $rn(G) = 3$, for almost all graphs $G$. $\square$

Of course, this result implies that almost every graph is reconstructible.

Myrvold [33] has also shown that $urn(G) = 3$, for almost all $G$. In addition, she has proved the following results on $rn(G)$. The proof of part (b) was corrected by Molina [31].

**Theorem 2.7.2** (Myrvold [34, 36]) The following results concerning reconstruction numbers hold:

(a) $rn(T) = 3$ for every tree $T$ of order 5 or more;

(b) the reconstruction number of a disconnected graph is 3, except in the case where all the components are isomorphic;

(c) if $G$ is a disconnected graph in which every component is isomorphic of order $p$, then $rn(G) \leq p + 2$ (the upper bound is attained when $G$ consists of $k$ isomorphic copies of $K_p$);

(d) if $G$ is an $r$-regular graph of order $n$, then $rn(G) \leq min\{r+3, n-r-2\} \leq \lfloor \frac{n}{2} \rfloor + 2$ (the upper bound is attained when $G$ is either $K_{p,p}$ or $2K_p$). □

Asciak and Lauri [2] further showed that the only graphs that attain the bound in (c) are those given in the theorem. In addition, Asciak [1] showed that $kK_{r+1}$ is the only $r$-regular graph with ally reconstruction number equal to $r + 3$.

Although interesting, $rn(G)$ does not give any idea of the degree of similarity between the deck of $G$ and the deck of any other non-isomorphic graph. To assess this, we must use $urn(G)$; that is, we must calculate the maximum value of $b(G, H)$ over all graphs $H$ that are not isomorphic to $G$.

The problem of finding the maximum number of common cards was first considered by Harary and Manvel [19]. They presented an infinite family of pairs of disconnected graphs of even order with $\frac{n}{2} + 1$ common cards. Twenty years later, Bondy [8] gave an infinite family of forests with $\frac{n+3}{2}$ common cards. Where $(8n + 9)$ is a square, Myrvold [33, 35] then presented an infinite family of pairs of disconnected graphs with $\frac{n}{2} + \frac{1}{8}(3 + \sqrt{8n+9})$ common cards and another infinite family of pairs of disconnected graphs of odd order with the same degree sequence having $\frac{n+1}{2}$ common cards. Myrvold's families are given below.

**Example 2.7.3** (Myrvold [33]) Let $p$ be an integer, $p \geq 1$. Then, for $n = (p+1)(2p-1)$, the following pair of graphs of order $n$ has $\frac{n}{2} + \frac{1}{8}(3 + \sqrt{8n+9})$ common cards:

$$\begin{aligned} G &= K_{p-1} \oplus (p-1)K_p \oplus pK_{p+1} \\ H &= (p+1)K_p \oplus (p-1)K_{p+1}. \end{aligned}$$

The removal of any vertex from a component of $G$ isomorphic to $K_p$ and any vertex in a component of $H$ isomorphic to $K_{p+1}$ gives isomorphic cards. So $b(G, H) = p(p+1)$. Solving for $p$ in terms of $n$ gives the result. □

**Example 2.7.4** (Myrvold [33]) Let $p$ be an integer, $p \geq 1$. Then, for $n = 6p+5$, the following pair of graphs of order $n$ with the same degree sequence has $\frac{n+1}{2}$ common cards:

$$\begin{aligned} G &= P_2 \oplus C_{3p+3} \oplus pK_3 \\ H &= P_{3p+2} \oplus (p+1)K_3. \end{aligned}$$

The removal of any vertex from the $C_{3p+3}$ component of $G$ and any vertex in a component of $H$ isomorphic to $K_3$ gives isomorphic cards. So $b(G, H) = 3(p+1) = \frac{n+1}{2}$. In addition, since both graphs have $6p+3$ vertices of degree 2, and two leaves, they have the same degree sequence. □

Myrvold [33, 35] conjectured that her first family had the maximum value of $b(G,\, H)$ for all pairs of non-isomorphic graphs $G$ and $H$ of order $n$, for large $n$; that is, $b(G,\, H) \leq \frac{n}{2} + \frac{1}{8}(3 + \sqrt{8n+9})$, for such $n$. In addition, she conjectured that her second family had the maximum value of $b(G,\, H)$ for pairs with the same degree sequence, for large $n$; that is, for such $n$, any pair with the same degree sequence has $b(G,\, H) \leq \frac{n+1}{2}$. These conjectures were repeated by Lauri [24].

For small values of $n$, there exist pairs of graphs with more common cards than the number implied by the conjecture: for example, for $n = 5$ there is a pair with 4 common cards and for $n = 6$ there is a pair with 5 common cards. Two examples of these are shown in Figures 2.4 and 2.5. In both cases, $G - v_i \cong H - w_i$. Baldwin [3] and McMullen [30] recently reported three pairs of graphs of order 8 with 6 common cards. Rivshin [38] improved on these results and presented four pairs of graphs of order 10 and six pairs of order 11 with 7 common cards.


Figure 2.4: A pair of graphs of order 5 with 4 common cards.

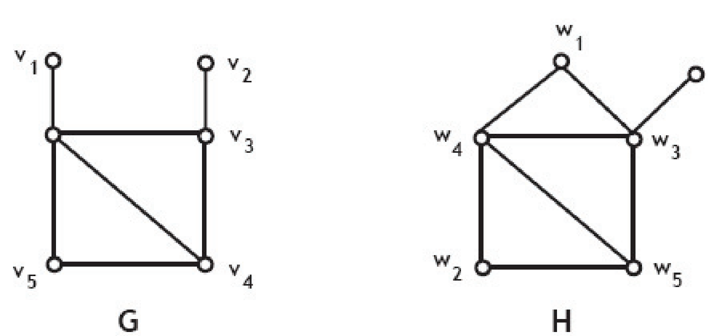


Figure 2.5: A pair of graphs of order 6 with 5 common cards.

## 2.8 Thesis Outline

In this thesis, we examine more thoroughly the question of the maximum number of common cards between a pair of graphs. We present various methodologies which, subject to specific criteria, enable us to place bounds on the number of active vertices of $G$ with respect to $H$, and vice versa. With the aid of these bounds, we then derive, and moreover solve, a multitude of equations that bound the number of common cards between $G$ and $H$ under various conditions. The bounds we prove are as follows:

(a) Theorem 3.2.5: When $G$ is connected and $H$ is disconnected then

$$b(G,\, H) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1;$$

(b) Lemma 4.1.8 and Theorem 4.2.30: When $G$ is a sunshine graph (a graph where the removal of all of its leaves reduces the graph to a single cycle) and $H$ is a tree then

$$b(G,\, H) \leq \left\lfloor \frac{2(n+1)}{5} \right\rfloor;$$

(c) Theorem 5.5.11: When $G$ and $H$ are a 2UC graph pair (a pair of graphs, in which after the iterative removal of all common isomorphic components, at least one of the resulting graphs is disconnected) then

$$b(G,\, H) \leq 2 \left\lfloor \frac{1}{3}(n-1) \right\rfloor.$$

A key idea we develop is whether a particular type of active vertex in $G$ (or $H$) *induces a distinct non-active vertex* in either $G$ (or $H$). For example, suppose that $G$ contains an active vertex $u$ that is a cut-vertex. If it can be shown that there is some component $X_u$ of $G - u$ that does not contain any active vertices, then $u$ can be thought of as inducing the set of non-active vertices $V(X_u) \subset V(G)$. Moreover, since $u$ is a cut-vertex, we can consider $u$ to uniquely induce this set of non-active vertices. Thus, if we could show that for a subset $S \subset A_H(G)$, there are disjoint subgraphs $X_u$ of non-active vertices in $G$ for each $u$ in $S$, then the number of active vertices of $G$ is at most

$$a_H(G) \leq n - \sum_{u \in S} |X_u|.$$

This notion first occurs in Lemma 3.2.1. There we show that for pairs of cut-vertices $u$ and $v$, we can always find two such disjoint subgraphs $X_u$ and $X_v$ of $G$. It is shown later in Chapter 3 that, if $u$ and $v$ are active, then, under certain conditions, these two disjoint subgraphs do not contain any active vertices; thus $u$ and $v$ can be thought of as inducing a collection of distinct non-active vertices in $G$. If this can be shown to be true for many pairs of active vertices, we can obtain a strong bound on $a_H(G)$, and thus $b(G, H)$.

This idea is extended further when we show that certain active vertices in $G$ induce non-active vertices in $H$. We construct an isomorphism between various subgraphs of $G$ and $H$ and look at the images in $H$ of the active vertices of $G$. We then show that, in certain cases, these images cannot be active in $H$. For example, suppose there is an isomorphism $\phi$ from a subgraph $U$ of $G$ to a subgraph $W$ of $H$. Then, if $S \subset U$ is a collection of distinct active vertices of $G$ such that every vertex of $\phi(S)$ is not active in $H$, it follows that

$$a_G(H) \leq n - |S|.$$

If we can find many such distinct subgraphs in $G$, again we can place a strong bound on $a_H(G)$. Since $b(G, H) \leq \min(a_H(G), a_G(H))$, this process will enable us to bound $b(G, H)$.

An easy way to assess whether certain vertices are active is to examine the possible degrees of pairs of associated vertices. This approach is key to the results of Chapter 4. There we combine knowledge of the degrees of these pairs with the isomorphisms described above to find bounds on the number of non-active vertices in both graphs.

Another useful approach is to consider what effect the existence of certain active vertices has on the structure of our graphs. We are often able to show that the presence of a certain number of active vertices in one graph is only possible if this graph contains a collection of leaf 2-paths. By examining the number and lengths of various leaf 2-paths in both graphs (again using the isomorphisms mentioned above), we prove that only certain pairs of graphs contain a large number of particular kinds of active vertices.

One thing that is key to all of our analyses, is that we can partition the active vertices into subsets with a common property and examine these subsets individually. For example, we consider the active vertices of certain degrees and investigate whether any of these induce non-active vertices in $G$ or $H$ or we examine all the active cut-vertices in one of the graphs. What is important to note is that in all of the pairs that we report on in this thesis, we are able to make useful partitions of the active vertices so that this case-by-case approach bears fruit.

In Chapters 3 and 5, we partition the active vertices of $G$ in terms of the isomorphism class of the component in which any associated vertex lies. In Chapter 5, we then further partition these sets to consider the subsets of cut-vertices and non cut-vertices of these (already partitioned) sets. In Chapter 4, on the other hand, we partition the active vertices by their degrees.

This approach additionally allows us to find families that attain these bounds in each of the cases we examine. Moreover, it allows us to show that the families we present are unique. The uniqueness is important since it gives an insight into the nature of any pairs of graphs that have a large number of common cards when $n$ is large. All the families that attain our bounds possess a large degree of symmetry and we would conjecture that this is the case for *all* families of pairs of graphs that have a large number of common cards. Moreover, this approach has allowed us to find other families of 2UC graph pairs with certain fixed parameters (for example the same number of edges) that have a large number of common cards.

In the case of the class of 2UC graph pairs, the bound for $b(G,\, H)$ is much larger than the bound in Myrvold's conjectures. So, since we are able to find a family that attains this bound, this shows that her first conjecture is false. We now present the unique family in Example 5.5.12 that attains the bound of $b(G,\, H) = \frac{2(n-1)}{3}$ when $n = 3p+1 \equiv 1 \pmod 3$:

$$\begin{aligned} G &\cong 2K_{p+1} \oplus K_{p-1} \\ H &\cong K_{p+1} \oplus 2K_p. \end{aligned} \tag{2.8}$$

The removal of a vertex from a component of $G$ isomorphic to $K_{p+1}$ and a vertex from a component of $H$ isomorphic to $K_p$ gives isomorphic cards. This example is easily extended to all value of $n$ (see Examples 5.5.13 and 5.5.14). The uniqueness of this example for 2UC graph pairs is shown in Theorem 5.5.11.

The uniqueness of this example is more interesting than the construction itself (it is perhaps surprising that it was not discovered by previous researchers). The class of 2UC graph pairs contains many disconnected graph pairs and, moreover, is the largest family to which the techniques outlined in this thesis can be readily applied to. Having said that, we believe, with some modifications, that the methods outlined can be applied to other classes of graphs.

What we are able to do, as developed in Chapter 6, is extend the given example to find other families of 2UC graph pairs with a large number of common cards. For example, by replacing the complete graphs in (2.8) with 1-stars, we obtain the following pair of graphs:

$$\begin{aligned} G &\cong 2S_{p+1} \oplus S_{p-1} \\ H &\cong S_{p+1} \oplus 2S_p. \end{aligned} \tag{2.9}$$

The removal of a leaf from a component of $G$ isomorphic to $S_{p+1}$ and a leaf from a component of $H$ isomorphic to $S_p$ gives isomorphic cards. So $b(G, H) = \frac{2(n-4)}{3}$ when $n \equiv 1 \pmod 3$. In addition, the pair of graphs are both forests with the same number of components, so have same number of edges.

This example is explained more fully in Theorem 6.2.2. We prove in Theorem 6.2.12 that, for large $n$, this is one of only two families of 2UC graph pairs (apart from (2.8) and the extensions above) having this many common cards.

In Example 6.2.13, we show how to construct a family of 2UC graph pairs with the same degree sequence, which for large $n$, have many common cards than the bound in Mryvold's second conjecture. This shows her second conjecture is incorrect as well. We briefly present this example here.

We recall from Section 1.7 that $S_q[K_p]$ denotes the graph of order $p(q+1)$ that consists of $K_p$ with $q$ leaves added to each of its vertices. We let $S'_q[K_p]$ denote the graph $S_q[K_p]$ with a single leaf removed, and let $S''_q[K_p]$ denote the graph $S_q[K_p]$ with two leaves, adjacent to different vertices, removed. For $n = 3p^2 - 2$, where $p \geq 3$, let $G$ and $H$ be the following pair of graphs:

$$\begin{aligned} G &\cong (S_{p-1}[K_p] \oplus S''_{p-1}[K_p]) \oplus (S_{p-1}[K_p]) \\ H &\cong (S'_{p-1}[K_p] \oplus S'_{p-1}[K_p]) \oplus (S_{p-1}[K_p]). \end{aligned}$$

The removal of any leaf from component of $G$ isomorphic to $S_{p-1}[K_p]$ and an appropriate leaf from a component of $H$ isomorphic to $S'_{p-1}[K_p]$ give isomorphic cards. So $b(G,\, H) = 2(p-1)^2 = \frac{2}{3}(n + 5 - 2\sqrt{3n+6})$. In addition, it is easy to see that $G$ and $H$ have the same degree sequence.

We conclude the thesis by showing how to construct infinite families of pairs of connected graphs with $2\left\lfloor \frac{1}{3}(n-1) \right\rfloor$ or only slightly fewer common cards. The easiest way to do this it to complement the disconnected graphs given in previous examples. However, we show in Theorem 6.3.3 that by using the *join* of two graphs, we can construct infinite families of pairs of graphs with $n$ vertices and connectivity $\kappa$ that have $2\left\lfloor \frac{1}{3}(n-\kappa-1) \right\rfloor$ common cards.

A final important example, presented in Theorem 6.3.4, is a family of pairs of trees with $2\left\lfloor \frac{1}{3}(n-5) \right\rfloor$ common cards. This we do by adding a vertex to each of the pair of forests in (2.9) and, adding three edges joining this additional vertex to the centres of the three stars. These, and other examples are explained in more detail in Section 6.3.

Our investigations suggest two important conjectures, both of which strengthen the Reconstruction Conjecture. We know of no counter-example to the first conjecture for $n \geq 13$, and none to the second for $n \geq 22$.

**Conjecture 6.3.5** For large enough $n$, every simple finite undirected graph is determined, up to isomorphism, by any $2\left\lfloor \frac{1}{3}(n-1) \right\rfloor + 1$ of its vertex-deleted subgraphs.

In other words, we conjecture that $urn(G) \leq 2\left\lfloor \frac{1}{3}(n-1) \right\rfloor + 1$ for large enough $n$.

We also conjecture that our families are unique. This conjecture is explained more fully in Section 6.3.

**Conjecture 6.3.6** For large enough $n$, the only pairs of graphs that attain the bound in Conjecture 6.3.5 are, up to isomorphism, the 18 families of pairs of graphs that can be constructed from Example 5.5.12, by any combination of complementing, and adding up to two isolated vertices or a component isomorphic to $K_2$.

# Chapter 3

# The Number of Common Cards between a Connected Graph and a Disconnected Graph

By Lemma 2.4.7, the connectedness of a graph is reconstructible. In this chapter we show that the maximum number of common cards between a connected graph and a disconnected graph is $\left\lfloor \frac{n}{2} \right\rfloor + 1$, and thus we can recognise the connectedness of a graph from any $\left\lfloor \frac{n}{2} \right\rfloor + 2$ cards of its deck. In addition, we show that this bound is only attained by three families of pairs of graphs and one "super-family", together with a few pairs of order at most 7.

## 3.1 Active Vertices in Disconnected Graphs

For the whole of this chapter, $G$ will denote a connected graph and $H$ a disconnected graph, both of order $n \geq 3$, where $H$ is expressed as in (1.2). We begin with the following definition.

Let $v$ be a vertex in $A_H(G)$. Then $v$ is $H_j$*-active* if some associated vertex is in a component of $H$ isomorphic to $H_j$. We denote the set of $H_j$-active vertices of $G$ by $A_{H_j}(G)$ and its cardinality by $a_{H_j}(G)$. Note that since $H_j$ is a representative of an isomorphism class, this definition is only meaningful in the context of the decomposition of $H$ given in (1.2). However, since it will always be clear from the context which two graphs we are discussing, there will be no confusion with this definition.

Suppose that $v$ is an $H_j$-active vertex and that $w$ is a vertex of $H$ associated with $v$, which is in some component $W$. Then, from (1.2) and (2.2),

$$G - v \cong H - w \cong (\textstyle\bigoplus_{k \neq j} \beta_k H_k) \oplus (\beta_j - 1) H_j \oplus (W - w), \tag{3.1}$$

where $W - w \cong H_j - w'$, for some $w'$ in $H_j$. We use (3.1) to show the following result.

**Lemma 3.1.1** Let $G$ be a connected graph and $H$ a disconnected graph. Then $\left\{ A_{H_j}(G) \,|\, 1 \leq j \leq r \right\}$ is a partition of $A_H(G)$, so

$$a_H(G) = \sum_{j=1}^{r} a_{H_j}(G).$$

*Proof* Let $v$ be an active vertex of $G$ and suppose that $w_1$ and $w_2$ are distinct vertices of $H$ associated with $v$. Let $w_1$ and $w_2$ be in (not necessarily distinct) components $W_1$ and $W_2$, respectively, where $W_1 \cong H_s$ and $W_2 \cong H_t$. By setting $j = s$ and $j = t$ in (3.1), clearly

$$\begin{aligned} G - v \cong H - w_1 &\cong (\textstyle\bigoplus_{k \neq s} \beta_k H_k) \oplus (\beta_s - 1) H_s \oplus (W_1 - w_1) \qquad \text{and} \\ G - v \cong H - w_2 &\cong (\textstyle\bigoplus_{k \neq t} \beta_k H_k) \oplus (\beta_t - 1) H_t \oplus (W_2 - w_2), \end{aligned}$$

so

$$H_t \oplus (W_1 - w_1) \cong H_s \oplus (W_2 - w_2).$$

Since $W_1 \cong H_s$, it follows that $s = t$ and therefore each active vertex of $G$ is $H_j$-active for precisely one $j$. The result then follows. $\square$

Suppose now that $W_1$ and $W_2$ are isomorphic components of $H$. Then for each vertex $w_1$ in $W_1$, we can choose a distinct vertex $w_2$ in $W_2$ such that $H - w_1 \cong H - w_2$. It follows that the number of active vertices of $H$ in $W_1$ must be identical to the number in $W_2$, that is, isomorphic components of $H$ contain the same number of active vertices. We therefore write $a_G(H_j)$ to denote the number of active vertices of $H$ that are in a single component isomorphic to $H_j$. We now extend Lemma 3.1.1 from active vertices to common cards.

Since $\left\{A_{H_j}(G) \,|\, 1 \leq j \leq r\right\}$ is a partition of $A_H(G)$, it follows that each edge of $B(G,\, H)$ joins an $H_j$-active vertex of $G$ and an active vertex of $H$ that lies in a component isomorphic to $H_j$, for some $j$. We therefore define $b(G,\, H_j)$ to be the size of a maximum matching of the subgraph of $B(G,\, H)$ induced by the set of all $H_j$-active vertices of $G$ and all active vertices of $H$ in components isomorphic to $H_j$; thus $b(G,\, H) = \sum_{j=1}^{r} b(G,\, H_j)$. Clearly, $b(G,\, H_j) \leq \min(a_{H_j}(G),\, \beta_j a_G(H_j))$, so we therefore obtain the following upper bounds on $b(G,\, H)$.

**Corollary 3.1.2** Let $G$ be a connected graph and $H$ a disconnected graph. Then

$$b(G,\, H) \leq \sum_{j=1}^{r} \min\left(a_{H_j}(G),\, \beta_j a_G(H_j)\right) \leq \sum_{j=1}^{r} \min\left(a_{H_j}(G),\, \beta_j h_j\right) \leq a_H(G).$$

So, when $H$ consists of precisely two non-isomorphic components, that is
$H = H_1 \oplus H_2$,

$$b(G,\, H) \leq a_{H_1}(G) + \min\left(a_{H_2}(G), h_2\right). \tag{3.2}$$

*Proof* This follows immediately from the above discussion, noting that $a_G(H_j) \leq h_j$, for all $j$. □

## 3.2 Bounding the Number of Common Cards between a Connected Graph and a Disconnected Graph

Before we prove that $b(G, H) \leq \lfloor \frac{n}{2} \rfloor + 1$ for $G$ connected and $H$ disconnected, we prove two simple results concerning the cards of $G$. These results are also integral to proving the bounds in Chapter 5. Note that, since any card is a subgraph of $G$, all the vertices and edges of a card of $G$ are also vertices and edges of the graph $G$ itself. Moreover, any component of a disconnected card of $G$ is a vertex-induced subgraph of $G$. We can thus talk about a component of a card of $G$ intersecting, or being contained in, a component of another card of $G$.

**Lemma 3.2.1** Let $G$ be a connected graph of order $n$ containing two distinct vertices $u$ and $v$. Let $X_{uv}$ be the component of $G - u$ that contains $v$, and $X_{vu}$ the component of $G - v$ that contains $u$. Then

(a) $(G - u) - X_{uv} \subset X_{vu}$ and $(G - v) - X_{vu} \subset X_{uv}$;

(b) $|V(X_{vu})| + |V(X_{uv})| \geq n$;

(c) $(G - u) - X_{uv}$ and $(G - v) - X_{vu}$ are disjoint.

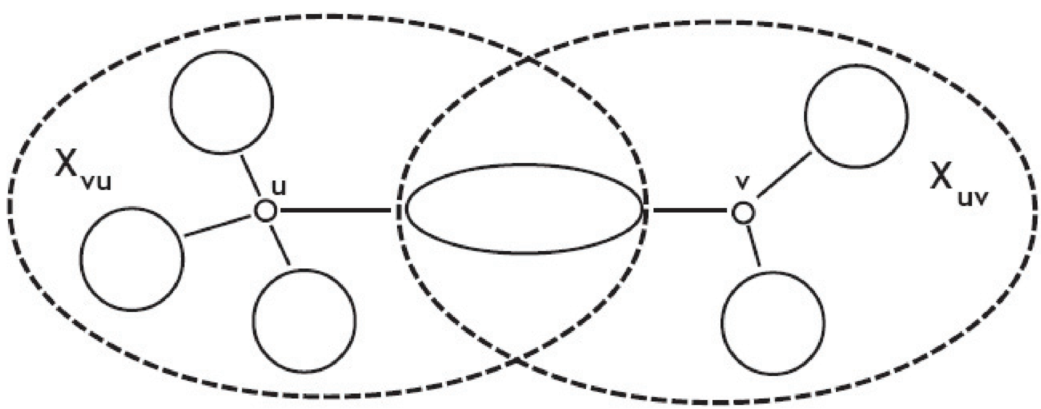


Figure 3.1: $X_{uv}$ and $X_{vu}$.

*Proof* If $u$ is not a cut-vertex then $X_{uv} = G - u$; similarly if $v$ is not a cut-vertex then $X_{vu} = G - v$. The results follow immediately in either case, so we may assume that both $u$ and $v$ are cut-vertices, and therefore $G - u$ and $G - v$ both contain at least two components.

(a) Suppose that $x$ is a vertex of $(G - u) - X_{uv}$. Then there is a path in $G$ from $x$ to $u$ that does not contain any vertex of $X_{uv}$; in particular, it does not contain $v$. Hence $x$ and $u$ are in the same component of $G - v$; so $x$ is in $X_{vu}$ and thus $(G - u) - X_{uv} \subset X_{vu}$. The second assertion follows by symmetry.

(b) Since $X_{vu}$ contains $u$, the result follows from part (a).

(c) Since $(G - u) - X_{uv}$ and $X_{uv}$ are disjoint, $(G - u) - X_{uv}$ and $(G - v) - X_{vu}$ are disjoint by part (a). $\square$

**Corollary 3.2.2** Let $G$ be a connected graph of order $n$, and let $S \subseteq V(G)$, with $|S| \geq 2$. Suppose that, for each vertex $u$ in $G$, $\mathcal{T}_u$ is the (possibly empty) collection of those components of $G - u$ that do not contain a vertex of $S$. Then

$$\sum_{u \in S} (|V(\mathcal{T}_u)| + 1) \leq n.$$

*Proof* Let $u$ and $v$ be in $S$, with $u \neq v$, and let $X_{uv}$ and $X_{vu}$ be as in Lemma 3.2.1. By part (c) of the lemma, $(G - u) - X_{uv}$ and $(G - v) - X_{vu}$ are disjoint; so, since $\mathcal{T}_u \subseteq (G - u) - X_{uv}$ and $\mathcal{T}_v \subseteq (G - v) - X_{vu}$, $\mathcal{T}_u$ and $\mathcal{T}_v$ are disjoint. Thus $\{\mathcal{T}_u \,|\, u \in S\}$ is a collection of disjoint subgraphs of $G$, and the result then follows since these subgraphs are also disjoint from $S$. $\square$

We now use Lemmas 3.2.1 and Corollary 3.2.2 to prove the bound on $b(G, H)$. We first prove the following lemma which relates the structure of $H$ to the active vertices of $G$.

**Lemma 3.2.3** Let $G$ be a connected graph and $H$ a disconnected graph, both of order $n$, with $a_H(G) \geq 2$. Let $u$ be an active vertex of $G$ and let $X$ be a component of $G - u$. We have the following results:

(a) $h_1 \geq \frac{n}{2}$;

(b) if $|V(X)| = h_1$, then $X$ contains every active vertex of $G$ except $u$;

(c) if $\beta_2 > 0$ and $|V(X)| = h_2 < h_1$, then $X$ contains no $H_1$-active vertices. Furthermore, $X$ contains no active vertices at all unless $h_1 + h_2 = n$.

*Proof* Let $v$ be any vertex in $A_H(G) - \{u\}$, and let $X_{uv}$ and $X_{vu}$ be as in Lemma 3.2.1. By part (b) of the lemma,

$$|V(X_{uv})| + |V(X_{vu})| \geq n. \tag{3.3}$$

Suppose that $X$ and $X_{uv}$ are two different components of $G - u$. Then $|V(X)| + |V(X_{uv})| \leq n - 1$; so $|V(X)| < |V(X_{vu})|$ by (3.3). Therefore, it follows that if $|V(X)| \geq |V(X_{vu})|$ then $X$ must be $X_{uv}$. Similarly, for any component $\widehat{X}$ of $G - v$, if $|V(\widehat{X})| \geq |V(X_{uv})|$, then $\widehat{X}$ is $X_{vu}$.

(a) By (3.1), $|V(X_{uv})| \leq h_1$ and $|V(X_{vu})| \leq h_1$. The result then follows by (3.3).

(b) Since $|V(X)| = h_1 \geq |V(X_{vu})|$, it follows that $X$ is $X_{uv}$. So $v$ is in $X$.

(c) Suppose first that $v$ is $H_1$-active. Then by (3.1), $G - v$ contains a component $\widehat{X}$ of order $h_2$, since $\beta_2 > 0$. Now, if $X$ is $X_{uv}$, then $|V(\widehat{X})| = |V(X)| = |V(X_{uv})|$, so $\widehat{X}$ is $X_{vu}$. Thus $2h_2 \geq n$ by (3.3), which is impossible since $h_2 < h_1$. Therefore $X$ cannot be $X_{uv}$, so $v$ is not in $X$.

Suppose instead that $v$ is active but not $H_1$-active. Then $G - v$ contains a component isomorphic to $H_1$, which must contain $u$ by (b); so $|V(X_{vu})| = h_1$. Now, if $v$ is in $X$, then $X$ is $X_{uv}$, so $|V(X_{uv})| = h_2$. Thus if $X$ contains any active vertices, $h_1 + h_2 \geq n$ by (3.3), and it follows that $h_1 + h_2 = n$ by (1.2). $\square$

**Lemma 3.2.4** Let $G$ be a connected graph and $H$ a disconnected graph, both of order $n$.

(a) If $H$ has at least two components of order $h_1$, then $a_H(G) \leq 2$.

(b) If $H$ contains at least two components of order less than $h_1$, then $a_H(G) \leq \left\lfloor \frac{n}{h_2+1} \right\rfloor$.

(c) If $H = H_1 \oplus H_2$ with $h_1 > h_2$, then $a_{H_1}(G) \leq \left\lfloor \frac{n}{h_2+1} \right\rfloor$ and $a_{H_2}(G) \leq \left\lfloor \frac{n}{h_2} \right\rfloor$.

*Proof* The results clearly hold if $a_H(G) \leq 1$, so we may assume that $a_H(G) \geq 2$. Thus, by Lemma 3.2.3(a), $h_1 \geq \frac{n}{2}$.

(a) Suppose that $H$ has two components of order $h_1$, so $h_1 = \frac{n}{2}$. Let $u$ be in $A_H(G)$. Then, by (3.1), $G - u$ contains a component $X$ of order $h_1$. Every vertex of $A_H(G)$ except $u$ is in $X$ by Lemma 3.2.3(b). We apply Corollary 3.2.2 with $S = A_H(G)$. Then, since $|V(\mathcal{T}_u)| = n - 1 - h_1 = \frac{n}{2} - 1$, it follows from this corollary that $\frac{n}{2} a_H(G) \leq n$. Therefore $a_H(G) \leq 2$.

(b) Suppose next that $H$ contains at least two components of order less than $h_1$, so $h_1 + h_2 + 1 \leq n$, by (1.2). Let $u$ be in $A_H(G)$. By (3.1), $G - u$ contains a component $X$ that is isomorphic to either $H_1$ or $H_2$. As in (a), we apply Corollary 3.2.2 with $S = A_H(G)$. Now if $X \cong H_1$, then $X$ contains every active vertex of $G$ except $u$ by Lemma 3.2.3(b), so $|V(\mathcal{T}_u)| = n - 1 - h_1 \geq h_2$. On the other hand, if $X \cong H_2$, then $X$ contains no active vertices by Lemma 3.2.3(c), so $|V(\mathcal{T}_u)| \geq h_2$. It therefore follows from the corollary that $a_H(G)(h_2 + 1) \leq n$, which yields the result.

(c) Finally, suppose that $H = H_1 \oplus H_2$ with $h_1 > h_2$. Suppose first that $u$ is in $A_{H_1}(G)$. By (3.1), $G - u$ has a component $X \cong H_2$, which contains no $H_1$-active vertices, by Lemma 3.2.3(c). Clearly we may assume that $a_{H_1}(G) \geq 2$, so we may apply Corollary 3.2.2 with $S = A_{H_1}(G)$. As in (b), $|V(\mathcal{T}_u)| \geq h_2$, so $a_{H_1}(G)(h_2 + 1) \leq n$. Thus $a_{H_1}(G) \leq \left\lfloor \frac{n}{h_2+1} \right\rfloor$.

Suppose instead that $u$ is in $A_{H_2}(G)$. By (3.1), $G-u$ has a component $X \cong H_1$, which contains every active vertex of $G$ except $u$, by Lemma 3.2.3(b). Clearly we may assume that $a_{H_2}(G) \geq 2$, so we again may apply Corollary 3.2.2, now with $S = A_{H_2}(G)$. In this case, clearly $a_{H_2}(G) \leq \left\lfloor \frac{n}{h_2} \right\rfloor$, since $|V(\mathcal{T}_u)| = n-1-h_1 = h_2-1$. □

We note that Lemma 3.2.4 covers every possible component structure for $H$.

**Theorem 3.2.5** Let $G$ be a connected graph and $H$ a disconnected graph, both of order $n$. Then

$$b(G,\, H) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \tag{3.4}$$

so the connectedness of a graph can be determined from any $\left\lfloor \frac{n}{2} \right\rfloor + 2$ of its cards. In addition, if equality holds in (3.4), then $H \cong H_1 \oplus H_2$ with $h_1 > h_2$.

*Proof* The result holds trivially for $n = 3$, so we assume that $n \geq 4$. Let $H$ be expressed as in (1.2). Since $b(G,\, H) \leq a_H(G)$, by Lemma 3.2.4(a) and (b), (3.4) holds with strict inequality unless $H \cong H_1 \oplus H_2$ with $h_1 > h_2$. In this case, by (3.2) and Lemma 3.2.4(c),

$$b(G,\, H) \leq \left\lfloor \frac{n}{h_2+1} \right\rfloor + \min\left( \left\lfloor \frac{n}{h_2} \right\rfloor,\, h_2 \right). \tag{3.5}$$

Thus the result is trivial for $h_2 = 1$ or $h_2 \geq 4$. For $h_2 = 2$ or $h_2 = 3$, the result holds by straightforward calculations. □

We note the bound (3.4) was first obtained by Myrvold in her doctoral thesis (see [33]).

## 3.3 Pairs that Attain the Bound of Theorem 3.2.5

We now characterise the graph pairs that attain the bound of Theorem 3.2.5. The theorem indicates that we only need to consider graphs where $H \cong H_1 \oplus H_2$ with $h_1 > h_2$. We begin by giving the only four pairs of graphs that attain the bound when $h_2 \geq 2$. Note that, in this case, every active vertex of $G$ must be a cut-vertex.

**Example 3.3.1** For $n \geq 4$, let $G = P_n$ and $H = P_{n-2} \oplus K_2$. Then the removal of either leaf-adjacent vertex from $G$, and either vertex from the $K_2$ component of $H$, gives the card $P_{n-2} \oplus K_1$; the removal of a vertex that is a distance of 2 from a leaf of $G$, and a leaf from the $P_{n-2}$ component of $H$, gives the card $P_{n-3} \oplus K_2$. There are thus 2 common cards for $n = 4$, 3 common cards for $n = 5$ and 4 common cards for $n \geq 6$. It follows that $b(G,\, H) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ for $n = 5$, 6 or 7. Figure 3.2 shows the case when $n = 6$. □


Figure 3.2: $P_6$ and $P_4 \oplus K_2$.

**Example 3.3.2** Let $G$ and $H$ be the pair of graphs in Figure 3.3. Then
$G - v_i \cong H - w_i$, for $1 \leq i \leq 4$; so $b(G,\, H) = \left\lfloor \frac{n}{2} \right\rfloor + 1 = 4$. □


Figure 3.3: Connected and disconnected graphs of order 7 with 4 common cards.

We now prove that these four pairs of graphs are the only pairs that attain the bound when $h_2 \geq 2$. Before we do so however, we make the following simple observation concerning the degrees of associated vertices. This observation will be useful in this and subsequent chapters.

**Lemma 3.3.3** Let $F$ and $U$ be a pair of graphs. Suppose that $v$ is an active vertex of $F$ and that $w$ is a vertex of $U$ associated with $v$. Then $d(v) = d(w) + |E(F)| - |E(U)|$.

*Proof* $|E(F)| - d(v) = |E(F - v)| = |E(U - w)| = |E(U)| - d(w)$, since $F - v \cong U - w$. This implies the result. □

**Lemma 3.3.4** Let $G$ be a connected graph and $H$ a disconnected graph, both of order $n$, where $H = H_1 \oplus H_2$ and $h_1 > h_2 \geq 2$. If $b(G,\, H) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, then $G$ and $H$ are one of the four pairs in Examples 3.3.1 and 3.3.2.

*Proof* Suppose that $b(G,\, H) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Then, by Lemma 3.2.4(c),

$$a_{H_1}(G) \leq \left\lfloor \frac{n}{h_2 + 1} \right\rfloor \qquad \text{and} \qquad a_{H_2}(G) \leq \left\lfloor \frac{n}{h_2} \right\rfloor. \tag{3.6}$$

So, by Corollary 3.1.2,

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 = b(G,\, H) \leq a_{H_1}(G) + \min(a_{H_2}(G),\, h_2) \leq \left\lfloor \tfrac{n}{h_2+1} \right\rfloor + \min(\left\lfloor \tfrac{n}{h_2} \right\rfloor,\, h_2). \tag{3.7}$$

Clearly, this cannot hold if $h_2 \geq 4$. In addition, by straightforward calculations, if $h_2 = 3$ then $n = 9$ and if $h_2 = 2$, then $n = 5$, 6, 7 or 9. Moreover, in all of these cases, equality holds throughout (3.7). It is then easy to show that, for these values of $h_2$ and $n$,

$$a_{H_1}(G) = \left\lfloor \frac{n}{h_2 + 1} \right\rfloor \qquad \text{and} \qquad a_{H_2}(G) \geq h_2. \tag{3.8}$$

Let $\{u_i\}$ be the vertices in $A_{H_1}(G)$ and $\{v_j\}$ be the vertices in $A_{H_2}(G)$. For each vertex $v_j$, $G - v_j$ contains a component $Y_j$ isomorphic to $H_1$ by (3.1), and the order of each subgraph $G - v_j - Y_j$ is equal to $h_2 - 1$. By Lemma 3.2.3(b), every active vertex of $G$ except $v_j$ is in $Y_j$. So, by Lemma 3.2.1(c), for each pair of distinct $H_2$-active vertices $v_j$ and $v_k$, the subgraphs $G - v_j - Y_j$ and $G - v_k - Y_k$ are disjoint.

Suppose first that $h_2 = 3$ and $n = 9$. Then $a_{H_2}(G) = 3$, by (3.6) and (3.8). Thus, $G$ contains three disjoint subgraphs $G - v_j - Y_j$ of order 2 that contain no active vertices. So $a_H(G) \leq 3$, which contradicts the fact that $b(G,\, H) = 5$. Hence this case is impossible.

We may therefore suppose that $h_2 = 2$. By (3.1), each $G - v_j \cong H_1 \oplus K_1$, so each $v_j$ is adjacent to precisely one leaf. In addition, each $G - u_i$ contains a component $X_i$ isomorphic to $K_2$. There are two possibilities for each $X_i$: either both vertices of $X_i$ are of degree two and adjacent to $u_i$, or one vertex is a leaf and the other vertex is of degree 2 and adjacent to $u_i$. Since $n \geq 5$, the $X_i$ are clearly disjoint.

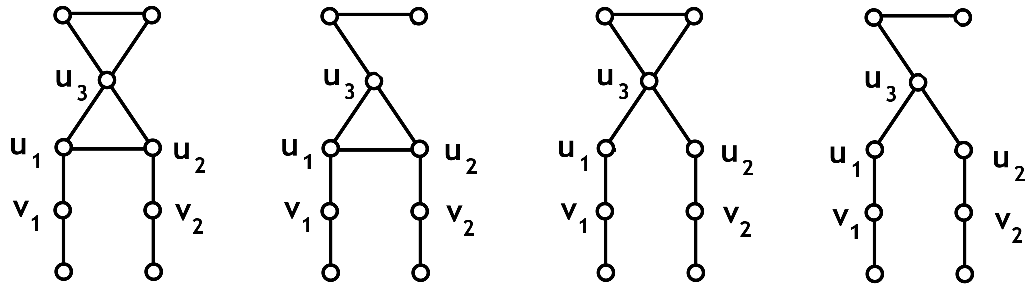


Figure 3.4: The four "possibilities" for $G$ with $n = 9$ and $h_2 = 2$.

Suppose that $n = 9$. Then, since $a_{H_1}(G) = 3$ by (3.8), it follows that $V(G) = X_1 \cup X_2 \cup X_3 \cup A_{H_1}(G)$. So since $d_1(v_j) = 1$, each $v_j$ must be contained in precisely one $X_i$ and $d(v_j) = 2$. We may therefore assume without loss of generality that $v_1$ is adjacent to $u_1$ and $v_2$ is adjacent to $u_2$. Now, $v_1$ is only adjacent to $u_1$ and $v_1^*$ and $v_2$ is only adjacent to $u_2$ and $v_2^*$. So since $G - v_1 \cong G - v_2 \cong H_1 \oplus K_1$, it follows that $d(u_1) = d(u_2)$. It is thus easy to see that the only possibilities for $G$ are the four graphs in Figure 3.4; in each case, $H$ is isomorphic to the graph obtained by deleting the edge $u_1v_1$ from $G$. By inspection, in each of these four cases, the image of $u_1$ is the only active vertex in $H_1$. So $a_G(H) \leq 3$, which contradicts the fact that $b(G, H) = 5$. So the case $n = 9$ cannot occur.

We recall that, in all cases, $G$ contains $v_1$ and $v_2$, where $d_1(v_1) = d_1(v_2) = 1$. If $n = 5$, then $G$ contains $u_1$ and is connected, so $G$ must be a path. Similarly, if $n = 6$, then $G$ contains $u_1$ and $u_2$, and again $G$ must be a path. Finally, if $n = 7$, then $G$ contains $u_1$, $u_2$, and an additional vertex which must be adjacent to $u_1$ or $u_2$, or both, and no other vertex. This additional vertex cannot be a leaf since $G - v_1 \cong G - v_2$. Thus if $n = 7$, then $G$ is either a path or the graph in Example 3.3.2. This completes the proof. □

We now turn our attention to when $h_2 = 1$. We begin by presenting the three families and one "super-family" of pairs of graphs that attain the bound. We recall from Section 1.1, that when $v$ is a vertex of $G$ with $d_1(v) = 1$, we denote the unique leaf adjacent to $v$ by $v^*$.

The first family is the unique family of pairs of graphs of even order that attain the bound.

**Example 3.3.5** Let $p$ be an integer, $p \geq 0$. Then, for $n = 2(p+1)$, the following pair of graphs of order $n$ has $\lfloor \frac{n}{2} \rfloor + 1$ common cards. Let $G$ be isomorphic to $S_1[S_p^1]$ and let $u_0$ be its central vertex. Let $H_1$ be isomorphic to $G - u_0^*$ and let $x_0$ be the central vertex of $H_1$. Now let $H \cong H_1 \oplus K_1$, and let $z$ be the isolated vertex of $H$. Let the other cut-vertices of $G$ and $H$ be $u_1, u_2, \ldots, u_p$ and $x_1, x_2, \ldots, x_p$, respectively. Clearly, $G - u_0 \cong H - x_0$, $G - u_0^* \cong H - z$ and $G - u_i \cong H - x_i^*$, for each $i \geq 1$. So $b(G, H) = p + 2 = \lfloor \frac{n}{2} \rfloor + 1$. Figure 3.5 shows these graphs for $p = 5$.

$\square$

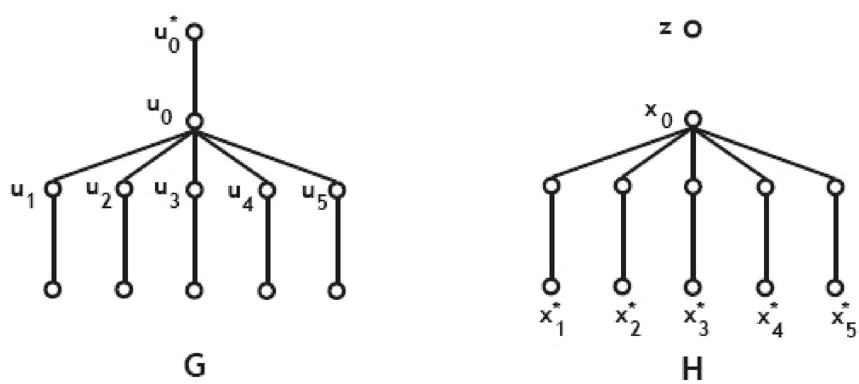


Figure 3.5: The pair of graphs in Example 3.3.5 of order 12 with 7 common cards.

There are three families of odd order that attain the bound. The first two are similar to the family in Example 3.3.5.

**Example 3.3.6** Let $p$ be an integer, $p \geq 0$. Then, for $n' = 2p + 3$, the following pair of graphs of order $n'$ has $\lfloor \frac{n'}{2} \rfloor + 1$ common cards. Let $G'$ and $H'$ be the graphs obtained from $G$ and $H$ in Example 3.3.5 by adding a single leaf to each graph, adjacent to $u_0$ and $x_0$, respectively. Clearly, $G'$ and $H'$ have the same number of common cards as $G$ and $H$. So $b(G', H') = p + 2 = \lfloor \frac{n'}{2} \rfloor + 1$. Figure 3.6 shows these graphs for $p = 5$.

$\square$

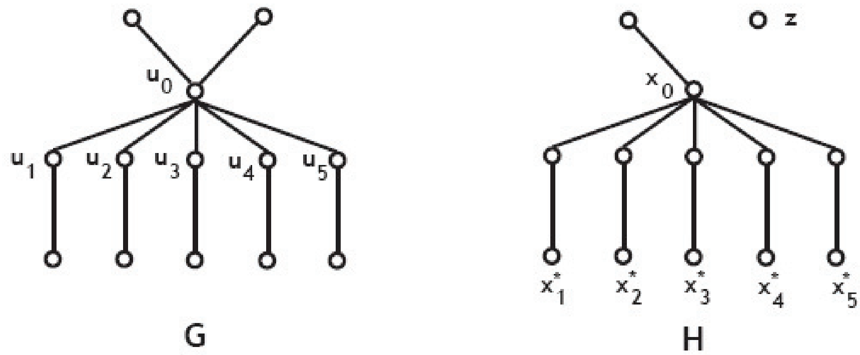


Figure 3.6: The pair of graphs in Example 3.3.6 of order 13 with 7 common cards.

**Example 3.3.7** Let $p$ be an integer, $p \geq 0$. Then, for $n'' = 4p+5$, the following pair of graphs of order $n''$ has $\left\lfloor \frac{n''}{2} \right\rfloor + 1$ common cards. Let $A$ and $B$ be disjoint graphs, both isomorphic to $S_1[S^1_{p+1}] - t^*_0$, where $t_0$ is the central vertex of $S_1[S^1_{p+1}]$. Let $v_0$ and $y_0$ be the central vertices of $A$ and $B$, respectively, and let $v_1, v_2, \ldots, v_{p+1}$ and $y_1, y_2, \ldots, y_{p+1}$ be their other cut-vertices. Now let $G''$ and $H''$ be the graphs obtained by adding the edges $u_0v_0$ and $x_0y_0$ to $G \oplus A$ and $H \oplus B$, respectively, where $G$ and $H$ are the graphs in Example 3.3.5. Then $G''-u_0 \cong H''-x_0$, $G''-u^*_0 \cong H''-z$, $G'' - u_i \cong H'' - x^*_i$, for all $i \geq 1$, and $G'' - v_j \cong H'' - y^*_j$, for all $j \geq 1$. So $b(G'', H'') = 2p+3 = \left\lfloor \frac{n}{2}'' \right\rfloor + 1$. Figure 3.7 shows these graphs for $p = 5$. □

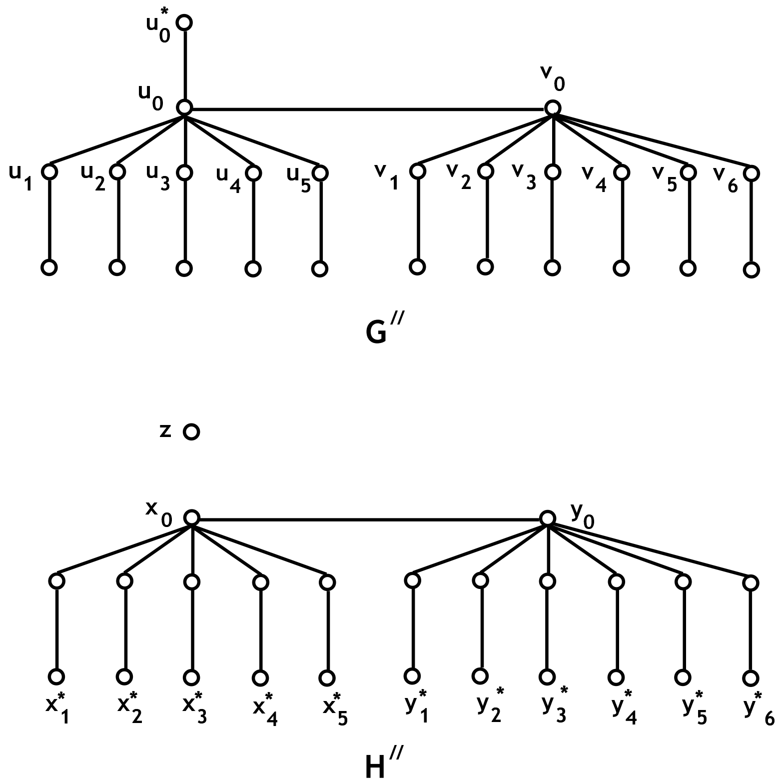


Figure 3.7: The pair of graphs in Example 3.3.7 of order 25 with 13 common cards.

The last family is a “super-family”, with many graph pairs for each odd $n$.

**Example 3.3.8** Let $p$ be an integer, $p \geq 2$. Then, for $n = 2p+1$, the following pair of graphs of order $n$ has $\lfloor \frac{n}{2} \rfloor + 1$ common cards. Let $T$ be any connected vertex-transitive graph of order $p+1$ and let $t$ be any vertex of $T$. Now let $G = S_1[T] - t^*$ and $H \cong S_1[T] - t \cong S_1[T-t] \oplus K_1$, and let $z$ be the isolated vertex of $H$. For any vertex $u \neq t$ in $T$, there is some automorphism $\phi_u$ of $T$ such that $\phi_u(u) = t$, since $T$ is vertex-transitive. Let $\phi_u(t)^*$ be the leaf of $G$ adjacent to $\phi_u(t)$.

Clearly $G - t \cong H - z$. In addition, $G - u \cong S_1[T] - \phi_u(t)^* - t \cong H - x$ for some leaf $x$ of $H$. We will show in Corollary 3.3.16 that we can find a distinct $\phi_u(t)^*$, and thus a distinct $x$, for each $u$ in $A_{H_1}(G)$. So $b(G, H) = p + 1 = \lfloor \frac{n}{2} \rfloor + 1$. □

A simple example of this construction is obtained when $T \cong K_{p+1}$, so $T - t \cong K_p$ (note that, in this case every leaf of $H$ is associated with every $H_1$-active vertex of $G$). Figure 3.8 shows these graphs for $T = K_8$.

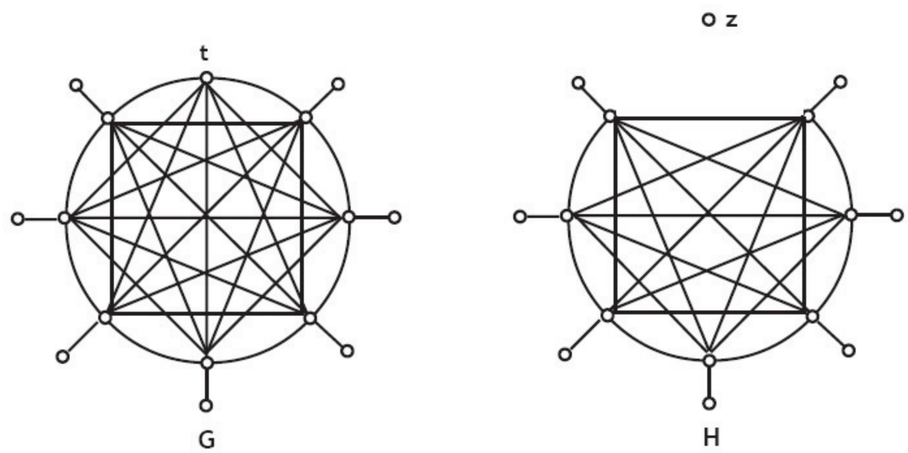


Figure 3.8: A member of the super-family in Example 3.3.8 when $T = K_8$.

We will show, in Lemmas 3.3.13 and 3.3.14, that when $h_2 = 1$, the only pairs attaining the bound that are not members of the families in Examples 3.3.5 to 3.3.8 are the following two pairs of small graphs.


Figure 3.9: Connected and disconnected graphs of order 5 with 3 common cards.

**Example 3.3.9** Let $G$ and $H$ be the pair of graphs in Figure 3.9. Then
$G - q \cong H - z$, $G - u_0 \cong G - s \cong H - y_0 \cong H - x_0$. So $b(G,\, H) = \left\lfloor \frac{n}{2} \right\rfloor + 1 = 3$. $\square$


Figure 3.10: Connected and disconnected graphs of order 7 with 4 common cards.

**Example 3.3.10** Let $G$ and $H$ be the pair of graphs in Figure 3.10. Then $G - q \cong H - z$, $G - u_0 \cong H - x_0$, $G - s \cong H - y_0$ and $G - v \cong H - w$. So $b(G, H) = \lfloor \frac{n}{2} \rfloor + 1 = 4$. □

We recall from Section 1.1 that, since $G$ is connected, a non-leaf of $G$ is any vertex of degree 2 or more.

**Lemma 3.3.11** Let $G$ be a connected graph and $H$ a disconnected graph, both of order $n$, $n \geq 4$, where $H = H_1 \oplus H_2$ and $h_2 = 1$. Suppose that $b(G, H) = \lfloor \frac{n}{2} \rfloor + 1$.

(a) If $u$ is in $A_{H_1}(G)$, then $u$ is a non-leaf and is adjacent to one more leaf than any vertex of $H$ associated with $u$.

(b) $a_{H_1}(G) = \lfloor \frac{n}{2} \rfloor$ and $a_{H_2}(G) \geq 1$.

(c) In any maximum matching of $B(G, H)$, every vertex of $A_{H_1}(G)$ is incident to some edge of the matching.

(d) Every vertex of $A_{H_2}(G)$ is not a cut-vertex and is of degree $|E(G)| - |E(H)|$.

*Proof* (a) Let $u$ be any vertex in $A_{H_1}(G)$ and let $x$ be a vertex of $H_1$ associated with $u$. Clearly, $u$ cannot be a leaf, and since $H$ contains precisely one isolated vertex, $u$ must be adjacent to precisely one more leaf than $x$.

(b) Since $b(G, H) = \lfloor \frac{n}{2} \rfloor + 1$, clearly, $a_{H_1}(G) \geq \lfloor \frac{n}{2} \rfloor$ by (3.2). From (a), it follows that $G$ contains at least $a_{H_1}(G)$ leaves, none of which can be $H_1$-active. Therefore $a_{H_1}(G) = \lfloor \frac{n}{2} \rfloor$, so $a_{H_2}(G) \geq 1$.

(c) This follows directly from (b) and (3.2).

(d) Any $H_2$-active vertex of $G$ is associated with the isolated vertex of $H$, so is therefore not a cut-vertex and in addition is of degree $|E(G)| - |E(H)|$ by Lemma 3.3.3. □

**Corollary 3.3.12** Let $G$ and $H$ be as in Lemma 3.3.11.

(a) If $n$ is even, then $d_1(G) = \frac{n}{2}$ and $G$ contains precisely $\frac{n}{2}$ $H_1$-active vertices, each adjacent to precisely one leaf.

(b) If $n$ is odd, then $G$ contains precisely $\frac{n-1}{2}$ $H_1$-active vertices, and either

(i) $d_1(G) = \frac{n+1}{2}$, one $H_1$-active vertex is adjacent to precisely two leaves, and the others are each adjacent to precisely one leaf;

or

(ii) $d_1(G) = \frac{n-1}{2}$, every $H_1$-active vertex is adjacent to precisely one leaf, and there is one non $H_1$-active vertex that is neither a leaf nor adjacent to a leaf.

*Proof* By parts (a) and (b) of Lemma 3.3.11, $G$ contains $\lfloor \frac{n}{2} \rfloor$ $H_1$-active vertices, each of which is adjacent to a leaf, from which the result easily follows. □

We first consider the case when $|E(G)| - |E(H)| = 1$; so by Lemma 3.3.11(b) and (d), $G$ contains an $H_2$-active leaf.

**Lemma 3.3.13** Let $G$ and $H$ be as in Corollary 3.3.12, but not either of the pairs in Examples 3.3.9 and 3.3.10. Suppose that $|E(G)| - |E(H)| = 1$, and let $q$ be an $H_2$-active leaf (so $G - q \cong H_1$), and let $u_0$ be the vertex of $G$ adjacent to $q$.

(a) Every vertex in $A_{H_1}(G) - \{u_0\}$ is of degree 2.

(b) One of the following three possibilities must hold:

(i) if $d_1(G) = \frac{n}{2}$, then $G$ and $H$ are the pair described in Example 3.3.5;

(ii) if $d_1(G) = \frac{n+1}{2}$, then $G$ and $H$ are the pair described in Example 3.3.6;

(iii) if $d_1(G) = \frac{n-1}{2}$, then $G$ and $H$ are the pair described in Example 3.3.7.

*Proof* Let $\phi$ be an isomorphism from $G-q$ to $H_1$. Clearly $G-u_0 \cong H-\phi(u_0)$, so $u_0$ is associated with $\phi(u_0)$. Thus, since by Lemma 3.3.11(c), in any maximum matching of $B(G, H)$, every vertex of $A_{H_1}(G)$ is incident to some edge of the matching, it follows that every vertex in $A_{H_1}(G) - \{u_0\}$ is associated with some active vertex of $H_1$ other than $\phi(u_0)$. Clearly, for any $u$ in $V(G) - \{u_0, q\}$,

$$d(\phi(u)) = d(u) \qquad \text{and} \qquad d_1(\phi(u)) \geq d_1(u), \tag{3.9}$$

noting that since $n \geq 4$, if $u$ is a leaf then $d_1(\phi(u)) = d_1(u) = 0$.

(a) Let $u$ be a vertex in $A_{H_1}(G) - \{u_0\}$ and let $\phi(v)$ be a vertex in $V(H_1) - \{\phi(u_0)\}$ associated with $u$. We shall show that $\phi(v)$ is always a leaf. Since $d(u) = d(\phi(v))+1$ by Lemma 3.3.3, the result will then follow. Following Corollary 3.3.12, we consider three cases: (I) $n$ is even and $d_1(G) = \frac{n}{2}$; (II) $n$ is odd and $d_1(G) = \frac{n+1}{2}$; and (III) $n$ is odd and $d_1(G) = \frac{n-1}{2}$.

(I) By Corollary 3.3.12(a), $d_1(u) = 1$; so $d_1(\phi(v)) = 0$ by Lemma 3.3.11(a). Moreover, by Corollary 3.3.12(a), every non-leaf of $G$ is adjacent to precisely one leaf, so by (3.9), every vertex of $H_1$, except possibly $\phi(u_0)$, is either a leaf or adjacent to a leaf. Therefore, $\phi(v)$ must be a leaf.

(II) By Corollary 3.3.12(b)(i), let $t$ be the $H_1$-active vertex of $G$ with $d_1(t) = 2$. In a similar manner to (I), it is easy to show that $\phi(v)$ is a leaf except when $u$ is $t$. So every vertex of $A_{H_1}(G) - \{u_0\}$ except $t$ is of degree 2 and adjacent to precisely one non-leaf. This proves the case when $t$ is $u_0$. We now show that the case when $t$ is not $u_0$ cannot exist.

Suppose then $t$ is not $u_0$ and let $\phi(r)$ be a vertex of $H$ associated with $t$. By (3.9), $r$ is not a leaf since $d(\phi(r)) = d(r)$ and $d_1(\phi(r)) = 1$; so $r$ is $H_1$-active by Corollary 3.3.12(b)(i). Clearly, $r \neq u_0$ since $d_1(\phi(u_0)) = 0$, and $r \neq t$ since $d_1(\phi(t)) \geq d_1(t) = 2$ by (3.9). Thus $r$ is in $A_{H_1}(G) - \{u_0, t\}$, so $d(\phi(r)) = d(r) = 2$. It follows that $d(t) = 3$, by Lemma 3.3.3, and thus every vertex in $A_{H_1}(G) - \{u_0\}$ must be adjacent to precisely one non-leaf. Therefore, since $G$ is connected, every such vertex (including $t$ and $r$) must be adjacent to $u_0$, so $d(u_0) \geq 3$. Thus $G - t$ does not contain a vertex adjacent to two or more leaves. This contradicts the fact that $G - t \cong H - \phi(r)$, since $\phi(t))$ is clearly adjacent to at least two leaves in $H - \phi(r)$. Therefore, $t$ must be $u_0$, and the result is proved for case (II).

(III) By Corollary 3.3.12(b)(ii), let $v_0$ be the non-leaf of $G$ that is not $H_1$-active. Then by that corollary and (3.9), every vertex of $H_1$, except possibly $\phi(u_0)$ and $\phi(v_0)$, is a leaf or adjacent to a leaf. Using a similar argument to that in (I), it is easy to see that $\phi(v)$ is a leaf unless $v$ is $v_0$. Thus if $\phi(v_0)$ is not active, the result follows. To complete the proof, we shall show that if $\phi(v_0)$ is active then $G$ and $H$ are one of the pairs in Examples 3.3.9 and 3.3.10.

So suppose that $\phi(v_0)$ is active and let $s$ be an $H_1$-active vertex of $G$ associated with $\phi(v_0)$. By Lemma 3.3.11(c), every vertex of $A_{H_1}(G) - \{u_0, s\}$ must be associated with some leaf of $H_1$ other than $\phi(u_0)$ and $\phi(v_0)$; so every such vertex is leaf-adjacent and of degree 2. Moreover, since $G$ is connected and $n \geq 5$, it follows that each of these vertices are adjacent to precisely one of $u_0$, $s$ and $v_0$.

Let $u_0$, $s$ and $v_0$ be adjacent to $\alpha$, $\beta$ and $\gamma$ vertices in $A_{H_1}(G) - \{u_0, s\}$, respectively (see Figure 3.11). Clearly each of $u_0$, $s$ and $v_0$ is adjacent to at least one of the other two since $G$ is connected. We use a dotted line in the diagram to indicate that the edge may or may not be present.

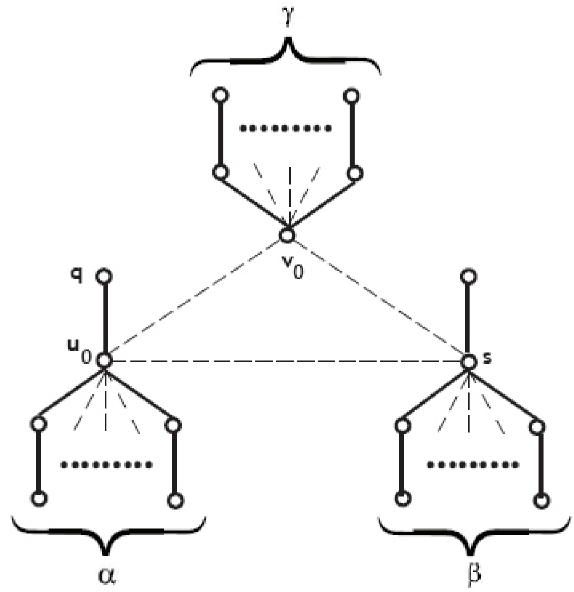


Figure 3.11: The graph $G$ when $s$ is associated to $\phi(v_0)$.

We first show that $\alpha = 0$. So suppose that $\alpha \geq 1$. Then $u_0$ is the only vertex of $G - s$ that is adjacent to both a leaf and a leaf-adjacent vertex of degree two unless $\alpha = 1$ and $v_0$ is not adjacent to $u_0$. Similarly, $\phi(s)$ is the only vertex of $H - \phi(v_0)$ that is adjacent to both a leaf and a leaf-adjacent vertex of degree 2, unless $\beta = 1$ and $s$ is not adjacent to $u_0$. Since $u_0$ must be adjacent to one of $v_0$ or $s$, it follows that $u_0$ and $\phi(s)$ are the only two such vertices in $G - s$ and $H - \phi(v_0)$, respectively. By counting the number of leaf-adjacent vertices of degree 2 adjacent to these vertices, it is easy to see that this implies that $\beta = \alpha \geq 1$. Thus, $H - \phi(v_0)$ contains precisely $\gamma$ components isomorphic to $K_2$. Similarly, $G - s$ contains precisely $\beta$ components isomorphic to $K_2$. Therefore, $\alpha = \beta = \gamma \geq 1$. It follows that no vertex of $G$ can be of degree greater than $\alpha + 3$.

Since $G$ is connected, and the fact that $G - s$ and $H - \phi(v_0)$ have an equal number of components, it easy to see that $u_0$ must be adjacent to both $v_0$ and $s$. It follows that if $v_0$ is not adjacent to $s$, then $u_0$ is the unique vertex of $G$ of maximum degree $\alpha + 3$, whereas if $v_0$ is adjacent to $s$, then both $s$ and $u_0$ are of maximum degree $\alpha + 3$.

Let $v'$ be a vertex of $A_{H_1}(G) - \{u_0, s\}$ that is adjacent to $v_0$, and let $\phi(u')$ be a leaf of $H$ associated with $v'$. Since $v'$ is not adjacent to either $u_0$ or $s$, clearly $d_{\alpha+3}(G - v') = d_{\alpha+3}(G)$. However, since $d(\phi(u_0)) = d(u_0) - 1$ and $d(\phi(x)) = d(x)$, for every other vertex $x$ of $G$ except $q$, it follows that $d_{\alpha+3}(H) = d_{\alpha+3}(G) - 1$. So

$$d_{\alpha+3}(H - \phi(u')) \leq d_{\alpha+3}(H) = d_{\alpha+3}(G) - 1 = d_{\alpha+3}(G - v') - 1,$$

which contradicts the fact that $G - v' \ncong H - \phi(u')$. So $\alpha = 0$, which implies that $d(u_0) \leq 3$.

Since $s$ is $H_1$-active, $d_1(s) = 1$. Thus $d_1(\phi(v_0)) = 0$, so $\phi(u_0)$ cannot be a leaf adjacent to $\phi(v_0)$. So, since $\alpha = 0$ and $G$ is connected, $u_0$ must be adjacent to $s$. Therefore, since $u_0$ and $s$ are the only possible leaf-adjacent vertices of degree 3 or more, $G - s$ cannot contain a leaf-adjacent vertex of degree greater than 3. So, $H - \phi(v_0)$ cannot contain such a vertex and it is easy to see that this implies that $\beta = 0$, thus $d(s) \leq 3$. Since $H - \phi(v_0)$ contains $\gamma$ components isomorphic to $K_2$, it follows that either $\gamma = 0$ and $u_0$ is adjacent to $v_0$, or $\gamma = 1$ and $u_0$ is not adjacent to $v_0$. Since $v_0$ is not a leaf, the first case is the pair in Example 3.3.9. The second case is the pair in Example 3.3.10, since $G$ is connected.

(b) Note that cases (i), (ii), and (iii) correspond to the three cases in Corollary 3.3.12, that is, cases (I), (II) and (III) from part (a). By (a), every vertex of $A_{H_1}(G) - \{u_0\}$ is of degree 2 and, in addition, is adjacent to a leaf by Corollary 3.3.12.

(i) Since $G$ is connected, every vertex of $A_{H_1}(G) - \{u_0\}$ must be adjacent to $u_0$. Noting that $H \cong (G - q) \oplus K_1$, it is easy to see that $G$ and $H$ are members of the family of pairs of graphs described in Example 3.3.5.

(ii) This follows in a similar manner to (i), since the additional leaf must be adjacent to $u_0$.

(iii) Since $v_0$ is not a leaf, every vertex of $A_{H_1}(G) - \{u_0\}$ must be adjacent to either $u_0$ or $v_0$, since $G$ is connected. By pairing the vertices of $A_{H_1}(G) - \{u_0\}$ with their associated leaves of $H_1$, it is easy to see that $G$ and $H$ must be members of the family of pairs of graphs in Example 3.3.7. $\square$

We now consider the case when $G$ contains an $H_2$-active vertex that is not a leaf. For any connected graph $A$, we denote the connected subgraph of $A$ obtained by removing all of its leaves by $S_1^{-1}[A]$; so $S_1^{-1}S_1[B] = B$ for any non-trivial connected graph $B$.

**Lemma 3.3.14** Let $G$ and $H$ be as in Lemma 3.3.11 and suppose that $|E(G)|-|E(H)| \geq 2$. Now let $T = S_1^{-1}[G]$. Then $T$ is a (connected) vertex-transitive graph of order $\frac{n+1}{2}$. In addition, for any vertex $t$ of $T$, $G \cong S_1[T] - t^*$ and $H \cong S_1[T] - t \cong S_1[T-t] \oplus K_1$.

*Proof* By Lemma 3.3.11(b) and (d), there is some $H_2$-active vertex $q$ of $G$ such that $G - q \cong H_1$, and $d(q) = |E(G)| - |E(H)| \geq 2$. So Corollary 3.3.12(b)(ii) must hold and $q$ must be the unique vertex of $G$ that is neither a leaf nor $H_1$-active (so not leaf-adjacent). Thus, since every vertex of $G$ except $q$ is either a leaf or adjacent to a leaf, it is easy to see that this is also true for $H_1$. It also follows that $T$ is the subgraph of order $\frac{n+1}{2}$ induced by $q$ and the vertices of $A_{H_1}(G)$.

Let $u$ be any vertex in $A_{H_1}(G)$ and let $x$ be a vertex of $H_1$ associated with $u$. By Corollary 3.3.12(b)(ii), $d_1(u) = 1$, so $d_1(x) = 0$ by Lemma 3.3.11(a), and therefore $x$ must be a leaf. So $d(u) = d(q) + 1 \geq 3$, by Lemma 3.3.3. Since this holds for every vertex $u$ in $A_{H_1}(G)$, it follows that $T$ is $d(q)$-regular.

If $d(q) = 2$, then $T$ is a cycle, since it is regular and connected. Thus $T$ is vertex-transitive, and it is easy to see that $G$ and $H$ have the required form.

So suppose that $d(q) \geq 3$. Then $d(u) \geq 4$, so neither $G$ nor $H_1$ contain any vertices of degree 2. Therefore, with the exception of $u^*$, a vertex is a leaf in $G$ if and only if it is a leaf in $G-u$. Thus $S_1^{-1}[G-u] \cong (S_1^{-1}[G]-u) \oplus K_1$, since $u^*$ is an isolated vertex in $G-u$. Similarly, with the exception of $x$, a vertex is a leaf in $H_1$ if and only if it is a leaf in $H_1 - x$. So $S_1^{-1}[H_1 - x] \cong S_1^{-1}[H_1]$, and thus $S_1^{-1}[H-x] \cong S_1^{-1}[H_1] \oplus K_1$. Therefore, since $G - u \cong H - x$, it follows that $T - u = S_1^{-1}[G] - u \cong S_1^{-1}[H_1]$. Since a similar approach shows that $T - q = S_1^{-1}[G] - q \cong S_1^{-1}[H_1]$, every card of $T$ is isomorphic.

For any pair of vertices $v_1$ and $v_2$ in $T$, $T - v_1 \cong T - v_2$. Moreover, $v_1$ and $v_2$ are adjacent in $T$ to every vertex of degree $d(q) - 1$ in $T - v_1$ and $T - v_2$, respectively. Therefore, the isomorphism between $T - v_1$ and $T - v_2$ can be extended naturally to an automorphism of $T$ that maps $v_1$ to $v_2$. So, $T$ must be vertex-transitive. If $t$ is $q$, then $G$ and $H$ are clearly of the required form. Moreover, since $T$ is vertex-transitive, this clearly holds for all $t$ in $T$. □

We now show that the converse of Lemma 3.3.14 holds, that is, the construction in Example 3.3.8 attains the bound for *any* connected vertex-transitive graph. We begin with a lemma concerning transitive permutation groups.

**Lemma 3.3.15** Let $A$ be a transitive permutation group on the set $R$, and let $t$ be in $R$. Then there exists a set of $|R|$ distinct permutations $\{\alpha_u \,|\, u \in R\} \subseteq A$, such that for every pair of distinct elements $u$ and $v$ in $R$,

(a) $\alpha_u(u) = t$;

(b) $\alpha_u(t) \neq \alpha_v(t)$.

*Proof* For any $u$ and $v$ in $R$, let $A_{vu} = \{\alpha \in A \,|\, \alpha(u) = v\}$ and let $S(u) = \{\alpha(t) \,|\, \alpha \in A_{tu}\}$. Since $A$ is transitive $A_{tu} \neq \emptyset$, so $S(u) \neq \emptyset$. Thus (a) holds for every permutation $\alpha_u \in A_{tu}$. We shall show that

$$\left|\bigcup_{u \in I} S(u)\right| \geq |I| \qquad \text{for all } I \subseteq R. \tag{3.10}$$

It will then follow by Hall's theorem [17] that there exists $\alpha_u \in A_{tu}$, for each $u \in R$, such that the elements $\alpha_u(t)$ are all distinct. This will complete the proof of the lemma.

$A_{tt}$ is clearly a subgroup of $A$. Let $\alpha$ be any element of $A_{tu}$. Then $A_{tu} \subseteq A_{tt}\alpha$, since $\beta\alpha^{-1}$ is in $A_{tt}$ for all $\beta$ in $A_{tu}$. As $A_{tt}\alpha \subseteq A_{tu}$, it follows that $A_{tu} = A_{tt}\alpha$, so $A_{tu}$ is a right coset of $A_{tt}$. Therefore $|A_{tu}| = |A_{tt}|$ for each $u$ and, by symmetry, $|A_{ut}| = |A_{tt}|$.

Let $I \subseteq R$ and let $S = \bigcup_{u\in I} S(u)$. We note that for each $u$ in $I$, if $\alpha$ is in $A_{tu}$ then $\alpha(t)$ is in $S$, so $\alpha$ is in $\bigcup_{v\in S} A_{vt}$. Therefore $\bigcup_{u\in I} A_{tu} \subseteq \bigcup_{v\in S} A_{vt}$. As the $A_{vt}$ are all mutually disjoint, it follows that

$$|I||A_{tt}| = |\bigcup_{u\in I} A_{tu}| \leq |\bigcup_{v\in S} A_{vt}| \leq |S||A_{tt}|,$$

since $|A_{tt}| = |A_{tu}| = |A_{vt}|$, for all $u$ and $v$. Thus (3.10) holds, which completes the proof. □

**Corollary 3.3.16** For any odd $n$, let $T$ be a connected vertex transitive graph of order $\frac{n+1}{2}$, and let $t$ be a vertex of $T$. Let $G = S_1[T] - t^*$ and $H \cong S_1[T] - t$, as in Lemma 3.3.14. Then $b(G, H) = \frac{n+1}{2}$.

*Proof* Let $u$ be a vertex of $T$ different from $t$, and let $\phi_u$ be an automorphism of $T$ for which $\phi_u(u) = t$. We extend $\phi_u$ to $S_1[T]$ in the natural way, so that $\phi_u(w^*) = \phi_u(w)^*$ for all $w$ in $T$. Clearly $\phi_u$ induces an isomorphism from $S_1[T]-t^*-u$ to $S_1[T]-\phi_u(t)^*-t$. This implies that there is an isomorphism from $G-u$ to $H-x$, for some leaf $x$ of $H$. Therefore $u$ is in $A_{H_1}(G)$. We next show that, moreover, for each $u$ in $V(T)-\{t\}$, we can select a distinct leaf $x$ of $H$ that is associated with $u$.

Since $T$ is vertex transitive, its automorphism group $Aut(T)$ is transitive. So, by Lemma 3.3.15 with $A = Aut(T)$, there is a distinct automorphism $\phi_u$ for each $u$ in $T$, such that $\phi_u(u) = t$, and $\phi_u(t) \neq \phi_v(t)$, and thus $\phi_u(t)^* \neq \phi_v(t)^*$, if $v \neq u$. So, for each of the $\frac{n-1}{2}$ vertices $u$ in $V(T)-\{t\}$, there is a distinct leaf $x$ in $H$ such that $G-u \cong H-x$. In addition, $G-t \cong H-z$, where $z$ is the isolated vertex of $H$. Thus $b(G, H) = \frac{n+1}{2}$. □

The following theorem characterises every pair of graphs that attain the bound of Theorem 3.2.5.

**Theorem 3.3.17** Let $G$ be a connected graph and $H$ a disconnected graph, both of order $n$. Then $b(G, H) = \lfloor \frac{n}{2} \rfloor + 1$ if and only if $G$ and $H$ are either members of one of the families given in Examples 3.3.5, 3.3.6, 3.3.7 and 3.3.8, or are one of the six exceptional pairs of graphs in Examples 3.3.1, 3.3.2, 3.3.9 and 3.3.10.

*Proof* Since the result holds by inspection, for $n = 2$ or $n = 3$, we assume that $n \geq 4$. Corollary 3.3.16 shows that the claim in Example 3.3.8 is true. Thus, every pair in each of these examples clearly attains the bound. We therefore need to show these are the only such pairs.

Now, by Theorem 3.2.5, the bound can only be attained when $H \cong H_1 \oplus H_2$, with $h_1 > h_2$. Suppose first that $h_2 \geq 2$. Then by Lemma 3.3.4, $G$ and $H$ are one of the four pairs in Examples 3.3.1 and 3.3.2. So suppose instead that $h_2 = 1$, and that $G$ and $H$ are not either of the pairs in Examples 3.3.9 and 3.3.10. Then if $|E(G)| - |E(H)| = 1$, $G$ and $H$ must be one of the pairs in Examples 3.3.5, 3.3.6 or 3.3.7, by Lemma 3.3.13(b). On the other hand, if $|E(G)| - |E(H)| \geq 2$, then by Lemma 3.3.14, $G$ and $H$ must be a member of the family given in Example 3.3.8. $\square$

# Chapter 4

# The Number of Common Cards between a Tree and a Connected Non-tree

In Chapter 3, we showed that the maximum number of common cards between a tree and a disconnected graph of order $n$ is $\left\lfloor \frac{n}{2} \right\rfloor + 1$. A natural question to ask is: what is the maximum number of common cards between a tree and a connected graph that is not a tree?

In this chapter, we partially answer this question. We first show that we only need to consider unicyclic graphs, and then show that, when $n \geq 62$, the maximum number of common cards between a tree and a *sunshine graph* is $\left\lfloor \frac{2(n+1)}{5} \right\rfloor$. (A sunshine graph is a unicyclic graph in which every leaf is adjacent to a vertex of the cycle). Moreover, we show that this bound is only attained when $n \equiv 4 \pmod 5$ and the pair of graphs belongs to a unique infinite family. For these values of $n$, this pair has a greater number of common cards than any previously published tree and a non-tree. Furthermore, since the tree in this maximal example is a *caterpillar graph*, this refutes the claim by Francalanza [13] that the number of common cards between a caterpillar graph and a sunshine graph is at most $\frac{n+10}{3}$.

Our work has led us to conjecture that, in fact, this family has the largest number of common cards between a tree and any connected non-tree. This, along with Theorem 3.2.5, would imply that a tree can be recognised from any $\lfloor \frac{n}{2} \rfloor + 2$ of its cards.

## 4.1 Common Cards between Trees and other Connected Graphs

Before we examine sunshine graphs, we first give some results concerning the maximum number of common cards between a tree and any non-tree.

**Lemma 4.1.1** Let $F$ be a graph of order $n$ that contains two or more cycles. Then at least $n-2$ of the cards in $\mathcal{D}(F)$ contain a cycle.

*Proof* Suppose that $u$ is a vertex of $F$ such that $F - u$ is acyclic. Then $u$ must lie on every cycle in $F$. It is easy to see that any two cycles of $F$ cannot have more than two vertices in common. This implies the result. $\square$

**Corollary 4.1.2** Let $F$ be a graph containing two or more cycles and let $T$ be a tree. Then $b(F, T) \leq a_T(F) \leq 2$.

*Proof* Since $T$ is a tree, every card of $T$ is acyclic. By Lemma 4.1.1, there are at most two acyclic cards in $\mathcal{D}(F)$, so the result follows. $\square$

By Corollary 4.1.2, to bound the maximum number of common cards between a tree and any other connected graph that is not a tree, it is sufficient to restrict our analysis to unicyclic graphs (connected graphs that contain precisely one cycle).

We first eliminate the case when the unicyclic graph is a cycle or the tree is a path.

**Lemma 4.1.3** Let $U$ be a unicyclic graph and $T$ be a tree, both of order $n$. Then

(a) $b(C_n, T) \leq 3$;

(b) $b(U, P_n) \leq 3$.

*Proof* (a) It is easy to see that there can be at most three cards in $\mathcal{D}(T)$ that are isomorphic to $P_{n-1}$, and this case only occurs when $n = 4$ and $T \cong S_3^1$. As noted in Section 2.1, every card of $C_n$ is isomorphic to $P_{n-1}$, so it follows that $b(C_n, T) \leq 3$.

(b) By (a), we may assume that $U \ncong C_n$. As noted in Section 2.1, every card of $P_n$ consists of either one or two components, both of which are paths of order less than $n$. By inspection, there can be at most 3 cards in $\mathcal{D}(U)$ that have this component structure, and this case only occurs when $U$ consists of a cycle plus precisely one path adjacent to a single vertex of the cycle. So $b(U, P_n) \leq 3$. $\square$

For the rest of this chapter, $U$ will denote a unicyclic graph and $T$ a tree, both of order $n$. By Lemma 4.1.3, we assume that $U \ncong C_n$ and $T \ncong P_n$, which implies that $n \geq 4$. Now, since $U$ contains a single cycle, for any edge $e$ of the cycle of $U$, $U - e$ is a tree. Thus $|E(U)| = n$ since, as noted in Section 1.6, $|E(T)| = n - 1$.

Let $C$ denote the unique cycle in $U$, where $C$ is of length $c$. Suppose that $v$ is a vertex of $U$ that does not lie on $C$. Clearly $C$ is a subgraph of $U - v$. So, since every card of $T$ is acyclic, $v$ cannot be in $A_T(U)$, thus $a_T(U) \leq c$. Myrvold (see Francalanza [13]) used this observation to prove the following (weak) bound on $b(U, T)$.

**Theorem 4.1.4 (Myrvold)** Let $U$ be a unicyclic graph and let $T$ be a tree. Then $b(U, T) \leq \lceil \frac{n}{2} \rceil + \lceil \frac{n}{2} \rceil^{\frac{1}{2}}$. $\square$

Suppose now that $v$ is an active vertex of $U$ and that $w$ is a vertex of $T$ associated with $v$. By Lemma 3.3.3, $d(v) = d(w) + 1$, since $|E(U)| = |E(T)| + 1$. In particular, since $T - w$ is connected if and only if $w$ is a leaf, $U - v$ is connected if and only if $d(v) = 2$. We concentrate first on those common cards that are connected.

We define $A_T^*(U)$ to be the set of active vertices $v$ of $U$ such that $U - v$ is connected, and denote its cardinality by $a_T^*(U)$. We similarly define $A_U^*(T)$ and $a_U^*(T)$, and let $b^*(U, T)$ denote the maximum number of connected common cards between $U$ and $T$. The above discussion yields the following result.

**Lemma 4.1.5** Let $U$ be a unicyclic graph with unique cycle $C$ and let $T$ be a tree. Let $\delta_i(U)$ denote the number of vertices of degree $i$ of $U$ that lie on $C$. Then $b^*(U, T) \leq \min(\delta_2(U), d_1(T))$.

*Proof* Every vertex in $A_T^*(U)$ is on $C$ and is of degree 2. Every vertex in $A_U^*(T)$ is a leaf. The result then follows since $b^*(U, T) \leq \min(a_T^*(U), a_U^*(T))$. $\square$

We now obtain some simple relations between the number of leaves of $U$ and $T$, in terms of the number of degree 2 vertices adjacent to any pair of associated vertices of $U$ and $T$. We recall the following result from Chapter 2.

**Lemma 2.4.6** Let $G$ be a graph and $v$ a vertex of $G$ where $d(v) = k$. Then

(a) $d_k(G - v) = d_k(G) + d_{k+1}(v) - d_k(v) - 1$;

(b) $d_i(G - v) = d_i(G) + d_{i+1}(v) - d_i(v)$, for $i \neq k$. $\square$

**Corollary 4.1.6** Let $U$ be a unicyclic graph and let $T$ be a tree. Suppose that $v$ is an active vertex of $U$ of degree 2 and that $w$ is a leaf of $T$ associated with $v$. Then

(a) $d_1(T) = d_1(U) + d_2(v) - d_2(w) + 1$;

(b) $d_1(U) \leq d_1(T) \leq d_1(U) + 3$.

*Proof* Since $U - v$ is connected, $d_1(v) = 0$. So, by Lemma 2.4.6(b), $d_1(U - v) = d_1(U) + d_2(v)$. In addition, since $w$ is a leaf, $d_1(T - w) = d_1(T) + d_2(w) - 1$, by part (a) of that lemma. Therefore, $d_1(T) = d_1(U) + d_2(v) - d_2(w) + 1$, since $d_1(T - w) = d_1(U - v)$. Part (b) follows immediately from part (a), since $d_2(w) \leq 1$ and $d_2(v) \leq 2$. □

**Corollary 4.1.7** Let $U$ be a unicyclic graph and let $T$ be a tree. Suppose that $v$ is an active vertex of $U$ of degree 2 and that $w$ is a leaf of $T$ associated with $v$.

(a) If $d_1(T) = d_1(U)$, then $d_2(v) = 0$ and $d_2(w) = 1$.

(b) If $d_1(T) = d_1(U) + 1$, then either $d_2(v) = d_2(w) = 1$, or $d_2(v) = d_2(w) = 0$.

(c) If $d_1(T) = d_1(U) + 2$, then either $d_2(v) = 2$ and $d_2(w) = 1$, or $d_2(v) = 1$ and $d_2(w) = 0$.

(d) If $d_1(T) = d_1(U) + 3$, then $d_2(v) = 2$ and $d_2(w) = 0$.

*Proof* These all follow directly from Corollary 4.1.6, using the fact that $d_2(v) \leq 2$ and $d_2(w) \leq 1$. □

Now, it is possible to show that if $b^*(U, T) = 0$ then $b(U, T) \leq \lfloor \frac{n}{3} \rfloor \leq \left\lfloor \frac{2(n+1)}{5} \right\rfloor$. We shall therefore assume $b^*(U, T) \geq 1$, so precisely one of Corollary 4.1.7(a) to (d) always holds. However, establishing an upper bound for $b(U, T)$ is still quite complicated in cases (b) and (c). So instead, following a suggestion by Myrvold (see Francalanza [13]), in this thesis we only consider the case when every vertex of $U$ that is not on $C$ is a leaf. Such a (unicyclic) graph is called a *sunshine* graph. We shall denote an arbitrary sunshine graph by $S$.

The motivation behind this approach is as follows. In order to maximise $b(U, T)$, we attempt to maximise $b^*(U, T)$. So, by Lemma 4.1.5, we need to ensure that both $\delta_2(U)$ and $d_1(T)$ are large relative to $n$. Since $d_1(U)$ and $d_1(T)$ do not differ by more than three, and $\delta_2(U) \leq c$, it follows that we must maximise $c$ and $d_1(U)$ relative to $n$. Sunshine graphs are precisely those graphs for which $c + d_1(U) = n$.

Let $S$ be a sunshine graph and let $v$ be a vertex of $S$. Then, since every vertex not on $C$ is a leaf, every vertex of $S$ is adjacent to at most two non-leaves. Moreover, it is easy to see that this also holds for every vertex of $S - v$.

Suppose now that $v$ is active and that $w$ is a vertex of $T$ associated with $v$. Then $d_1(v) = d_1(w)$, since neither $S$ nor $T$ contain any isolated vertices. Thus, since $v$ is on $C$ and $d(w) = d(v) - 1$, it follows that $d_1(w) = d_1(v) = d(v) - 2 = d(w) - 1$, and $w$ is therefore adjacent to precisely one non-leaf. So, since every vertex of $T - w$ is adjacent to at most two non-leaves, it is easy to see that every vertex of $T$, except possibly one exceptional vertex $y_0$, is also adjacent to at most two non-leaves. This exceptional vertex would be adjacent to exactly three non-leaves and, moreover, precisely one of the following must occur:

(a) $w$ is a non-leaf that is adjacent to $y_0$;

(b) $y_0$ is adjacent a degree two vertex $x_0$ that is also adjacent to $w$.

There are therefore two possibilities: either $T$ contains such an exceptional vertex $y_0$, or every vertex of $T$ is adjacent to at most two non-leaves. Any tree that is of the latter type is called a *caterpillar graph*, and consists of a path and a collection of leaves adjacent to some of the non-leaves of this path. We shall denote an arbitrary caterpillar graph by $CT$. The above discussion leads to the following result.

**Lemma 4.1.8** Let $S$ be a sunshine graph and let $T$ be a tree. Suppose that $T$ is not a caterpillar graph. Then $b(S, T) \leq 6$.

*Proof* Let $w$ be an active vertex of $T$. Then, every vertex of $T-w$ must be adjacent to at most two non-leaves and, in addition, $d_1(w) = d(w) - 1$. Since $T$ is not a caterpillar, $T$ contains precisely one exceptional vertex $y_0$ that is adjacent to three non-leaves. Moreover, precisely one of the cases (a) and (b) above must occur. It is easy to see that there are at most six such vertices $w$ of $T$ (three of case (a) and three of case (b)). Therefore, $b(S, T) \leq a_S(T) \leq 6$. □

Myrvold and Francalanza [13] presented the family of pairs with $b(S, CT) = \frac{n+7}{3}$ described in Example 4.1.9. Moreover, Francalanza claimed a proof that $b(S, CT) \leq \left\lfloor \frac{n+10}{3} \right\rfloor$ for any such pair. Myrvold went on to conjecture that her family had the maximum value of $b(U, T)$ for any unicyclic graph $U$ and tree $T$. We show in the next section that, when $n \geq 62$, the bound is in fact $b(S, CT) \leq \left\lfloor \frac{2(n+1)}{5} \right\rfloor$. In addition, we show that for these values of $n$, there is a unique family of graph pairs with $b(S, CT) = \frac{2(n+1)}{5}$, when $n \equiv 4 \pmod 5$. Moreover, we conjecture that this pair has the maximum number of common cards between any tree and any unicyclic graph for large $n$. We state this conjecture more formally as Conjecture 4.2.31 at the end of the chapter.

**Example 4.1.9** Let $p$ be an integer, $p \geq 1$. Then for $n = 3p + 5$, the following pair of graphs has $\frac{n+7}{3}$ common cards. Let $S$ be the sunshine graph obtained from the cycle $v_1, v_2, \ldots, v_{2p+4}, v_1$ by adding a single leaf to $v_{2j+1}$, for $1 \leq j \leq p+1$, and let $CT$ be the caterpillar graph obtained from the path $w_1, w_2, \ldots, w_{2p+5}$ by adding a single leaf to $w_{2j+2}$, for $1 \leq j \leq p$. $S - v_{2j+2} \cong CT - w^*_{2j+2}$, for $1 \leq j \leq p$. In addition $S - v_2 \cong CT - w_1$, $S - v_{2p+4} \cong CT - w_{2p+5}$, $S - v_3 \cong CT - w_2$ and $S - v_{2p+3} \cong CT - w_{2p+4}$. So $b(G, H) = p + 4 = \frac{n+7}{3}$. Figure 4.1 shows these graphs for $p = 4$. $\square$

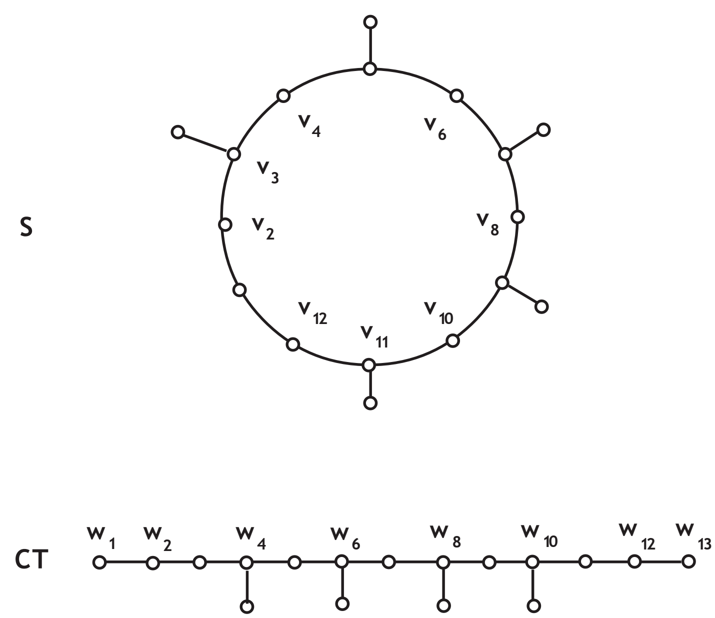


Figure 4.1: The pair of graphs in Example 4.1.9 of order 17 with 8 common cards.

## 4.2 Sunshine and Caterpillar Graphs

For the rest of this chapter, we let $S$ denote a sunshine graph and $CT$ a caterpillar graph, both of order $n$. We further let $C$ be the unique cycle in $S$, and suppose that $C$ is of length $c$. In addition, we suppose that $CT$ consists of a path $P = y_1, y_2, \ldots, y_r$ and a collection of leaves adjacent to some of the non-leaves of $P$ (that is any of the vertices of $P$ except $y_1$ and $y_r$). Clearly, $P$ is a longest path of $CT$ and of length $r - 1$. Note that, we assume following Lemma 4.1.3 that $S \ncong C_n$ and $CT \ncong P_n$.

Note that if $r = 3$, then $CT \cong S^1_{n-1}$, the 1-star of order $n$. By inspection, a 1-star can have at most 2 common cards with any unicyclic graph, except $C_4$. In light of this, we shall therefore assume that $r \geq 4$. So, since $CT$ is not a path, clearly $n \geq 5$. In this case, it is easy to see that for $3 \leq i \leq r - 2$, $d_1(y_i) = d(y_i) - 2$ and, in addition, $d_1(y_2) = d(y_2) - 1$, $d_1(y_{r-1}) = d(y_{r-1}) - 1$ (and $d_1(y_1) = d_1(y_r) = 0$). Thus, $y_2$ and $y_{r-1}$ are the only possible leaf-adjacent vertices of $CT$ of degree 2.

We recall that $a^*_S(CT)$ is the number of active leaves of $CT$, and $b^*(S, CT)$ is the number of connected common cards of $S$ and $CT$.

**Lemma 4.2.1** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph. Then $y_2$ and $y_{r-1}$ are the only possible active cut-vertices of $CT$ with respect to $S$.

*Proof* As noted near the end of Section 4.1, every active vertex of $CT$ is adjacent to precisely one non-leaf. Since $y_2$ and $y_{r-1}$ are the only such cut-vertices of $CT$, the result follows. □

**Corollary 4.2.2** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph. Then $a_S(CT) \leq a^*_S(CT) + 2$, and $b(S, CT) \leq b^*(S, CT) + 2$.

*Proof* This follows immediately from Lemma 4.2.1. □

**Lemma 4.2.3** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph. Suppose that $y_2$ is an active cut-vertex of $CT$ and that $u$ is a cut-vertex of $S$ associated with $y_2$. Then $d_1(CT) \leq d_1(S) + 2$. Moreover, equality only holds if $d_2(y_2) = 0$ and $d_2(u) = 2$.

*Proof* Since $y_2$ and $u$ are cut-vertices, $d(y_2) \geq 2$ and $d(u) \geq 3$. Thus, by Lemma 2.4.6(b), $d_1(CT - y_2) = d_1(CT) + d_2(y_2) - d_1(y_2)$ and $d_1(S - u) = d_1(S) + d_2(u) - d_1(u)$. So, since $d_1(y_2) = d_1(u)$ and $CT - y_2 \cong S - u$, it follows that $d_1(CT) = d_1(S) + d_2(u) - d_2(y_2)$. The result then follows since $d_2(y_2) \leq 1$ and $d_2(u) \leq 2$. □

It follows from Lemma 4.2.3, that if $U$ has an active cut-vertex, then $d_1(CT) \leq d_1(S) + 2$. The above results yield the following corollary.

**Corollary 4.2.4** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph.

(a) If $d_1(CT) > d_1(S) + 3$, then $b(S, CT) = 0$.

(b) If $d_1(CT) = d_1(S) + 3$, then $b(S, CT) = b^*(S, CT)$.

(c) If $d_1(CT) = d_1(S)$, then $b(S, CT) \leq 4$.

(d) If $d_1(CT) < d_1(S)$ then $b(S, CT) \leq 2$.

*Proof* If $S$ contains an active vertex of degree 2, then, by Corollary 4.1.6(b), $d_1(S) \leq d_1(CT) \leq d_1(S) + 3$. In addition, if $S$ contains an active cut-vertex, then, by Lemma 4.2.3, $d_1(S) \leq d_1(CT) + 2$. Thus, (a) and (b) follow immediately. Now, by Corollary 4.2.2, $b(S, CT) \leq b^*(S, CT) + 2$. This implies (d). So finally, suppose that $d_1(S) = d_1(CT)$. Then, by Corollary 4.1.7(a), $d_2(w) = 1$ for any active leaf $w$ of $CT$. The only possible leaf-adjacent vertices of $CT$ of degree 2 are $y_2$ and $y_{r-1}$; so $a^*_S(CT) \leq 2$. Therefore, $b(S, CT) \leq b^*(S, CT) + 2 \leq 4$. □

For the rest of this chapter, we shall assume, in light of Corollary 4.2.2 and Corollary 4.2.4 that $d_1(S) + 1 \leq d_1(CT) \leq d_1(S) + 3$ and, in addition, that $CT$ contains some active leaf.

We recall from Section 1.3, that a 2-path of length $s \geq 1$ in a graph is a path $v_1, v_2, \ldots, v_{s+1}$ in which $d(v_i) = 2$ for $2 \leq i \leq s$, $d(v_1) \geq 3$ and $d(v_{s+1}) \neq 2$. If $d(v_{s+1}) \geq 3$, then this 2-path is called a cut 2-path and, if $d(v_{s+1}) = 1$, then it is called a leaf 2-path. We denote the number of cut 2-paths of lengths $i$ in a graph $F$ by $c_i(F)$ and the number of leaf 2-paths by $l_i(F)$. Note that, a leaf 2-path of length 1 is simply an edge joining a leaf to a vertex of degree 3 or more.

Suppose that $S$ contains $\gamma \geq 1$ cut-vertices. Then the unique cycle $C$ in $S$ consists of $\gamma$ adjacent cut 2-paths of $S$. Moreover, it is easy to see that every vertex of degree 2 of $S$ is an interior vertex of a unique cut 2-path in $S$.

We now partition the vertices of degree 2 of $S$, and so on $C$, according to how many of their neighbours are of degree 2. Let $\mathcal{A}_i(S) = \{v \in S \mid d(v) = 2 \text{ and } d_2(v) = i\}$. The following equation relates the size of the $\mathcal{A}_i$ to $\gamma$ and $d_1(S)$:

$$n = |\mathcal{A}_0(S)| + |\mathcal{A}_1(S)| + |\mathcal{A}_2(S)| + \gamma + d_1(S). \tag{4.1}$$

**Lemma 4.2.5** Let $S$ be a sunshine graph with $\gamma$ cut-vertices. Then $\gamma \geq |\mathcal{A}_0(S)| + \frac{1}{2}|\mathcal{A}_1(S)|$.

*Proof* Let $Z$ be a cut 2-path of length $k$ on $C$. If $k = 1$, then $Z$ has no interior vertices; if $k = 2$, then the unique interior vertex of $Z$ must be in $\mathcal{A}_0(S)$; finally if $k \geq 3$, then there are precisely two interior vertices of $Z$ in $\mathcal{A}_1(S)$ and $k - 3$ interior vertices in $\mathcal{A}_2(S)$. Since $S$ contains precisely $\gamma$ cut 2-paths, all of which are on $C$, the result follows. □

We choose some maximum matching of $B(S, CT)$, the choice of which is irrelevant. Then we define $|\overline{\mathcal{A}(S)}|$ to be the number of vertices of degree 2 of $S$ that are not incident to an edge of this matching, so

$$|\overline{\mathcal{A}(S)}| = |\mathcal{A}_0(S)| + |\mathcal{A}_1(S)| + |\mathcal{A}_2(S)| - b^*(S, CT). \tag{4.2}$$

Clearly, $|\overline{\mathcal{A}(S)}|$ is at least as large as the number of non-active vertices on $C$ of degree 2. In addition, since $b^*(S,\, CT) \leq d_1(CT)$, by Lemma 4.1.5, we can rearrange (4.1) to

$$
\begin{aligned}
n &= b^*(S,\, CT) + \gamma + d_1(S) + |\overline{\mathcal{A}(S)}| && (4.3)\\
&= b^*(S,\, CT) + d_1(CT) + \gamma + (d_1(S) - d_1(CT)) + |\overline{\mathcal{A}(S)}| && (4.4)\\
&\geq 2b^*(S,\, CT) + \gamma + (d_1(S) - d_1(CT)) + |\overline{\mathcal{A}(S)}|. && (4.5)
\end{aligned}
$$

We use these equations to bound $b^*(S,\, CT)$, and thus $b(S,\, CT)$.

We first consider the case when $d_1(CT) = d_1(S) + 1$.

**Lemma 4.2.6** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph, both of order $n \geq 5$. Suppose that $d_1(CT) = d_1(S) + 1$. Then $b(S,\, CT) \leq \left\lfloor \frac{n+8}{3} \right\rfloor$.

*Proof* By Corollary 4.1.7(b), $S$ has no active vertices in $\mathcal{A}_2(S)$, so $a^*_{CT}(S) \leq |\mathcal{A}_0(S)| + |\mathcal{A}_1(S)|$. It also follows from that corollary that any active vertex in $\mathcal{A}_1(S)$ is associated with some leaf $w$, for which $d_2(w) = 1$. The only two possible such leaves in $CT$ are $y_1$ and $y_r$, so $b^*(S,\, CT) \leq \mathcal{A}_0(S) + \min(2,\, |\mathcal{A}_1(S)|)$. Now, by Lemma 4.2.5, $\gamma \geq |\mathcal{A}_0(S)| + \frac{|\mathcal{A}_1(S)|}{2}$. Thus,

$$b^*(S,\, CT) \leq \gamma - \frac{|\mathcal{A}_1(S)|}{2} + \min(2,\, |\mathcal{A}_1(S)|) \leq \gamma + 1,$$

with equality only if $|\mathcal{A}_1(S)| = 2$. Therefore, by (4.5) and Corollary 4.2.2,

$$n \geq 2b^*(S,\, CT) + \gamma - 1 \geq 3b^*(S,\, CT) - 2 \geq 3b(S,\, CT) - 8,$$

and the result follows. $\square$

Note that, Example 4.1.9 fits the criteria of Lemma 4.2.6 and almost attains this bound. We believe that, for large $n$, this example has the maximum number of common cards between a caterpillar graph $C$ and a sunshine graph $S$ with $d_1(CT) = d_1(S) + 1$.

Before we consider the two cases $d_1(CT) = d_1(S) + 2$ or $d_1(CT) = d_1(S) + 3$, we first prove some relationships between the number of cut 2-paths and leaf 2-paths of $S$ and $S - v$, and $CT$ and $CT - w$.

**Lemma 4.2.7** Let $S$ be a sunshine graph and let $v$ be a vertex of $S$ of degree 2. Suppose that $d_3(v) = 0$ and, in addition, that $v$ lies on a cut 2-path of length $k \geq 3$, at a distance of $x$ from one of the end-vertices of this cut 2-path. Then

(a) $c_k(S - v) = c_k(S) - 1$, and $c_i(S - v) = c_i(S)$ for all $i \neq k$,

(b) $l_i(S - v) = l_i(S)$ for all $i \neq x - 1,\ k - x - 1$. In addition,

- (i) if $d_2(v) = 2$ and $x = \frac{k}{2}$, then $l_{x-1}(S - v) = l_{x-1}(S) + 2$;
- (ii) if $d_2(v) = 2$ and $x \neq \frac{k}{2}$, then $l_{x-1}(S - v) = l_{x-1}(S) + 1$ and $l_{k-x-1}(S - v) = l_{k-x-1}(S) + 1$;
- (iii) if $d_2(v) = 1$ then $l_{k-2}(S - v) = l_{k-2}(S) + 1$.

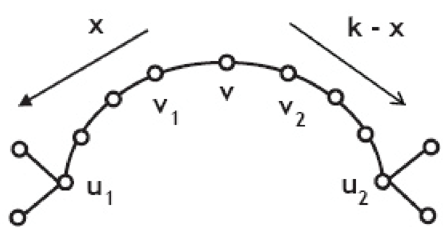


Figure 4.2: The breaking of a cut 2-path on $S$.

*Proof* Note that $d_2(v) = 1$ or $d_2(v) = 2$, since $k \geq 3$.

Now, the removal of $v$ from $S$ destroys the cut 2-path in $S$ on which $v$ lies. However, since $d_3(v) = 0$, its removal does not affect any other cut 2-path in $S$. Thus (a) holds.

Since $d_3(v) = 0$, the removal of $v$ from $S$ does not destroy any leaf 2-paths of $S$. Let $u_1$ and $u_2$ be the end-vertices of the cut 2-path that contains $v$, where $u_1$ is a distance of $x$ from $v$. Suppose that $d_2(v) = 2$, and let $v_1$ and $v_2$ be the two vertices adjacent to $v$, as in Figure 4.2. Then the removal of $v$ creates two new leaf 2-paths, one from $u_1$ to $v_1$ of length $x - 1$, and another from $u_2$ to $v_2$ of length $k - x - 1$. The removal of $v$ does not create any other leaf 2-paths, since $d_3(v) = 0$, so either (b)(i) or (b)(ii) holds in this case. Suppose instead that $d_2(v) = 1$. Then, if $v_2$ is the vertex of degree 2 adjacent to $v$, the removal of $v$ creates a leaf 2-path of length $k - 2$ from $u_2$ to $v_2$. Since $d_3(v) = 0$, the removal of $v$ does not create any other leaf 2-paths, so (b)(iii) holds. □

We note that this lemma shows that if an active vertex $v$ of $S$ of degree 2 is not adjacent to any vertex of degree 3, then $\sum_{i \geq 1} c_i(S - v) = \sum_{i \geq 1} c_i(S) - 1$.

**Lemma 4.2.8** Let $CT$ be a caterpillar graph, and let $w$ be a leaf of $CT$. Suppose that $w$ is adjacent to a $y_s$, where $d(y_s) = k$, for some $k \neq 3$. Then
$c_i(CT - w) = c_i(CT)$, for all $i$. Furthermore, if $k \geq 4$, then $l_j(CT - w) = l_j(CT)$ for all $j \geq 2$.

*Proof* Suppose first that $d(y_s) = 2$. Then either $s = 2$ or $s = r - 1$ and, moreover, $y_s$ is an interior vertex of a leaf 2-path in $CT$. So the removal of $w$ from $CT$ does not create or destroy any cut 2-paths. Suppose, on the other hand, that $d(y_s) \geq 4$. Then, with the exception of precisely two 2-paths, every 2-path in $CT$ of which $y_s$ is an end-vertex is a leaf 2-path of length 1. Since $y_s$ is of degree at least 3 in $CT - w$, clearly the removal of $w$ from $CT$ does not affect these two 2-paths. Thus, since the removal of $w$ can clearly only affect a 2-path in $CT$ in which $y_s$ is an end-vertex, it follows that $CT$ and $CT - w$ must have the same number of cut 2-paths of every length and leaf 2-paths of length 2 or more. □

We now prove three general results that will be useful in our analysis of the final two cases. The first two are easy corollaries of Lemma 2.4.6.

**Corollary 4.2.9** Let $S$ be a sunshine graph and let $v$ be a vertex of $S$ of degree 2 that is adjacent to one vertex of degree 2 and another of degree $p \geq 2$.

(a) If $p = 2$, then $d_1(S-v) = d_1(S)+2$, $d_2(S-v) = d_2(S)-3$, and $d_i(S-v) = d_i(S)$ for all $i \geq 3$.

(b) If $p = 3$, then $d_1(S-v) = d_1(S)+1$, $d_2(S-v) = d_2(S)-1$, $d_3(S-v) = d_3(S)-1$, and $d_i(S-v) = d_i(S)$ for all $i \geq 4$.

(c) If $p \geq 4$, then $d_1(S-v) = d_1(S)+1$, $d_2(S-v) = d_2(S)-2$, $d_{p-1}(S-v) = d_{p-1}(S)+1$, $d_p(S-v) = d_p(S)-1$, and $d_i(S-v) = d_i(S)$, for all $i \neq 1,\, 2,\, p-1,\, p$.

*Proof* Since $d(v) = 2$, these all follow directly from Lemma 2.4.6. □

**Corollary 4.2.10** Let $CT$ be a caterpillar graph and let $w$ be leaf of $CT$ that is adjacent to a vertex of degree $q$.

(a) If $q = 2$, then $d_2(CT-w) = d_2(CT)-1$ and $d_j(CT-w) = d_j(CT)$ for all $j \neq 2$.

(b) If $q = 3$, then $d_1(CT-w) = d_1(CT)-1$, $d_2(CT-w) = d_2(CT)+1$, $d_3(CT-w) = d_3(CT)-1$, and $d_j(CT-w) = d_j(CT)$ for all $j \geq 4$.

(c) If $q \geq 4$, then $d_1(CT-w) = d_1(CT)-1$, $d_{q-1}(CT-w) = d_{q-1}(CT)+1$, $d_q(CT-w) = d_q(CT)-1$, and $d_j(CT-w) = d_j(CT)$ for all $j \neq 1,\, q-1,\, q$.

*Proof* Since $d(w) = 1$, these all follow directly from Lemma 2.4.6. □

For the rest of this chapter only, we now denote the number of leaves of $CT$ that are adjacent to a vertex of degree 2 by $\lambda_2$, the number that are adjacent to a vertex of degree 3 by $\lambda_3$, and the number that are adjacent to a vertex of degree 4 or more by $\lambda^*$; so $d_1(CT) = \lambda_2 + \lambda_3 + \lambda^*$. We recall that $y_1$ and $y_r$ are leaves of $CT$ adjacent to $y_2$ and $y_{r-1}$, respectively.

**Lemma 4.2.11** Let $CT$ be a caterpillar graph. Suppose that $d_2(y_1) + d_2(y_r) = s$ and $d_3(y_1) + d_3(y_r) = t$; so $s + t \leq 2$. Then

(a) $\lambda_2 = s$;

(b) $\lambda_3 = d_3(CT) + t$;

(c) $\lambda^* = \sum_{i \geq 4} (i-2)d_i(CT) + (2 - s - t)$.

*Proof* We note first that $y_2$ and $y_{r-1}$ are the only possible leaf-adjacent vertices of degree 2 in $CT$, so $\lambda_2 = s$. Now, for $3 \leq i \leq r-2$, every non-leaf $y_i$ of $CT$ is adjacent to $d(y_i) - 2$ leaves. The vertices $y_2$ and $y_{r-1}$ are adjacent to $d(y_2) - 1$ and $d(y_{r-1}) - 1$ leaves, respectively. It is therefore easy to see that $\lambda_3 = d_3(CT) + t$ and, moreover, $d_1(CT) = \sum_{i \geq 3} (i-2)d_i(CT) + 2$. It then follows immediately that $\lambda^* = \sum_{i \geq 4} (i-2)d_i(CT) + (2 - s - t)$. □

We now begin our analysis of the final two cases. We first bound $b(S, CT)$ when every active leaf of $CT$ is adjacent to a vertex of degree 3.

**Lemma 4.2.12** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph, both of order $n \geq 5$, where $d_1(CT) = d_1(S) + 2$ or $d_1(CT) = d_1(S) + 3$. Suppose that $d_3(w) = 1$ for every active leaf $w$ of $CT$. If $d_1(CT) = d_1(S) + 2$, then $b^*(S, CT) \leq \left\lfloor \frac{n+5}{3} \right\rfloor$, otherwise $b^*(S, CT) \leq \left\lfloor \frac{n+6}{3} \right\rfloor$.

*Proof* Since every active leaf of $CT$ is adjacent to a degree 3 vertex, $a^*_S(CT) \leq \lambda_3 \leq d_3(CT) + 2$, by Lemma 4.2.11(b). Let $v$ be an active vertex of $S$ of degree 2 and let $w$ be a leaf of $CT$ associated with $v$. By Corollary 4.2.10(b), $d_3(CT - w) = d_3(CT) - 1$ since $d_3(w) = 1$. Thus, since $CT - w \cong S - v$,

$$a^*_S(CT) \leq d_3(CT) + 2 \leq d_3(CT - w) + 3 = d_3(S - v) + 3. \tag{4.6}$$

By Corollary 4.1.7(c) and (d), $d_2(v) \geq 1$. Suppose that $d_4(v) \neq 1$. Then, by Corollary 4.2.9, $d_3(S-v) \leq d_3(S) \leq \gamma$. Suppose, on the other hand, that $d_4(v) = 1$. Then by the same corollary, $d_3(S-v) = d_3(S)+1 \leq \gamma$, since $d_4(S) \geq 1$. Thus, in either case, $b^*(S, T) \leq a^*_S(CT) \leq \gamma + 3$ by (4.6). Therefore, by (4.5),

$$n \geq 3b^*(S, CT) - 3 + (d_1(S) - d_1(CT)) + |\overline{\mathcal{A}(S)}|,$$

which implies the result. $\square$

We now consider the case when $d_1(CT) = d_1(S) + 3$ and every active leaf of $CT$ is adjacent to a vertex of degree 4 or more.

**Lemma 4.2.13** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph, both of order $n \geq 5$. Suppose that $d_1(CT) = d_1(S) + 3$. Suppose that every active leaf of $CT$ is adjacent to a vertex of degree 4 or more. Then $b^*(S, CT) \leq \left\lfloor \frac{2(n+3)}{7} \right\rfloor$.

*Proof* Let $w$ be any active leaf of $CT$. Since $d_3(w) = d_2(w) = 0$, it follows from Lemma 4.2.8, that $CT-w$ and $CT$ contain precisely the same number of cut 2-paths and leaf 2-paths of every length greater than 1. Since this holds for each active leaf $w$ of $CT$, it must hold for each active vertex $v$ of $S$ of degree 2. Therefore, $S-v$ contains precisely the same number of cut 2-paths and leaf 2-paths of every length greater than 1, for every such $v$.

Now, let $v$ be such an active vertex of $S$ of degree 2. By Corollary 4.1.7(d), $d_2(v) = 2$, that is, $v$ is in $\mathcal{A}_2(S)$, and so $v$ is an interior vertex of a cut 2-path of length $k \geq 4$, a distance of at least 2 from each end-vertex of this cut 2-path. Thus, by Lemma 4.2.7(a), $c_i(S-v) = c_i(S)$, unless $i = k$, in which case $c_k(S-v) = c_k(S) - 1$. It therefore follows from the above that every such active $v$ of $S$ must be an interior vertex of a cut 2-path of $S$ of length $k$.

Suppose that $k = 4$ or $k = 5$. Then, for each cut 2-path of length $k$, there is at most one or two active vertices on this cut 2-path. Suppose instead that $k \geq 6$, and that $v$ is a distance of $x$ from one of the end-vertices of this cut 2-path. Then, by Lemma 4.2.7(b), $l_i(S - v) = l_i(S)$, unless $i = x - 1$ or $i = k - x - 1$. So, since at least one of $x - 1$ or $k - x - 1$ must be greater than 1, it follows that every active vertex must be a distance of either $i = x$ or $i = k - x$ from the end-vertex of the cut 2-path on which it lies. There are clearly only two possible such vertices on any cut 2-path of length $k$.

Now, every active vertex of $S$ of degree 2 is an interior vertex on a cut 2-path of length $k \geq 4$. Moreover, by the above argument, any such 2-path can contain at most two active vertices; therefore, $a_{CT}(S) \leq 2\gamma$. In addition, since $k \geq 4$, any such 2-path must contain precisely two vertices in $\mathcal{A}_1$; so $a_{CT}(S) \leq \mathcal{A}_1$. By Corollary 4.1.7(d), no vertex in $\mathcal{A}_1(S)$ is active, thus $|\overline{\mathcal{A}(S)}| \geq |\mathcal{A}_1| \geq a_{CT}(S)$. Therefore, by (4.5),

$$n \geq 2b^*(S,T) + \gamma - 3 + |\overline{\mathcal{A}(S)}| \geq 2b^*(S,T) + \tfrac{a_{CT}(S)}{2} + a_{CT}(S) - 3 \geq \tfrac{7b^*(S,CT)}{2} - 3,$$

so $b^*(S,\, CT) \leq \left\lfloor \frac{2(n+3)}{7} \right\rfloor$. □

Using the above two results, we now complete the case when $d_1(CT) = d_1(S) + 3$.

**Lemma 4.2.14** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph. Suppose that $d_1(CT) = d_1(S) + 3$. Then every vertex that is adjacent to an active leaf of $CT$ is of the same degree $k$, where $k \geq 3$.

*Proof* By Corollary 4.2.4(b), $S$ contains no active cut-vertices. Let $v$ be an active vertex of $S$. Then, by Corollary 4.1.7(d), $d_2(v) = 2$. Therefore, since $d(v) = 2$, it follows that the degree sequence of $S - v$ must be identical for every active vertex $v$ of $S$. Thus the degree sequence of $CT - w$ must be identical for any active leaf $w$ of $CT$. The result then follows immediately from Corollary 4.2.10, noting that $d_2(w) = 0$, by Corollary 4.1.7(d). □

**Corollary 4.2.15** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph, both of order $n \geq 5$. Suppose that $d_1(CT) = d_1(S) + 3$. Then $b(S, CT) \leq \lfloor \frac{n+6}{3} \rfloor$.

*Proof* By Corollary 4.2.4(b), $b^*(S, CT) = b(S, CT)$. In addition, by Lemma 4.2.14, every active vertex of $CT$ is adjacent to a vertex of the same degree $k \geq 3$. So the bound of either Lemma 4.2.12 or 4.2.13 must hold. Therefore, $b(S, CT) \leq \lfloor \frac{n+6}{3} \rfloor$, since $\lfloor \frac{n+6}{3} \rfloor \geq \left\lfloor \frac{2(n+3)}{7} \right\rfloor$. □

We now turn our attention to the case when $d_1(S) = d_1(CT)+2$. In light of Corollary 4.2.2 and Lemma 4.2.12, we assume from now on that $CT$ contains an active leaf adjacent to a vertex of degree 4 or more.

**Lemma 4.2.16** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph with $d_1(CT) = d_1(S)+2$. Then, $d_3(v)-d_3(w) = d_2(CT)-d_2(S)+2$, for all active vertices $v$ of $S$ of degree 2 and all leaves $w$ of $CT$ associated with $v$.

*Proof* Let $v$ be an active vertex of $S$ of degree 2 and let $w$ be a leaf of $CT$ associated with $v$. Then, by parts (a) and (b) of Lemma 2.4.6,
$d_2(S - v) = d_2(S) + d_3(v) - d_2(v) - 1$ and $d_2(CT - w) = d_2(CT) + d_3(w) - d_2(w)$. Now, by Corollary 4.1.7(c), $d_2(v) = d_2(w) + 1$. So, since $S - v \cong CT - w$, it follows that

$$d_3(v) - d_3(w) = d_2(CT) + d_2(v) + 1 - d_2(S) - d_2(w) = d_2(CT) - d_2(S) + 2.$$

□

We note that, each cut-vertex of $S$ is adjacent to at most two vertices in $\mathcal{A}_1(S)$. This observation will be useful in the following few lemmas. We recall that we denote the number of leaves of $CT$ that are adjacent to a degree 3 vertex by $\lambda_3$.

**Lemma 4.2.17** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph, both of order $n \geq 5$, where $d_1(CT) = d_1(S)+2$. Suppose that $d_3(v)+d_3(w) = 1$, for some active vertex $v$ of $S$ of degree 2 and some leaf $w$ of $CT$ associated with $v$. Then $b^*(S, CT) \leq \lfloor \frac{n+4}{3} \rfloor$.

*Proof* By Lemma 4.2.16, $d_3(v) - d_3(w)$ is a constant for all active vertices $v$ of degree 2 and all leaves $w$ of $CT$ associated with $v$. Thus, since $d_3(v) + d_3(w) = 1$ for some such pair of active vertices, it follows that one of the following must occur: either (i) $d_3(v) = 1$ and $d_3(w) = 0$ for *every* such pair of active vertices, or (ii) $d_3(v) = 0$ and $d_3(w) = 1$ for *every* such pair of active vertices. We note that $d_3(S-v) = d_3(CT-w)$, since $S - v \cong CT - w$.

(i) Suppose first that $d_3(v) = 1$ and $d_3(w) = 0$. Then, by Corollary 4.2.9(b) and Corollary 4.2.10(c),

$$d_3(S) - 1 = d_3(S - v) = d_3(CT - w) \leq d_3(CT) + 1,$$

since $S - v \cong CT - w$. So $d_3(S) \leq d_3(CT) + 2$. Now, since no active leaf of $CT$ is adjacent to a degree 3 vertex, $a^*_S(CT) \leq d_1(CT) - \lambda_3 \leq d_1(CT) - d_3(CT)$, by Lemma 4.2.11(b). Thus, since $d_1(S) = d_1(CT) - 2$, it follows that
$a^*_S(CT) \leq d_1(S) - d_3(S) + 4$. Therefore, since every vertex of $S$ of degree 3 is adjacent to at most two vertices of degree 2, $a^*_S(CT) \leq d_1(S) - \frac{1}{2}a^*_{CT}(S) + 4$. So since $d_3(S) \leq \gamma$, it follows from (4.3) that

$$n \geq b^*(S,\, CT) + \tfrac{b^*(S, CT)}{2} + \tfrac{3a^*_S(CT)}{2} - 4 \geq 3b^*(S,\, CT) - 4,$$

so $b^*(S,\, CT) \leq \left\lfloor \frac{n+4}{3} \right\rfloor$.

(ii) Suppose instead that $d_3(w) = 1$ and $d_3(v) = 0$. Then, by Corollary 4.2.9(c) and Corollary 4.2.10(b),

$$d_3(CT) - 1 = d_3(CT - w) = d_3(S - v) \leq d_3(S) + 1,$$

since $S - v \cong CT - w$. So $d_3(CT) \leq d_3(S) + 2$. Now, since every active leaf of $CT$ is adjacent to a degree 3 vertex, $a^*_S(CT) \leq \lambda_3 \leq d_3(CT) + 2$, by Lemma 4.2.11(b). In addition, since every vertex of $S$ of degree 4 or more is adjacent to at most two vertices of degree 2, $a^*_{CT}(S) \leq 2(\gamma - d_3(S))$. Therefore,

$$3b^*(S,\, CT) \leq 2a^*_S(CT) + a^*_{CT}(S) \leq 2(\gamma - d_3(S)) + 2(d_3(S) + 4) \leq 2\gamma + 8.$$

So by (4.5),

$$n \geq 2b^*(S,\, CT) + (\tfrac{3b^*(S, CT)}{2} - 4) - 2 \geq \tfrac{7b(S, CT)}{2} - 6.$$

The result then follows since $\left\lfloor \frac{2n+6}{7} \right\rfloor \leq \left\lfloor \frac{n+4}{3} \right\rfloor$. $\square$

**Corollary 4.2.18** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph, both of order $n \geq 5$, where $d_1(CT) = d_1(S) + 2$. Suppose that $b^*(S, CT) > \left\lfloor \frac{n+4}{3} \right\rfloor$. Then $d_3(v) = d_3(w)$, for all active vertices $v$ of degree 2 of $S$ and all leaves of $CT$ associated with $v$.

*Proof* Let $v$ be an active vertex of $S$ of degree 2 and let $w$ be a leaf of $CT$ associated with $v$. Suppose that $d_3(v) \neq d_3(w)$. Then, by Corollary 4.1.7(c), $d_3(v) \leq 1$, so $d_3(v) + d_3(w) = 1$. However, in this case, by Lemma 4.2.17, $b^*(S, CT) \leq \left\lfloor \frac{n+4}{3} \right\rfloor$, which is a contradiction. $\square$

In light of Corollary 4.2.18, we now assume that $d_3(v) = d_3(w)$, for any active vertex $v$ of $S$ of degree 2 and any leaf $w$ of $CT$ associated with $v$. We now prove the following two important results.

**Lemma 4.2.19** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph with $d_1(CT) = d_1(S) + 2$. Suppose that $u$ is an active cut-vertex of $S$ that is associated with the vertex $y_2$ of $CT$. Then $CT$ contains at least one more cut 2-path than $S$ of length 1.

*Proof* By Lemma 4.2.3, $d_2(y_2) = 0$. So $y_2$ is an end-vertex of precisely one cut 2-path, and this cut 2-path is of length 1. Thus, $CT - y_2$ must contain at least one less cut 2-path of length 1 than $CT$. By the same lemma, $d_2(u) = 0$, so $u$ is the end-vertex of precisely two cut 2-paths, neither of which is of length 1. So, $S - u$ must contain the same number of cut 2-paths of length 1 as $S$. Therefore, since $S - u \cong CT - y_2$, it follows that $CT$ must have at least one more cut 2-path of length 1 than $S$. $\square$

**Lemma 4.2.20** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph with $d_1(CT) = d_1(S) + 2$. Suppose that $v$ is an active vertex of $S$ of degree 2 and that $w$ is an active leaf of $CT$ associated with $v$. Suppose further that $v$ and $w$ are both adjacent to a vertex of degree 4 or more. Then $S$ contains no active cut-vertices.

*Proof* Suppose that $S$ contains an active cut-vertex. Then, by Lemma 4.2.19, $CT$ contains at least one more cut 2-path of length 1 than $S$.

Now, by Corollary 4.1.7(c), $v$ is an interior vertex of cut 2-path of length $k \geq 3$. So, since $v$ is not adjacent to a vertex of degree 3, by Lemma 4.2.7(a), $S - v$ and $S$ contain the same number of cut 2-paths of length 1. Similarly, since $w$ is not adjacent to a vertex of degree 3, then, by Lemma 4.2.8, $CT$ and $CT - w$ contain the same number of cut 2-paths of length 1, that is at least one more than $S$. This is impossible since $S - v \cong CT - w$. Therefore, $S$ does not contain an active cut-vertex.

□

We now show that if two vertices of an active pair are both adjacent to a vertex of the same degree, then the degree sequences of graphs are identical except for their leaves and degree 2 vertices.

**Lemma 4.2.21** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph with $d_1(CT) = d_1(S) + 2$. Suppose that $v$ is an active vertex of $S$ of degree 2 and that $w$ is an active leaf of $CT$ associated with $v$. Then $v$ and $w$ are adjacent to a vertex of the same degree if and only if $d_2(S) = d_2(CT) + 2$ and $d_i(S) = d_i(CT)$, for all $i \geq 3$.

*Proof* By Corollary 4.1.7(c), either $d_2(v) = 2$ and $d_2(w) = 1$, or $d_2(v) = 1$ and $d_q(w) = 1$, for some $q \geq 3$. We first consider the case when $d_2(v) = 2$ and $d_2(w) = 1$. By Corollary 4.2.9(a), $d_2(S - v) = d_2(S) - 3$ and $d_i(S - v) = d_i(S)$ for all $i \geq 3$. Similarly, by Corollary 4.2.10(a), $d_2(CT - w) = d_2(CT) - 1$ and $d_j(S - v) = d_j(S)$ for all $j \geq 3$. So, $d_2(S) = d_2(CT) + 2$, and $d_j(S) = d_j(CT)$, for all $j \geq 3$. A similar proof using parts (b) and (c) of the same corollaries shows the result holds for the other case. This shows sufficiency.

The necessity is immediate in the first case. In the second case, necessity follows by parts (b) and (c) of Corollary 4.2.9 and Corollary 4.2.10, noting that $d_q(CT - w) = d_q(CT) - 1$, and that $d_i(S - v) = d_i(CT - w)$ for all $i$. □

We can combine the above lemma with Lemma 4.2.11 to give a useful bound on $b^*(S, CT)$. We recall that $\lambda^*$ denotes the number of leaves of $CT$ adjacent to a vertex of degree 4 or more.

**Lemma 4.2.22** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph with $d_1(CT) = d_1(S) + 2$, $d_2(S) = d_2(CT) + 2$, and $d_i(S) = d_i(CT)$ for all $i \geq 3$. Now let $\mathcal{B}_j$ be the set of vertices of $S$ of degree 4 or more that are adjacent to $j \leq 2$ active vertices of degree 2. Suppose, as in Lemma 4.2.11, that $d_2(y_1) + d_2(y_r) = s$ and $d_3(y_1) + d_3(y_r) = t$. Then

$$b^*(S, CT) \leq d_3(S) + |\mathcal{B}_1| + 2|\mathcal{B}_2| + t + s \leq 2\gamma - d_3(S) + t + s. \tag{4.7}$$

*Proof* Let $v$ be an active vertex of $S$ of degree 2 and let $w$ be a leaf of $CT$ associated with $v$. By Lemma 4.2.21, $v$ and $w$ are adjacent to a vertex of the same degree. So, since $d_1(CT) = \lambda_2 + \lambda_3 + \lambda^*$, it follows that

$$b^*(S, CT) \leq \lambda_2 + \lambda_3 + \min(|\mathcal{B}_1| + 2|\mathcal{B}_2|, \lambda^*). \tag{4.8}$$

Now, by parts (a) and (b) of Lemma 4.2.11, $\lambda_2 = s$ and $\lambda_3 = d_3(CT) + t$. Therefore, since $d_3(S) = d_3(CT)$, it follows from (4.8) that $b^*(S, CT) \leq d_3(S) + |\mathcal{B}_1| + 2|\mathcal{B}_2| + t + s$. Moreover, since $\gamma \geq d_3(S) + |\mathcal{B}_1| + |\mathcal{B}_2|$, clearly,

$$d_3(S) + |\mathcal{B}_1| + 2|\mathcal{B}_2| + t + s \leq 2\gamma - d_3(S) + t + s,$$

so (4.7) holds. □

We now consider the case when both $y_2$ and $y_{r-1}$ are of degree 3, so $d_1(y_2) = d_1(y_{r-1}) = 2$. First we make the following observation, recalling that any vertex $v$ of $S$ on $C$ is adjacent to precisely $d(v) - 2$ leaves.

**Lemma 4.2.23** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph, where $d(y_2) = d(y_{r-1}) = 3$. Suppose that $v$ is an active vertex of $S$ of degree 2 and that $w$ is a leaf of $CT$ associated with $v$. Suppose further that $d_3(v) = d_3(w)$. Then $v$ is adjacent to precisely one vertex of degree 2. Moreover, if $x$ is this vertex of degree 2, then $d_3(x) = 1$.

*Proof* Since $d(y_2) = d(y_{r-1}) = 3$, there are no leaf-adjacent vertices in $CT$ of degree 2. Therefore, by Corollary 4.1.7(c), $d_2(v) = 1$. So, let $x$ and $u$ be the neighbours of $v$, where $d(x) = 2$ and $d(u) \geq 3$. Let $y$ be the other vertex adjacent to $x$. Since $x$ is a leaf in $S - v$, $y$ must be adjacent to $d(y) - 1$ leaves in this card. In addition, in $S - v$, clearly $u$ is of degree $d(u) - 1$ and, moreover, is adjacent to $d(u) - 2$ leaves. So, since $v$ is only adjacent to $x$ and $u$, it follows that $u$ and $y$ are the only vertices in $S - v$ that are adjacent to precisely one non-leaf.

Suppose that $w$ is not adjacent to either $y_2$ or $y_{r-1}$. Then $CT - w$ contains precisely two vertices of degree 3 adjacent to two leaves. So, $S - v$ contains exactly two such vertices. By the above reasoning, these two vertices must be $u$ and $y$, thus $d(y) = 3$, so $d_3(x) = 1$.

Suppose instead that $w$ is adjacent either $y_2$ or $y_{r-1}$. Then, $CT - w$ contains exactly one vertex of degree 3 adjacent to two leaves. So, $S - v$ contains precisely one such vertex. Now, $d(u) = 3$, since $d_3(v) = d_3(w)$. Therefore, this vertex of degree 3 must be $y$, so $d_3(x) = 1$. □

**Corollary 4.2.24** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph, both of order $n$ with $d_1(CT) = d_1(S) + 2$. Suppose that $d(y_2) = d(y_{r-1}) = 3$ and that $n \geq 8$. Then $b^*(S, CT) \leq \left\lfloor \frac{4(n+3)}{11} \right\rfloor$.

*Proof* Let $v$ be an active vertex of $S$ of degree 2 and let $w$ be a leaf associated with $v$. Since $\left\lfloor \frac{4(n+3)}{11} \right\rfloor \geq \left\lfloor \frac{n+4}{3} \right\rfloor$, when $n \geq 8$, we may assume by Corollary 4.2.18, that $d_3(v) = d_3(w)$. So, by Lemma 4.2.23, $d_2(v) = 1$ and, moreover, $v$ is adjacent to some degree 2 vertex $x$ with $d_3(x) = 1$. Thus, for every pair of active vertices of $S$ of degree 2, there is at least one degree 3 vertex in $S$, so $a^*_{CT}(S) \leq 2d_3(S)$.

Suppose first that every active vertex of $S$ of degree 2 is adjacent to a vertex of degree 4 or more. Then, since any vertex of $S$ of degree 4 or more is adjacent to at most two degree 2 vertices, $a^*_{CT}(S) \leq 2(\gamma - d_3(S)) \leq 2\gamma - a^*_{CT}(S)$, so $a^*_{CT}(S) \leq \gamma$. Therefore, by (4.5),

$$n \geq 2b^*(S,\, CT) + a^*_{CT}(S) - 2 \geq 3b^*(S,\, CT) - 2,$$

so $b^*(S,\, CT) \leq \lfloor \frac{n+2}{3} \rfloor \leq \left\lfloor \frac{4(n+3)}{11} \right\rfloor$.

Suppose instead that $v$ is adjacent to some vertex of degree 3. Then $w$ is also adjacent to a vertex of degree 3, so Lemma 4.2.21 holds, thus
$b^*(S,\, CT) \leq 2\gamma - d_3(S) + 2$ by (4.7). Therefore, since $a^*_{CT}(S) \leq 2d_3(S)$, it follows from (4.5), that

$$n \geq 2b^*(S,\, CT) + (\tfrac{3b^*(S,CT)}{4} - 1) - 2 \geq \tfrac{11b^*(S,CT)}{4} - 3.$$

So, $b^*(S,\, CT) \leq \left\lfloor \frac{4(n+3)}{11} \right\rfloor$. □

This bound is attained by the following pair of graphs of small order. Note that in this case, $b(G,\, H) = b^*(G,\, H)$.

**Example 4.2.25** Let $S$ and $CT$ be the pair of graphs of order 8 in Figure 4.3. Then $S - v_i \cong CT - w_i$, for $1 \leq i \leq 4$. So $b(G,\, H) = \left\lfloor \frac{4(n+3)}{11} \right\rfloor = 4$. □

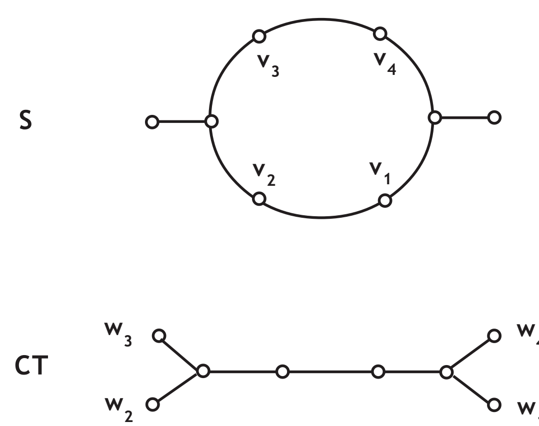


Figure 4.3: A caterpillar and sunshine graph with $\frac{4(n+3)}{11}$ common cards.

We now consider the case when either $y_2$ or $y_{r-1}$ is of degree 2.

**Lemma 4.2.26** Let $S$ and $CT$ be a caterpillar graph with $d_1(CT) = d_1(S)+2$. Let $v$ be an active vertex of $S$ of degree 2 and let $w$ be an active leaf of $CT$ associated with $v$. Suppose that $d_3(v) = d_3(w) = 0$ and that $v$ lies on a cut 2-path of length $k$. Then $k \geq 4$. In addition, if $v'$ is some other active vertex of $S$ of degree 2 and $w'$ is a leaf of $CT$ associated with $v'$ with $d_3(v') = d_3(w') = 0$, then $v'$ also lies on a cut 2-path of length $k$.

*Proof* We first note that, since $d(y_2) = 2$, $CT$ contains a leaf 2-path of length 2 or more. Now, since $d_3(v) = 0$, by Lemma 4.2.7(a), $S - v$ contains one less cut 2-path than $S$ of length $k$, and the same number of cut 2-paths of every other length. Moreover, this also holds for $CT - w$, since $S - v \cong CT - w$.

Suppose that $d_2(v) = 2$, so $k \geq 4$. Then, by Corollary 4.1.7(c), $d_2(w) = 1$, so by Lemma 4.2.8, $CT - w$ and $CT$ contain the same amount of cut 2-paths of every length. Thus, $CT$ contains one less cut 2-path of length $k$ than $S$, and the same number of cut 2-paths of every other length.

Suppose instead that $d_2(v) = 1$. Then, by Lemma 4.2.7(b)(iii), $S - v$ contains one more leaf 2-path of length $k-2$ than $S$, and the same number of leaf 2-paths of every other length. Moreover, this also holds from $CT - w$. Now, by Corollary 4.1.7(c), $d_q(w) = 1$ for some $q \geq 4$. So, by Lemma 4.2.8, $CT - w$ and $CT$ contain the same number of cut 2-paths of every length and the same number of leaf 2-paths of every length greater than 1. Therefore, $S - v$ contains some leaf 2-path of length 2 or more, thus $k \geq 4$. Moreover, $CT$ contains one less cut 2-path of length $k$ than $S$, and the same number of cut 2-paths of every other length.

Finally, suppose that $v'$ is some other active vertex of $S$ of degree 2 and that $w'$ is a leaf of $CT$ associated with $v'$, with $d_3(v') = d_3(w') = 0$. Then, if $v'$ lies on a cut 2-path of length $k'$, the same argument as above will show that $CT$ contains one less cut 2-path of length $k'$ than $S$; so $k' = k$, and the lemma is proved. $\square$

**Corollary 4.2.27** Let $S$ and $CT$ be a caterpillar graph, both of order $n \geq 5$ with $d_1(CT) = d_1(S) + 2$. Suppose that $d(y_2) = 2$. Then $b^*(S, CT) \leq \lfloor \frac{n+6}{3} \rfloor$.

*Proof* Let $v$ be an active vertex of $S$ of degree 2 and let $w$ be a leaf associated with $v$. By Corollary 4.2.18 we may assume that $d_3(v) = d_3(w)$.

As in Lemma 4.2.22, let $\mathcal{B}_1$ and $\mathcal{B}_2$ denote the set of vertices of $S$ of degree 4 or more adjacent to precisely one, and two active vertices, respectively. By Lemma 4.2.26, every vertex of $\mathcal{B}_1$ and $\mathcal{B}_2$ is an end-vertex of one or two cut 2-paths of length $k$, respectively, for some $k \geq 4$. Thus, it is easy to see that, for each pair of vertices in $\mathcal{B}_1$, and for each vertex in $\mathcal{B}_2$, there must be at least one distinct cut 2-path in $S$ of length $k$. Now, for each such cut 2-path there must be precisely $k - 3$ distinct vertices in $\mathcal{A}_2(S)$. It therefore follows that $|\mathcal{A}_2(S)| \geq \frac{k-3}{2}(|\mathcal{B}_1| + 2|\mathcal{B}_2|)$.

As in Lemma 4.2.22, we let $d_2(y_1)+d_2(y_r) = s$, and $d_3(y_1)+d_3(y_r) = t$, so $1 \leq s \leq 2$ and $0 \leq t \leq 1$. Now, by Corollary 4.1.7(c), $y_1$ and $y_r$ are the only possible vertices in $CT$ associated with a vertex in $\mathcal{A}_2(S)$. Thus, in any maximum matching of $B(S, CT)$, there are at most $s$ vertices in $\mathcal{A}_2(S)$ that are incident to any edge of this matching. So, since there are at least $\frac{k-3}{2}(|\mathcal{B}_1| + 2|\mathcal{B}_2|)$ vertices in $\mathcal{A}_2(S)$, it follows that $|\overline{\mathcal{A}(S)}| \geq \frac{(k-3)}{2}(|\mathcal{B}_1| + 2|\mathcal{B}_2|) - s$.

Suppose first that there are no active leaves in $CT$ adjacent to a vertex of degree 3. Then $b^*(S, CT) \leq |\mathcal{B}_1| + 2|\mathcal{B}_2| + s$. Therefore, by (4.5),

$$\begin{aligned} n &\geq 2b^*(S, CT) + (|\mathcal{B}_1| + |\mathcal{B}_2|) + \tfrac{(k-3)}{2}(|\mathcal{B}_1| + 2|\mathcal{B}_2|) - s - 2 \\ &\geq 2b^*(S, CT) + (|\mathcal{B}_1| + 2|\mathcal{B}_2| + s) - (2s + 2) \geq 3b^*(S, CT) - 6, \end{aligned}$$

since $|\mathcal{B}_1| + |\mathcal{B}_2| \leq \gamma$. So $b^*(S, CT) \leq \lfloor \frac{n+6}{3} \rfloor$.

So suppose instead that $v$ is adjacent to vertex of degree 3. Then, $w$ is also adjacent to a vertex of degree 3, so Lemma 4.2.21 holds, thus
$b^*(S, CT) \leq 2m - d_3(S) + t + s$ by (4.7). Therefore, by (4.5),

$$
\begin{aligned}
n &\geq 2b^*(S,\, CT) + (|\mathcal{B}_1| + |\mathcal{B}_2| + d_3(S)) + \tfrac{(k-3)}{2}(|\mathcal{B}_1| + 2|\mathcal{B}_2|) - s - 2 \\
&\geq 2b^*(S,\, CT) + (|\mathcal{B}_1| + |\mathcal{B}_2| + d_3(S) + t + s) - (2s + t + 2) \geq 3b^*(S,\, CT) - 6,
\end{aligned}
$$

since $|\mathcal{B}_1| + 2|\mathcal{B}_2| + d_3(S) \leq \gamma$. Thus $b^*(S,\, CT) \leq \left\lfloor \frac{n+6}{3} \right\rfloor$. □

By symmetry, the only remaining case to consider is when $d_2(y_2) \geq 4$ and $d_2(y_{r-1}) \geq 3$. For simplicity, we only consider pairs of graphs of order $n \geq 57$.

**Lemma 4.2.28** Let $S$ be a sunshine graph and let be $CT$ be a caterpillar graph, both of order $n \geq 57$ with $d_1(CT) = d_1(S)+2$. Suppose that there is some active leaf of $CT$ that is adjacent to vertex of degree 4 or more. Suppose further that $d(y_2) \geq 4$ and $d(y_{r-1}) \geq 3$. Then $b(S,\, CT) \leq \left\lfloor \frac{2(n+1)}{5} \right\rfloor$. In addition, if $b(S,\, CT) = \frac{2(n+1)}{5}$, then the following conditions are satisfied:

(a) $d_3(S) = d_3(CT) = 1$, $d_4(S) = d_4(CT) = \gamma - 1$ and $d_i(CT) = d_i(S) = 0$, for all $i \geq 4$;

(b) $y_{r-1}$ is the unique degree in $CT$ of degree 3;

(c) every cut-vertex of $S$ is adjacent to two vertices of degree 2;

(d) $|\mathcal{A}_0(S)| = |\mathcal{A}_2(S)| = 0$.

*Proof* Let $v$ be an active vertex of $S$ and let $w$ be a leaf associated with $v$. Since $n \geq 57$, we may assume by Corollary 4.2.18, that $d_3(v) = d_3(w)$. So, since both $y_2$ and $y_{r-1}$ are of degree 3 or more, by Corollary 4.1.7(c), $d_2(v) = 1$ and $d_2(w) = 0$.

Now, by assumption there is some active leaf of $CT$ that is adjacent to vertex of degree 4 or more. So we may initially assume that $w$ is this leaf. Then $v$ must be adjacent to a vertex of degree four or more, and it follows from Lemma 4.2.20 that $S$ does not contain an active cut-vertex. Therefore $b(S,\, CT) = b^*(S,\, CT)$ for these values of $n$.

We fix some maximum matching of $B(S, CT)$ (the choice of which is irrelevant), and $\alpha$ be the number of cut-vertices of $S$ that are adjacent to some vertex that is either not active or not incident to an edge of this matching. Since every cut-vertex of $S$ is adjacent to at most two vertices in $\mathcal{A}_1(S)$, we have $b^*(S, CT) \leq 2\gamma - \alpha$. Let $\beta = d_1(CT) - b^*(S, CT)$. Then, since $|\overline{\mathcal{A}(S)}| \geq |\mathcal{A}_0(S)| + |\mathcal{A}_2(S)|$, and $b(S, CT) = b^*(S, CT)$, rearranging (4.4), we have

$$\begin{aligned} n &\geq 2b(S, CT) + \frac{b(S, CT) + \alpha}{2} + |\mathcal{A}_0(S)| + |\mathcal{A}_2(S)| + (\beta - 2) \\ &\geq \frac{5b(S, CT)}{2} + (\frac{\alpha}{2} + |\mathcal{A}_0(S)| + |\mathcal{A}_2(S)| + \beta - 2). \end{aligned} \tag{4.9}$$

We show that $\frac{\alpha}{2} + \beta \geq 1$, with equality only when conditions (a) to (d) hold. The result will then follow. Let $d_3(y_1) + d_3(y_r) = t$, so $0 \leq t \leq 1$. Then, since $d_2(y_1) = d_2(y_r) = 0$, by parts (b) and (c) of Lemma 4.2.11, $\lambda_2 = 0$, $\lambda_3 = d_3(CT) + t$ and $\lambda^* = \sum_{i \geq 4}(i-2)d_i(CT) + (2-t)$.

Suppose first that no active leaf of $CT$ is adjacent to a degree 3 vertex, so $a_S^*(CT) \leq \lambda^*$. Clearly, if either $d(y_{r-1}) = 3$ or $d_3(CT) \geq 2$, then $\beta \geq 2$, and $\frac{\alpha}{2} + \beta \geq 2$ as required. We may therefore assume that $d(y_{r-1}) \geq 4$ and $d_3(CT) \leq 1$. Now, since every active vertex of $S$ of degree 2 is adjacent to a vertex of degree 4 or more, using parts (c) of Corollaries 4.2.9 and 4.2.10, it is easy to show that $\gamma = \sum_{i \geq 3} d_i(S) = \sum_{i \geq 3} d_i(CT)$. Now, if $d_3(CT) = 0$, then

$$\beta = d_1(CT) - b(S, CT) \geq 2\gamma + 2 - b(S, CT) \geq 2,$$

thus $\frac{\alpha}{2} + \beta \geq 2$ as required. So, we may therefore assume that $d_3(CT) = 1$, so $\beta \geq 1$ and $\alpha = 0$.

Suppose therefore, that $d_3(CT) = 1$, so $\sum_{i\geq 4} d_i(CT) = \gamma - 1$. Since $\alpha = 0$, every cut-vertex of $S$ must be adjacent to a pair of active vertices. In particular, it follows that $d_3(S) = 0$. Suppose that $w$ is adjacent to a vertex of degree 4. Then, by Corollary 4.2.10(c), $d_3(CT - w) = 2$. However, since $d_3(S - v) \leq 1$, by Corollary 4.2.9, this contradicts the fact that $w$ is associated with $v$. It therefore follows that every active leaf of $CT$ is adjacent to a vertex of degree 5 or more. Clearly, $\beta \geq d_3(CT) + 2d_4(CT) \geq 2$, if $d_4(CT) \geq 1$. However, if $d_4(CT) = 0$, then since $b(S, CT) \leq 2\gamma$, it follows that

$$\beta = d_1(CT) - b(S,\, CT) \geq \sum_{i\geq 5}(i-2)d_i(CT) + 3 - b(S,\, CT) \geq 3(\gamma-1) + 3 - 2\gamma \geq 2,$$

unless $\gamma = 1$. The result holds trivially in this case.

We are therefore left with the case when $d_3(w) = d_3(v) = 1$; so by Lemma 4.2.21, $d_i(S) = d_i(CT)$ for all $i \geq 3$. Thus,

$$d_1(CT) = d_3(S) + \sum_{i\geq 4}(i-2)d_i(S) + 2 = 2\gamma - d_3(S) + \sum_{i\geq 4}(i-4)d_i(S) + 2.$$

In addition, $b(S, CT) \leq 2\gamma - d_3(S) + t$ by (4.7).

Suppose first that $d(y_{r-1}) \geq 4$. Then $t = 0$, thus $\beta = d_1(CT) - b(S, CT) \geq 2$. So, suppose instead that $d(y_{r-1}) = 3$. Then $t = 1$, so

$$\beta = (2\gamma - d_3(S) + \sum_{i\geq 4}(i-4)d_i(S) + 2) - b(S,\, CT) \geq \sum_{i\geq 4}(i-4)d_i(S) + 1. \quad (4.10)$$

Therefore, $\beta \geq 1$, and the bound holds.

Finally, we note that $b(S, CT) = \frac{2(n+1)}{5}$, only when $(\frac{\alpha}{2} + |\mathcal{A}_0(S)| + |\mathcal{A}_2(S)| + \beta) = 1$ in (4.9). This can only occur when $\beta = 1$, so by (4.10), $d_i(S) = 0$ for all $i \geq 5$. In addition, in this maximum case, clearly $\alpha = |\mathcal{A}_0(S)| = |\mathcal{A}_2(S)| = 0$. Since $\alpha = 0$ implies that every cut-vertex of $S$ is adjacent to two vertices of degree 2, it follows that $b(S, CT) = 2\gamma$ and thus $d_3(S) = d_3(CT) = 1$. Therefore, conditions (a) to (d) hold in the maximum case. □

The bound is attained by the following infinite family of pairs of graphs.

**Example 4.2.29** Let $p$ be an integer, $p \geq 2$. Then, for $n = 5p-1$, the following pair of graphs has $\frac{2(n+1)}{5}$ common cards. Let $S$ be the sunshine graph obtained from the cycle $v_1, v_2, \ldots, v_{3p}, v_1$ by adding a pair of leaves to each $v_{3j+1}$, for $1 \leq j \leq p-1$, and a single leaf to $v_1$. Let $CT$ be the caterpillar graph obtained from the path $w_1, w_2, \ldots, w_{3p}$ by adding a pair of leaves to each $w_{3j-1}$, for $1 \leq j \leq p-1$, and a single leaf to $w_{3p-1}$. For $2 \leq j \leq p-1$, the removal of either of the leaves adjacent to $w_{3j-1}$ gives a card isomorphic to both $S - v_{3j}$ and $S - v_{3(p+1-j)-1}$. In addition, the removal of any of the leaves adjacent to $w_2$ gives a card isomorphic to both $S - v_3$ and $S - v_{3p-1}$. Finally, the removal of either of the leaves adjacent to $w_{3p-1}$ gives a card isomorphic to both $S - v_2$ and $S - v_{3p}$. So $b(G, H) = 2(p-2) + 4 = \frac{2(n+1)}{5}$. Figure 4.4 shows these graphs for $p = 4$. □

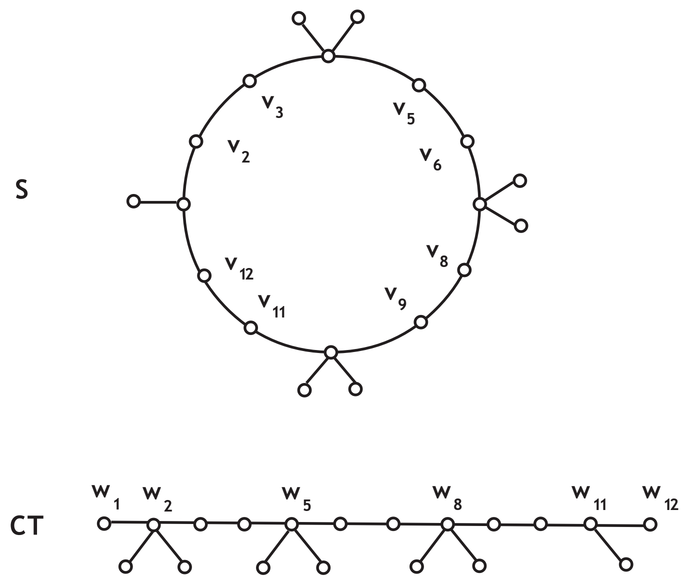


Figure 4.4: The pair of graphs in Example 4.2.29 of order 19 with 8 common cards.

**Theorem 4.2.30** Let $S$ be a sunshine graph and let $CT$ be a caterpillar graph, both of order $n$, $n \geq 62$. Then $b(S, CT) \leq \left\lfloor \frac{2(n+1)}{5} \right\rfloor$. Moreover, for these values of $n$, if $b(S, CT) = \frac{2(n+1)}{5}$, then $S$ and $CT$ are a member of the family of pairs of graphs in Example 4.2.29.

*Proof* By Corollary 4.2.4, Lemma 4.2.6 and Corollary 4.2.15, we may assume that $d_1(S) = d_1(CT) + 2$, since $n \geq 62$. In addition, for these values of $n$, we may assume from Lemma 4.2.12, that there is some active leaf of $CT$ adjacent to a vertex of degree 4 or more. Now if $d(y_2) = d(y_{r-1}) = 3$, then the bound follows from Corollary 4.2.24; if $d(y_2) = 2$ or $d(y_{r-1}) = 2$, the bound then follows from Corollary 4.2.27; otherwise the bound follows by Lemma 4.2.28. Furthermore, to attain the bound for these values of $n$, conditions (a) to (d) in Lemma 4.2.28 must be satisfied. It is easy to see that these conditions define the family in Example 4.2.29. $\square$

Our work has led us to conjecture that this bound is, in fact, the best possible for a tree and a connected non-tree. If the pairs contain active cut-vertices, they all seem to have many fewer common cards. We therefore make the following conjecture, of which we know no counter-example when $n \geq 62$.

**Conjecture 4.2.31** Let $T$ be a tree and let $U$ be a connected non-tree, both of order $n \geq 62$. Then $b(U, T) \leq \left\lfloor \frac{2(n+1)}{5} \right\rfloor$, with equality only if $n \equiv 4 \pmod 5$ and moreover, $U$ and $T$ are the pair in Example 4.2.29.

We note that if this conjecture is correct, then combined with Theorem 3.2.5, this would imply the following conjecture.

**Conjecture 4.2.32** Whether a graph is a tree or not can be determined from any $\left\lfloor \frac{n}{2} \right\rfloor + 2$ of its vertex-deleted subgraphs.

# Chapter 5

# The Number of Common Cards between a $2$UC Graph Pair

In this chapter, we introduce a new class of pairs of graphs called $2$*UC graph pairs*. We show that, if $G$ and $H$ are a 2UC graph pair, then $b(G,\, H) \leq \left\lfloor \frac{2}{3}(n+1) \right\rfloor$. In addition, we show if $n \geq 13$, then $b(G,\, H) \leq 2\left\lfloor \frac{1}{3}(n-1) \right\rfloor$, and further, when $n \geq 22$, that this bound is only attained by one of four families of 2UC graph pairs. For pairs of this order, these families have a greater number of common cards than any previously published pair of graphs.

## 5.1 $2$UC Graph Pair Definition

Let $G$ and $H$ be non-isomorphic graphs of order $n$. We express $G$ and $H$ as
$G = \mathcal{G} \oplus \mathcal{P}_G$ and $H = \mathcal{H} \oplus \mathcal{P}_H$, where

(i) $\mathcal{G}$ and $\mathcal{H}$ are non-empty collections of components of $G$ and $H$, respectively, such that no component of $\mathcal{G}$ is isomorphic to any component of $\mathcal{H}$;

(ii) $\mathcal{P}_G$ and $\mathcal{P}_H$ are (possibly empty) collections of components of $G$ and $H$, respectively, such that $\mathcal{P}_G \cong \mathcal{P}_H$.

We call the components of $\mathcal{G}$ and $\mathcal{H}$ the *unmatched* components of $G$ and $H$, and the components of $\mathcal{P}_G$ and $\mathcal{P}_H$ the *matched* components of $G$ and $H$. Note that $\mathcal{G}$ and $\mathcal{H}$ must be non-empty since $G \ncong H$. In addition, $G$ and $H$ may have different numbers of components, and furthermore, a component of $\mathcal{G}$ (or $\mathcal{H}$) may be isomorphic to a component of $\mathcal{P}_G$ (or $\mathcal{P}_H$). Thus the decompositions of $G$ and $H$ are unique only up to isomorphism.

Suppose that the Reconstruction Conjecture is false and that $A$ and $B$ are two non-isomorphic connected graphs, both of order $n-1$, with identical decks. If $G = A \oplus K_1$ and $H = B \oplus K_1$, then $b(G, H) = n - 1$. It follows that it is as difficult to find the maximum number of common cards for pairs with only one unmatched component as it is in general for connected graph pairs. Therefore, we only consider pairs of graphs where $G$ or $H$ has at least two ***U***nmatched ***C***omponents (so at least one of $\mathcal{G}$ or $\mathcal{H}$ is disconnected). We call a pair of graphs with this property a *2UC graph pair*. Note that, if $G$ is connected and $H$ is disconnected, then $G$ and $H$ are a 2UC graph pair; in this case $\mathcal{P}_G$ is empty and $\mathcal{G}$ has only one component.

Both of Myrvold's examples are families of 2UC graph pairs: Example 2.7.3 can be expressed as

$$\begin{aligned} G &= (K_{p+1} \oplus K_{p-1}) \oplus ((p-1)K_{p+1} \oplus (p-1)K_p) \\ H &= (K_p \oplus K_p) \oplus ((p-1)K_{p+1} \oplus (p-1)K_p), \end{aligned} \tag{5.1}$$

and Example 2.7.4 as

$$\begin{aligned} G &= (C_{3k+3} \oplus P_2) \oplus (kK_3) \\ H &= (P_{3k+2} \oplus K_3) \oplus (kK_3). \end{aligned} \tag{5.2}$$

Indeed, an examination of the properties of these families motivated our 2UC graph pair definition. Note that, when describing examples of 2UC graph pairs, we use brackets to differentiate the unmatched and the matched components of the graphs.

Let $\mathcal{P}$ be a graph that is isomorphic to $\mathcal{P}_G$ (and thus $\mathcal{P}_H$). We divide the components of $\mathcal{P}$ into three groups: $\mathcal{G}^*$, $\mathcal{H}^*$ and $\mathcal{F}$, where every component of $\mathcal{G}^*$, respectively $\mathcal{H}^*$, is isomorphic to some component $\mathcal{G}$, respectively $\mathcal{H}$, and $\mathcal{F}$ consists of the remaining components of $\mathcal{P}$. Then $G$ and $H$ can be expressed as

$$\begin{aligned} G &\cong \mathcal{G} \oplus (\mathcal{G}^* \oplus \mathcal{H}^* \oplus \mathcal{F}) \\ H &\cong \mathcal{H} \oplus (\mathcal{G}^* \oplus \mathcal{H}^* \oplus \mathcal{F}). \end{aligned} \tag{5.3}$$

Suppose that $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{F}$ contain $r$, $s$ and $t$ distinct isomorphism classes, $G_1, G_2 \ldots G_r$, $H_1, H_2 \ldots H_s$ and $F_1, F_2 \ldots F_t$, respectively, and let $g_i = |V(G_i)|$, $h_j = |V(H_j)|$ and $f_k = |V(F_k)|$. Then we call these $r + s + t$ isomorphism classes the *isomorphism classes of the components* of $G$ and $H$ and order them so that $g_{i+1} \leq g_i$, $h_{j+1} \leq h_j$ and $f_{k+1} \leq f_k$. Suppose further that $\mathcal{G}$ and $\mathcal{G}^*$ contain $\alpha_i$ and $\lambda_i$, respectively, isomorphic copies of each $G_i$, that $\mathcal{H}$ and $\mathcal{H}^*$ contain $\beta_j$ and $\mu_j$, respectively, isomorphic copies of each $H_j$, and that $\mathcal{F}$ contains $\gamma_k$ isomorphic copies of each $F_k$. Then in a similar manner to (1.2), we express the component structure of $G$ and $H$ as

$$\begin{aligned} G &\cong (\textstyle\bigoplus_{i=1}^{r} \alpha_i G_i) \oplus (\bigoplus_{i=1}^{r} \lambda_i G_i \bigoplus_{j=1}^{s} \mu_j H_j \bigoplus_{k=1}^{t} \gamma_k F_k) \\ H &\cong (\textstyle\bigoplus_{j=1}^{s} \beta_j H_j) \oplus (\bigoplus_{i=1}^{r} \lambda_i G_i \bigoplus_{j=1}^{s} \mu_j H_j \bigoplus_{k=1}^{t} \gamma_k F_k), \end{aligned} \tag{5.4}$$

where each $\alpha_i$, $\beta_j$ and $\gamma_k$ is positive, and each $\lambda_i$ and $\mu_j$ is non-negative. We define $\alpha_i = \lambda_i = 0$ for $i > r$, $\beta_j = \mu_j = 0$ for $j > s$, and $\gamma_k = 0$ for $k > t$.

## 5.2 Active Vertices in 2UC Graph Pairs

Throughout the rest of this chapter, we assume that $G$ and $H$ are a 2UC graph pair, both of order $n \geq 3$, expressed as in (5.4). In addition, we assume that $\mathcal{H}$ contains at least two components, so $\beta_1 + \beta_2 \geq 2$.

We begin with the following definition, which is an extension of the one given in Chapter 3.

Let $G$ and $H$ be a 2UC graph pair and let $Z$ be an isomorphism class of the components of $G$ and $H$. A vertex $v$ in $A_H(G)$ is *$Z$-active* if some associated vertex is in a component of $H$ isomorphic to $Z$. We denote the set of $Z$-active vertices of $G$ by $A_Z(G)$, and its cardinality by $a_Z(G)$. So, for example, if $Z = H_b$, then $v$ is $H_b$-active, the set of $H_b$-active vertices of $G$ is denoted by $A_{H_b}(G)$, and $a_{H_b}(G)$ is the number of $H_b$-active vertices of $G$. We define $A_Z(H)$ and $a_Z(H)$, similarly.

We extend this definition to the components of $G$ and $H$ as follows. Suppose that $U_1$ is a component of $G$. Then we denote the set of active vertices of $G$ with respect to $H$ in $U_1$ by $A_H(U_1)$ and its cardinality by $a_H(U_1)$. Similarly, we denote the set of $Z$-active vertices of $G$ in $U_1$ by $A_Z(U_1)$ and the cardinality of this set by $a_Z(U_1)$.

Now suppose that $U_2$ is a component of $G$ isomorphic to $U_1$. Then by (2.2) and (5.4), for each vertex $u_1$ in $U_1$, we can choose a distinct vertex $u_2$ in $U_2$ such that $G - u_1 \cong G - u_2$; so $a_Z(U_1) = a_Z(U_2)$, thus $a_H(U_1) = a_H(U_2)$. It follows that if $Y$ is an isomorphism class of the components of $G$ and $H$, then every component of $G$ isomorphic to $Y$ contains the same number of $Z$-active (and thus active) vertices. We therefore denote the number of $Z$-active vertices in a component of $G$ isomorphic to $Y$ by $a_Z(Y, G)$, and the total number of active vertices with respect to $H$ in any such component by $a_H(Y)$. So, for example, $a_{H_b}(G_a, G)$ is the number of $H_b$-active vertices in any component of $G$ isomorphic to $G_a$, and $a_H(G_a)$ is the total number of active vertices with respect to $H$ in such a component. $a_Z(Y, H)$ is similarly defined.

Note that, since $Z$ is a representative of an isomorphism class, the definition $Z$-active is only meaningful in terms of the decompositions of $G$ and $H$ given in (5.4). However, since it will always be clear from the context which two graphs that we are discussing, there will be no confusion with this definition.

We now extend Lemma 3.1.1 to 2UC graph pairs. Note, for simplicity, we write $\bigoplus_i \alpha_i G_i$ instead of $\bigoplus_{i=1}^{r} \alpha_i G_i$, and similarly for the other isomorphic components of $G$ and $H$.

**Lemma 5.2.1** Let $G$ and $H$ be a 2UC graph pair. Suppose that $u$ is an active vertex in some component $U$ of $G$ and that $w$ is a vertex of $H$ associated with $u$, which is in some component $W$. Then precisely one of the following holds.

(a) $u$ and $w$ are both $F_c$-active for a unique $c$. Moreover,

$$\begin{aligned} U - u &\cong \textstyle\bigoplus_{j=1}^{s} \beta_j H_j \oplus \mathcal{S} \\ W - w &\cong \textstyle\bigoplus_{i=1}^{r} \alpha_i G_i \oplus \mathcal{S}, \end{aligned} \tag{5.5}$$

where $\mathcal{S}$ is isomorphic to a (possibly empty) collection of components of both $U - u$ and $W - w$.

(b) $u$ is $H_b$-active and $w$ is $G_a$-active for a unique $a$ and a unique $b$. Moreover,

$$\begin{aligned} U - u &\cong \textstyle\bigoplus_{j \neq b} \beta_j H_j \oplus (\beta_b - 1) H_b \oplus \mathcal{R} \\ W - w &\cong \textstyle\bigoplus_{i \neq a} \alpha_i G_i \oplus (\alpha_a - 1) G_a \oplus \mathcal{R}, \end{aligned} \tag{5.6}$$

where $\mathcal{R}$ is isomorphic to a (possibly empty) collection of components of both $U - u$ and $W - w$ .

*Proof* We examine the three possible cases for $w$: (i) $w$ is $H_b$-active; (ii) $w$ is $F_c$-active; (iii) $w$ is $G_a$-active.

(i) Suppose first that $w$ is $H_b$-active, for some $b$; so $U \cong H_b$. Then by (2.2) and (5.4), $G - u$ contains precisely $\mu_b - 1$ components isomorphic to $H_b$, whereas $H - w$ must contain at least $\beta_b + \mu_b - 1$ components isomorphic to $H_b$. Since $\beta_b \geq 1$, it follows that $G - u$ contains fewer components isomorphic to $H_b$ than $H - w$, contradicting the fact that $G - u \cong H - w$. Therefore $w$ is not $H_b$-active, for any $b$. By symmetry, $u$ is not $G_a$-active, for any $a$.

(ii) Suppose instead that $w$ is $F_c$-active, for some $c$; so $U \cong F_c$. Then again by (2.2) and (5.4), $U - u$ contains precisely $\gamma_c - 1$ components isomorphic to $F_c$, so $H - w$ must also contain precisely $\gamma_c - 1$ components isomorphic to $F_c$. The same equations then show that $W \cong F_c$; that is, $u$ is $F_c$-active also. Moreover,

$$G - u \cong (\textstyle\bigoplus_i \alpha_i G_i) \oplus (\bigoplus_i \lambda_i G_i \bigoplus_j \mu_j H_j \bigoplus_{k \neq c} \gamma_k F_k) \oplus (\gamma_c - 1) F_c \oplus (U - u)$$

and $$H - w \cong (\textstyle\bigoplus_j \beta_j H_j) \oplus (\bigoplus_i \lambda_i G_i \bigoplus_j \mu_j H_j \bigoplus_{k \neq c} \gamma_k F_k) \oplus (\gamma_c - 1) F_c \oplus (W - w).$$

So, since $G - u \cong H - w$, it follows that

$$\bigoplus_{i=1}^{r} \alpha_i G_i \oplus (U - u) \cong \bigoplus_{j=1}^{s} \beta_j H_j \oplus (W - w)$$

and (5.5) holds, since $U \cong W \cong F_c$. By symmetry, if $u$ is $F_c$-active then $w$ is $F_c$-active, and it follows that $c$ is unique.

(iii) Finally, suppose that $w$ is $G_a$-active, for some $a$; so $U \cong G_a$. Then, by the above arguments, $v$ must be $H_b$-active, for some $b$. Moreover,

$$\begin{aligned} G - u &\cong (\textstyle\bigoplus_{i \neq a} \alpha_i G_i) \oplus (\bigoplus_i \lambda_i G_i \bigoplus_j \mu_j H_j \bigoplus_k \gamma_k F_k) \oplus (\alpha_a - 1) G_a \oplus (U - u) \\ \text{and } H - w &\cong (\textstyle\bigoplus_{j \neq b} \beta_j H_j) \oplus (\bigoplus_i \lambda_i G_i \bigoplus_j \mu_j H_j \bigoplus_k \gamma_k F_k) \oplus (\beta_b - 1) H_b \oplus (W - w). \end{aligned} \tag{5.7}$$

So $H - w$ contains precisely $\beta_b + \mu_b - 1$ components isomorphic to $H_b$. Thus $G - u$ must contain precisely $\beta_b + \mu_b - 1$ components isomorphic to $H_b$ and it follows from (2.2) and (5.4) that $b$ is unique. By symmetry, $a$ must be unique also. Finally, from (5.7), clearly

$$(\textstyle\bigoplus_{i \neq a} \alpha_i G_i) \oplus (\alpha_a - 1) G_a \oplus (U - u) \cong (\bigoplus_{j \neq b} \beta_j H_j) \oplus (\beta_b - 1) H_b \oplus (W - w)$$

and (5.6) holds. □

We note that, by Lemma 5.2.1, it follows that if a component isomorphic to any $F_k$ contains any active vertices, then $f_k > \sum_{i=1}^{r} \alpha_i g_i = \sum_{j=1}^{s} \beta_j h_j$.

Since, for all $i$ and $j$, $G$ contains no $G_i$-active vertices, we write $a_{H_j}(G_i)$, instead of $a_{H_j}(G_i,\, G)$, for the number of $H_j$-active vertices in any component of $G$ isomorphic to $G_i$. Similarly, since $H$ contains no $H_j$-active vertices, we write $a_{G_i}(H_j)$, instead of $a_{G_i}(H_j,\, H)$, for the number of $G_i$-active vertices in any component of $H$ isomorphic to $H_j$. We have the following corollary of Lemma 5.2.1.

**Corollary 5.2.2** Let $G$ and $H$ be a pair of 2UC graphs. Then

$$\begin{aligned} a_H(G) &= \sum_{i=1}^{r} (\alpha_i + \lambda_i) \sum_{j=1}^{s} a_{H_j}(G_i) + \sum_{k=1}^{t} \gamma_k a_{F_k}(F_k,\, G) \\ a_G(H) &= \sum_{j=1}^{s} (\beta_j + \mu_j) \sum_{i=1}^{r} a_{G_i}(H_j) + \sum_{k=1}^{t} \gamma_k a_{F_k}(F_k,\, H). \end{aligned} \tag{5.8}$$

*Proof* Any active vertex of $G$ is either $H_b$-active, for a unique $b$, or $F_c$-active, for a unique $c$, by Lemma 5.2.1. Similarly, any active vertex of $H$ is either $G_a$-active, for a unique $a$, or $F_c$-active, for a unique $c$. The result then follows from (5.4). □

As in Chapter 3, we extend Corollary 5.2.2 to common cards. Each edge of $B(G, H)$ either joins a vertex of $G$ in a component isomorphic to $G_i$ to a vertex of $H$ in a component isomorphic to $H_j$, or a pair of vertices in two components isomorphic to $F_k$. We therefore define $b(G_i, H_j)$ to be the size of a maximum matching of the subgraph of $B(G, H)$ induced by the set of all $H_j$-active vertices of $G$ and all $G_i$-active vertices of $H$. In other words, $b(G_i, H_j)$ is the maximum number of common cards that are formed by the removal of a pair of vertices from a component of $G$ that is isomorphic to $G_i$ and a component of $H$ that is isomorphic to $H_j$. We further define $b(F_k, F_k)$ to be the size of a maximum matching of the subgraph of $B(G, H)$ induced by the set of all $F_k$-active vertices of $G$ and $H$.

It is clear that $b(G_i, H_j) \leq \min\big((\alpha_i + \lambda_i)a_{H_j}(G_i), (\beta_j + \mu_j)a_{G_i}(H_j)\big)$ and $b(F_k, F_k) \leq \min\left((\gamma_k a_{F_k}(F_k, G), \gamma_k a_{F_k}(F_k, H)\right))$. In addition, $b(G, H) = \sum_{i=1}^{r}\sum_{j=1}^{s} b(G_i, H_j) + \sum_{k=1}^{t} b(F_k, F_k)$. We therefore obtain the following upper bounds on $b(G, H)$.

**Corollary 5.2.3** Let $G$ and $H$ be a pair of 2UC graphs. Then

$$\begin{aligned} b(G, H) &\leq \sum_{i=1}^{r}\sum_{j=1}^{s} \min((\lambda_i + \alpha_i)a_{H_j}(G_i), (\mu_j + \beta_j)a_{G_i}(H_j)) \\ &+ \sum_{k=1}^{t} \min(\gamma_k a_{F_k}(F_k, G), \gamma_k a_{F_k}(F_k, H)). \end{aligned} \tag{5.9}$$

*Proof* This follows directly from the above discussion. □

## 5.3 Preliminary Lemmas for the 2UC Bound

In this section, we prove many results that are used to place bounds on the number of common cards between a 2UC graph pair under various conditions. We begin this analysis with a simple observation about the number of active vertices in $\mathcal{G}$ and $\mathcal{H}$.

**Lemma 5.3.1** Let $G$ and $H$ be a 2UC graph pair. Suppose that at least two components of $\mathcal{H}$ contain active vertices. Then $\alpha_1 = 1$, and $g_1 > h_1 > g_2$. In addition, any component of $G$ that contains an $H_j$-active vertex is isomorphic to $G_1$.

*Proof* Let $w_1$ and $w_2$ be two active vertices of $H$ in two distinct components $W_1$ and $W_2$, respectively, of $\mathcal{H}$, where $W_1 \cong H_b$ and $W_2 \cong H_q$. Let $u_1$ and $u_2$ be two (not necessarily distinct) vertices of $G$ associated with $w_1$ and $w_2$, respectively. Suppose that $u_1$ is in the component $U_1$ and $u_2$ is in the component $U_2$. Then by Lemma 5.2.1, there are $a$ and $p$ such that $U_1 \cong G_a$ and $U_2 \cong G_p$. So, by (5.6),

$$\begin{aligned}
U_1 - u_1 &\cong \textstyle\bigoplus_{j\neq b} \beta_j H_j \oplus (\beta_b - 1)H_b \oplus \mathcal{R}_1 \\
W_1 - w_1 &\cong \textstyle\bigoplus_{i\neq a} \alpha_i G_i \oplus (\alpha_a - 1)G_a \oplus \mathcal{R}_1 \\
U_2 - u_2 &\cong \textstyle\bigoplus_{j\neq q} \beta_j H_j \oplus (\beta_q - 1)H_q \oplus \mathcal{R}_2 \\
W_2 - w_2 &\cong \textstyle\bigoplus_{i\neq p} \alpha_i G_i \oplus (\alpha_p - 1)G_p \oplus \mathcal{R}_2,
\end{aligned}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are two (possibly empty) collection of components.

It is easy to see that $U_1 - u_1$ contains a component isomorphic to $H_q$. For if $b \neq q$, then this is clearly the case, whereas if $b = q$, then $\beta_q = \beta_b \geq 2$, since $W_1$ and $W_2$ are distinct components of $\mathcal{H}$. Thus for $i \leq a$,

$$g_i \geq g_a = |V(U_1)| > |V(U_1 - u_1)| \geq h_q = |V(W_2)|.$$

Therefore, $W_2 - w_2$ cannot contain a component isomorphic to $G_i$, for any $i \leq a$. A similar argument shows that $W_1 - w_1$ cannot contain a component isomorphic to $G_i$ for any $i \leq p$; so $p = a = 1$, $\alpha_1 = 1$, and thus $U_1 = U_2 \cong G_1$. Finally, we note that $|V(W_1)| \geq g_2$, since $W_1 - w_1$ must contain a component isomorphic to $G_2$, if $\alpha_2 \geq 1$. Therefore, since at least one of $U_1 - u_1$ or $U_1 - u_2$ contains a component isomorphic to $H_1$, it follows that $g_1 = |V(U_1)| > h_1 \geq |V(W_1)| > g_2$. □

For any component $U$ of $G$, we define a vertex $u$ of $U$ to be a *component cut-vertex* (of $G$) if the graph $U - u$ is disconnected. Clearly, if $U$ contains such a vertex, then $|V(U)| \geq 3$. Note that, if $G$ is connected, then $u$ is simply a cut-vertex of $G$.

We recall two results from Chapter 3, noting that we are using $U$ instead of $G$ to denote the connected graph. We repeat the diagram here to assist the reader with the proof of Lemma 5.3.2.

**Lemma 3.2.1** Let $U$ be a connected graph containing two distinct vertices $u$ and $v$. Let $X_{uv}$ be the component of $U - u$ that contains $v$, and $X_{vu}$ the component of $U - v$ that contains $u$. Then

(a) $(U - u) - X_{uv} \subset X_{vu}$ and $(U - v) - X_{vu} \subset X_{uv}$;

(b) $|V(X_{vu})| + |V(X_{uv})| \geq |V(U)|$;

(c) $(U - u) - X_{uv}$ and $(U - v) - X_{vu}$ are disjoint.

□

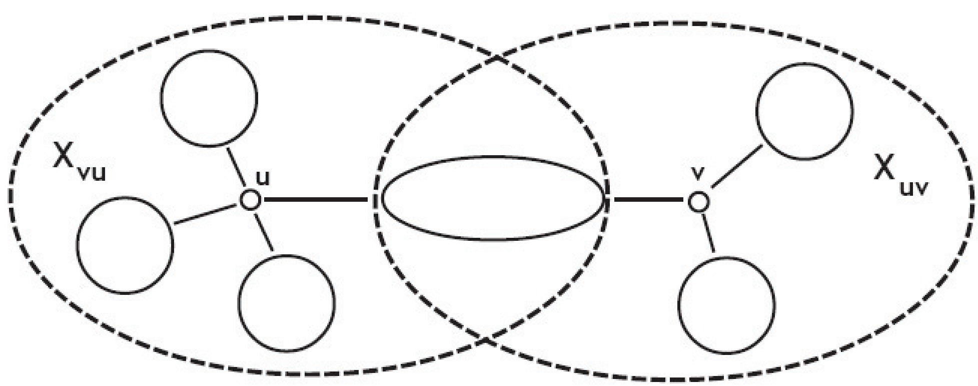


Figure 3.1: $X_{uv}$ and $X_{vu}$.

**Corollary 3.2.2** Let $U$ be a connected graph and let $S \subseteq V(U)$, with $|S| \geq 2$. Suppose that, for each vertex $u$ in $U$, $\mathcal{T}_u$ is the (possibly empty) collection of those components of $U - u$ that do not contain a vertex of $S$. Then

$$\sum_{u \in S}(|V(\mathcal{T}_u)| + 1) \leq |V(U)|.$$

□

Since a component cut-vertex in a component $U$ of $G$ is simply a vertex $u$ such that $U - u$ is disconnected, we can use these results to find bounds for the number of active component cut-vertices in the various components of a 2UC graph pair. Note that, if $U$ contains two or more component cut-vertices, then $|V(U)| \geq 4$.

**Lemma 5.3.2** Let $G$ and $H$ be a 2UC graph pair and let $U$ be a component of $G$. Suppose that, for every $u$ in $A_H(U)$, $U - u$ contains two components $X_u$ and $\widehat{X_u}$, where both $X_u$ and $\widehat{X_u}$ are isomorphic to components of $\mathcal{H}$. Then $a_H(U) \leq \frac{|V(U)|}{h_s+1}$.

*Proof* Since $|V(U)| \geq 2h_s + 1$, the result is true if $a_H(U) = 1$, so we may assume that $a_H(U) \geq 2$. Let $u$ be a vertex in $A_H(U)$ and, for some $a \leq b$, suppose that $U - u$ contains two components $X_u$ and $\widehat{X_u}$, isomorphic to $H_a$ and $H_b$, respectively. We shall show that $\widehat{X_u}$ does not contain any vertex of $A_H(U)$. Applying Corollary 3.2.2 with $S = A_H(U)$, it will then follow immediately that $A_H(U) \leq \frac{|V(U)|}{h_s+1}$, since $|V(\mathcal{T}_u)| \geq h_b \geq h_s$.

So let $v$ be any other vertex in $A_H(U)$, and let $X_{uv}$ and $X_{vu}$ be as in Lemma 3.2.1. Suppose that $\widehat{X_u}$ is $X_{uv}$, so $(U - u) - X_{uv}$ contains the component $X_u$. By part (a) of that lemma, $X_{vu}$ contains every component of $U - u$, except $X_{uv}$, so $|V(X_{vu})| > |V(X_u)| = h_a$. In addition, $X_{uv}$ contains every component of $U - u$, except $X_{vu}$, thus $h_a \geq h_b = |X_{uv}| > |V((U - v) - X_{vu})|$. However, by (5.5) and (5.6), $U - v$ must contain some component isomorphic to either $H_a$ or $H_b$, which is clearly impossible. So $\widehat{X_u}$ is not $X_{uv}$ and the result holds. □

We use the previous lemma to obtain bounds on the size of some subsets of the active vertices of $G$ and $H$.

**Corollary 5.3.3** Let $G$ and $H$ be a 2UC graph pair. Then we have the following results.

(a) Every $F_k$-active vertex of $G$ is a component cut-vertex, and $a_H(F_k) \leq \frac{f_k}{h_s+1} \leq \frac{f_k}{2}$, for all $k$.

(b) If $\beta_1 + \beta_2 + \beta_3 \geq 3$, then every active vertex of $G$ is a component cut-vertex, and $a_H(G_i) \leq \frac{g_i}{h_s+1} \leq \frac{g_i}{2}$, for all $i$.

*Proof* (a) Let $u$ be an active vertex of $G$ in a component $U$ isomorphic to $F_k$, for some $k$. Since $\beta_1 + \beta_2 \geq 2$, by (5.5), there are two components in $U - u$ that are isomorphic to components of $\mathcal{H}$. Thus $u$ is a component cut-vertex of $G$, and by Lemma 5.3.2, $a_H(U) = a_H(F_k) \leq \frac{f_k}{h_s+1}$.

(b) Suppose that $\beta_1+\beta_2+\beta_3 \geq 3$, and let $u$ be an active vertex of $U$ in a component isomorphic to $G_i$, for some $i$. By (5.6), there are two components in $U - u$ that are isomorphic to components of $\mathcal{H}$. Thus $u$ is a component cut-vertex of $G$, and by Lemma 5.3.2, $a_H(U) = a_H(G_i) \leq \frac{g_i}{h_s+1}$. □

Since no active vertex of $G$ is in a component isomorphic to any $H_j$, the above lemma shows that if $\mathcal{H}$ contains three or more components, then $b(G, H) \leq \frac{n}{2}$. By symmetry, it also follows that $b(G, H) \leq \frac{n}{2}$, if $\mathcal{G}$ contains three or more components. Therefore, for the rest of this section, we assume that $\mathcal{G}$ contains at most two components and $\mathcal{H}$ contains precisely two components; that is $r, s \leq 2$, $\alpha_1 + \alpha_2 \leq 2$ and $\beta_1+\beta_2 = 2$. In addition, in light of Lemma 5.3.1, we assume that the component isomorphic to $G_1$ is the only component of $\mathcal{G}$ that contains active vertices.

We have the following corollary of (5.6).

**Corollary 5.3.4** Let $G$ and $H$ be a 2UC graph pair and let $U$ and $W$ be components of $G$ and $H$, respectively, where $U \cong G_1$ and $W \cong H_j$, for some $j$. Suppose that $u$ and $w$ are a pair of associated vertices in $A_H(U)$ and $A_G(W)$, respectively. Then we have the following possibilities for $U - u$ and $W - w$:

(a) if $u$ is $H_1$-active and $\beta_1 = 2$, then $U - u \cong H_1 \oplus \mathcal{R}$;

(b) if $u$ is $H_1$-active and $\beta_1 = \beta_2 = 1$, then $U - u \cong H_2 \oplus \mathcal{R}$;

(c) if $u$ is $H_2$-active (so $\beta_1 = \beta_2 = 1$), then $U - u \cong H_1 \oplus \mathcal{R}$;

(d) if $\alpha_1 = 2$, then $W - w \cong G_1 \oplus \mathcal{R}$;

(e) if $\alpha_1 = \alpha_2 = 1$, then $W - w \cong G_2 \oplus \mathcal{R}$;

(f) if $\alpha_1 = 1$ and $\alpha_2 = 0$, then $W - w \cong \mathcal{R}$,

where $\mathcal{R}$ is again isomorphic to a (possibly empty) collection of components of both $W - w$ and $U - u$ (so is of order at most $\min(g_1 - 1,\, h_1 - 1)$, when $W \cong H_1$, and $\min(g_1 - 1,\, h_2 - 1)$, when $W \cong H_2$).

*Proof* This follows immediately from (5.6). $\square$

The above corollary shows that if $U$ is a component of $G$ isomorphic to $G_1$, then for every vertex $u$ in $A_H(U)$, $U - u$ contains precisely one component isomorphic to a component of $\mathcal{H}$.

We now use techniques similar to those above to obtain bounds for $a_{H_j}(G_1)$, when all the $H_j$-active vertices of $G$ are component cut-vertices.

**Lemma 5.3.5** Let $G$ and $H$ be a 2UC graph pair and let $U$ be a component of $G$ isomorphic to $G_1$. Suppose that $u$ is a vertex in $A_{H_j}(U)$ and that $u$ is a component cut-vertex of $G$. Then every $H_j$-active vertex of $G$ in a component isomorphic to $G_1$ is a component cut-vertex. In addition, if $X$ is the component of $U - u$ that is isomorphic to a component of $\mathcal{H}$, we have the following results:

(a) if $X \cong H_1$, then every active vertex in $U$, except $u$, is in $X$;

(b) if $X \cong H_2$, and $h_2 \geq \frac{g_1}{2}$, then every $H_1$-active vertex in $U$, except $u$, is in $X$;

(c) if $X \cong H_2$, and $h_2 < \frac{g_1}{2}$, then $X$ contains no $H_1$-active vertices.

*Proof* Suppose that $u$ is $H_1$-active and $\beta_1 = 2$. Then by Corollary 5.3.4(a), $U - u \cong H_1 \oplus \mathcal{R}$. Since $u$ is a component cut-vertex, $\mathcal{R}$ is not of order 0, so $h_1 \leq g_1 - 2$. Now suppose that $v$ is another $H_1$-active vertex of $G$ in a component $V$ isomorphic to $G_1$. Then again by Corollary 5.3.4(a), $V - v$ contains a component isomorphic to $H_1$. Since $h_1 \leq g_1 - 2$, this component is not the whole of $V - v$. So, $v$ must be a cut-vertex of $V$, and thus a component cut-vertex of $G$. A similar argument would show that $v$ is a component cut-vertex if $\beta_1 = 1$ and either $u$ is $H_1$-active or $u$ is $H_2$-active.

$g_1 \geq 3$ since $u$ is a component cut-vertex of $G$. Parts (a) to (c) are straightforward if $U$ contains only one active component cut-vertex. So let $v$ be another vertex in $A_H(U)$, and let $X_{uv}$ and $X_{vu}$ be as in Lemma 3.2.1. By part (a) of that lemma, $X_{vu}$ contains every component of $U - u$ except $X_{uv}$, and $X_{uv}$ contains every component of $U - v$ except $X_{vu}$.

(a) By Corollary 5.3.4(a) and (c), $X_{vu}$ is of order at most $h_1$, so $(U-u)-X_{uv}$ cannot contain a component of order $h_1$. Thus $X$ must be $X_{uv}$, and therefore $X$ contains every active vertex in $U$, except $u$.

For (b) and (c), we suppose that $v$ is $H_1$-active, so $U - v$ contains a component isomorphic to $H_2$.

(b) If $h_2 \geq \frac{g_1}{2}$, then by Corollary 5.3.4(b), the component of largest order in $U - v$ is the one isomorphic to $H_2$, since $|U| = g_1$; thus $X_{vu}$ cannot contain $X$. So $X$ is not in $(U - u) - X_{uv}$. Hence $X$ is $X_{uv}$, and therefore every $H_1$-active vertex, except $u$, is in $X$.

(c) Clearly, $X$ cannot contain a component of order $h_2$. So if $X$ is $X_{uv}$, then $|V(X_{vu})| = h_2 < \frac{g_1}{2}$. However, by Lemma 3.2.1(b), $|V(X_{uv})| + |V(X_{vu})| \geq g_1$, and so this is impossible. Thus $X$ is not $X_{uv}$, and therefore $X$ does not contain any $H_1$-active vertices. $\square$

Note that the proof of the previous lemma shows that if every $H_2$-active vertex of $G$ is a component cut-vertex, then every $H_1$-active vertex is also a component cut-vertex (since $h_2 \leq h_1$).

**Corollary 5.3.6** Let $G$ and $H$ be as in Lemma 5.3.5. Then $a_{H_j}(G_1) \leq \left\lfloor \frac{g_1}{2} \right\rfloor$.

*Proof* Let $U$ be a component of $G$ isomorphic to $G_1$ and let $u$ be a vertex in $A_{H_j}(U)$. We apply Corollary 3.2.2 with $S = A_{H_j}(U)$. By Corollary 5.3.4, precisely one of Lemma 5.3.5(a), (b) or (c) must hold for all such $u$. Let $X$ be the component of $U - u$ from Lemma 5.3.5. Note that, if parts (b) or (c) of the lemma hold, then $u$ is $H_1$-active and moreover, $h_2 \geq 1$ and $g_1 - h_2 - 1 \geq 1$.

Suppose that part (a) of the lemma holds. Then $X$ contains every $H_j$-active vertex except $u$, so $|V(\mathcal{T}_u)| = g_1 - h_1 - 1 \geq 1$. Similarly, if part (b) of the lemma holds, then $X$ contains every $H_1$-active vertex, so again, $|V(\mathcal{T}_u)| = g_1 - h_2 - 1 \geq 1$. Finally, if part (c) of the lemma holds, then $X$ contains no $H_1$-active vertices, so $|V(\mathcal{T}_u)| \geq h_2 \geq 1$. Therefore, applying Corollary 3.2.2, it follows that $a_{H_j}(G_1) \leq \left\lfloor \frac{g_1}{2} \right\rfloor$, in all cases. □

**Corollary 5.3.7** Let $G$ and $H$ be a 2UC graph pair with $a_H(G_1) > \frac{g_1}{2}$. Suppose that $U$ is a component of $G$ isomorphic to $G_1$. If every active vertex of $U$ is a component cut-vertex, then $\beta_1 = \beta_2 = 1$. Moreover, $G$ contains both $H_1$-active and $H_2$-active vertices.

*Proof* Since $a_H(G_1) > \frac{g_1}{2}$, this follows immediately from Corollary 5.3.6. □

Now if $G$ contains $H_1$ and $H_2$-active vertices, then $\alpha_1 = 1$, by Lemma 5.3.1. Therefore by Corollary 5.3.7, we only need to consider the two cases: $\alpha_1 = \alpha_2 = 1$, and $\alpha_1 = 1$ and $\alpha_2 = 0$.

**Lemma 5.3.8** Let $G$ and $H$ be a 2UC graph pair, with $\alpha_1 = \alpha_2 = 1$, and $a_H(G_1) > \frac{g_1}{2}$. Suppose that $U$ is a component of $G$ isomorphic to $G_1$ and that every active vertex in $U$ is a component cut-vertex. Then $a_G(H_1) \leq \lfloor \frac{h_1}{2} \rfloor$ and $a_G(H_2) \leq \lfloor \frac{h_2}{2} \rfloor$.

*Proof* By Corollary 5.3.7, $\beta_1 = \beta_2 = 1$, and in addition, $G$ contains both $H_1$-active and $H_2$-active vertices. So, by Lemma 5.3.1, $H$ contains no $G_2$-active vertices.

Suppose that $u$ is in $A_{H_1}(U)$ and that $w$ is a vertex of $H$ associated with $u$, which is in a component $W$ that is isomorphic to $H_1$. Then by Corollary 5.3.4(b) and (e), $U - u \cong H_2 \oplus \mathcal{R}$ and $W - w \cong G_2 \oplus \mathcal{R}$. Since $u$ is a component cut-vertex of $G$, $\mathcal{R}$ is not of order 0, so $w$ is a component cut-vertex of $H$. By symmetry, we may apply Lemma 5.3.5(b) and (c) to $W$. Thus $a_{G_1}(H_1) \leq \lfloor \frac{h_1}{2} \rfloor$ by Corollary 5.3.6, and since $H$ contains no $G_2$-active vertices, it follows that $a_G(H_1) \leq \lfloor \frac{h_1}{2} \rfloor$. A similar argument shows that $a_G(H_2) \leq \lfloor \frac{h_2}{2} \rfloor$. □

**Lemma 5.3.9** Let $G$ and $H$ be a 2UC graph pair, with $\alpha_1 = 1$, $\alpha_2 = 0$, and $a_H(G_1) > \frac{g_1}{2}$. Suppose that $U$ is a component of $G$ isomorphic to $G_1$ and that every active vertex in $U$ is a component cut-vertex. Then $\mathcal{H} \cong H_1 \oplus H_2$, where $2 \leq h_2 \leq 3$ and $h_2 < h_1$. In addition, $a_{H_1}(G) \leq \left\lfloor \frac{g_1}{h_2+1} \right\rfloor$ and $a_{H_2}(G) \leq \left\lfloor \frac{g_1}{h_2} \right\rfloor$.

*Proof* By Corollary 5.3.7, $\beta_1 = \beta_2 = 1$, and in addition, $G$ contains both $H_1$-active and $H_2$-active vertices. Let $u$ be in $A_H(U)$ and let $w$ be a vertex of $H$ associated to $u$, which is in a component $W$ of $H$. Now if $u$ is $H_2$-active, then $W \cong H_2$, so by Corollary 5.3.4(c) and (f), $U - u \cong H_1 \oplus (W - w)$. Similarly, if $u$ is $H_1$-active, then $W \cong H_1$ and by Corollary 5.3.4(b) and (f), $U - u \cong H_2 \oplus (W - w)$. It follows that calculating the number of $H_1$ and $H_2$-active vertices in $U$ is the same as calculating the number of such vertices if $U$ was a connected graph and $\mathcal{H}$ was a disconnected graph. We may therefore apply the results from Chapter 3. So the conclusions of the lemma follow from Lemma 3.2.4, noting that if $h_2 \geq 4$, clearly $a_H(G_1) \leq \frac{g_1}{2}$. □

We show in Section 5.4 that if every active vertex of $G$ is a component cut-vertex, then $a_H(G) \leq \lfloor \frac{n}{2} \rfloor$, unless $G$ and $H$ are one of the four exceptional graph pairs in Examples 3.3.1 and 3.3.2. To enable us to do this, we must prove the relationships given in Lemmas 5.3.17 and 5.3.18. First, we make the following two observations that are also used later.

We recall from Chapter 1, that a leaf 2-path of length $k \geq 1$ in a graph $F$ is a path $v_1, v_2, \ldots, v_{k+1}$, where $v_1$ is of degree 3 or more, $v_{k+1}$ is a leaf and every other vertex on the path is of degree 2. We call $v_1$ the root, and $v_{k+1}$ the end-leaf of this leaf 2-path. All other vertices on the leaf 2-path are interior vertices. Note that, by definition, $P_n$ cannot contain a leaf 2-path.

For any connected graph $F$, we denote the number of leaf 2-paths of length $k$ in $F$ by $l_k(F)$. For the purposes of Lemma 5.3.10 and Corollary 5.3.11, we let $P_0$ be the path of zero length, that is the null graph.

**Lemma 5.3.10** Let $F$ be a connected graph that contains some leaf 2-path $v_1, v_2, \ldots, v_{k+1}$ of length $k$, rooted at $v_1$. Then

(a) for $2 \leq i \leq k+1$, $F - v_i \cong A \oplus P_{k+1-i}$;

(b) for $3 \leq i \leq k+1$, $l_k(A) = l_k(F) - 1$, $l_{i-2}(A) = l_{i-2}(F) + 1$ and $l_j(A) = l_j(F)$, for all other $j$,

where $A$ is some connected graph.

*Proof* For $2 \leq i \leq k+1$, $v_i$ lies on a unique leaf 2-path of length $k$. So $F - v_i$ consists of some component $A$ and a path of length $k+1-i$, thus (a) holds. Clearly, the removal of $v_i$ destroys this leaf 2-path, and creates a new leaf 2-path rooted at $v_1$ of length $i-2$. It is easy to see that when $i \geq 3$, this new leaf 2-path is the only leaf 2-path that is created by the removal of $v_i$. (b) then follows. $\square$

**Corollary 5.3.11** Let $F$ be a connected graph with two distinct leaf 2-paths of length $k$. Suppose that $u$ and $v$ are two vertices on two distinct leaf 2-paths of $F$ of length $k \geq 1$, a distance of $i \geq 2$ and $j \geq 2$, from their respective roots. Then $(F - u) - v \cong A \oplus P_{k-i} \oplus P_{k-j}$ and $l_k(A) = l_k(F) - 2$, where $A$ is some connected graph. Furthermore

(a) if $i = j$, then $l_{i-1}(A) = l_{i-1}(F) + 2$ and $l_q(A) = l_q(F)$, for all $q \neq i-1, k$;

(b) if $i \neq j$, then $l_{i-1}(A) = l_{i-1}(F) + 1$, $l_{j-1}(A) = l_{j-1}(F) + 1$ and $l_q(A) = l_q(F)$, for all $q \neq i-1, j-1, k$.

*Proof* Since $u$ and $v$ are not on the same leaf 2-path, this follows by repeated application of Lemma 5.3.10, noting that an interior vertex that is a distance $i$ from its root corresponds to the vertex $v_{i+1}$ in that lemma. □

From Lemma 5.3.12 to Lemma 5.3.18, we now place the following further restrictions on $G$ and $H$. We assume that $\mathcal{G} \cong G_1$, $\mathcal{H} \cong H_1 \oplus H_2$, where $h_1 > h_2 = 2$. In addition, we let $U$ be a component of $G$ isomorphic to $G_1$, and suppose throughout that $U \ncong P_k$, for any $k$ . We first consider the $H_2$-active vertices of $G$.

**Lemma 5.3.12** Let $v$ be an $H_2$-active vertex of $G$. Then $d(v) = E(G_1) - E(H_1)$ and $d_1(v) = 1$. So every $H_2$-active vertex of $G$ is a single leaf-adjacent vertex of the same degree.

*Proof* $|E(G)| - |E(H)| = |E(G_1)| - |E(H_1)| - 1$, since $|E(H_2)| = 1$ and $\alpha_2 = 0$. Suppose that $w$ is a vertex of $H$ associated with $v$, which is in a component $W$. Then $w$ is a leaf, since $W \cong K_2$, so by Lemma 3.3.3, $d(v) = |E(G_1)| - |E(H_1)|$. In addition, by Corollary 5.3.4(c) and (f), $U - v \cong H_1 \oplus K_1$, therefore $d_1(v) = 1$. □

Since an isomorphism of graphs is a bijection between the vertex sets that preserves adjacency, then for any $k$, two isomorphic graphs have the same number of leaf 2-paths of length $k$. We use this observation to prove the following result.

**Corollary 5.3.13** Suppose that every $H_2$-active vertex of $G$ is of degree 2. Then for some $k \geq 2$, every $H_2$-active vertex of $G$ is an interior vertex of a leaf 2-path of length $k$. Moreover, every such vertex is a distance of $k-1$ from the root of its leaf 2-path.

*Proof* Let $u$ and $v$ be two vertices in $A_{H_2}(U)$. By Lemma 5.3.12, $d_1(u) = d_1(v) = 1$. So, since $d(u) = d(v) = 2$, both $u$ and $v$ are interior vertices of leaf 2-paths of lengths $k \geq 2$ and $l \geq 2$, respectively, a distance of $k-1$ and $l-1$ from their respective roots. It remains for us to show that $k = l$. We therefore suppose that $u \neq v$.

Suppose first that $k = 2$, so $u$ is adjacent to a leaf and some vertex of degree $r \geq 3$. Then by Lemma 2.4.6(b), $d_r(U-u) = d_r(U) - 1$. In addition, by the same lemma, $d_r(U-v) \geq d_r(U) - 1$, with equality only if $v$ is also adjacent to a vertex of degree $r$. So since $U - u \cong U - v$, clearly $v$ must be adjacent to such a vertex, thus $l = 2$, and the result holds in this case. So suppose instead that $k \geq 3$ and $l \geq 3$. Both $U - u$ and $U - v$ must have the same number of leaf 2-paths of every length, since $U - u \cong U - v$. By applying Lemma 5.3.10(b) to both $U - u$ and $U - v$, it is easy to see that $l = k$ in this case we well. $\square$

We next consider the $H_1$-active vertices of $G$.

**Lemma 5.3.14** Let $u$ be a vertex in $A_{H_1}(U)$. Then $u$ is a component cut-vertex and $U - u$ contains some component $X$ isomorphic to $H_2$ that either contains no active vertices or precisely one $H_2$-active vertex. Moreover, if the latter case holds, then every $H_2$-active vertex is of degree 2, so $|E(G_1)| - |E(H_1)| = 2$.

*Proof* First note that, since $h_1 > h_2 = 2$, then $g_1 = h_1 + h_2 \geq 5$. By Corollary 5.3.4(b), $U - u$ contains some component $X$ isomorphic to $H_2$. So $u$ is a component cut-vertex. By Lemma 5.3.5(c), $X$ does not contain any $H_1$-active vertices, since $g_1 \geq 5$. In addition, since every $H_2$-active vertex is adjacent to precisely one leaf by Lemma 5.3.12, $X$ can contain at most one $H_2$-active vertex. Moreover, if $X$ contains such a vertex, then this vertex must be of degree 2. Since by Lemma 5.3.12, every $H_2$-active vertex is of degree $|E(G_1)| - |E(H_1)|$, the lemma is proved. $\square$

We now examine the $H_1$-active vertices that are adjacent to an $H_2$-active vertex. We recall that if $v$ is a vertex of a graph with $d_1(v) = 1$, then we denote the unique leaf adjacent to $v$ by $v^*$.

**Lemma 5.3.15** Let $\mathcal{A}$ be the subset of $A_{H_1}(U)$, such that for every $v$ in $\mathcal{A}$, $U - v$ contains a component $X$ isomorphic to $H_2$ that contains an $H_2$-active vertex. Then every vertex of $\mathcal{A}$ is of the same degree, and is adjacent to precisely the same number of leaves.

*Proof* Let $u_1$ and $u_2$ be two distinct vertices of $\mathcal{A}$ and let $X_1$ and $X_2$ be the two components of $U - u_1$ and $U - u_2$, respectively, of order 2 that contain an $H_2$-active vertex. $X_1$ and $X_2$ are clearly disjoint, so there are two distinct $H_2$-active vertices $v_1$ and $v_2$ that are adjacent to $u_1$ and $u_2$, respectively. By Lemmas 5.3.12 and 5.3.14, $d(v_1) = d(v_2) = 2$, and $d_1(v_1) = d_1(v_2) = 1$.

By Corollary 5.3.4(c) and (f), $U - v_1 \cong U - v_2 \cong H_1 \oplus K_1$, so there is some isomorphism $\phi$ from $U - v_1 \cong U - v_2$. Clearly $\phi(v_1^*) = v_2^*$, so $\phi(u_1)$ must be $u_2$, since $v_1$ is only adjacent to $u_1$ and $v_1^*$ and $v_2$ is only adjacent to $u_2$ and $v_2^*$. Therefore $d(u_1) = d(u_2)$, since $u_1$ is of degree $d(u_1) - 1$ in $U - v_1$ and $u_2$ is of degree $d(u_2) - 1$ in $U - v_2$. In addition, the removal of neither $v_1$ nor $v_2$ affects the number of leaves adjacent to either $u_1$ or $u_2$. The result then follows. □

**Lemma 5.3.16** Let $\mathcal{A}$ be as Lemma 5.3.15 and let $W$ be a component of $H$ isomorphic to $H_1$. Suppose that $v$ is an $H_2$-active vertex of $U$ that is adjacent to a vertex $u$ of $\mathcal{A}$ and that $\phi$ is some isomorphism from $U - v$ to $W \oplus K_1$. Suppose further that $x$ is a leaf of $U$ such that $\phi(x)$ is associated with some vertex $u'$ of $\mathcal{A} - \{u\}$. Then $x$ is not adjacent to an $H_2$-active vertex.

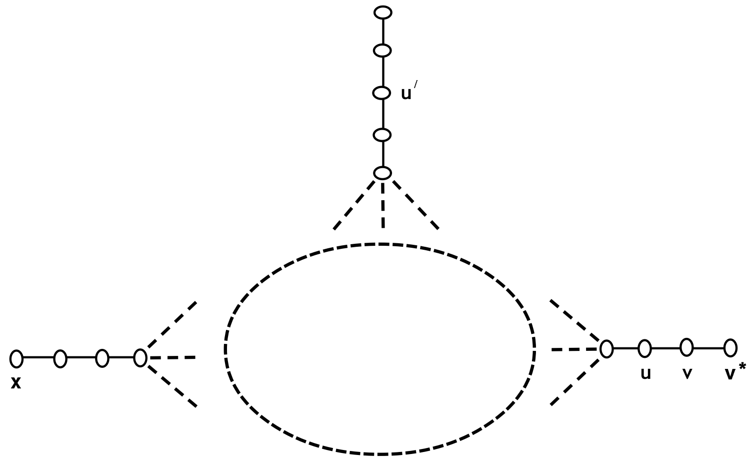


Figure 5.2: $U$ with the vertices $u$, $v$, $v^*$, $x$ and $u'$ marked.

*Proof* Since $x$ is not $v^*$, clearly $\phi(x)$ must also be a leaf of $W$. Thus by Lemmas 3.3.3 and 5.3.14, $d(u') = d(\phi(x)) + |E(G_1)| - |E(H_1)| - 1 = 2$. Therefore, by Lemma 5.3.15, $d(u) = 2$ also, so $v$ must be an interior vertex of a leaf 2-path of length $k \geq 3$, and adjacent to $u$. By Corollary 5.3.13, every $H_2$-active vertex in $U$ is on a leaf 2-path of length $k \geq 3$. In addition, since $U - v \cong W \oplus K_1$, by applying Lemma 2.4.6 to $u$ and $v$, it follows that $d_1(H_1) = d_1(G_1)$, $d_2(H_1) = d_2(G_1) - 2$ and $d_i(H_1) = d_i(G_1)$, for all other $i$.

Suppose for a contradiction that $x$ is adjacent to an $H_2$-active vertex. Then $x$ is the end-leaf of a leaf 2-path of length $k$. Since $x$ is not $v^*$, it follows that $x$ and $v$ are on two distinct leaf 2-paths of lengths 3 or more. Thus since
$(U - v) - x \cong W - \phi(x) \oplus K_1$, it follows from Corollary 5.3.11, that $W - \phi(x)$ contains precisely two less leaf 2-paths of length $k$ than $U$, and two more of lengths less than $k$. Now, by Lemma 2.4.6, $d_2(W - \phi(x)) = d_2(H_1) - 1 = d_2(G_1) - 3$, and $d_i(W - \phi(x)) = d_i(H_1)$, for all other $i$. Thus since $d(u') = 2$, it follows that $d_2(u') = 2$, so since $u'$ is adjacent to a 1-leaf adjacent vertex of degree 2, $u'$ must lie on some leaf 2-path of length $r \geq 4$. Thus by Lemma 5.3.10(b), the component of $U - u'$ that is isomorphic to $W - \phi(x)$ contains one less leaf 2-path of length $r$ than $U$, one more of length $r - 3$ and the same number of leaf 2-paths of every other length. This clearly cannot happen since $W - \phi(x)$ contains two less leaf 2-paths of length $k$ than $U$. This contradiction shows that $x$ cannot be adjacent to an $H_2$-active vertex.

□

We now use the above results to find a relationship between some of the common cards of $G$ and $H$ and the order of $G_1$. The first result holds irrespective of the number of isomorphic copies of $H_1$ in the pair.

**Lemma 5.3.17** Suppose that $G$ and $H$ are a 2UC graph pair with the stated restrictions. Then $g_1 - a_H(G_1) \geq \max(a_{H_1}(G_1),\, a_{H_2}(G))$.

*Proof* Every $H_2$-active vertex in $U$ is adjacent to a leaf by Lemma 5.3.12. So, if $a_{H_2}(G_1) \geq a_{H_1}(G_1)$, the result clearly holds, since no leaf in $U$ is active. So suppose that $a_{H_1}(G_1) > a_{H_2}(G_1)$ and let $u$ be a vertex in $A_{H_1}(U)$. By Lemma 5.3.14, $U - u$ contains some component $X$ isomorphic to $K_2$ that contains at most one active vertex. Moreover, any such vertex is $H_2$-active. So we may therefore assume that $a_{H_1}(G_1) \geq 2$ and let $v$ be another vertex in $A_{H_1}(U)$. Then, $U - v$ must also contain some component $\widehat{X}$ that contains a non-active vertex and in addition, contains no $H_1$-active vertex. Since $v$ is not in $X$ and $u$ is not in $\widehat{X}$, by Lemma 3.2.1(c), $X$ and $\widehat{X}$ are disjoint. So, for each vertex in $A_{H_1}(U)$, there is a distinct non-active vertex. Therefore, $a_{H_1}(G_1) \leq g_1 - a_H(G_1)$ and the result follows. $\square$

We finally consider the case when there is only one component in $G$ and $H$ that is isomorphic to $H_1$. We recall from Section 5.2, that we denote by $b(G_1,\, H_j)$, the size of a maximum matching of the subgraph of $B(G,\, H)$, in which all the vertices are adjacent to an $H_j$-active vertex of $G$ and a $G_1$-active vertex of $H$.

**Lemma 5.3.18** Suppose that $G$ and $H$ are a 2UC graph pair with the stated restrictions, and in addition, with $\mu_1 = 0$ and $a_H(G_1) > \frac{g_1}{2}$. Then we have the following inequalities:

(a) if $b(G_1,\, H_1) \geq 4$, then $(\lambda_1 + 1)(g_1 - a_H(G_1)) \geq (\lambda_1 + 1)a_{H_1}(G_1) + \frac{b(G_1, H_1) - 1}{3}$;

(b) if $a_{H_2}(G_1) \geq a_{H_1}(G_1)$ and $b(G_1,\, H_1) \geq 4$, then
$(\lambda_1 + 1)(g_1 - a_H(G_1)) \geq (\lambda_1 + 1)a_{H_1}(G_1) + \frac{b(G_1, H_1) + 2}{3}$;

(c) if $b(G_1,\, H_1) = 3$, then $(\lambda_1 + 1)(g_1 - a_H(G_1)) \geq (\lambda_1 + 1)a_{H_1}(G_1) + \frac{b(G_1, H_1) - 1}{2}$;

(d) if $a_{H_2}(G_1) \geq a_{H_1}(G_1)$ and $b(G_1,\, H_1) = 3$, then
$(\lambda_1 + 1)(g_1 - a_H(G_1)) \geq (\lambda_1 + 1)a_{H_1}(G_1) + \frac{b(G_1, H_1) + 1}{2}$.

*Proof* By Lemma 5.3.8, we may assume that $G$ contains both $H_1$ and $H_2$-active vertices. Let $u$ and $v$ be vertices in $U$ that are $H_1$ and $H_2$-active, respectively. By Lemma 5.3.12, $d_1(v) = 1$, and by Lemma 5.3.14, $u$ is a component cut-vertex. Let $W$ be the component of $H$ isomorphic to $H_1$, so there is some isomorphism $\phi$ from $U - v$ to $W \oplus K_1$. Note that $b(G_1, H_1) \leq a_G(W)$, since $\mu = 0$.

Let $\mathcal{A}$ be as in Lemma 5.3.15 and let $\mathcal{B} = A_{H_1}(U) - \mathcal{A}$. For each vertex $u'$ in $\mathcal{A}$ there is a distinct $H_2$-active vertex adjacent to $u'$ and a non-active leaf. In addition, by Lemma 5.3.14, for each vertex $r$ of $\mathcal{B}$, there is a component of $U - r$ of order 2 that contains no active vertices. Let $\mathcal{T}_u$ and $\mathcal{T}_v$ be the collection of components of $U - u$ and $U - v$, respectively, that contain no active vertices. Clearly, $|\mathcal{T}_v| \geq 1$, and if $u$ is not in $\mathcal{A}$, then $|\mathcal{T}_u| \geq 2$. Applying Corollary 3.2.2 with $S = A_H(U)$ and $\mathcal{T}_u$ and $\mathcal{T}_v$ as given, it is easy to see that

$$g_1 \geq 3|\mathcal{B}| + 2|\mathcal{A}| + a_{H_2}(G_1) + \max(a_{H_2}(G_1) - |\mathcal{A}|, 0). \tag{5.10}$$

Since $a_H(G_1) > \frac{g_1}{2}$, it follows that $\mathcal{A} \neq \emptyset$, so we may therefore assume that $u$ is adjacent to $v$. In addition if $d_1(u) = 1$, then by Lemma 5.3.15, every vertex in $\mathcal{A}$ is adjacent to a leaf, and $g_1 \geq 3|\mathcal{B}|+3|\mathcal{A}|+a_{H_2}(G_1)+\max(a_{H_2}(G_1)-|\mathcal{A}|, 0) \geq 2a_H(G_1)$. So we additionally assume that $d_1(u) = 0$.

We may clearly associate $u$ and $\phi(u)$. So since $\mu_1 = 0$, we can choose a maximum matching of $B(G, H)$, in which we associate any vertex of $\mathcal{A}$ other than $u$ with some vertex of $V(W) - \{\phi(u)\}$. Let $\mathcal{A}^*$ be the vertices of $A_G(W)$ that are associated with some vertex of $\mathcal{A} - \{u\}$, and in addition, are incident to an edge of this matching. So, $b(G_1, H_1) \leq (\lambda_1 + 1)|\mathcal{B}| + |\mathcal{A}^*| + 1$. Now if $\mathcal{A}^* = \emptyset$, then by (5.10),
$(\lambda_1+1)g_1 \geq (\lambda_1+1)(a_H(G_1)+a_{H_1}(G_1))+b(G_1, H_1)-1$, thus (a) to (d) hold, since $b(G_1, H_1) \geq 3$. So let $\phi(x)$ be a vertex in $\mathcal{A}^*$ and suppose that $u'$ is a vertex in $\mathcal{A} - \{u\}$ associated with $\phi(x)$.

By Lemma 5.3.15, every vertex in $\mathcal{A}$ is of the same degree. So, every vertex of $\mathcal{A}^*$ must also be of the same degree, thus $d(\phi(x)) = d(\phi(u)) = d(u)-1$. Hence, since $x$ is not adjacent to $v$, $x$ is not in $\mathcal{A}$. In addition, since $d_1(u') = 0$, $d_1(x) = d_1(\phi(x)) = 0$, then $x$ is not in $A_{H_2}(U)$. Finally, by Lemma 5.3.16, $x$ is not a leaf adjacent to an $H_2$-active vertex. Therefore if $(A_{H_2}(U))^*$ is the set of leaves of $U$ that are adjacent to an $H_2$-active vertex, $x$ must be in $(V(U) - \mathcal{A} - A_{H_2}(U) - (A_{H_2}(U))^*)$. Let $R$ be this subset of $V(U)$, so $|R| \geq |\mathcal{A}^*|$, and moreover,

$$g_1 = |R| + 2|\mathcal{A}| + a_{H_2}(U) + \max(a_{H_2}(G_1) - |\mathcal{A}|, 0). \tag{5.11}$$

Now if $\mathcal{B} = \emptyset$, then $|R| \geq b(G_1, H_1) - 1$, and (a) to (d) clearly follow from (5.11), since $b(G_1, H_1) \geq 3$. We may therefore assume that $\mathcal{B} \neq \emptyset$. Now as stated above, for each vertex $r$ in $\mathcal{B}$, there is a component $X$ in $U - r$ of order 2 that contains no active vertices. Clearly, there is one vertex $s$ of $X$ of degree 2 in $U$ that is adjacent to $r$, and another vertex $t$ that is either the vertex $s^*$, or is only adjacent to $s$ and $r$. Suppose that $d(s) = d(t) = 2$. Then $x = s$ or $x = t$ if and only if $\phi(r)$ is not in $\mathcal{A}^*$, since $d(r) \geq 3$. Suppose on the other hand that $t = s^*$. Then $x \neq s$, and in addition, $x = t$ if and only if $\phi(r)$ is not in $\mathcal{A}^*$, since $d(r) \geq 2$. We may therefore partition the vertices of $\mathcal{B}$ as follows.

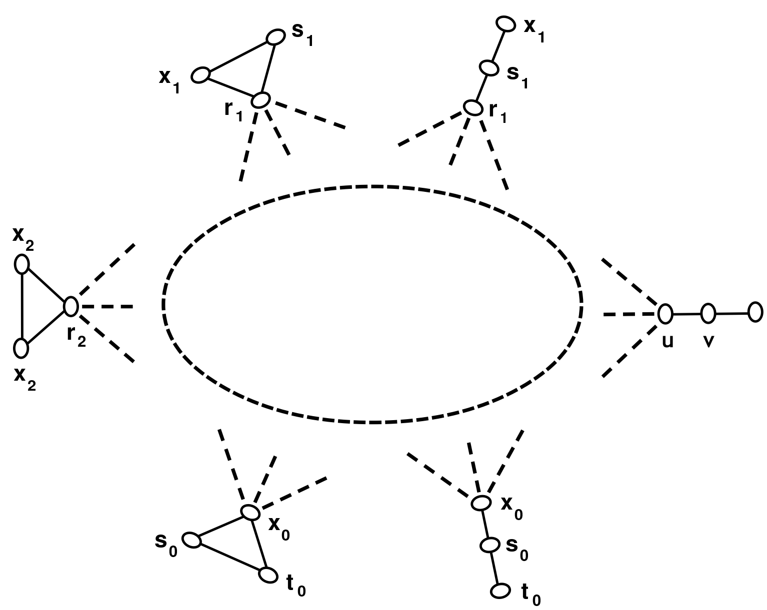


Figure 5.3: The different vertices in $A_{H_1}(U)$.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the subsets of $\mathcal{B}$ such that these components of order 2 contain one vertex, or two or more vertices, respectively, whose image under $\phi$ is in $\mathcal{A}^*$. Let $\mathcal{B}_0$ and $\mathcal{B}_0^*$ be the subsets of $\mathcal{B}$ such that these components contain no such vertices, and in addition, the image of every vertex in $\mathcal{B}_0^*$ is in $\mathcal{A}^*$ and the image of every vertex in $\mathcal{B}_0$ is not in $\mathcal{A}^*$. In Figure 5.3, any vertex labelled $x_0$, $x_1$ and $x_2$ has an image under $\phi$ in $\mathcal{A}^*$, and any vertex labelled $r_1$, $r_2$, $s_0$, $s_1$ or $t_0$ has an image under $\phi$ that is *not* in $\mathcal{A}^*$. In addition, any vertex labelled $x_0$, $r_1$ and $r_2$ is in $\mathcal{B}_0^*$, $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. It is easy to see that $|R| \geq 3|\mathcal{B}_0| + 2|\mathcal{B}_0^*| + 2|\mathcal{B}_1| + |\mathcal{B}_2| + |\mathcal{A}^*|$ and $|\mathcal{A}^*| \geq |\mathcal{B}_0^*| + |\mathcal{B}_1| + 2|\mathcal{B}_2|$.

Suppose that $|\mathcal{B}_2| \geq 1$, so $b(G_1, H_1) \geq 4$. Then $|R| \geq \frac{7|\mathcal{B}|}{3} + \frac{|\mathcal{A}^*|}{3}$. Thus, since $b(G_1, H_1) \leq (\lambda_1 + 1)|\mathcal{B}| + |\mathcal{A}^*| + 1$, it follows that

$$(\lambda_1+1)(|R|+2|\mathcal{A}|) \geq (\lambda_1+1)(2|\mathcal{B}|+2|\mathcal{A}|+\tfrac{|\mathcal{B}|+|\mathcal{A}^*|}{3}) \geq 2(\lambda_1+1)a_{H_1}(G_1)+\tfrac{(b(G_1, H_1)-1)}{3}. \tag{5.12}$$

Suppose on the other hand that $|\mathcal{B}_2| = 0$. Then $|R| \geq \frac{5|\mathcal{B}|}{2} + \frac{|\mathcal{A}^*|}{2}$ and again since $b(G_1, H_1) \leq (\lambda_1 + 1)|\mathcal{B}| + |\mathcal{A}^*| + 1$,

$$(\lambda_1+1)(|R|+2|\mathcal{A}|) \geq (\lambda_1+1)(2|\mathcal{B}|+2|\mathcal{A}|+\tfrac{|\mathcal{B}|+|\mathcal{A}^*|}{2}) \geq 2(\lambda_1+1)a_{H_1}(G_1)+\tfrac{(b(G_1, H_1)-1)}{2}. \tag{5.13}$$

Now, if $a_{H_2}(G_1) \geq a_{H_1}(G_1)$, then $\max(a_{H_2}(G_1) - |\mathcal{A}|, 0) \geq 1$ since $\mathcal{B} \neq \emptyset$. (a) and (b) then follow from (5.12) and (5.11), and (c) and (d) follow from (5.13) and (5.11). $\square$

The above results give us all the information necessary to bound $b(G, H)$ when all the active vertices of $G$ are component cut-vertices. We now examine the case when $G$ contains an active vertex that is not a component cut-vertex. Note that by Corollary 5.3.3(a), the only such active vertices of $G$ are in components isomorphic to $G_1$ or $G_2$. Note further that, by Lemma 5.3.5, either every $H_j$-active vertex in every component $U$ that is isomorphic to $G_1$ is a component cut-vertex, or no such vertex is.

The following result is immediate from Corollary 5.3.4.

**Corollary 5.3.19** Let $G$ and $H$ be a 2UC graph pair and let $U$, $W$, $u$ and $w$ be as in Corollary 5.3.4. Suppose that $u$ is not a component cut-vertex. Then $w$ is also not a component cut-vertex. In addition,

(a) if $u$ is $H_1$-active and $\beta_1 = 2$, then $U - u \cong H_1$, so $g_1 = h_1 + 1$;

(b) if $u$ is $H_1$-active and $\beta_1 = \beta_2 = 1$, then $U - u \cong H_2$, so $g_1 = h_2 + 1$;

(c) if $u$ is $H_2$-active (so $\beta_1 = \beta_2 = 1$), then $U - u \cong H_1$, so $g_1 = h_1 + 1$;

(d) if $\alpha_1 = 2$, then $W - w \cong G_1$, so $|V(W)| = g_1 + 1$;

(e) if $\alpha_1 = \alpha_2 = 1$, then $W - w \cong G_2$, so $|V(W)| = g_2 + 1$;

(f) if $\alpha_1 = 1$ and $\alpha_2 = 0$, then $W \cong K_1$.

*Proof* Let $\mathcal{R}$ be as in Corollary 5.3.4. Since $u$ is not a cut-vertex of $U$, then $U - u$ contains precisely one component, so $\mathcal{R}$ is the null graph. Therefore, $W - w$ contains one component, so $w$ is also not a cut-vertex of $W$. The six cases follow immediately, noting that in case (f), $W - w$ is of order 0 if and only if $W \cong K_1$. □

**Corollary 5.3.20** Let $G$ and $H$ be a 2UC graph pair, and let $U$ and $u$ be as in Corollary 5.3.19. If $\beta_2 = 2$ or $u$ is $H_2$-active, then $d(u) = |E(G_1)| - |E(H_1)|$. If $\beta_2 = 1$ and $u$ is $H_1$-active, then $d(u) = |E(G_1)| - |E(H_2)|$.

*Proof* The result follows by Corollary 5.3.19(a) to (c), since the degree of $u$ is equal to the difference in the number of edges of $U$ and $H_1$, respectively $H_2$. □

We now consider 2UC graph pairs in which $G$ contains both $H_1$ and $H_2$-active vertices and in addition, the $H_1$-active vertices are cut-vertices and the $H_2$-active vertices are not. We begin with the following three results, the first of which is similar to Corollary 3.2.2.

**Corollary 5.3.21** Let $U$ be a connected graph of order $n$, and let $S \subset V(U)$ and $R \subset V(U)$, where $|S| \geq 2$. For each vertex $s$ in $S$, let $\mathcal{T}_s$ denote the collection of those components of $U - s$ that do not contain a vertex of $S$. Suppose that each $\mathcal{T}_s$ contains some vertex of $R$. Then $|R| \geq |S|$.

*Proof* By Lemma 3.2.1(c), $\{\mathcal{T}_s \mid s \in S\}$ is a collection of disjoint subgraphs of $U$. So each $\mathcal{T}_s$ contains a distinct vertex of $R$. Since there are precisely $|S|$ subgraphs $\mathcal{T}_s$ in $U$, the result follows. $\square$

The next two lemmas are necessary to identify subgraphs of $U$ that contain no active vertices.

**Lemma 5.3.22** Let $U$ be a connected graph of order 5 or more with a cut-vertex $u$ such that $U - u$ contains a component $X$, where $2 \leq |V(X)| < \frac{|V(U)|}{2}$. Suppose that $s$ and $v$ are two vertices in $X$. Then $(U - s) - v$ contains one component of order at least equal to $|V(U)| - |V(X)|$, and every other component is of order at most $|V(X)| - 2$. In particular, $(U - s) - v$ does not contain a component of order $|V(X)| - 1$.

*Proof* Since $u$ is a cut-vertex, every vertex in $V(U) - V(X)$ is in the same component of $(U - s) - v$. This component is of order at least equal to $|V(U)| - |V(X)| > |V(X)|$, since $|V(X)| < \frac{|V(U)|}{2}$. Furthermore, any other component of $(U - s) - v$ is contained in $X$, so must be of order at most $|V(X)| - 2$, since $s$ and $v$ are both in $X$. $\square$

**Lemma 5.3.23** Let $U$ be a connected graph of order 5 or more with a cut-vertex $u$ such that $U - u$ contains a component $X$, where $1 \leq |V(X)| < \frac{|V(U)|}{2}$. Suppose that for every $v$ in $X$, $U - v$ is isomorphic to the same connected graph. Then every such $v$ is adjacent to $u$, so of degree $d(v) - 1$ in $U - u$.

*Proof* The result is trivial if $|V(X)| = 1$, so we assume that $|V(X)| \geq 2$. Suppose that $U - u$ contains precisely $k$ components of order $|V(X)| - 1 \geq 1$. Since $X$ is connected, $X$ contains at least one vertex that is not a cut-vertex (of $X$). Let $v$ be such a vertex of $U$, so $(U - u) - v$ contains $k + 1$ components of order $|V(X)| - 1$.

For any subgraph $Z$ of $U$, let $\mathcal{B}(Z)$ be a set of vertices of $U$ such that $z$ is in $\mathcal{B}(Z)$ if and only if $Z - z$ contains precisely $k+1$ components of order $|V(X)| - 1$. Clearly, $u$ is in $\mathcal{B}(U - v)$ but not in $\mathcal{B}(U)$. We shall first show that $\mathcal{B}(U - v) = \mathcal{B}(U) \cup \{u\}$. Moreover, we shall show that every vertex in $\mathcal{B}(U - v)$, except possibly $u$, is of the same degree in both $u$ and $U - v$.

Let $s$ be some vertex of $V(U) - \{u\}$. Suppose first that $s$ is not a cut-vertex of $U$, so $s$ is not in $\mathcal{B}(U)$. Now if $s$ is in $X$, then by Lemma 5.3.22, $s$ is not in $\mathcal{B}(U - v)$. On the other hand, if $s$ is not in $X$, then since $u$ is a cut-vertex and $v$ is not a cut-vertex, $s$ and $v$ cannot be a cut-pair, so again, $s$ is not in $B(U - v)$. Thus, if $s$ is not a cut-vertex, then $s$ is not in either $\mathcal{B}(U)$ or $\mathcal{B}(U - v)$.

So suppose instead that $s$ is a cut-vertex, so $s$ is not in $X$. Since $u$ is a cut-vertex, every vertex of $X$ must be in some component $Y$ of $U - s$ that also contains $u$. Clearly, $v$ cannot be a cut-vertex of $Y$, since $v$ is in $X$ (so not a cut-vertex of $U$) and $u$ is a cut-vertex of $U$. So $Y - v$ must be a (connected) component of $(U - v) - s$, and hence $(U-v)-s$ and $U-s$ contain the same number of components. Moreover, since $|V(Y)| \geq |V(X)| + 1$, it follows that any component of either $(U - v) - s$ or $U-s$ of order $|V(X)|-1$ must be in the subgraphs isomorphic to $(U-s)-Y$ of these two graphs. So, $(U-v)-s$ and $U-s$ must contain the same number of components of order $|V(X)| - 1$. Therefore, $s$ is in $\mathcal{B}(U)$ if and only if $s$ is in $\mathcal{B}(U - v)$, and it follows that $\mathcal{B}(U - v) = \mathcal{B}(U) \cup \{u\}$. Moreover, since $s$ is in $\mathcal{B}(U)$ only if $s$ is not in $X$ then, since $u$ is a cut-vertex of $U$, $s$ cannot be adjacent to $v$. Therefore, $s$ is of the same degree in both $U$ and $U - v$.

Now, let $t$ be some other vertex in $X$. Then $|\mathcal{B}(U-t)| = |\mathcal{B}(U-v)|$ since $U-t \cong U-v$. Moreover, the number of vertices in $\mathcal{B}(U-t)$ and $\mathcal{B}(U-v)$ of degree $d(u)-1$ must be identical.

Suppose that $t$ is a cut-vertex of $X$, so $(U-u)-t$ contains only $k$ components of order $|V(X)|-1$. Then, using a similar argument to the above, it is easy to see that for any $s$ in $V(U)-\{u\}$, $s$ is in $\mathcal{B}(U)$ if and only if $s$ is in $\mathcal{B}(U-t)$. So, since $u$ is not in either $\mathcal{B}(U)$ or $\mathcal{B}(U-t)$, clearly $|\mathcal{B}(U-t)| = |\mathcal{B}(U)| \neq |\mathcal{B}(U-v)|$. This contradiction shows that $t$ is not a cut-vertex of $X$, and it follows that $\mathcal{B}(U-t) = \mathcal{B}(U-v) = \mathcal{B}(U) \cup \{u\}$ for all $t$ in $X$. In particular, this holds for all $t$ in $X$ adjacent to $u$. Since for any such $t$, $u$ is of degree $d(u)-1$ in $U-t$, the number of vertices of degree $d(u)-1$ in $\mathcal{B}(U-t)$ is therefore one greater than the number of such vertices in $\mathcal{B}(U)$. Since the number of such vertices must be the same for all $v$ in $X$, it follows that every $v$ in $X$ must be adjacent to $u$. This completes the proof. $\square$

For the rest of this section, we now place the following restrictions on $G$ and $H$. We assume that $\beta_1 = \beta_2 = 1$ and that $G$ contains both $H_1$-active and $H_2$-active vertices, so that $\alpha = 1$ and $g_1 > h_1 \geq h_2$ by Lemma 5.3.1. We further assume that every $H_1$-active vertex of $G$ is a component cut-vertex, and every $H_2$-active is not a component cut-vertex.

For ease of notation, in all the following lemmas and corollaries, we let $U$ be a component of $G$ isomorphic to $G_1$, and let $W_1$ and $W_2$ be two components of $H$ isomorphic to $H_1$ and $H_2$, respectively. In addition, we suppose that $u$ and $v$ are two distinct vertices in $U$, and that $u$ is an $H_1$-active and $v$ is $H_2$-active. We further suppose that $w$ is a vertex in $W_2$ associated with $v$.

By Corollary 5.3.4(b), $U-u$ contains some component isomorphic to $H_2$. We shall denote this component by $X_u$. In addition, by Corollary 5.3.19(c), $U-v \cong H_1$, so we let $\phi : U-v \longrightarrow W_1$ be an isomorphism. Note that, by Corollary 5.3.20, every $H_2$-active vertex is of degree $|E(G_1)| - |E(H_1)|$.

The following results determine some relationships between $a_{H_1}(G_1)$, $a_{H_2}(G_1)$, $a_G(H_1)$ and $a_G(H_2)$, in the following three cases (which cover all possibilities for the order of $h_2$ in relation to $g_1$): when $2 \le h_2 < \frac{g_1}{2}$; when $h_2 = 1$; when $\frac{g_1}{2} \le h_2 \le g_1 - 1$. These relationships are used in Section 5.4 to place bounds on $b(G, H)$ when $G$ contains $H_1$-active vertices that are component cut-vertices and $H_2$-active vertices that are not. We begin when $2 \le h_2 < \frac{g_1}{2}$.

**Lemma 5.3.24** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition, with $2 \le h_2 < \frac{g_1}{2}$. Suppose that every vertex of $X_u$ is an $H_2$-active vertex of $G$. Suppose further that $s$ is a vertex in $V(U) - \{u\}$ such that $\phi(s)$ is an active vertex of $H$ in $W_1$. Then $s$ is a component cut-vertex and, moreover, there is some component $X_s$ of $U - s$ of order $g_2$ that contains at least one non-active vertex. In addition, no vertex in $V(\phi(X_s))$ is an active vertex of $H$.

*Proof* Since every vertex of $X_u$ is $H_2$-active, we may assume that $v$ is in $X_u$. By Corollary 5.3.20, every vertex of $X_u$ is of degree $|E(G_1)| - |E(H_1)|$ in $U$, and by Lemma 5.3.23, every vertex of $X_u$ is of degree $|E(G_1)| - |E(H_1)| - 1$ in $U - u$. So since $X_u \cong H_2$ in $U - u$, it follows that every vertex of $H_2$ is regular of degree $|E(G_1)| - |E(H_1)| - 1$.

Since $\phi(s)$ is active and $g_2 = h_2 - 1 \ge 1$, by Corollary 5.3.4(e), $W_1 - \phi(s)$ contains some component isomorphic to $G_2$. Thus, since $(U - v) - s \cong W_1 - \phi(s)$, it follows that $(U - v) - s$ contains some component isomorphic to $G_2$. Since $u$ is a cut-vertex, and $|V(X_u)| = h_2 < \frac{g_1}{2}$, by Lemma 5.3.22, $(U - v) - t$ does not contain any component of order $|V(X_u)| - 1$, for any $t$ in $X_u$. So $s$ is not in $X_u$, since $|V(X_u)| - 1 = g_2$. Thus $s$ and $v$ are not in the same component of $G - u$, and it follows that $s$ must be a cut-vertex of $U$. Therefore, since $s \ne u$, clearly $U - s$ contains some component $X_s$ isomorphic to $G_2$ (see Figure 5.4).

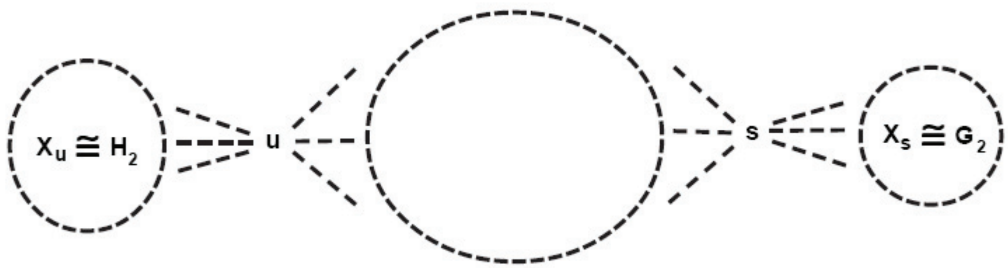


Figure 5.4: $X_u$ and $X_s$.

Now since $s$ is a cut-vertex and $g_2 < \frac{g_1}{2}$, it is easy to see that for every vertex $u'$ in $X_s$, $U - u'$ contains precisely one component of order at least $g_1 - g_2 > h_2$ and every other component is of order at most $g_2 - 1$. Thus, $X_s$ cannot contain any $H_1$-active vertices. So suppose that every vertex in $X_s$ was $H_2$-active. Then by the above reasoning, every vertex of $X_s$ must be of degree $|E(G_1)| - |E(H_1)| - 1$ in $U - s$. But this is impossible since $W_2 \cong H_2$ and $W_2 - w \cong G_2$. Therefore, $X_s$ contains at least one non-active vertex. Finally, since $v$ is not in $X_s$, then for all $u'$ in $X_s$, there is no component of order $g_2$ in $(U - v) - u'$, so no vertex in $V(\phi(X_s))$ can be an active vertex of $H$. $\square$

**Corollary 5.3.25** Let $G$ and $H$ be as in Lemma 5.3.24. Then $g_1 \geq a_H(U) + a_G(W_1) - 1$.

*Proof* We may clearly assume that $W_1$ contains at least two active vertices. So since $U - v \cong W_1$, there is some vertex $s \neq u$ such that $\phi(s)$ is active. By Lemma 5.3.24, $s$ is a component cut-vertex and, moreover, there is some component $X_s$ of $U - s$ of order $g_2 = h_2 - 1$ that contains some non-active vertex. In addition, the same lemma tells us that there is no vertex $t$ in $X_s$ such that $\phi(t)$ is active.

Now if $a_G(W_1) = 2$, then $g_1 \geq a_H(U) + |X_s| \geq a_H(U) + 1 \geq a_H(U) + a_G(W_1) - 1$. We may therefore assume that $a_G(W_1) \geq 3$ and let $S = \{s \in U \mid \phi(s) \in A_G(W_1) \text{ and } s \neq u\}$. Now let $R = V(U) - A_H(U)$ and let $\mathcal{T}_s$ be as in Corollary 5.3.21. Clearly each $X_s \subset \mathcal{T}_s$. Therefore, since each $X_s$ contains at least one vertex of $R$, applying that corollary gives $|V(U) - A_H(U)| \geq |S|$, so $g_1 - a_H(U) \geq a_G(W_1) - 1$. $\square$

**Corollary 5.3.26** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition, with $2 \leq h_2 < \frac{g_1}{2}$. Then at least one of the following holds:

(a) $g_1 - a_H(G_1) \geq a_{H_1}(G_1)$;

(b) $g_1 - a_H(G_1) \geq a_G(H_1) - 1$.

*Proof* We may clearly assume that $a_{H_1}(G_1) \geq 1$. Now by Corollary 5.3.4(b), for every vertex $s$ in $A_{H_1}(U)$, there is some component $X_s$ of $U - s$ isomorphic to $H_2$. By Lemma 5.3.5(c), no vertex in this component is $H_1$-active. Moreover, each of the $X_s$ are disjoint by Lemma 3.2.1(c). Suppose first there is no such $s$ such that every vertex of $X_s$ is $H_2$-active. (a) clearly holds immediately if $a_{H_1}(G_1) = 1$. On the other hand, if $a_{H_1}(G_1) \geq 2$, then by applying Corollary 5.3.21, with $S = A_{H_1}(U)$ and $R = V(U) - A_H(U)$, gives $g_1 - a_H(G_1) \geq a_{H_1}(G_1)$, and again (a) holds. So suppose instead that there is such an $s$ such that every vertex of $X_s$ is $H_2$-active. Then by Corollary 5.3.25, (b) holds. This completes the proof. □

We now deal with the case when $H_2 \cong K_1$, so $\alpha_1 = 1$ and $\alpha_2 = 0$.

**Lemma 5.3.27** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition, with $h_2 = 1$. Suppose that $x$ is a vertex in $W_1$ associated with $u$. Then $d_1(u) = d_1(x) + 1$, so every $H_1$-active vertex in $U$ is adjacent to at least one leaf.

*Proof* $H$ contains one more component isomorphic to $K_1$ than $G$, so $x$ is adjacent to one less leaf than $u$. □

**Corollary 5.3.28** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition, with $h_2 = 1$. Suppose that every leaf adjacent to an $H_1$-active vertex in $U$ is not active. Then $g_1 - a_H(G_1) \geq a_{H_1}(G_1)$. Note that, this result holds in particular, when the $H_2$-active vertices are not leaves.

*Proof* Every $H_1$-active vertex in $U$ is adjacent to a leaf by Lemma 5.3.27. Thus for each $H_1$-active vertex of $U$, there is a unique non-active leaf in $U$. This implies the result. □

We now consider the case when $v$ is a leaf. Note that, $|E(U)| = |E(W_1)| + 1$, since $U - v \cong H_1$.

**Lemma 5.3.29** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition, with $h_2 = 1$. Suppose that $v$ is a leaf and that $v$ is adjacent to some vertex $s$.

(a) If $d(s) \geq 3$, then $d_1(U) = d_1(W_1)+1$, whereas if $d(s) = 2$, then $d_1(U) = d_1(W_1)$.

(b) Every $H_2$-active leaf in $U$ is adjacent to a $d_1(s)$-leaf adjacent to a vertex of degree $d(s)$.

*Proof* (a) By Lemma 2.4.6(a), $d_1(U - v) = d_1(U) + d_2(v) - 1$. So if $d(s) = 2$, then $d_1(U - v) = d_1(U)$, otherwise $d_1(U - v) = d_1(U) - 1$. (a) follows immediately, since $U - v \cong W_1$.

(b) Suppose now that $v'$ is another $H_2$-active leaf in $U$ and that $v'$ is adjacent to some vertex $s'$. The result is trivial if $g_1 \leq 3$, so we assume that $g_1 \geq 4$. Now, if $d(s) = 2$, then $d_1(U) = d_1(W_1)$ by (a). So since by Corollary 5.3.19(c), $W_1 \cong U - v \cong U - v'$, applying again (a) to $s'$ shows that $d(s') = 2$ also. Clearly, $d_1(s) = d_1(s') = 1$, since $g_1 \geq 4$. Therefore, both $s$ and $s'$ are 1-leaf adjacent vertices of degree 2 and (b) holds when $d(s) = 2$.

So suppose that $d(s) \geq 3$. Then every vertex of $U$, except $s$, is adjacent to the same number of leaves in both $U$ and $U - v$; $s$ is adjacent to one less in $U - v$ than in $U$. It follows that $U - v$ contains one less $d_1(s)$-leaf adjacent vertex of degree $d(s)$ than $U$, one more $(d_1(s) - 1)$-leaf adjacent vertex of degree $d(s) - 1$, and the same number of $i$-leaf adjacent vertices of degree $j$, for all $i$ and $j$. Now by (a), $d_1(U - v) = d_1(W_1) = d_1(U) - 1$. So since $U - v' \cong U - v$, applying (a) again to $U - v'$, clearly $d(s') \geq 3$ also. Therefore, $U - v'$ contains one less $d_1(s')$-leaf adjacent vertex of degree $d(s')$ than $U'$, one more $(d_1(s') - 1)$-leaf adjacent vertex of degree $d(s') - 1$, and the same number of $i$-leaf adjacent vertices of degree $j$, for all $i$ and $j$. Since $U - v \cong U - v'$, clearly $d(s) = d(s')$ and $d_1(s) = d_1(s')$, so (b) must hold when $d(s) \geq 3$ also. $\square$

We now show that if the image of every active leaf of $U$ is active in $H$, then, except in one exceptional case $G_1$, and hence $H_1$, must be a path.

**Lemma 5.3.30** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition with $h_2 = 1$. Suppose that $v$ is a leaf and that $v$ is adjacent to $u$. Suppose further that there exists some other $H_2$-active leaf $s$ in $U$ such that $\phi(s)$ is an active vertex of $H$. Then one of the following must occur:

(a) $d(u) = 3$ and $s$ is adjacent to $u$ (so $d_1(u) = 2$);

(b) $d(u) = 2$ and $G_1$ is a path.

*Proof* Suppose that such an $s$ exists and let $q$ be its adjacent vertex. Then $d(q) = d(u)$ by Lemma 5.3.29(b). Since $s$ is a leaf of $U$, $s$ is a leaf of $U - v$, so $\phi(s)$ is a leaf of $W_1$. Now if $d_1(\phi(s)) = 1$, then clearly $u = q$ and $G_1 \cong P_3$. We therefore assume that $d_1(\phi(s)) = 0$.

Let $t$ be a vertex in $U$ associated with $\phi(s)$. Then by Corollary 5.3.4(b) and (f), $U - t \cong (W_1 - \phi(s)) \oplus K_1$. $d_1(t) = 1$ by Lemma 5.3.27(a). In addition, since $|E(W_1)| = |E(U)| - 1$, by Corollary 5.3.20, it follows that $|E(U) - t| = |E(W_1) - \phi(s)| = |E(U)| - 2$, therefore, $d(t) = 2$.

Suppose that $d(u) \geq 3$, so $d(q) \geq 3$. Then by Lemma 5.3.29(a), $d_1(W_1) = d_1(U) - 1$. In addition, $d_1(W_1 - \phi(s) \oplus K_1) = d_1(U) - 2$, unless $d(\phi(q)) = 2$, in which case, $d_1(W_1 - \phi(s) \oplus K_1) = d_1(U) - 1$. Since $d_1(t) = 1$, clearly $d_1(U - t) \geq d_1(U) - 1$. Therefore, $t$ is not associated to $\phi(s)$, unless $d(\phi(q)) = 2$, that is, $q$ is $u$ and $d(u) = 3$.

Suppose instead that $d(u) = d(q) = 2$. Then by Lemma 5.3.29(a), $d_1(W_1) = d_1(U)$. Moreover, since $d(\phi(q)) = 2$, it follows that $d_1(W_1 - \phi(s)) = d_1(U)$ also. In addition, by Lemma 2.4.6(b), $d_1(U - t) = d_1(U) + d_2(t) - 1$. So since $U - t \cong W_1 - \phi(s) \oplus K_1$, it follows that $d_2(t) = 1$, thus $t$ is adjacent to a degree 2 vertex. We now show that $U$ must be a path.

Suppose that $U$ is not a path. Then since $v$ is a leaf adjacent to a degree 2 vertex, $v$ is an end-leaf of a leaf 2-path of length $m \geq 2$. So by Lemma 5.3.10(b), $U - v$ contains one less leaf 2-path of length $m$, and one more of length $m-1$, and the same number of leaf 2-paths of every other length. Since $U - v \cong U - s$, by applying the same lemma to $U - s$, it is easy to see that $s$ must be the end-leaf of another leaf 2-path of length $m \geq 2$. So by Corollary 5.3.11, $(U - v) - s \cong W_1 - \phi(s)$ must contain two less leaf 2-paths of length $m$ than $U$ and two more of length $m - 1$. Now since $d(t) = 2$ and $d_2(t) = d_1(t) = 1$, $t$ must be on some leaf 2-path of length $l \geq 3$. So, by Lemma 5.3.10(b), the non-path component of $U - t$ contains one less leaf 2-path than $U$ of length $l$, and one more of length $l - 2$, and the same number of leaf 2-paths of every other length. It follows that $U - t$ and $(U - v) - s$ contain a different number of leaf 2-paths of length $m$. Since isomorphic graphs must have the same number of leaf 2-paths of every length, $t$ cannot be associated with $\phi(s)$. This contradiction shows that $U$ must be a path. $\square$

Note that, since a path only contains two leaves, it follows from the above lemma that if $v$ is a leaf adjacent to $u$, then $s$ is the only $H_2$-active in $U$ such that $\phi(s)$ is active.

**Lemma 5.3.31** Let $G$ and $H$ be a 2UC graph pair as in Lemma 5.3.30. Suppose that $G_1$ is not a path and that $H_1 \ncong P_4$. Then there is some non-active leaf $y$ in $G_1$ such that $\phi(y)$ is not active.

*Proof* Since $G_1$ is not a path, the lemma shows that $d(u) = 3$ and $s$ is adjacent to $u$. Let $t$ be a vertex in $U$ associated with $\phi(s)$, so by Corollary 5.3.4(b) and (f), $U - t \cong (W_1 - \phi(s)) \oplus K_1$. As in Lemma 5.3.30, it is easy to show that $d(t) = 2$ and $d_1(t) = 1$. Let $x$ be the non-leaf adjacent to $t$ (existence guaranteed since $g_1 \geq 4$). Then, since $d(u) = 3$ and $U - t \cong (W_1 - \phi(s)) \oplus K_1 \cong (U - v) - s$, it follows from Lemma 2.4.6(b) that $d(x) = 3$.

Since $d(u) = 3$ and $d_1(u) = 2$, clearly $W_1 \cong U - v$ contains two less leaf 2-path of length one than $U$. So, since $d(\phi(u)) = 2$, it follows that $U$ contains at least one more leaf 2-path of length one than $W_1 - \phi(s)$. Therefore, since isomorphic graphs must contain the same number of leaf 2-paths of length one, the removal of $t$ from $U$ must reduce the number of leaf 2-paths in $U$ by at least one. Since $t$ is an interior vertex of a leaf 2-path of length two, the only way this can happen is if $d_1(x) \geq 1$.

Now, if $d_1(x) = 2$, then $g_1 = 5$, $u$ is $x$ and $U - v \cong H_1 \cong P_4$; so $d_1(x) = 1$. Therefore, since $d_1(u) = 2$, $x^*$, the leaf adjacent to $x$, is not $H_2$-active by Lemma 5.3.29(b), so not active. In addition, since $u$ is not $x$, it is easy to show using the argument from Lemma 5.3.30 that $\phi(x)$ is also not active. Setting $x^* = y$ in the statement of the lemma gives the result. □

**Corollary 5.3.32** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition with $h_2 = 1$. Suppose that every $H_2$-active vertex of $G$ is a leaf. Suppose further that $G_1$ is not a path and $H_1 \ncong P_4$. If $a_{H_1}(G_1) > g_1 - a_H(G_1)$ then $h_1 - a_G(H_1) \geq a_{H_2}(G) - 1$.

*Proof* Suppose that $a_{H_1}(G_1) > g_1 - a_H(G_1)$. Then by Corollary 5.3.28, there must be some $H_2$-active leaf that is adjacent to an $H_1$-active vertex. We may therefore assume that $v$ is adjacent to $u$. Now, if there is no $H_2$-active vertex $s$ in $U$ such that $\phi(s)$ is active in $W_1$, then the number of non-active vertices in $W_1$ is at least equal to $a_{H_2}(G) - 1$ and the result holds. So suppose that such an $s$ exists. Then, since $G_1$ is not a path, by Lemma 5.3.30, $d(u) = 3$, and $s$ is adjacent to $u$. Clearly, the image of every vertex of $A_{H_2}(U) - \{v, s\}$ is not active. In addition, since $H_1 \ncong P_4$, by Lemma 5.3.31, there must be some non-active leaf $y$ in $U$ such that $\phi(y)$ is not active in $W_1$. Therefore, the number of non-active vertices in $W_1$ must be at least equal to $|A_{H_2}(U) - \{v, s\}| + 1$. So $h_1 - a_G(H_1) \geq a_{H_2}(G) - 1$. □

We use the above corollary to form a relation between the number of non-active vertices in $G_1$ and $H_1$, and the order of $g_1$.

**Corollary 5.3.33** Let $G$ and $H$ be a 2UC graph pair as in Corollary 5.3.32. Suppose that $a_{H_1}(G_1) > g_1 - a_H(G_1)$. Then $g_1 - 2 \leq 2(g_1 - a_H(G_1)) + 2(h_1 - a_G(H_1)$.

*Proof* Every $H_1$-active vertex of $G$ in $U$ is adjacent to a leaf by Lemma 5.3.27. Let $\mathcal{A}$ be the set of $H_1$-active vertices in $U$ that are adjacent to an $H_2$-active leaf. Then the number of non-active vertices of $G$ is at least equal to $a_{H_1}(G_1) - |\mathcal{A}|$. So since $a_{H_2}(G_1) \geq |\mathcal{A}|$, it follows that $a_{H_1}(G_1) - a_{H_2}(G_1) \leq g_1 - a_H(G)$, thus $g_1 - 2a_{H_2}(G_1) \leq 2(g_1 - a_H(G))$, since $a_H(G_1) = a_{H_1}(G_1) + a_{H_2}(G_1)$. Now by Corollary 5.3.32, $2(a_{H_2}(G_1) - 1) \leq 2(h_1 - a_G(H_1))$. Combining the two expressions, yields the result. □

We now consider the case when there is an $H_1$-active vertex of $G$ whose image under $\phi$ is active in $H$.

**Lemma 5.3.34** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition with $h_2 = 1$. Suppose that $v$ is a leaf and that $v$ is adjacent to $u$. Let $s \neq u$ be an $H_1$-active vertex in $U$ such that $\phi(s)$ is an active vertex of $H$. If $s$ is a $d_1(u)$-leaf adjacent vertex of degree $d(u)$, then any vertex in $U$ associated with $\phi(s)$, except possibly $u$, is adjacent to some non-active leaf.

*Proof* Let $t \neq u$ be a vertex in $U$ associated with $\phi(s)$. By Lemma 3.3.3, $d(t) = d(\phi(s)) + 1 = d(s) + 1$ since $|E(U)| - |E(W_1)| = 1$. Thus by 5.3.29(b), no leaves adjacent to $t$ can be active. Since $d_1(t) \geq 1$ by Lemma 5.3.27, the result follows. □

**Corollary 5.3.35** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition with $h_2 = 1$ and $\mu_1 = 0$ (so $H$ contains only one component isomorphic to $H_1$). Suppose that every $H_2$-active vertex of $G$ is a leaf and that $G_1$ is not a path. Then $(\lambda_1 + 1)(g_1 - a_H(G_1)) \geq b(G_1, H_1) - 1$.

*Proof* Suppose that there is no $H_1$-active vertex in $U$ adjacent to an $H_2$-active leaf. Then by Corollary 5.3.28, $g_1 - a_H(G_1) \geq a_{H_1}(G_1)$, and the result holds since $(\lambda_1+1)a_{H_1}(G_1) \geq b(G_1, H_1)$. We may therefore assume that $v$ is adjacent to $u$. Since $u$ is clearly associated with $\phi(u)$, we choose a maximum matching of $B(G, H)$, in which $u$ and $\phi(u)$ are adjacent. We may clearly assume that $b(G_1, H_1) \geq 2$.

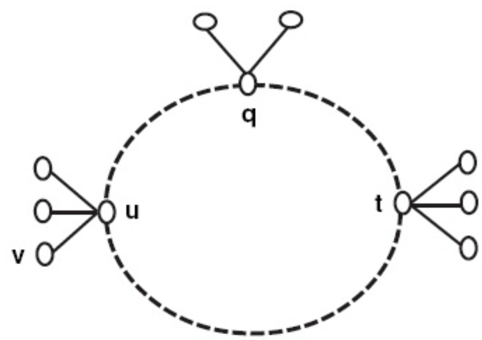


Figure 5.5: $U$ with the vertices $u$, $v$, $q$ and $t$ marked.

Since $\mu_1 = 0$, there is some vertex $t \neq u$ of $U$ such that $q\phi(t)$ is an edge of this matching, for some vertex $q \neq u$ of $G$. Suppose that $t$ is $H_2$-active. Then since $G_1$ is not a path, by Lemma 5.3.30, $d(u) = 3$, and $t$ is a leaf adjacent to $u$. Clearly $d(q) = 2$ and $d_1(q) = 1$, so the leaf-adjacent to $q$ cannot be active by Lemma 5.3.29(b). Suppose, on the other hand that $t$ is $H_1$-active, so $t$ must be adjacent to a leaf. Now if every leaf adjacent to $t$ is active, then by Lemma 5.3.29(b), $d(t) = d(u)$ and $d_1(t) = d_1(u)$. But in this case, by Lemma 5.3.34, $q$ is adjacent to a non-active leaf.

It follows that in both cases, either $t$ or $q$ is adjacent to a non-active leaf. Therefore, for each edge of this matching, that is incident to an $H_1$-active vertex of $G$ and a vertex in $W_1$, except possibly $u\phi(u)$, there is a distinct non-active leaf in $G$. Since the number of non-active vertices in $G$ is equal to $(\lambda_1 + 1)(g_1 - a_H(G_1))$, the result follows. $\square$

Finally we consider the case when $h_2 \geq \frac{g_1}{2}$ (and so $g_2 \geq 1$).

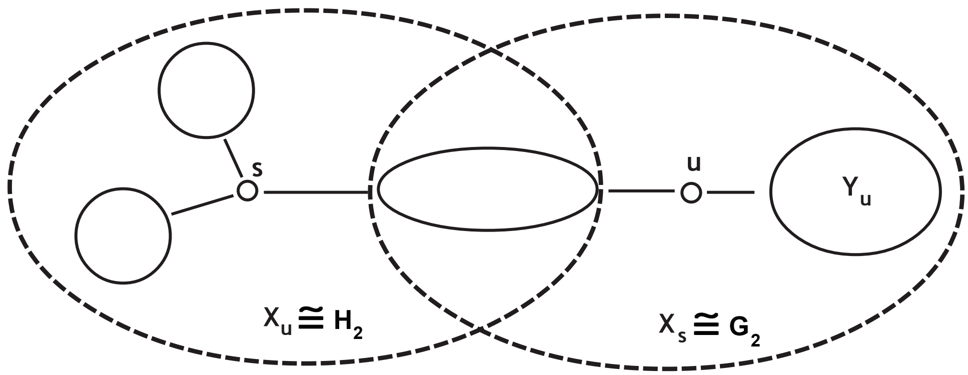


Figure 5.6: $U$ when $h_2 \geq \frac{g_1}{2}$.

**Lemma 5.3.36** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition, with $h_2 \geq \frac{g_1}{2}$. Suppose that $v$ is in some component $Y_u \neq X_u$ in $U - u$. Then for every active vertex in $W_1$, except possibly $\phi(u)$, there is a cut vertex in $W_2$.

*Proof* Let $s \neq u$ be a vertex of $U$ such that $\phi(s)$ is active. Then by Corollary 5.3.4(e), $W_1 - \phi(s)$, and thus $(U - v) - s$, contains a component isomorphic to $G_2$. $|V(U)| - |V(X_u)| > |V(Y_u)|$, since $X_u$ and $Y_u$ are distinct components in $U - u$. Thus since $|V(X_u)| = h_2$, it follows that $|Y_u| < \frac{g_1}{2}$. Now by Lemma 5.3.22, for all $t$ in $Y_u$, $(U - v) - t$ contains one component of order at least equal to $|V(U)| - |V(Y_u)|$ and no other component of order greater than $|V(Y_u)| - 2$. Thus it follows that for all such $t$, $(U - v) - t$, cannot contain a component of order $g_2$. Therefore, $s$ is not in $Y_u$, so $s$ must be a cut-vertex of $U$.

Now since $h_2 - 1 = g_2 \geq \frac{g_1}{2} - 1$, there is no component in $U - s$ of greater order than $g_2$. Let $X_{us}$ and $X_{su}$ be as in Lemma 3.2.1. Then by part (c) of that lemma, $X_{su}$ must contain every component of $U - s$, except $X_{us}$. So since $X_u$ is of order $h_2 = g_2 + 1$, clearly $X_{us} = X_u$, so $s$ is in $X_u$. Since $s$ is a cut-vertex of $U$, it is easy to see that $s$ must also be a cut-vertex of $X_u$. Therefore, for every active vertex in $W_1$, except possibly $\phi(u)$, there is a cut-vertex in $X_u$. Since $X_u \cong W_2$, the result follows. $\square$

The above lemma allows us to bound the number of active vertices in $U$ or $W_2$, when $\mu_1 = 0$ and $h_2 \geq \frac{g_1}{2}$.

**Corollary 5.3.37** Let $G$ and $H$ be a 2UC graph pair with the stated restrictions, and in addition with $h_2 \geq \frac{g_1}{2}$, and $\mu_1 = 0$. Then at least one of the following hold:

(a) $g_1 - a_H(G_1) \geq a_{H_1}(G_1)$;

(b) $h_2 - a_G(H_2) \geq a_G(H_1) - 1$.

*Proof* We may clearly assume that $a_{H_1}(G_1) \geq 1$. Now by Corollary 5.3.4(b), for every vertex $s$ in $A_{H_1}(U)$, there is some component $X_s$ of $U - s$ isomorphic to $H_2$. By Lemma 5.3.5(b), every $H_1$-active vertex in $U$ except $s$ is in $X_s$. Thus for every such $s$, there is some component $Y_s$ of $U - s$ that contains no $H_1$-active vertices. Moreover, by Lemma 3.2.1(c), each of the $Y_s$ are disjoint. Suppose first there is no such $s$ such that every vertex of $Y_s$ is $H_2$-active. (a) clearly holds immediately if $a_{H_1}(G_1) = 1$. On the other hand, if $a_{H_1}(G_1) \geq 2$, then by applying Corollary 5.3.21, with $S = A_{H_1}(U)$ and $R = V(U) - A_H(U)$ gives $g_1 - a_H(G_1) \geq a_{H_1}(G_1)$, and again (a) holds. We may therefore assume that $Y_u$ contains an $H_2$-active vertex. Then by Lemma 5.3.36, $W_1$ contains at least $a_G(H_1) - 1$ cut-vertices. Since by Corollary 5.3.19(e), no active vertex in $W_2$ is a cut-vertex, (b) holds, which completes the proof. $\square$

## 5.4 Bounding the Number of Common Cards between a $2$UC Graph Pair

We now use the results from Sections 5.2 and 5.3 to place upper bounds on $b(G, H)$ for any 2UC graph pair $G$ and $H$. By Corollary 5.3.3, $b(G, H) \le \lfloor \frac{n}{2} \rfloor$, if either $\mathcal{H}$ or $\mathcal{G}$ contains three or more components. So, since we wish to find 2UC graph pairs with a large number of common cards, we assume, as in Section 5.3, that $\alpha_1 + \alpha_2 \le 2$ and $\beta_1 + \beta_2 = 2$.

By Corollary 5.3.3(a), $a_H(\mathcal{F}, G) = \sum_k \gamma_k a_H(F_k, G) \le \frac{|\mathcal{F}|}{2}$. We therefore assume that $\mathcal{G}$ contains at least one active vertex. Moreover, we do not need to express $\mathcal{F}$ in terms of its components.

Now, by Lemma 5.3.1, if both components of $\mathcal{H}$ contain active vertices, then only one component of $\mathcal{G}$ contains active vertices. We therefore assume, without loss of generality, that both components of $\mathcal{H}$ can contain active vertices but only one component of $\mathcal{G}$ can contain active vertices. This implies immediately that $\alpha_1 = 1$ and all the active vertices in $\mathcal{H}$ are $G_1$-active. Moreover, since we are assuming that $H$ does not contain any $G_2$-active vertices, we assume that $\lambda_2 = 0$.

In light of these assumptions, we write $G$ and $H$ as

$$\begin{aligned} G &\cong (G_1 \oplus \alpha_2 G_2) \oplus (\lambda_1 G_1 \oplus \mu_1 H_1 \oplus \mu_2 H_2 \oplus \mathcal{F}) \\ H &\cong (\beta_1 H_1 \oplus \beta_2 H_2) \oplus (\lambda_1 G_1 \oplus \mu_1 H_1 \oplus \mu_2 H_2 \oplus \mathcal{F}), \end{aligned} \tag{5.14}$$

where $0 \le \alpha_2 \le 1$, $\beta_1 + \beta_2 = 2$ and $1 \le \beta_1 \le 2$. Thus,

$$\begin{aligned} n &= (1+\lambda_1)g_1 + \alpha_2 g_2 + \mu_1 h_1 + \mu_2 h_2 + |V(\mathcal{F})| \\ &= (\beta_1 + \mu_1)h_1 + (\beta_2 + \mu_2)h_2 + \lambda_1 g_1 + |V(\mathcal{F})|. \end{aligned} \tag{5.15}$$

We now show that, for any fixed maximum matching of $B(G, H)$, we can express $n$ in terms of $b(G, H)$, $g_1$, and the total number of vertices in $G$ and $H$ that are not incident to any edge of this matching.

Let $\overline{a_H(G_1)} = (g_1 - a_H(G_1))$, $\overline{a_G(H_1)} = (h_1 - a_G(H_1))$, $\overline{a_G(H_2)} = (h_2 - a_G(H_2))$ and $\overline{a_H(\mathcal{F})} = (|V(\mathcal{F})| - a_H(\mathcal{F}))$. With this notation, we rearrange (5.15) to give

$$\begin{aligned} n &= (1+\lambda_1)a_{H_1}(G_1) + (1+\lambda_1)a_{H_2}(G_1) + (\beta_1+\mu_1)a_G(H_1) + (\beta_2+\mu_2)a_G(H_2) \\ &+ (1+\lambda_1)\overline{a_H(G_1)} + (\beta_1+\mu_1)\overline{a_G(H_1)} + (\beta_2+\mu_2)\overline{a_G(H_2)} + a_H(\mathcal{F}) + \overline{a_H(\mathcal{F})} \\ &+ \alpha_2 g_2 - \beta_1 h_1 - \beta_2 h_2. \end{aligned} \tag{5.16}$$

We fix some maximum matching of $B(G,\, H)$ (the choice of which is irrelevant), and let $b_1 = b(G_1,\, H_1)$, $b_2 = b(G_1,\, H_2)$ and $b_{\mathcal{F}} = \sum_{k=1}^{t} b(F_k,\, F_k)$, so that $b(G,\, H) = b_1 + b_2 + b_{\mathcal{F}}$. We denote the active vertices of $G$ and $H$ that are not incident to any edge of this matching as follows:

(a) $\overline{b_1(G)} = (1+\lambda_1)a_{H_1}(G_1) - b_1$;

(b) $\overline{b_2(G)} = (1+\lambda_1)a_{H_2}(G_1) - b_2$;

(c) $\overline{b_1(H)} = (\beta_1+\mu_1)a_G(H_1) - b_1$;

(d) $\overline{b_2(H)} = (\beta_2+\mu_2)a_G(H_2) - b_2$;

(e) $\overline{b_{\mathcal{F}}(G)} = a_H(\mathcal{F}) - b_{\mathcal{F}}$.

These give us the following relations:

$$(1+\lambda_1)g_1 = b_1 + b_2 + \overline{b_1(G)} + \overline{b_2(G)} + (1+\lambda_1)\overline{a_H(G_1)} \tag{5.17}$$

$$(\beta_1+\mu_1)h_1 = b_1 + \overline{b_1(H)} + (\beta_1+\mu_1)\overline{a_G(H_1)} \tag{5.18}$$

$$(\beta_2+\mu_2)h_2 = b_2 + \overline{b_2(H)} + (\beta_2+\mu_2)\overline{a_G(H_2)} \tag{5.19}$$

$$b_1 = (1+\lambda_1)a_{H_1}(G_1) - \overline{b_1(G)} = (\beta_1+\mu_1)a_G(H_1) - \overline{b_1(H)} \tag{5.20}$$

$$b_2 = (1+\lambda_1)a_{H_2}(G_1) - \overline{b_2(G)} = (\beta_2+\mu_2)a_G(H_1) - \overline{b_2(H)}. \tag{5.21}$$

Finally, using the fact that $b(G,\, H) = b_1 + b_2 + b_{\mathcal{F}}$ and $g_1 + \alpha_2 g_2 = \beta_1 h_1 + \beta_2 h_2$, we substitute (a) to (e) into (5.16) to express $n$ as

$$\begin{aligned} n &= 2b(G,\, H) + \overline{b_1(G)} + \overline{b_2(G)} + \overline{b_1(H)} + \overline{b_2(H)} + (1+\lambda_1)\overline{a_H(G_1)} \\ &+ (\beta_1+\mu_1)\overline{a_G(H_1)} + (\beta_2+\mu_2)\overline{a_G(H_2)} + (\overline{a_H(\mathcal{F})} + \overline{b_{\mathcal{F}}(G)} - b_{\mathcal{F}}) - g_1, \end{aligned} \tag{5.22}$$

noting that by Corollary 5.3.3(a), $\overline{a_H(\mathcal{F})} - b_{\mathcal{F}} \geq 0$.

We begin with a simple observation from above.

**Lemma 5.4.1** Let $G$ and $H$ be a 2UC graph pair, both of order $n \geq 3$. If $\lambda_1 \geq 1$, then $b(G,\, H) \leq \left\lfloor \frac{(1+\lambda_1)n}{1+2\lambda_1} \right\rfloor \leq \left\lfloor \frac{2n}{3} \right\rfloor$. Moreover, when $n \geq 11$, equality can only hold if $\lambda_1 = 1$.

*Proof* $b(G,\, H) = b_1 + b_2 + b_{\mathcal{F}}$. So, by (5.17),
$g_1 = \frac{1}{1+\lambda_1}(b(G,\, H) + \overline{b_1(G)} + \overline{b_2(G)} - b_{\mathcal{F}})) + \overline{a_H(G_1)}$. Thus, by substituting for $g_1$ in (5.22),

$$\begin{aligned} n &= \frac{(1+2\lambda_1)b(G,\, H)}{1+\lambda_1} + \frac{(\lambda_1(\overline{b_1(G)} + \overline{b_2(G)}) + b_{\mathcal{F}}}{1+\lambda_1} \\ &+ \lambda_1\overline{a_H(G_1)} + \overline{b_1(H)} + \overline{b_2(H)} + (\beta_1 + \mu_1)\overline{a_G(H_1)} + (\beta_2 + \mu_2)\overline{a_G(H_2)} \\ &+ (\overline{a_H(\mathcal{F})} - b_{\mathcal{F}}) + \overline{b_{\mathcal{F}}}(G). \end{aligned} \tag{5.23}$$

Therefore, since $\overline{a_H(\mathcal{F})} - b_{\mathcal{F}} \geq 0$, it follows that $n \geq \frac{(1+2\lambda_1)b(G,\, H)}{1+\lambda_1}$, which implies the bound. When $n \geq 11$, straightforward calculations show that equality holds only if $\lambda_1 \geq 1$. □

We now show that if one of the components of $\mathcal{H}$ does not contain any active vertices, the bound on $b(G,\, H)$ is much tighter for all values of $\lambda_1$. We recall from Lemma 5.3.5 that either every $H_j$-active vertex of $G$ is a component cut-vertex, or no $H_j$-active vertex of $G$ is a component cut-vertex.

**Lemma 5.4.2** Let $G$ and $H$ be a 2UC graph pair, both of order $n \geq 3$. Suppose that $\beta_2 = 1$ and that $G$ contains no $H_2$-active vertices. Then $b(G,\, H) \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, with equality only if $\mu_2 = 0$. Moreover, this bound is attained for all $n$.

*Proof* By Corollary 5.3.3(a), $a_H(\mathcal{F}) \leq \left\lfloor \frac{|V(\mathcal{F})|}{2} \right\rfloor$. Suppose first that every $H_1$-active vertex of $G$ is a component cut-vertex. Then, by Corollary 5.3.6, $a_H(G_1) \leq \left\lfloor \frac{g_1}{2} \right\rfloor$. So, since $a_{H_2}(G) = 0$,

$$b(G,\, H) \leq (1+\lambda_1)a_{H_1}(G) + a_H(\mathcal{F}) \leq (1+\lambda_1)\left\lfloor \frac{g_1}{2} \right\rfloor + \left\lfloor \frac{|V(\mathcal{F})|}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor,$$

and the result follows.

So suppose instead that no $H_1$-active vertex of $G$ is a component cut-vertex. Then, by Corollary 5.3.19(b), $g_1 = h_2 + 1$. Thus, since $\overline{a_G(H_2)} = h_2$, by (5.22),

$$n \geq 2b(G,\, H) + (1 + \mu_2)h_2 - g_1 \geq 2b(G,\, H) - 1,$$

and equality holds in this expression only if $\mu_2 = 0$. We will present a family of graph pairs in Example 5.5.1 that shows this bound is attained for all $n$. □

We note that, an identical proof would show that we have the same bound for $b(G,\, H)$ if $\beta_2 = 1$ and that $G$ contains no $H_1$-active vertices. In light of this, for the rest of this section we only consider 2UC graph pairs where either $\beta_1 = 2$, or $\beta_1 = 1$ and $G$ contains both $H_1$ and $H_2$-active vertices.

We now prove a bound on $b(G,\, H)$ when there exists precisely one component isomorphic to $G_1$ in $G$ and, in addition, every active vertex in this component is not a cut-vertex.

**Lemma 5.4.3** Let $G$ and $H$ be a 2UC graph pair, both of order $n \geq 3$. Suppose that none of the active vertices in components isomorphic to $G_1$ are cut-vertices. If $\lambda_1 = 0$, then $b(G,\, H) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$. Moreover, this bound is attained for all $n$.

*Proof* By Lemma 5.4.2, we may assume that if $\beta_1 = \beta_2 = 1$, then $G$ contains both $H_1$ and $H_2$-active vertices. It therefore follows by Corollary 5.3.19, that if $\beta_2 = 0$, then $h_1 = g_2 + 1$ and $g_1 = h_1 + 1$, and if $\beta_2 = 1$, then $h_1 = h_2 = g_2 + 1$ and $g_1 = h_1 + 1$. So $g_1 = g_2 + 2$ in either case, noting that by part (f) of that corollary, this relationship still holds if $\alpha_2 = 0$. Therefore, since by (5.17), $b_1 + b_2 \leq g_1$ and by Corollary 5.3.3(a), $|V(\mathcal{F})| \geq 2b_{\mathcal{F}}$, it follows from (5.15) that,

$$n \geq g_1 + \alpha_2 g_2 + |V(\mathcal{F})| \geq b(G,\, H) + b(G,\, H) - 2 = 2b(G,\, H) - 2,$$

and the result follows. As noted by Harary and Manvel [19], the bound is attained by the pair $G = K_{p+1} \oplus K_{p-1}$ and $H = K_p \oplus K_p$. □

We now prove tighter bounds on $b(G, H)$ when some active vertex in a component isomorphic to $G_1$ is a component cut-vertex. Note that, as explained following Lemma 5.3.5, if $G$ contains an $H_2$-active component cut-vertex then all the active vertices of $G$ must be cut-vertices. We first give the bound when all the active vertices of $G$ are cut-vertices.

**Lemma 5.4.4** Let $G$ and $H$ be a 2UC graph pair of order $n \geq 3$. Suppose that all the active vertices of $G$ are component cut-vertices. Then $b(G, H) \leq \lfloor \frac{n}{2} \rfloor$ unless $G$ and $H$ are one of the four exceptional graph pairs given in Examples 3.3.1 and 3.3.2 (in which case, $b(G, H) = \lfloor \frac{n}{2} \rfloor + 1$).

*Proof* Suppose that $b_1 + b_2 \leq \left\lfloor \frac{n-|V(\mathcal{F})|}{2} \right\rfloor$. Then, since $b(G, H) = b_1 + b_2 + b_{\mathcal{F}}$ and $b_{\mathcal{F}} \leq \left\lfloor \frac{|V(\mathcal{F})|}{2} \right\rfloor$, it follows that $b(G, H) \leq \lfloor \frac{n}{2} \rfloor$. We may therefore assume that

$$\begin{aligned} 2(b_1 + b_2) > n - |V(\mathcal{F})| &= (1 + \lambda_1)g_1 + \mu_1 h_1 + \mu_2 h_2 \\ &= (1 + \mu_1)h_1 + (1 + \mu_2)h_2 + \lambda_1 g_1. \end{aligned} \tag{5.24}$$

Clearly, (5.24) does not hold if $\overline{a_H(G)} \leq \lfloor \frac{g_1}{2} \rfloor$, since $b_1 + b_2 \leq (1 + \lambda_1)a_H(G)$. Therefore, by Corollary 5.3.7, we may also assume that $\beta_1 = \beta_2 = 1$ and, moreover, $G$ contains both $H_1$ and $H_2$-active vertices (so $b_1 \geq 1$ and $b_2 \geq 1$).

Suppose first that $\alpha_1 + \alpha_2 = 2$. Then, by Lemma 5.3.8, $a_G(H_1) \leq \lfloor \frac{h_1}{2} \rfloor$ and $a_G(H_2) \leq \lfloor \frac{h_2}{2} \rfloor$, so (5.24) does not hold, since $b_1 + b_2 \leq (1 + \mu_1)a_G(H_1) + (1 + \mu_2)a_G(H_2)$. We may therefore assume that $\alpha_2 = 0$, so $g_1 = h_1 + h_2$. So, by Lemma 5.3.9, $h_1 > h_2$, $2 \leq h_2 \leq 3$ and, moreover,

$$a_{H_1}(G_1) \leq \left\lfloor \frac{g_1}{h_2 + 1} \right\rfloor \qquad \text{and} \qquad a_{H_2}(G_1) \leq \left\lfloor \frac{g_1}{h_2} \right\rfloor. \tag{5.25}$$

We now determine another relation between some of the variables in (5.22) and the orders of $g_1$ and $h_2$, for these particular types of 2UC graph pairs.

Suppose that

$$(1+\lambda_1)\overline{a_H(G_1)}+2\overline{b_1(G)}+\overline{b_2(G)}+\overline{b_2(H)}+\mu_1 g_1 \geq (1+\mu_1)h_2+(1+\lambda_1)a_{H_1}(G_1). \tag{5.26}$$

Then

$$\begin{aligned}(1+\lambda_1)\overline{a_H(G_1)}+\overline{b_1(G)}+\overline{b_2(G)}+\overline{b_1(H)}+\overline{b_2(H)} &\quad + \\ ((1+\mu_1)(g_1-h_2)-(1+\lambda_1)a_{H_1}(G_1)+\overline{b_1(G)}-\overline{b_1(H)}) &\quad \geq \quad g_1.\end{aligned}$$

Now, since $g_1 = h_1 + h_2$ and $h_1 = a_G(H_1) + \overline{a_G(H_1)}$, it follows from (5.20) that

$$(1+\mu_1)\overline{a_G(H_1)} = (1+\mu_1)(h_1 - a_G(H_1)) = (1+\mu_1)(g_1-h_2)-(1+\lambda_1)a_{H_1}(G_1)+\overline{b_1(G)}-\overline{b_1(H)},$$

thus

$$(1+\lambda_1)\overline{a_H(G_1)}+\overline{b_1(G)}+\overline{b_2(G)}+\overline{b_1(H)}+\overline{b_2(H)}+(1+\mu_1)\overline{a_G(H_1)} \geq g_1.$$

Therefore, if (5.26) holds, $b(G,\, H) \leq \left\lfloor \frac{n}{2} \right\rfloor$ by (5.22). We show the assumption (5.24) implies that (5.26) holds unless $\lambda_1 = \mu_1 = \mu_2 = 0$. We consider the three cases: (I) $h_2 = 3$; (II) $h_2 = 2$ and $G_1$ is a path; (III) $h_2 = 2$ and $G_1$ is not a path. Note that in case (I), $g_1 \geq 7$ and in cases (II) and (III), $g_1 \geq 5$.

(I) Suppose that $h_2 = 3$. Then by (5.25), $a_{H_1}(G) \leq \left\lfloor \frac{g_1}{4} \right\rfloor$ and $a_{H_2}(G) \leq \left\lfloor \frac{g_1}{3} \right\rfloor$, so $\overline{a_H(G_1)} \geq g_1 - (\left\lfloor \frac{g_1}{4} \right\rfloor + \left\lfloor \frac{g_3}{3} \right\rfloor)$. Simple calculations show that (5.26) holds unless $\lambda_1 = \mu_1 = 0$ and $g_1 = 8$, 9 or 12. So suppose that this is the case. Then since $n - |V(\mathcal{F})| = g_1 + 3\mu_2$, and $b_1 + b_2 \leq \left\lfloor \frac{g_1}{4} \right\rfloor + \left\lfloor \frac{g_1}{3} \right\rfloor$, it is easy to see that $2(b_1 + b_2) \leq n - |V(\mathcal{F})|$ for any of these values of $g_1$, unless $\mu_2 = 0$. So (5.24), does not hold unless $\mu_2 = 0$ also.

We now deal with cases (II) and (III). Note that, if $U$ is a component of $G$ isomorphic to $G_1$, then for any $H_1$-active vertex $u$ in $U$, $U - u$ contains some component isomorphic to $K_2$, and for any $H_1$-active vertex $v$ in $U$, $U - v \cong H_1 \oplus K_1$. In addition, clearly $\overline{a_G(H_2)} = 0$.

(II) Suppose that $h_2 = 2$ and $G_1 \cong P_k$, for $k \geq 5$. Then, since there are only two leaf-adjacent vertices in $G_1$, it follows that $H_1 \cong P_{k-2}$ and $a_G(H_2) = a_{H_2}(G_1) = 2$, thus $b_2 \leq \min(2(1+\lambda_1),\, 2(1+\mu_2))$. In addition, it is easy to see that $a_{H_1}(G_1) = 1$ for $k = 5$, and $a_{H_1}(G_1) = 2$ for $k \geq 6$. So $\overline{a_H(G_1)} \geq a_G(H_1)$ for all values of $k$. Therefore, if $\mu_1 \geq 1$, the inequality (5.26) holds immediately. We may therefore assume that $b_1 \leq (\lambda_1 + 1)a_{H_1}(G) \leq 2$.

Now since $\mu_1 = 0$,

$$\begin{aligned}
(\lambda_1 + 1)g_1 + 2\mu_2 &= (k-2)(1+\lambda_1) + 2(1+\lambda_1) + 2\mu_2 \\
&\geq (k-2)(1+\lambda_1) + b_2 + (b_2 - 2) + 2(b_1 - 2) \\
&\geq (k-2)(1+\lambda_1) + 2(b_1 + b_2) - 6. \qquad (5.27)
\end{aligned}$$

So (5.24) does not hold when $\lambda_1 \geq 1$. But if $\lambda_1 = 0$, then $b_2 \leq 2$, and it is easy to see that (5.24) cannot hold in this case unless $\mu_2 = 0$ also.

(III) Suppose now that $h_2 = 2$ and $G_1$ is not a path. By Lemma 5.3.17,

$$(1+\lambda_1)\overline{a_H(G)} \geq (1+\lambda_1)\max(a_{H_1}(G),\, a_{H_2}(G)) = \max(b_1(G) + \overline{b_1(G)},\, b_2 + \overline{b_2(G)}).$$

So, if either $\mu_1 \geq 1$, $\overline{b_1(G)} \geq 1$ or $\overline{b_2(H)} \geq 2$, then since $h_2 \geq 2$ and $g_1 \geq 5$, (5.26) holds. We therefore assume that none of these conditions apply. In this case, since $b_2 \leq 2(1+\mu_2)$ and $h_2 = 2$,

$$n - |V(\mathcal{F})| = (1+\lambda_1)g_1 + 2\mu_2 \geq b_1 + 2b_2 - 2 + \max(b_1,\, b_2 + \overline{b_2(G)}).$$

So by (5.24), we only need to consider the case where $b_2 \leq b_1 + 1$. We recall that by (5.25), $a_{H_1}(G) \leq \left\lfloor \frac{g_1}{3} \right\rfloor$.

Suppose that $b_2 = 2$. Then,

$$2(b_1 + b_2) \leq 2(1+\lambda_1)\left\lfloor \tfrac{g_1}{3} \right\rfloor + 4 \leq (1+\lambda_1)g_1 + 2\mu_2,$$

unless $(1+\lambda_1)g_1 + 6\mu_2 \leq 11$. So (5.24) does not hold unless this condition is met. However, straightforward calculations show that this equality only holds when $\lambda_1 = \mu_2 = 0$, so we are done in this case. We are therefore left to consider the case when $b_2 \geq 3$, so $b_1 \geq 2$ and $\mu_2 \geq 1$.

Suppose now that $b_1 \geq 5$. Then by Lemma 5.3.18(a),

$$(1+\lambda_1)\overline{a_H(G_1)} \geq (1+\lambda_1)a_{H_1}(G_1) + \tfrac{b_1-1}{3} \geq (1+\lambda_1)a_{H_1}(G_1) + 2.$$

Thus, the inequality (5.25) clearly holds.

Suppose next that $b_1 = 4$, so $(1+\lambda_1)g_1 \geq (1+\lambda_1)a_{H_1}(G) \geq 12$ and thus $n - |V(\mathcal{F})| \geq 14$, since $\mu_2 \geq 1$; so we may assume that $b_2 = 4$ or $b_2 = 5$. By Lemma 5.3.18(b),

$$(1+\lambda_1)\overline{a_H(G_1)} \geq (1+\lambda_1)a_{H_1}(G_1) + \tfrac{b_1+2}{3} = 6.$$

Thus, it follows that when $b_2 = 4$, $(1+\lambda_1)g_1 + 2\mu_2 \geq b_1 + b_2 + (1+\lambda_1)\overline{a_H(G)} + 2\mu_2 \geq 16$ and, similarly, when $b_2 = 5$ (so $\mu_2 = 2$), $(1+\lambda_1)g_1 + 2\mu_2 \geq 19$. Since both of these cases would contradict (5.24), the case $b_1 = 4$ cannot occur.

Suppose now that $b_1 = (1+\lambda_1)a_{H_1}(G) = 3$. Then, since $b_2 \geq 3$, we may apply Lemma 5.3.18(d). Thus, $(1+\lambda_1)\overline{a_H(G_1)} \geq (1+\lambda_1)a_{H_1}(G_1) + \frac{b_1+1}{2} \geq 5$, so (5.26) holds.

Finally, suppose that $b_1 = 2$ and $b_2 = 3$. Then, again by Lemma 5.3.18(d), $(1+\lambda_1)\overline{a_H(G_1)} \geq 3$, so $(1+\lambda_1)g_1 \geq b_1 + b_2 + (1+\lambda_1)\overline{a_H(G_1)} \geq 8$, and thus $(1+\lambda_1)g_1 + \mu_2 \geq 10$. This again contradicts (5.24).

This completes the three cases; that is we have shown that when any of $\lambda_1$, $\mu_1$ or $\mu_2$ are not zero, $b(G, H) \leq \left\lfloor \frac{n}{2} \right\rfloor$. So to complete the proof, we now suppose that $\lambda_1 = \mu_1 = \mu_2 = 0$. In this case, $G \cong \mathcal{G} \oplus \mathcal{F}$, $H \cong \mathcal{H} \oplus \mathcal{F}$, and moreover, $b_1 + b_2$ is the number of common cards between a connected and disconnected graph, in which neither of the components of the disconnected graph is an isolated vertex. By Lemma 3.3.4, $b_1 + b_2 \leq \left\lfloor \frac{g_1}{2} \right\rfloor$, unless $\mathcal{G}$ and $\mathcal{H}$ are one of the four exceptional graph pairs. Moreover, $b_1 + b_2 = \left\lfloor \frac{g_1}{2} \right\rfloor + 1$, in any of these exceptional cases. Since $b_{\mathcal{F}} \leq \frac{|V(\mathcal{F})|}{2}$, it follows that $b(G, H) \leq \left\lfloor \frac{n}{2} \right\rfloor$ in all cases, and moreover, the bound is only attained when $\mathcal{F}$ is the null graph. This completes the proof. $\square$

The above result shows that the highest number of common cards between a 2UC graph pair in which every active vertex is a component cut-vertex occurs when $G$ is connected and $H$ is disconnected. Moreover, there are only four pairs of graphs of order at most seven with $b(G, H) > \left\lfloor \frac{n}{2} \right\rfloor$. We now consider the case when every $H_1$-active vertex of $G$ is a component cut-vertex, and every $H_2$-active vertex of $G$ is not a component cut-vertex.

**Lemma 5.4.5** Let $G$ and $H$ be a 2UC graph pair, both of order $n \geq 3$. Suppose that each component of $G$ isomorphic to $G_1$ contains some active vertices that are cut-vertices and some active vertices that are not. Suppose further that $G_1$ is not isomorphic to either $P_4$ or $P_3$. Then $b(G,\, H) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and, moreover, this bound is attained for all $n$.

*Proof* By Lemma 5.3.5, $\beta_1 = \beta_2 = 1$ and, in addition, $G$ contains $H_1$-active vertices and $H_2$-active vertices. Furthermore, as noted following that lemma, if the $H_2$-active vertices are cut-vertices then so are the $H_1$-active vertices. Therefore it follows that the $H_1$-active vertices must be cut-vertices and the $H_2$-active vertices cannot be cut-vertices.

We show that

$$\begin{aligned} \overline{b_1(G)} + \overline{b_2(G)} + \overline{b_1(H)} + \overline{b_2(H)} + (1+\lambda_1)\overline{a_H(G_1)} \quad & + \\ (1+\mu_1)\overline{a_G(H_1)} + (1+\mu_2)\overline{a_G(H_2)} + \overline{b_{\mathcal{F}}(G)} + \overline{a_H(\mathcal{F})} \quad & \geq \quad g_1 - 2, \qquad (5.28) \end{aligned}$$

unless $G_1$ is either $P_3$ or $P_4$. The result will then follow from (5.22). Note that $g_1 \geq 4$, since $G_1$ contains a component cut-vertex and $G_1 \not\cong P_3$.

Suppose first that $G_1 \cong P_k$, for $k \geq 5$. Then $H_1 \cong P_{k-1}$, $H_2 \cong K_1$ and $a_{H_1}(G) = a_{H_2}(G) = a_G(H_1) = 2$. So $\overline{a_H(G_1)} = k - 4$ and $\overline{a_G(H_1)} = k - 2$, and since $g_1 = k$, it is easy to see that the inequality (5.28) holds. We may therefore assume that $G_1$ is not a path of length 5 or more.

We now consider the two cases: (I) $\alpha_2 = 0$ and (II) $\alpha_2 = 1$. By Corollary 5.3.19, in case (I), $h_2 = 1$, and in case (II), $h_2 = g_2 + 1 \geq 2$; in both cases, $g_1 = h_1 + 1$. Note that, if $\mu_1 \geq 1$ and $\overline{a_G(H_1)} \geq \frac{h_1}{2}$, then (5.28) clearly holds; we therefore assume that this is never the case. We make frequent use of (5.20) and (5.21).

(I) $\alpha_2 = 0$ and $h_2 = 1$.

Suppose first that no $H_2$-active vertex is a leaf. Then by Corollary 5.3.28, $\overline{a_H(G_1)} \geq a_{H_1}(G_1)$, thus by (5.20),

$$\begin{aligned}(1+\lambda_1)\overline{a_H(G_1)} &\geq (1+\lambda_1)a_{H_1}(G_1)\\ &\geq (1+\mu_1)a_G(H_1) - \overline{b_1(H)} + \overline{b_1(G)}\\ &\geq (1+\mu_1)(h_1 - \overline{a_G(H_1)}) - \overline{b_1(H)} + \overline{b_1(G)}.\end{aligned}$$

Therefore,

$$(1+\lambda_1)\overline{a_H(G_1)} + (1+\mu_1)\overline{a_G(H_1)} + \overline{b_1(H)} \geq (1+\mu_1)h_1 \geq g_1 - 1, \qquad (5.29)$$

so (5.28) holds.

Suppose instead that every $H_2$-active vertex is a leaf. Now if $H_1 \cong P_4$, then since $G_1$ is not a path, it is easy to see that $G_1$ consists of a path of length four with an additional leaf adjacent to one of the leaf-adjacent vertices. In this case, $\overline{a_H(G_1)} = 1$ and $\overline{a_G(H_1)} = \overline{a_G(H_2)} = 0$. Since $g_1 = 5$, the inequality (5.28) holds. We may therefore assume that $H_1 \not\cong P_4$.

Now, by Corollary 5.3.33, $2(\overline{a_H(G_1)} + \overline{a_G(H_1)}) \geq g_1 - 2$. Thus, (5.28) clearly holds if both $\lambda_1 \geq 1$ and $\mu_1 \geq 1$. So suppose that $\lambda_1 = 0$ and $\mu_1 \geq 1$, so $\overline{a_G(H_1)} \leq \frac{h_1}{2}$. Then $a_{H_1}(G_1) \leq \frac{g_1}{2}$, by Corollary 5.3.6. So since $g_1 = a_{H_1}(G_1) + a_{H_2}(G_1) + \overline{a_H(G_1)}$, it follows that
$(a_{H_2}(G_1) + \overline{a_H(G_1)}) \geq a_{H_1}(G)$, so

$$\overline{b_1(H)} + (a_{H_2}(G_1) + \overline{a_H(G_1)}) \geq (1+\mu_1)a_G(H_1).$$

Now if $\overline{a_H(G_1)} \geq a_{H_1}(G_1)$, then (5.28) will hold, using a similar proof to that involved in (5.29). So we may assume that this is not the case, thus by Corollary 5.3.32, $\overline{a_G(H_1)} \geq a_{H_2}(G) - 1$. Therefore, since $\mu_1 \geq 1$ and $a_G(H_1) \geq \frac{h_1}{2}$,

$$\overline{b_1(H)} + (\overline{a_G(H_1)} + \overline{a_H(G_1)}) - 1 \geq (1+\mu_1)a_G(H_1) - 1 \geq h_1 - 1.$$

So again, (5.28) holds.

Finally, suppose that $\mu_1 = 0$. Then by Corollary 5.3.35, $\overline{a_H(G_1)} \geq b_1 - 1$. Therefore,

$$(1+\lambda_1)\overline{a_H(G_1)} + \overline{b_1(H)} + \overline{a_G(H_1)} \geq a_G(H_1) + \overline{a_G(H_1)} - 1 = h_1 - 1,$$

and again (5.28) holds.

(II) $\alpha_2 = 1$ and $h_2 = g_2 + 1$.

By Corollaries 5.3.26 and 5.3.37, at least one of the following hold:
(i) $\overline{a_H(G_1)} \geq a_{H_1}(G_1)$; (ii) $\overline{a_H(G_1)} \geq a_G(H_1) - 1$; (iii) $\overline{a_G(H_2)} \geq a_G(H_1) - 1$. In case (i), it is easy to show that (5.28) holds as in Case (I). In case (ii),

$$(1+\lambda_1)\overline{a_H(G_1)} + (1+\mu_1)\overline{a_G(H_1)} \geq h_1 - 1,$$

whilst in case (iii),

$$(1+\mu_2)\overline{a_G(H_2)} + (1+\mu_1)\overline{a_G(H_1)} \geq h_1 - 1.$$

Thus, in any of these three cases, (5.28) holds, which completes the proof.

We present a family of graph pairs in Example 5.5.2 that shows this bound is attained for all $n$. □

We now consider the exceptional case, that is when $G_1 \cong P_4$ or $G_1 \cong P_3$. In the former case, $H_1 \cong P_3$ whilst in the latter case, $H_1 \cong K_2$; in both cases $H_2 \cong K_1$.

**Lemma 5.4.6** Suppose that $G$ and $H$ are either of the following families of 2UC graph pairs, both of order $n \geq 3$:

(a) $G \cong (P_4) \oplus (\lambda_1 P_4 \oplus \mu_1 P_3 \oplus \mu_2 K_1)$ and $H \cong (P_3 \oplus K_1) \oplus \lambda_1 P_4 \oplus \mu_1 P_3 \oplus \mu_2 K_1)$;

(b) $G \cong (P_3) \oplus (\lambda_1 P_3 \oplus \mu_1 K_2 \oplus \mu_2 K_1)$ and $H \cong (P_2 \oplus K_1) \oplus (\lambda_1 P_3 \oplus \mu_1 K_2 \oplus \mu_2 K_1)$.

Then $b(G, H) \leq \left\lfloor \frac{n+3}{2} \right\rfloor$. Furthermore, $b(G, H) = \frac{n+3}{2}$ if and only if

(i) $G = (P_4) \oplus (K_1)$ and $H = (P_3 \oplus K_1) \oplus (K_1)$;

(ii) $G = (P_3) \oplus ((2\beta_1 + 1)P_3 \oplus \beta_1 K_2 \oplus (4\beta_1 + 3)K_1)$
$H = (K_2 \oplus K_1) \oplus ((2\beta_1 + 1)P_3 \oplus \beta_1 K_2 \oplus (4\beta_1 + 3)K_1)$, for any $\beta \geq 0$.

*Proof* (a) In any component of $G$ isomorphic to $G_1$, the $H_1$-active vertices are its leaf-adjacent vertices, and the $H_2$-active vertices are its leaves. The active vertices of $H$ are the leaves of the components isomorphic to $H_1$, plus the vertices of the components isomorphic to $K_1$. So $\overline{a_H(G_1)} = \overline{a_G(H_2)} = 0$ and $\overline{a_G(H_1)} = 1$. Thus,
$b_1 = \min(2(\lambda_1+1),\, 2(\mu_1+1))$, $b_2 = \min(2(\lambda_1+1),\, \mu_2+1)$, and it follows that $\overline{b_1(G)} + \overline{b_1(H)} = |2\lambda_1 - 2\mu_1|$ and $\overline{b_2(G)} + \overline{b_2(H)} = |2\lambda_1 + 1 - \mu_2|$. Therefore by (5.22),

$$n = 2b(G,\, H) + (\mu_1 + 1) + |2\lambda_1 - 2\mu_1| + |2\lambda_1 + 1 - \mu_2| - 4.$$

So the bound holds for (a), with equality if and only if $\lambda_1 = \mu_1 = 0$ and $\mu_2 = 1$.

(b) In any component of $G$ isomorphic to $G_1$, the $H_1$-active vertex is its leaf-adjacent vertex, and the $H_2$-active vertices are its leaves. The active vertices of $H$ are the vertices of the components isomorphic to $H_1$, plus the vertices of the components isomorphic to $K_1$. So $\overline{a_H(G_1)} = \overline{a_G(H_1)} = \overline{a_G(H_2)} = 0$. Thus,
$b_1 = \min((\lambda_1+1),\, 2(\mu_1+1))$, $b_2 = \min(2(\lambda_1+1),\, \mu_2+1)$, and it follows that $\overline{b_1(G)} + \overline{b_1(H)} = |\lambda_1 - 2\mu_1 - 1|$ and $\overline{b_2(G)} + \overline{b_2(H)} = |2\lambda_1 + 1 - \mu_2|$. Therefore, by (5.22),

$$n = 2b(G,\, H) + |\lambda_1 - 2\mu_1 - 1| + |2\lambda_1 + 1 - \mu_2| - 3. \tag{5.30}$$

The bound in (b) thus holds, with equality if and only if
$|\lambda_1 - 2\mu_1 - 1| = |2\lambda_1 + 1 - \mu_2| = 0$; that is $\lambda_1 = 2\mu_1 + 1$ and $\mu_2 = 2\lambda_1 + 1 = 4\mu_1 + 3$.
□

This completes the bounds for the $b(G,\, H)$ when $G$ contains an $H_1$-active component cut-vertex. We now have bounds for all 2UC graph pairs, depending on whether they contain a component cut-vertex or not, and the value of $\lambda_1$. We now concentrate on finding 2UC graph pairs that attain the various bounds.

## 5.5 2UC Graph Pairs that Attain the Bound

In this section, we give some examples of graphs that attain the bounds of the previous section. In particular, we show that for $n \geq 10$, no 2UC graph pair has $b(G, H) \leq 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$ and, for $n \geq 22$, that this bound is attained by one of four graph pairs, up to isomorphism. Since we are only interested in pairs with $b(G, H) > \left\lfloor \frac{n}{2} \right\rfloor$, we again assume that $G$ and $H$ are of the form given in (5.14).

We begin by presenting examples of 2UC graph pairs that attain the bounds of Lemmas 5.4.2 and 5.4.5.

**Example 5.5.1** Let $p$ be an integer, $p \geq 1$. Then, for $n = 2p + 1$, the following 2UC graph pair has $\frac{n+1}{2}$ common cards, so attains the bound of Lemma 5.4.2:

$$\begin{aligned} G &= (K_{p+1}) \oplus (pK_1) \\ H &= (K_p \oplus K_1) \oplus (pK_1). \end{aligned} \tag{5.31}$$

The removal of any vertex of the $K_{p+1}$ component of $G$ and any of the isolated vertices of $H$ gives isomorphic cards. There are $p + 1$ such cards, so
$b(G, H) = p + 1 = \frac{n+1}{2}$. Figure 5.7 shows these graphs for $p = 5$. □

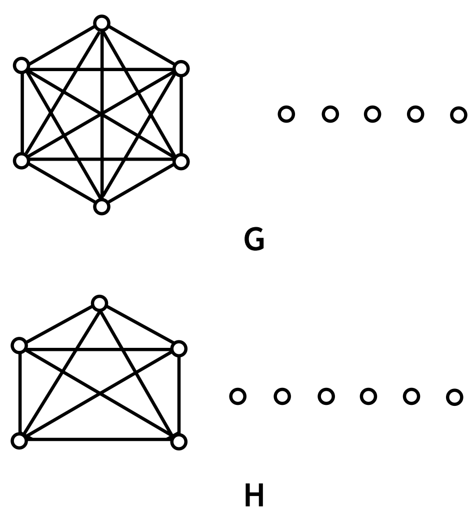


Figure 5.7: The pair of graphs in Example 5.5.1 of order 11 with 6 common cards.

**Example 5.5.2** Let $p$ be an integer, $p \geq 1$. Then, for $n = 2(p+1)$, the following 2UC graph pair has $\frac{n}{2} + 1$ common cards, so attains the bound of Lemma 5.4.5:

$$\begin{aligned} G &= (S^1_{p+1}) \oplus (pK_1) \\ H &= (S^1_p \oplus K_1) \oplus (pK_1). \end{aligned} \tag{5.32}$$

The removal of any leaf of the $S^1_{p+1}$ component of $G$ and any of the isolated vertices of $H$ gives isomorphic cards. In addition, the removal of the cut-vertex of the $S^1_{p+1}$ component of $G$ and the cut-vertex of the $S^1_p$ component of $H$ gives an isomorphic card. Since $G$ contains $p+1$ leaves, $b(G, H) = p + 2 = \frac{n}{2} + 1$. Figure 5.8 shows these graphs for $p = 5$. □

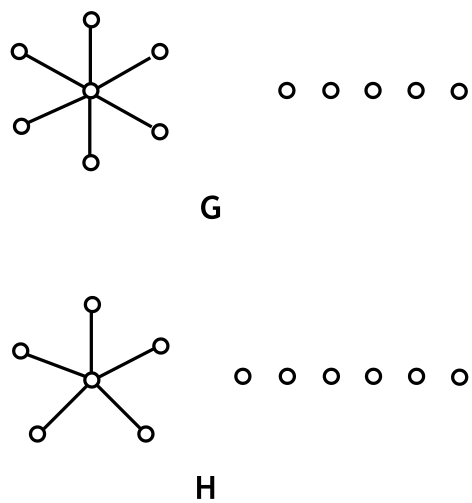


Figure 5.8: The pair of graphs in Example 5.5.2 of order 12 with 7 common cards.

The next few results show that we need only consider certain 2UC graph pairs when $n \geq 22$.

**Corollary 5.5.3** Let $G$ and $H$ be a 2UC graph pair of order $n$. Suppose that $\beta_2 = 1$ and that $G$ contains no $H_2$-active vertices.

(a) If $n \geq 10$, then $b(G, H) \leq 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$.

(b) If $n \geq 16$, then $b(G, H) < 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$.

*Proof* By Lemma 5.4.2, $b(G, H) \leq \left\lfloor \frac{(n+1)}{2} \right\rfloor$. The result then follows by simple calculations. □

**Corollary 5.5.4** Let $G$ and $H$ be a 2UC graph pair of order $n$. Suppose that $G$ contains an $H_1$-active component cut-vertex.

(a) If $n \geq 13$, then $b(G,\,H) \leq 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$.

(b) If $n \geq 22$, then $b(G,\,H) < 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$.

*Proof* By Corollary 5.5.3, we may assume that if $\beta_2 = 1$, then $G$ contains both $H_1$ and $H_2$-active vertices. So $b(G,\,H) \leq \left\lfloor \frac{(n+3)}{2} \right\rfloor$, by Lemmas 5.4.4, 5.4.5 and 5.4.6. Both (a) and (b) follow by simple calculations, unless $n = 15$ and $b(G,\,H) = \frac{(n+3)}{2}$. However, for this value of $n$, Lemma 5.4.6 shows that no such 2UC graph pair exist.

□

In light of the above two results, we now concentrate on 2UC graph pairs when no active vertex in a component of $G$ isomorphic to $G_1$ is a component cut-vertex and, if $\beta_2 = 1$, when $G$ contains both $H_1$ and $H_2$-active vertices. By Corollary 5.3.19, $g_1 = h_1 + 1$ (and $h_2 = h_1$ if $\beta_2 = 1$). So, by (5.17), (5.18) and (5.19), it follows that

$$\begin{aligned} h_1(\mu_1 + \mu_2 + 1 - \lambda_1) &= \overline{b_1(H)} + \overline{b_2(H)} + (\beta_1 + \mu_1)\overline{a_G(H_1)} + (\beta_2 + \mu_2)\overline{a_G(H_2)} \\ &- \overline{b_1(G)} - \overline{b_2(G)} - (\lambda_1 + 1)\overline{a_H(G_1)} + (\lambda_1 + 1). \end{aligned} \tag{5.33}$$

We rearrange (5.23) to get

$$b(G,\,H) = \frac{1}{2\lambda_1 + 1}\left\{ \begin{array}{c} (\lambda_1 + 1)\left(n - \overline{b_1(H)} - \overline{b_2(H)} - (\beta_1 + \mu_1)\overline{a(H_1)} - (\beta_2 + \mu_2)\overline{a(H_2)}\right) \\ -\lambda_1\left(\overline{b_1(G)} + \overline{b_2(G)}) + (\lambda_1 + 1)\overline{a(G_1)}\right) \\ -(\lambda_1 + 1)(\overline{b_{\mathcal{F}}}(G) + \overline{a_H(\mathcal{F})}) + \lambda_1 b_{\mathcal{F}}(G) \end{array} \right\} \tag{5.34}$$

and let

$$\begin{aligned} R(H) &= \overline{b_1(H)} + \overline{b_2(H)} + (\beta_1 + \mu_1)\overline{a_G(H_1)} + (\beta_2 + \mu_2)\overline{a_G(H_2)} \\ R(G) &= \overline{b_1(G)} + \overline{b_2(G)} + (\lambda_1 + 1)\overline{a_H(G_1)} \\ R(\mathcal{F}) &= (\lambda_1 + 1)(\overline{b_{\mathcal{F}}}(G) + \overline{a_H(\mathcal{F})}) - \lambda_1 b_{\mathcal{F}}(G), \end{aligned} \tag{5.35}$$

so that (5.33) can be expressed as $h_1(\mu_1 + \mu_2 + 1 - \lambda_1) = R(H) - R(G) + (\lambda_1 + 1)$ and (5.34) can be expressed as $b(G,\,H) = \frac{1}{2\lambda_1+1}\left((\lambda_1 + 1)n - (\lambda_1 + 1)R(H) - \lambda_1 R(G) - R(\mathcal{F})\right)$.

Note that since every $F_k$-active vertex is a component cut-vertex and $\overline{a_H(\mathcal{F})} \geq a_H(\mathcal{F})$, it follows that if $|V(\mathcal{F})| \geq 2$, then $R(\mathcal{F}) \geq 2$, otherwise $R(\mathcal{F}) = |V(\mathcal{F})| = 1$. We first consider the case when $\lambda_1 \geq 2$.

**Lemma 5.5.5** Let $G$ and $H$ be as in 2UC graph pair, both of order $n \geq 10$, such that no active vertex of $G$ is a component cut-vertex and, if $\beta_2 = 1$, then $G$ contains both $H_1$ and $H_2$-active vertices. If $\lambda_1 \geq 2$, then $b(G, H) \leq 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$, with equality only if $n \leq 22$.

*Proof* Suppose first that $\lambda_1 \geq 3$. Then by (5.34), $b(G, H) \leq \lfloor \frac{4n}{7} \rfloor \leq 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$, for $n \geq 10$. In addition, equality holds only if $n \leq 21$, so the result is true in this case.

Suppose instead that $\lambda_1 = 2$ and let $K = 3R(H) + 2R(G) + R(\mathcal{F})$. Then by (5.34), $b(G, H) = \frac{3n-K}{5} \leq 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$ for $n \geq 10$ and $K \geq 2$. In addition, when $n \geq 22$, straightforward calculations show that equality never holds if $K \geq 3$. So the result holds immediately unless $R(H) = 0$, and either $R(G) = 1$ and $R(\mathcal{F}) = 0$, or $R(G) = 0$ or $R(\mathcal{F}) \leq 2$. Now when $K \leq 2$, it is easy to see that $1 \leq h_1 \leq 3$, since $h_1(\mu_1 + \mu_2 - 1) = R(H) - R(G) + 3$ by (5.33). We can thus find all values of $b(G, H)$, $\mu_1 + \mu_2$ and $n$ in this case. The results are summarised in the table below. The result then follows immediately.

| $h_1$ | $\mu_1 + \mu_2$ | $\min n$ | $\max(b(G, H))$ | $\max n$ s.t. $\max(b(G, H)) \geq 2\left\lfloor \frac{n-1}{3} \right\rfloor$ |
|---|---|---|---|---|
| 1 | 3 | 9 | 5 | 9 |
| 1 | 4 | 10 | 6 | 12 |
| 2 | 2 | 13 | 8 | 15 |
| 3 | 2 | 20 | 12 | 21 |

□

An example of a 2UC graph pair with $b(G, H) = 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$ when $\lambda_1 = 2$ and $n = 20$ is presented here.

**Example 5.5.6** For $n = 20$, the following 2UC graph pair has $b(G, H) = 12$:

$$\begin{aligned} G &= (K_4 \oplus K_2) \oplus (2K_4 \oplus 2K_3) \\ H &= (K_3 \oplus K_3) \oplus (2K_4 \oplus 2K_3). \end{aligned} \tag{5.36}$$

The removal of any vertex of $G$ in a component isomorphic to $K_4$ and any vertex of $H$ in a component isomorphic to $K_3$ gives isomorphic cards. So $b(G, H) = 12$. □

By Lemma 5.4.3, $b(G, H) \leq \left\lfloor \frac{(n+1)}{2} \right\rfloor$ when $\lambda_1 = 0$. So, the only case left to consider is when $\lambda_1 = 1$.

**Lemma 5.5.7** Let $G$ and $H$ be as in 2UC graph pair, both of order $n \geq 11$, such that no active vertex of $G$ is a component cut-vertex and, if $\beta_2 = 1$, then $G$ contains both $H_1$ and $H_2$-active vertices. Suppose that $\lambda_1 = 1$. Then $b(G, H) \leq 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$. In addition, when $n \geq 22$, equality holds only if $\mu_1 = \mu_2 = \overline{a_G(H_1)} = \overline{a_G(H_2)} = \overline{b_1(H)} + \overline{b_2(H)} = 0$, and moreover, either $\overline{a_H(G_1)} = 1$ and $\overline{b_1(G)} + \overline{b_2(G)} = 0$, or $\overline{b_1(G)} + \overline{b_2(G)} = 2$ and $\overline{a_H(G_1)} = 0$.

*Proof* Let $K = 2R(H)+R(G)+R(\mathcal{F})$. Then by (5.34), $b(G, H) = \frac{2n-K}{3} \leq 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$, for $n \geq 11$ and $K \geq 4$. In addition, when $n \geq 22$, straightforward calculations show that equality holds only if $K \leq 6$. We therefore assume that $K \leq 6$, so $R(H) \leq 3$. Note that, if $n \geq 22$, and $K = 5$ or $K = 6$, then $b(G, H) < 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$, unless $n \equiv 0 \pmod 3$.

Now by (5.33), $h_1(\mu_1 + \mu_2) = R(H) - R(G) + 2$. So since $R(H) \leq 3$, it follows that $0 \leq h_1(\mu_1 + \mu_2) \leq 5$. Moreover, $h_1(\mu_1 + \mu_2) = 0$ only if $R(G) = R(H) + 2$. We therefore calculate all the possible values for $\max(b(G, H))$, when $1 \leq h_1 \leq 5$ and $\mu_1 + \mu_2 \neq 0$. The results are summarised in the table below. This shows that the result holds immediately except when $R(G) = R(H) + 2$.

| $h_1$ | $\mu_1 + \mu_2$ | $\min n$ | $\max(b(G,\, H))$ | $\max n$ s.t. $\max(b(G,\, H)) \geq 2\left\lfloor \frac{n-1}{3} \right\rfloor$ |
|---|---|---|---|---|
| 1 | 1 | 5 | 3 | 7 |
| 1 | 2 | 6 | 4 | 9 |
| 2 | 1 | 9 | 6 | 12 |
| 3 | 1 | 13 | 8 | 15 |
| 4 | 1 | 17 | 10 | 18 |
| 5 | 1 | 21 | 12 | 21 |

So suppose that $R(G) = R(H)+2$, so $K = 3R(H)+2+R(\mathcal{F})$ and $\mu_1+\mu_2 = 0$. Then it is easy to show that $b(G,\, H) > 2\left\lfloor \frac{n-1}{3} \right\rfloor$, unless $R(H) = 0$, or $R(H) = 1$, noting that in the latter case that $n \equiv 0 \pmod 3$, when $n \geq 22$. Now since $\mu_1 + \mu_2 = 0$, it follows that $n = 3h_1 + 1 + |V(\mathcal{F})|$. So when $n \equiv 0 \pmod 3$, $|V(\mathcal{F})| = 2$. Thus if $R(H) = 1$ and $n \equiv 0 \pmod 3$, $K = 8$ and the bound is not attained. Therefore, the bound is only attained when $R(H) = 0$ and $R(G) = 2$. This completes the proof. □

An example of a 2UC graph pair with $b(G,\, H) = 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$ when $\lambda_1 = 1$ and $n = 21$ is presented here.

**Example 5.5.8** For $n = 21$, the following 2UC graph pair has $b(G,\, H) = 12$:

$$\begin{aligned} G &= (K_6 \oplus K_4) \oplus (K_6 \oplus K_5) \\ H &= (K_5 \oplus K_5) \oplus (K_6 \oplus K_5). \end{aligned} \tag{5.37}$$

The removal of any vertex of $G$ in a component isomorphic to $K_6$ and any vertex of $H$ in a component isomorphic to $K_5$ gives isomorphic cards. So $b(G,\, H) = 12$. □

We now show that only when the components are complete graphs is the bound attained for $n \geq 22$. First we prove the following result.

**Lemma 5.5.9** Let $F$ be a connected graph of order $q$ and let $S \subseteq V(F)$. Suppose that, for every vertex $v$ in $S$, $d(v) = k$, and that $F - v$ is regular. Then precisely one of the following holds.

(a) $F \cong K_q$, $S = V(F)$, so $F - v \cong K_{q-1}$ for all $v$ in $S$;

(b) $|S| \leq \lfloor \frac{q}{2} \rfloor + 1$.

*Proof* Since any connected graph of order 2 or less is complete, we may assume that $F$ is of order 3 or more. In addition, since (b) clearly holds if $|S| = 1$, we may assume that $|S| \geq 2$. We show that (i) if $F \ncong K_q$ and any pair of vertices in $S$ are adjacent then (b) holds, and that (ii), (b) holds if any pair of vertices in $S$ are not adjacent. This implies the result. Let $u$ and $v$ be two vertices in $S$.

(i) Suppose then that $F$ is not complete and that $v$ is adjacent to $u$. Then since $d(u) = k - 1$ in $F - v$, and $F - v$ is regular, the degree of every vertex in $F - v$ must be equal to $k - 1$. So every vertex of $F$ adjacent to $v$ is of degree $k$, and every other vertex of $F$ is of degree $k - 1$. Thus since $d(v) = k$, it follows that there are precisely $q - k - 1$ vertices of $F$ of degree $k - 1$. Therefore, since every vertex of $S$ is of degree $k$, it follows that $q - k - 1 \leq q - |S|$. Now, since $F$ is not complete, $k \neq q - 1$, so there must be at least one vertex of degree $k - 1$ in $F$. Since such a vertex can clearly not be adjacent to any vertex in $S$, it follows that $k - 1 \leq q - |S|$. Thus $|S| \leq \lfloor \frac{q}{2} \rfloor + 1$, and (b) holds.

(ii) Now suppose that $u$ and $v$ are not adjacent, so $d(u) = k$ in $F - v$. Then since $F - v$ is regular, the degree of every vertex in $F - v$ is equal to $k$, and it follows that every vertex of $F$ adjacent to $v$ is of degree $k + 1$, and every other vertex is of degree $k$. Now since $d(v) = k$, there are precisely $k$ vertices of $F$ of degree $k + 1$ and $q - k$ vertices of degree $k$, thus $|S| \leq q - k$. Now since $F$ is connected, $k \geq 1$, so there is at least one vertex of degree $k + 1$. Clearly, any such vertex must be adjacent to every vertex of $S$, so $k + 1 \geq |S|$. Therefore, $|S| \leq \frac{q+1}{2}$ and again (b) holds. □

**Corollary 5.5.10** Let $G$ and $H$ be as a 2UC graph pair such that no active vertex of $G$ is a component cut-vertex and, if $\beta_2 = 1$, then $G$ contains both $H_1$ and $H_2$-active vertices. Suppose that $g_1 \geq 5$, $\overline{a_H(G_1)} \leq 1$ and $\overline{a_G(H_1)} = \overline{a_G(H_2)} = 0$. Then $G_1 \cong K_{g_1}$, $H_1 \cong K_{g_1 - 1}$ and $\beta_1 = 2$.

*Proof* $a_H(G_1) > \left\lfloor \frac{g_1}{2} \right\rfloor + 1$, since $g_1 \geq 5$ and $\overline{a_H(G_1)} \leq 1$. Let $U$ be a component of $G$ isomorphic to $G_1$ and let $W_1$ and $W_2$ be the two components in $\mathcal{H}$. Then for all vertices $w_1$ and $w_2$ in $W_1$ and $W_2$, respectively, $W_1 - w_1 \cong W_2 - w_2 \cong G_2$ by Corollary 5.3.19, so both $W_1$ and $W_2$ must be regular. Moreover, since $\mathcal{D}(W_1)$ and $\mathcal{D}(W_2)$ are identical, it follows from Lemma 2.4.2 that $W_1$ and $W_2$ must have the same degree sequence, so $W_1$ and $W_2$ are both regular of the same degree. Now again by Corollary 5.3.19, for active vertex $v$ in $U$, either $U - v \cong W_1$ or $U - v \cong W_2$. Thus every active vertex in $U$ is of the same degree. Therefore, setting $S = A_H(U)$ in Lemma 5.5.9, it follows $U \cong K_{g_1}$. So, since every card of a complete graph with $g_1$ vertices is a complete graph with $g_1 - 1$ vertices, $H_{g_1-1}$ and $\beta_2 = 2$, so the result holds. □

The above results give the following theorem.

**Theorem 5.5.11** Let $G$ and $H$ be a 2UC graph pair, both of order $n$.

(a) For $n \geq 13$, the maximum value of $b(G, H)$ is $2\left\lfloor \frac{1}{3}(n-1) \right\rfloor$. Moreover, this bound is attained for all such $n$.

(b) Suppose that $n \geq 22$ and $b(G, H) = 2\left\lfloor \frac{1}{3}(n-1) \right\rfloor$. If $n \equiv 1$ or $2 \pmod 3$ then $G$ and $H$ are unique (see Examples 5.5.12 and 5.5.13), whereas if $n \equiv 0 \pmod 3$ then $G$ and $H$ are one of precisely two pairs of graphs (see Example 5.5.14).

(c) For $n \leq 12$, there are a small number of 2UC graph pairs exceeding the bound in (a), but in all cases $b(G, H) \leq \left\lfloor \frac{2}{3}(n+1) \right\rfloor$.

*Proof* By Corollary 5.3.3, we may assume that $G$ and $H$ can be expressed as (5.14).

(a) This follows from Lemma 5.4.3, Corollaries 5.5.3 and 5.5.4 and Lemmas 5.5.5 and 5.5.7. Examples 5.5.12, 5.5.13 and 5.5.14 show that the bound is attained for all such $n$.

(b) The same results show that for $n \geq 22$, $b(G, H) = 2\left\lfloor \frac{1}{3}(n-1) \right\rfloor$ only when $\lambda_1 = 1$, $\mu_1 = \mu_2 = 0$, $\overline{a_H(G_1)} \leq 1$ and $\overline{a_G(H_1)} = \overline{a_G(H_2)} = 0$. Since $n \geq 22$, it follows that $g_1 \geq 5$, so by Corollary 5.5.10, $G_1 \cong K_{g_1}$, $H_1 \cong K_{g_1-1}$ and $\beta_1 = 2$. Simple calculations show that $|V(\mathcal{F})|$ must be of order 0, when $n \equiv 1 \pmod 3$, of order 1 when $n \equiv 2 \pmod 3$, and of order 2 when $n \equiv 0 \pmod 3$.

(c) Noting that $\left\lfloor \frac{n+3}{2} \right\rfloor \leq \left\lfloor \frac{2}{3}(n+1) \right\rfloor$, (c) holds by Lemmas 5.4.1 to 5.4.6. $\square$

**Example 5.5.12** Let $p$ be an integer, $p \geq 1$. Then, for $n = 3p+1$, the following 2UC graph pair has $\frac{2(n-1)}{3}$ common cards, so attains the bound of Theorem 5.5.11:

$$\begin{aligned} G &\cong (K_{p+1} \oplus K_{p-1}) \oplus (K_{p+1}) \\ H &\cong (K_p \oplus K_p) \oplus (K_{p+1}). \end{aligned}$$

The removal of any vertex from a component of $G$ isomorphic to $K_{p+1}$, and any vertex from a component of $H$ isomorphic to $K_p$ gives isomorphic cards. So $b(G, H) = 2p = \frac{2(n-1)}{3}$. Figure 5.9 shows these graphs for $p = 6$. $\square$

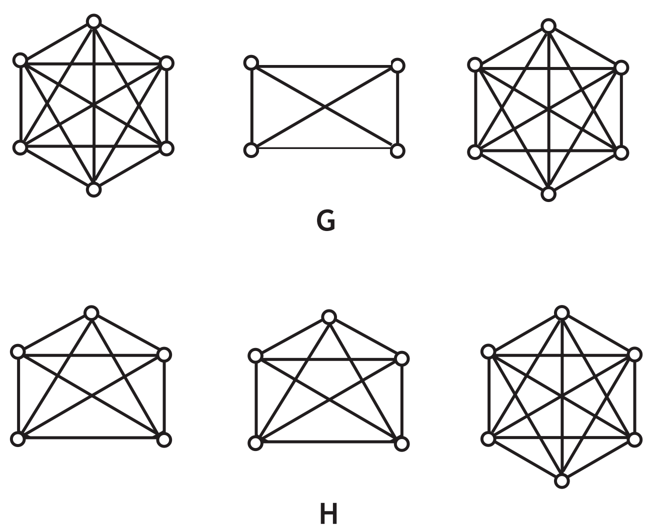


Figure 5.9: The pair of graphs in Example 5.5.12 of order 16 with 10 common cards.

For $n \not\equiv 1 \pmod 3$, we must ensure $|V(\mathcal{F})| \leq 2$. In each of the examples, the common cards are formed in an identical manner to those are in Example 5.5.12.

**Example 5.5.13** Let $p$ be an integer, $p \geq 1$. Then, for $n = 3p + 2$, the following 2UC graph pair has $\frac{2(n-2)}{3}$ common cards, so attains the bound of Theorem 5.5.11:

$$\begin{aligned} G &\cong (K_{p+1} \oplus K_{p-1}) \oplus (K_{p+1} \oplus K_1) \\ H &\cong (K_p \oplus K_p) \oplus (K_{p+1} \oplus K_1). \end{aligned}$$

□

**Example 5.5.14** Let $p$ be an integer, $p \geq 1$. Then, for $n = 3p+3$, the following two 2UC graph pairs have $\frac{2(n-3)}{3}$ common cards, so both attain the bound of Theorem 5.5.11:

$$\begin{aligned} G &\cong (K_{p+1} \oplus K_{p-1}) \oplus (K_{p+1} \oplus 2K_1) \\ H &\cong (K_p \oplus K_p) \oplus (K_{p+1} \oplus 2K_1), \end{aligned}$$

and

$$\begin{aligned} G &\cong (K_{p+1} \oplus K_{p-1}) \oplus (K_{p+1} \oplus K_2) \\ H &\cong (K_p \oplus K_p) \oplus (K_{p+1} \oplus K_2). \end{aligned}$$

□

Our investigations have shown that the unique 2UC graph pair (up to isomorphism) with $b(G, H) = \frac{2}{3}(n+1)$, is the pair given in Lemma 5.4.6(b)(i). In addition, the pair given in Lemma 5.4.6(b)(ii) has $b(G, H) = \left\lfloor \frac{2}{3}(n+1) \right\rfloor$, when $n = 9$. Another example that has $b(G, H) = \left\lfloor \frac{2}{3}(n+1) \right\rfloor$, when $n = 9$, is the following pair of graphs.

**Example 5.5.15** For $n = 9$, the following pair of graphs has
$b(G, H) = \left\lfloor \frac{2}{3}(n+1) \right\rfloor = 6$:

$$\begin{aligned} G &= (K_3 \oplus K_1) \oplus (K_3 \oplus K_2) \\ H &= (K_2 \oplus K_2) \oplus (K_3 \oplus K_2). \end{aligned}$$

□

As a coda to this chapter, we show that the number of components of a graph can be determined from $\lfloor \frac{n+5}{2} \rfloor$ of its cards.

**Theorem 5.5.16** Let $G$ and $H$ be a pair of graphs, both of order $n \geq 3$, that contain a different number of components. Then

$$b(G,\, H) \leq \left\lfloor \frac{n+3}{2} \right\rfloor . \tag{5.38}$$

So the number of components of a pair of graphs is recognisable by $\lfloor \frac{n+5}{2} \rfloor$ of its cards.

*Proof* Since $G$ and $H$ contain a different number of components, they are a 2UC graph pair, with $\sum_{i=1}^{r} \alpha_i \neq \sum_{j=1}^{s} \beta_j$. Now if either graph contains three or more components, then by Corollary 5.3.3(b), $b(G,\, H) \leq \lfloor \frac{n}{2} \rfloor$. We may therefore assume that $\alpha_1 = 1$, $\alpha_2 = 0$, and $\beta_1 + \beta_2 = 2$. In addition, by Lemma 5.4.2, we may assume that if $\beta_2 \geq 1$, then $G$ contains both $H_1$-active and $H_2$-active vertices.

Suppose that $G_1$ does not contain any active component cut-vertices. Then since $\alpha_2 = 0$, by Corollary 5.3.19(f), both components of $\mathcal{H}$ must be isomorphic to $K_1$ and $G_1 \cong K_2$. Thus $G \cong (\lambda_1 + 1)K_2 \oplus \mu_1 K_1 \oplus \mathcal{F}$, and it is easy to show that $b(G,\, H) \leq \frac{n+2}{2} \leq \lfloor \frac{n+3}{2} \rfloor$. On the other hand, if $G_1$ contains an active component cut-vertex, then $b(G,\, H) \leq \lfloor \frac{n+3}{2} \rfloor$ by Lemmas 5.4.5 and 5.4.6. $\square$

Finally, we note that by Lemmas 5.4.4, 5.4.5 and 5.4.6, if $G$ and $H$ contain a different number of components, then $b(G,\, H) = \frac{n+3}{2}$ only if $G$ and $H$ are members of the family of pairs of paths presented in Lemma 5.4.6(b)(ii) or are the exceptional pair in Lemma 5.4.6(b)(i).

# Chapter 6

# Extending the $2$UC Results

In this chapter, we extend some of the results of the previous chapter. We show that, for $n \geq 46$, there are only two other families of 2UC graph pairs of order $n$, with $2\left\lfloor\frac{(n-4)}{3}\right\rfloor$ or more common cards, that are not constructed from Example 5.5.12. For one of these families, the pair of graphs are both forests, which shows that there exists a 2UC graph pair with the same number of edges and approximately $\frac{2n}{3}$ common cards.

For appropriate values of $n$, we also present an infinite family of pairs of graphs with the same degree sequence having $\frac{2}{3}(n+5-2\sqrt{3n+6})$ common cards. For large $n$ (certainly $n \geq 200$), this family has the highest number of common cards yet published, amongst all pairs of graphs with the same degree sequence. However, we make no claims on whether there are other families of pairs of graphs with the same degree sequence and a higher number of common cards.

Finally, we present infinite families of pairs of connected graphs with $2\left\lfloor\frac{1}{3}(n-1)\right\rfloor$, or slightly fewer, common cards. These are obtained by simple transformations of the 2UC examples given in Sections 5.5 and 6.1. In particular, we show how to construct infinite families of pairs of graphs with arbitrary connectivity $\kappa$ that have $2\left\lfloor\frac{1}{3}(n-\kappa-1)\right\rfloor$ common cards and, in addition, we show how to construct a family of trees with $2\left\lfloor\frac{(n-5)}{3}\right\rfloor$ common cards. Amongst all pairs of trees, this family of trees has a greater number of common cards than any other published pair of trees, for large $n$.

Throughout this chapter, any 2UC graph pair $G$ and $H$ is assumed to be expressed as in (5.4). We begin with a few observations on other families discussed in the previous chapters.

## 6.1 Observations on Families Previously Discussed

The families of graph pairs presented by Harary and Manvel [19], Bondy [8] and Myrvold are clearly 2UC [33]. Thus, using the methodologies given in Chapter 5, we can ascertain the number of common cards between each pair.

The family of graph pairs presented by Bondy [8] is the collection of paths given in Lemma 5.4.6(b). The conclusion of that lemma is that the number of common cards between the two graphs is $\left\lfloor\frac{n+3}{2}\right\rfloor$. The family presented by Harary and Manvel [19] is the pair $G = K_{p+1} \oplus K_{p-1}$ and $H = 2K_p$. So $\alpha_1 = \alpha_2 = 1$, $\beta_1 = 2$ and every other coefficient equal to zero. In addition, no active vertex of $G$ is a cut-vertex. By Lemma 5.4.3, the maximum number of common cards between such a pair is $\left\lfloor\frac{n}{2}\right\rfloor + 1$.

As stated at the beginning of Chapter 5, both of Myrvold's families of pairs of graphs are 2UC. In her first family (Example 2.7.3), $\alpha_1 = \alpha_2 = 1$, $\beta_1 = 2$, $\lambda_1 = \mu_1 = p-1$ and $\beta_2 = \lambda_2 = \mu_2 = 0$. Since no active vertex of $G$ is a cut-vertex, by (5.34) $b(G,\, H) \leq \frac{pn}{2p-1}$. Since $n = (p+1)(2p-1)$, we can express $p$ in terms of $n$ to obtain the bound. In her second family (Example 2.7.4), $G$ does not contain any $H_1$-active vertices. By Lemma 5.4.2, any such pair has at most $\left\lfloor\frac{n+1}{2}\right\rfloor$ common cards.

Finally, since any graph pair in which $G$ is connected and $H$ is disconnected is 2UC, we can find the maximum number of common cards between such a pair. If $H$ has three or more components then by Corollary 5.3.3, $b(G, H) \leq \left\lfloor \frac{n}{2} \right\rfloor$. On the other hand if $H$ has only two components, then either $b(G, H) = 1$ or there must be some active vertex of $G$ that is a cut-vertex. By Lemmas 5.4.4 and 5.4.5, $b(G, H) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$, for such pairs.

## 6.2 2UC Graph Pairs with Specific Parameters

For each of the 2UC graph pairs discussed in Section 6.1, $b(G, H)$ is much less than the upper bound of $2\left\lfloor \frac{n-1}{3} \right\rfloor$, for large $n$. This is because in each case, one of the following possibilities occurs: $\beta_2 = 1$ but $G$ contains only $H_1$-active vertices; $G_1$ contains active cut-vertices; $\lambda_1 \neq 1$. So, to find other families of 2UC graph pairs that have $b(G, H)$ close to $2\left\lfloor \frac{n-1}{3} \right\rfloor$, we should look for pairs where $\lambda_1 = 1$, none of the active vertices of $G$ are cut-vertices and, if $\beta_2 = 1$, then $G$ contains both $H_1$ and $H_2$-active vertices. In addition, since we wish to find infinite families of graph pairs, it is necessary to not limit the size of $h_1$; so by (5.33), we look for pairs where $\mu_1 = \mu_2 = 0$ also. We now show how to construct three families in this manner. Moreover, we show that for large $n$, two of our families are the unique families of 2UC graph pairs with $b(G, H) = 2\left\lfloor \frac{(n-4)}{3} \right\rfloor$, that are not extensions of Example 5.5.12.

In trying to maximise $b(G, H)$, a reasonable strategy would be to maximise the number of active vertices in the subgraphs $\mathcal{G}$ and $\mathcal{H}$. We now explain an approach to accomplish this.

Let $U$ be a component of $G$ isomorphic to $G_1$ and let $W$ be a component of $H$ isomorphic to $H_1$. Suppose that $\beta_1 = 2$ and that no active vertex in $U$ is a cut-vertex (so that no active vertex in $W$ is a cut-vertex either). Let $u$ be an $H_1$-active vertex in $U$ and $w$ be vertex in $W$ associated with $u$. By Corollary 5.3.19, $U - u$ is isomorphic to $W$ and $W - w \cong G_2$. Therefore, there must be some other vertex $v$ in $U$ such that $(U - u) - v \cong G_2$.

Now, since we wish to maximise the number of active vertices in $U$ and $W$, we must minimise the number of $u$ such that $U - u \ncong W$ and the number of $w$ such that $W - w \ncong G_2$. Thus we must minimise the number of pairs of vertices $u$ and $v$ in $U$ such that $(U-u)-v \ncong G_2$. Since similar arguments would give the same conclusion, if $\beta_2 = 1$ and $u$ is $H_2$-active, it follows that to find a family of 2UC graph pairs such that $\mathcal{G}$ and $\mathcal{H}$ contain a large number of active vertices, we are required to find a connected graph $U$ that satisfies the following criterion: for as many pairs of vertices $u$ and $v$ of $U$ as possible, $(U-u)-v$ is isomorphic to the same connected graph.

For $p \geq 2$, we consider the 1-star of order $p$, $S^1_{p-1}$. Since as commented in Section 2.1, $\mathcal{D}(S^1_{p-1}) = \{(S^1_{p-2}; p-1), (p-1)K_1\}$, it follows that for any pair of leaves $u$ and $v$ in $S^1_{p-1}$, $(S^1_{p-1} - u) - v \cong S^1_{p-3}$. So, since every vertex of $S^1_{p-1}$ except one is a leaf, $S^1_{p-1}$ satisfies the stated criteria. It follows that the 2UC graph pair obtained by setting $\beta_1 = 2$, $G_1 \cong S^1_{p+1}$ and $H_1 \cong S^1_p$, will have a large number of common cards. In addition, since $G_1$ and $H_1$ are both trees, $G$ and $H$ are both forests, and therefore have the same number of edges.

**Example 6.2.1** Let $p$ be an integer, $p \geq 2$. Then for $n = 3p+4$, the following 2UC graph pair has the same number of edges and $\frac{2}{3}(n-4)$ common cards:

$$\begin{aligned} G &= (S^1_{p+1} \oplus S^1_{p-1}) \oplus (S^1_{p+1}) \\ H &= (S^1_p \oplus S^1_p) \oplus (S^1_{p+1}). \end{aligned}$$

The removal of any leaf from a component of $G$ isomorphic to $S^1_{p+1}$ and any leaf from a component of $H$ isomorphic to $S^1_p$ gives isomorphic cards. So
$b(G, H) = 2p = \frac{2}{3}(n-4)$. Since $G$ and $H$ are forests with the same number of components, they have the same number of edges. Figure 6.1 shows these graphs for $p = 5$. $\square$

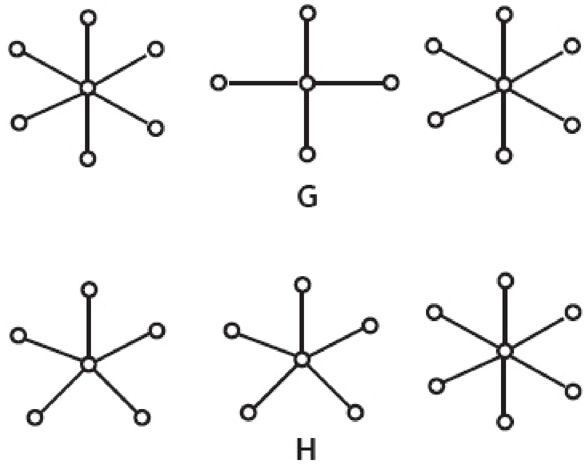


Figure 6.1: The pair of forests in Example 6.2.1 of order 19 with 10 common cards.

We can extend this example as in Example 5.5.12 to give the following result.

**Theorem 6.2.2** For all $n \geq 7$, there exist pairs of non-isomorphic graphs with the same number of edges that have $b(G, H) \geq 2\left\lfloor \frac{1}{3}(n-4) \right\rfloor$. Moreover, these graphs are forests.

*Proof* For $n \equiv 1$, the pair in Example 6.2.1 attains the bound. For $n \equiv 0$ or 2 (mod 3), we add components of total order 1 and 2, respectively (as in Examples 5.5.13 and 5.5.14). We note that, if we set $p = 1$ in the example, then $b(G, H) = 4$, so we can extend the theorem to all values of $n \geq 7$. □

Before we give the next example, we make the following three observations, the first of which will be useful here and in Section 6.3. We recall from Section 1.1, that if $F$ is a graph, then the complement of $F$ is the graph $F^C$ with vertex set $V(F)$, such that for any pair of vertices $u$ and $v$ of $F$, $uv$ in $E(F^C)$ if and only if $uv$ is not in $E(F)$.

**Lemma 6.2.3** Let $F$ be a graph and let $S \subset V(F)$. Then $(F-S)^C = F^C - S$.

*Proof* $(F-S)^C$ and $F^C - S$ both have the same vertex-set, $V(F-S)$. Let $u$ and $v$ be a pair of distinct vertices of $F-S$. Then, since $V(F-S)$ does not contain any edges incident to a vertex in $S$, $uv$ is in $E((F-S)^C)$ if and only if $uv$ is not in $E(F)$, that is, if and only if $uv$ is in $E(F^C)$. So since $u$ and $v$ are not in $S$, it follows that $uv$ is in $E((F-S)^C)$ if and only if $uv$ is in $E(F^C - S)$. So $E((F-S)^C) = E(F^C - S)$, and the result follows. $\square$

The above lemma yields the following corollary.

**Corollary 6.2.4** Let $F$ and $U$ be a pair of graphs. Then $b(F^C, U^C) = b(F, U)$.

*Proof* Suppose that $v$ is an active vertex of $F$ and that $w$ is a vertex of $U$ associated with $v$. Then since $F - v \cong U - w$, it follows from Lemma 6.2.3 that $F^C - v \cong (F-v)^C \cong (U-w)^C \cong U^C - w$. So $v$ is an active vertex of $F^C$ and $w$ is a vertex of $U^C$ associated with $v$. Therefore, $B(F, U) = B(F^C, U^C)$, so $b(F, U) = b(F^C, U^C)$.

$\square$

We use Corollary 6.2.4 in Section 6.3 to find families of connected graph pairs with a large number of common cards. Here we use these observations to present another family of graph pairs with $b(G, H) = \frac{2(n-4)}{3}$.

**Lemma 6.2.5** Let $F$ be a connected $(n-2)$-regular graph of order $n \geq 4$. Then $F^C \cong \frac{n}{2}K_2$. So $n$ is even and, moreover, $F$ is unique up to isomorphism.

*Proof* Let $v$ be a vertex of $F$. Then $v$ is adjacent to every vertex of $F$, except one. Thus $v$ is only adjacent to one vertex of $F^C$, so $d(v) = 1$ in $F^C$. It follows that every vertex in $F^C$ is of degree 1, so $F^C \cong \frac{n}{2}K_2$, since $n \geq 2$. Clearly $n$ is even, and since $K_2$ is unique up to isomorphism, so is $F$. $\square$

In light of Lemma 6.2.5, for $p \geq 1$, we let $VT_{2(p-1)}$ denote the $2(p-1)$-regular graph of order $2p$. In addition, we denote the graph constructed from $VT_{2(p-1)}$ by adding a single vertex adjacent to every vertex of $VT_{2(p-1)}$ by $VT'_{2(p-1)}$, and the graph constructed from $VT_{2(p-1)}$ by adding two vertices adjacent to every vertex of $VT_{2(p-1)}$, and additionally to each other, by $VT''_{2(p-1)}$. It is easy to see that for $p \geq 3$, $((p-1)K_2 \oplus K_1)^C \cong VT'_{2(p-2)}$ and $((p-2)K_2 \oplus 2K_1)^C \cong VT''_{2(p-3)}$. Note that, $VT'_{2(p-2)}$ contains $2(p-1)$ vertices of degree $2p-3$ and one vertex of degree $2p-2$. The following result is immediate.

**Lemma 6.2.6** Let $p$ be an integer, $p \geq 4$, and let $v$ be any vertex of $VT_{2(p-1)}$. Then $VT_{2(p-1)} - v \cong VT'_{2(p-2)}$. In addition, $(VT_{2(p-1)} - v) - u \cong VT''_{2(p-3)}$, for every vertex $u$ that is adjacent to $v$.

*Proof* By Lemma 6.2.5, $(VT_{2(p-1)})^C \cong pK_2$, so we can identify the vertices of $VT_{2(p-1)}$ with the vertices of $pK_2$. Let $u$ and $v$ be adjacent vertices of $(VT_{2(p-1)})^C$. Then $u$ and $v$ are not adjacent in $pK_2$. Clearly, $pK_2 - v \cong (p-1)K_2 \oplus K_1$, and $(pK_2 - v) - u \cong (p-2)K_2 \oplus 2K_1$. Therefore, by Lemma 6.2.3,
$VT_{2(p-1)} - v \cong (pK_2 - v)^C \cong ((p-1)K_2 \oplus K_1)^C \cong VT'_{2(p-2)}$, and
$(VT_{2(p-1)} - v) - u \cong ((pK_2 - v) - u)^C \cong ((p-2)K_2 \oplus 2K_1)^C \cong VT''_{2(p-3)}$. □

**Corollary 6.2.7** Let $p$ be an integer, $p \geq 4$. Then $VT'_{2(p-2)} - w \cong VT''_{2(p-3)}$ for every vertex $w$ of $VT'_{2(p-2)}$, except the unique vertex of degree $2p-2$.

*Proof* This follows immediately from Lemma 6.2.6, noting that $d(v) = 2p-2$, for every vertex $v$ of $VT_{2(p-1)}$. □

We now use the above results to construct another family with $b(G, H) = \frac{2}{3}(n-4)$.

**Example 6.2.8** For $n = 6p-2$, where $p \geq 3$, the following 2UC graph pair has $\frac{2}{3}(n-4)$ common cards:

$$\begin{aligned} G &= (VT_{2(p-1)} \oplus VT''_{2(p-3)}) \oplus (VT_{2(p-1)}) \\ H &= (VT'_{2(p-2)} \oplus VT'_{2(p-2)}) \oplus (VT_{2(p-1)}). \end{aligned}$$

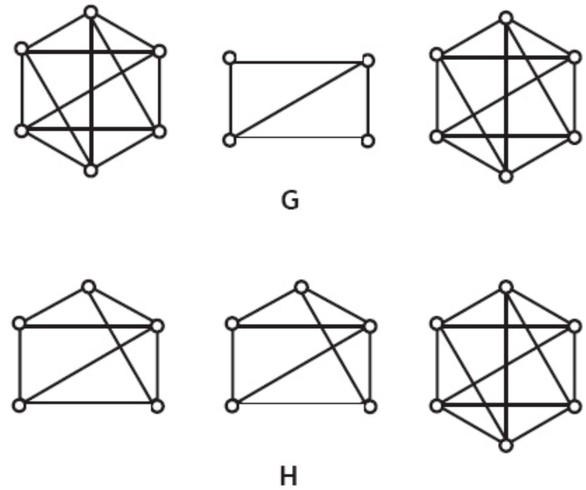


Figure 6.2: The pair of graphs in Example 6.2.8 of order 16 with 8 common cards.

By Lemma 6.2.6 and Corollary 6.2.7, the removal of any vertex from a component of $G$ isomorphic to $VT_{2(p-1)}$ and any vertex of degree $2p-3$ from a component of $H$ isomorphic to $VT'_{2(p-2)}$ gives isomorphic cards. There are $4p-4$ such vertices in $H$, so $b(G,\, H) = 4p - 4 = \frac{2}{3}(n-4)$. Figure 6.2 shows these graphs for $p = 3$. $\square$

We may clearly extend this example to all values of $n$ in a similar manner to Example 6.2.1.

Now, using the notation from Section 5.4, in Example 6.2.1, $\overline{a_H(G_1)} = \overline{a_G(H_1)} = 1$ and $\overline{b_1(G)} = 2$, and in Example 6.2.8, $\overline{b_1(G)} = 4$ and $\overline{a_G(H_1)} = 1$; in both examples, $\lambda_1 = 1$ and $\mu_1 = \mu_2 = \beta_2 = 0$. Thus the fact that $b(G,\, H) = \frac{2}{3}(n-4)$ is directly calculable by (5.34). We shall prove that the these two families (and their extensions) are, for $n \geq 46$, the only 2UC graph pairs that have $b(G,\, H) = 2\left\lfloor \frac{(n-4)}{3} \right\rfloor$ and are not constructed from Example 5.5.12. We first prove an interesting result about any graph in which all but one of the cards in the deck are isomorphic.

**Lemma 6.2.9** Let $F$ be a non-regular connected graph of order $q \geq 3$. Suppose that $u$ is a vertex of $F$ such that all cards in $\mathcal{D}(F)$ are isomorphic, except $F - u$. Then $d(u) = q - 1$, and $F - u$ is a vertex-transitive graph of order $q - 1$. Moreover, under these conditions, $F$ can be uniquely reconstructed from any of its cards.

*Proof* All cards in the deck of $F$ are isomorphic, except $F-u$. So, every vertex of $F$ except $u$ must be of the same degree, since $F$ is not regular. Let $v \neq u$ be a vertex of $F$ and let $k = d(v)$. Now if $u$ and $v$ are adjacent then $d_{k-1}(F-v) = k-1$, since $d(u) \neq k$. On the other hand, if $u$ and $v$ are not adjacent, then $d_{k-1}(F-v) \geq k$. Thus, since every card in $\mathcal{D}(F)$ except $F-u$ is isomorphic, it follows that every vertex of $F$ (except $u$) is adjacent to $u$ or no vertex is. Since $F$ is connected, $u$ must be adjacent to at least one vertex of $F$. Therefore, $u$ is adjacent to every vertex of $F$, so $d(u) = q-1$. It follows that $F$ can be uniquely reconstructed from $F-u$.

Since $u$ is adjacent to every vertex of $F$ and all the cards in $\mathcal{D}(F)$ are isomorphic, except $F-u$, it is easy to see that $(F-v)-u$ is isomorphic, for each $v \neq u$ in $F$. Thus, every card in $\mathcal{D}(F-u)$ is isomorphic, so $F-u$ is regular and, moreover, vertex-transitive. Therefore, as noted in the proof of Theorem 2.5.1, $F-u$ can be uniquely reconstructed from any of its cards. Now, since $d(u) = q-1$, $u$ is adjacent to every vertex of $F-v$. Thus, the removal of any vertex of degree $q-2$ from $F-v$ gives a graph isomorphic to $(F-u)-v$. Hence, for any $v \neq u$, we can always form a graph isomorphic to $(F-u)-v$ from $F-v$, so we can uniquely construct the card $F-u$ from $F-v$. Since we can uniquely reconstruct $F$ from $F-u$, therefore we can uniquely reconstruct $F$ from $F-v$. □

**Lemma 6.2.10** Let $F$ be as in Lemma 6.2.9. Suppose that $U$ is a graph of order $n \geq 6$, such that every card in $\mathcal{D}(U)$, is isomorphic to $F$, except at most two. If $U$ is regular, then $n = 2p$ and $U \cong VT_{2(p-1)}$; otherwise $U \cong S^1_{n-1}$.

*Proof* Since all cards in $\mathcal{D}(U)$ are isomorphic, except at most two, it follows that every vertex of $U$, except at most two, is of the same degree. Let $v$ be a vertex of $U$ such that $U-v \cong F$, and let $u$ be the unique vertex of $U$ such that $(U-v)-u$ is not isomorphic to $(U-v)-w$, for all other $w$ in $U$. By Lemma 6.2.9, $u$ is of degree $n-2$ in $U-v$, and since $U-v \cong F$, every other vertex of $U-v$ must be of degree $k$, for some $k \leq n-3$.

Suppose first that $d(v) = 1$. Then $k = 1$ and every vertex of $U$ except at most two must be a leaf. Since $d(u) \geq n - 2$ in $U - v$, there are only two possibilities: either $v$ is adjacent to $u$, and every vertex of $U$ except $u$ is a leaf; or $v$ is adjacent to some vertex $x$ where $d(x) = 2$ and $d(u) = n - 2$. In the latter case, the removal of any leaf $w$ from $U$ except $v$ gives a card that contains a vertex of degree two (that is $x$). Since this is impossible, the former case must occur. Therefore, $U$ must be the 1-star of order $n$.

We may therefore assume that $d(v) \geq 2$. We first consider the case when $u$ is adjacent to $v$, so $d(u) = n - 1$ in $U$. Since $U - v$ contains only one vertex of degree $n - 2$, every other vertex of $U$ is of degree at most $n - 2$ in $U$.

Now, if $d(v) = k$, then $v$ cannot be adjacent to any vertices of degree $d(v)$, so $d(v) = k = 2$, thus every vertex of $U$ except at most two is of degree two. Since $u$ is the only vertex of $U$ of degree $n - 1$, it is easy to see that there must be pair of adjacent vertices of degree two. But the cards of either of these vertices would contain a leaf, so this case cannot occur. It follows that $d(v) \neq k$, so every vertex of $U$ of degree $d(v)$ must be adjacent to $v$ and $d(v) = k + 1$. Thus $d(v) = n - 2$ and $k = n - 3$. Therefore, there must be a unique vertex that is not adjacent to any of the $n - 2$ vertices of degree $n - 2$. So $k = 1$, which contradicts the fact that $n \geq 6$.

We may therefore assume that $u$ is not adjacent to $v$ in $U$, so $d(u) = n - 2$ in $U$. Every vertex $w$ such that $U - w \cong F$ is adjacent to $u$. Thus $k = n - 3 \geq 3$ since $n \geq 6$. So $v$ is adjacent to every vertex in $U - w$ except $u$; therefore, $d(v) = n - 3$ if $w$ is not adjacent to $v$ and $d(v) = n - 2$ otherwise. In the former case, $v$ cannot be adjacent to any vertex of degree $d(v)$. Since there are at least four vertices in $U$ of degree $n - 3$, this cannot occur. Therefore $d(v) = n - 2$, and $U$ contains at least $n - 1$ vertices of degree $n - 2$. It is easy to see that $U$ must contain $n$ vertices of degree $n - 2$. Therefore, every vertex of $U$ is of degree $n - 2$ and $U \cong VT_{2(p-1)}$, where $n = 2p$. $\square$

**Lemma 6.2.11** Let $F$ be a connected graph of order $n$, and let $S$ and $T$ be two disjoint subsets of $V(F)$, both of size 4 or more. Suppose that every vertex $u$ in $S$, is of the same degree and furthermore, that for each such $u$, $F - u$ is $d$-regular, for some $d$. Suppose further that every vertex $v$ in $T$ is of the same degree and, in addition, for each such $v$, every vertex of $F - v$, except at most two, is of degree $d'$, for some $d'$. If $|S| + |T| \geq n - 2$, then $F \cong K_n$.

*Proof* Suppose that $|S| + |T| \geq n - 2$, and let $u$ and $v$ be vertices in $S$ and $T$, respectively, where $d(u) = k$ and $d(v) = l$. Since every vertex in $S$ must be of degree $k$, either $d = k - 1$ or $d = k$. Moreover, if $d = k - 1$, then either $l = k - 1$ or $l = k$, whereas if $d = k$, then either $l = k$ or $l = k + 1$.

Suppose first that $d = k - 1$. Then $F$ contains precisely $k + 1$ vertices of degree $k$ and $n - k - 1$ vertices of degree $k - 1$. Moreover, no vertex of $F$ of degree $k - 1$ can be adjacent to any vertex in $S$. So, if $l = k - 1$, then $F - v$ contains at least four vertices of degree $k$ and at least three vertices of degree $k - 1$ or less, that is, all the other vertices of $T$, which contradicts our assumption on $F - v$. So we may therefore assume that $l = k$, so all the vertices in $S$ and $T$ are of the same degree. Now since $|S| + |T| \geq n - 2$, $F$ contains at most two vertices of degree $k - 1$; so $n - k - 1 \leq 2$, thus $k \geq n - 3$. It follows that any vertex of degree $k - 1$ in $F$ must be adjacent to at least $n - 4$ vertices not in $S$. But since $|S| \geq 4$, no such vertex can exist. Therefore, $n - k - 1 = 0$ and $F \cong K_n$.

Suppose instead that $d = k$. Then $F$ contains precisely $k$ vertices of degree $k + 1$ and $n - k$ vertices of degree $k$. Moreover, every vertex of $F$ of degree $k + 1$ must be adjacent to every vertex of $S$. Now if $l = k + 1$, then $F - v$ contains at least four vertices of degree $k - 1$ and at least three vertices of degree $k$ or more, that is, all the other vertices of $T$. This again contradicts our assumption on $F - v$. On the other hand, if $l = k$, then since $|S| + |T| \geq n - 2$, $F$ can contain at most two vertices of degree $k + 1$, so $k \leq 2$. But no vertex of degree 3 or less can be adjacent to every vertex of $S$, since $|S| \geq 4$, again contradicting our assumptions. $\square$

It is easy to see we can construct a 2UC graph pair with $b(G,\, H) = 2\left\lfloor \frac{1}{3}(n-4) \right\rfloor$ by the addition of components of small orders to an appropriate pair from the family in Example 5.5.12. We now use Lemmas 6.2.9 to 6.2.11 to show that the only two families of 2UC graph pairs with this many common cards, that are not constructed in this family are constructed from the families in Examples 6.2.1 and 6.2.8.

As in Section 5.5, we let

$$\begin{aligned} R(H) &= \overline{b_1(H)} + \overline{b_2(H)} + (\beta_1 + \mu_1)\overline{a_G(H_1)} + (\beta_2 + \mu_2)\overline{a_G(H_2)}, \\ R(G) &= \overline{b_1(G)} + \overline{b_2(G)} + (\lambda_1 + 1)\overline{a_H(G_1)}, \\ R(\mathcal{F}) &= (\lambda_1 + 1)(\overline{b_{\mathcal{F}}(G)} + \overline{a_H(\mathcal{F})}) - b_{\mathcal{F}}(G), \end{aligned} \tag{6.1}$$

and

$$b(G,\, H) = \frac{1}{2\lambda_1 + 1}\left((\lambda_1 + 1)n - (\lambda_1 + 1)R(H) - \lambda_1 R(G) - R(\mathcal{F})\right). \tag{6.2}$$

**Theorem 6.2.12** For $n \geq 46$, let $G$ and $H$ be a 2UC graph pair of order $n$, that is not constructed from Example 5.5.12 by the addition of components of small order. Suppose that $b(G,\, H) \geq 2\left\lfloor \frac{1}{3}(n-4) \right\rfloor$. Then $G$ and $H$ are isomorphic to the pair in either Example 6.2.1 or 6.2.8 or their extensions.

*Proof* Suppose that $G$ and $H$ are a 2UC graph pair of order $n$, $n \geq 46$, such that $b(G,\, H) \geq 2\left\lfloor \frac{(n-4)}{3} \right\rfloor$. As in Theorem 5.5.11, we may assume that $G$ and $H$ are expressed as in (5.14). Moreover, since $n \geq 46$, by Corollaries 5.5.3 and 5.5.4, we may assume that no active vertex of either $G$ or $H$ is a cut-vertex, and if $\beta_1 = 1$, that $G$ contains both $H_1$ and $H_2$-active vertices.

Now, using the notation of (6.1), if $(1 + \lambda_1)R(H) + \lambda_1 R(G) + R(\mathcal{F}) > 12$, then $b(G,\, H) < 2\left\lfloor \frac{1}{3}(n-4) \right\rfloor$, by (6.2). Thus, using an identical technique to that used in Lemma 5.5.5 and 5.5.7, it is easy to show that since $n \geq 46$, $\lambda_1 = 1$, $\mu_1 = \mu_2 = 0$ and $R(G) = R(H) + 2$. Furthermore, since if $(1+\lambda_1)R(H) + \lambda_1 R(G) + R(\mathcal{F}) \geq 11$, then $b(G,\, H) \geq 2\left\lfloor \frac{1}{3}(n-4) \right\rfloor$, only when $|V(\mathcal{F})| = 2$, we only need consider the cases when $R(G) \leq 4$ and $R(H) \leq 2$, so $\overline{a_H(G_1)} \leq 2$.

Suppose first that $\beta_2 = 1$, and let $W_1$ and $W_2$ be components of $H$ isomorphic to $H_1$ and $H_2$, respectively. Let $w_1$ be an active vertex of $H$ in $W_1$, and let $w_2$ be an active vertex of $H$ in $W_2$. By Corollary 5.3.19, $W_1 - w_1 \cong W_2 - w_2 \cong G_2$. Now if $W_1$ is regular, then as noted in the proof of Theorem 2.5.1, $W_1$ can be uniquely reconstructed from $G_2$. A similar observation holds for $W_2$. So since $W_1 \ncong W_2$, clearly at least one of $W_1$ and $W_2$ must be not regular, thus at least one $\overline{a_G(H_1)}$ or $\overline{a_G(H_2)}$ must be non-zero. Moreover, since $R(H) \leq 2$, by (5.35), we may assume that one of the following holds: (i) $\overline{a_G(H_1)} = \overline{a_G(H_2)} = 1$; (ii) $\overline{a_G(H_1)} = 0$ and $1 \leq \overline{a_G(H_2)} \leq 2$; (iii) $\overline{a_G(H_2)} = 0$ and $1 \leq \overline{a_G(H_1)} \leq 2$. Note that, if $a_{H_1}(G_1) < 4$, then $b(G,\, H) \leq h_2 + 6 < 2\left\lfloor \frac{1}{3}(n-4) \right\rfloor$, for these values of $n$. Since a similar observation holds for $a_{H_2}(G_1)$, we may clearly assume that $a_{H_1}(G_1) \geq 4$ and $a_{H_2}(G_1) \geq 4$.

(i) Suppose that $\overline{a_G(H_1)} = \overline{a_G(H_2)} = 1$. Then every card in the decks of both $W_1$ and $W_2$ is isomorphic, except at most one in each deck. So, by Lemma 6.2.9, $W_1$ and $W_2$ can be uniquely reconstructed from $G_2$. But this is impossible since $W_1 \ncong W_2$.

(ii) Suppose instead that $\overline{a_G(H_1)} = 0$ and $1 \leq \overline{a_G(H_2)} \leq 2$. Let $U$ be a component of $G$ isomorphic to $G_1$ and let $u$ and $v$ be vertices in $U$, where $u$ is $H_1$-active and $v$ is $H_2$-active. By Corollary 5.3.19, $U - u \cong H_2$ and $U - v \cong H_1$. Since $\overline{a_G(H_1)} = 0$, $H_1$ and thus $U - v$ is regular. Similarly, since $\overline{a_G(H_2)} \leq 2$, the degree of every vertex of $U - u$, except at most two, must be the same. By Corollary 5.3.20, the degree of every $H_1$-active vertex is the same, and the degree of every $H_2$-active vertex is the same.

Since $a_{H_1}(G_1) \geq 4$ and $a_{H_2}(G_1) \geq 4$, we let $S = A_{H_1}(U)$ and $T = A_{H_2}(U)$ in Lemma 6.2.11. Then since $\overline{a_H(G_1)} \leq 2$, clearly $|S| + |T| \geq g_1 - 2$, so it follows from the lemma that $U$ is complete. This contradiction shows that case (ii) cannot occur.

(iii) This clearly holds by symmetry.

We are left to consider the case when $\beta_1 = 2$. Clearly $\overline{a_G(H_1)} \leq 1$, since $R(H) \leq 2$. In addition, since $G_1$ is not complete, by Corollary 5.5.10, $\overline{a_G(H_1)} = 1$. Now, if $H_1$ is regular, then for any component $U$ of $G$ isomorphic to $G_1$, by setting $S = A_H(U)$ in Lemma 5.5.9, it is easy to show that $R(G) \geq 6$. So we may therefore assume that $H_1$ is not regular. Then, since $\overline{a_G(H_1)} = 1$ and $\overline{a_H(G)} \leq 2$, setting $H_1$ to be the graph $F$ in Lemma 6.2.10, either $U \cong F_{\frac{g_1}{2}}$ or $U \cong S^1_{g_1-1}$. Noting that we can extend these families as shown in Examples 6.2.1 and 6.2.8 completes the proof. $\square$

The above theorem shows that the family of forests in Example 6.2.1 has, for large enough $n$, the highest number of common cards for any 2UC graph pair with the same number of edges. We now show how to construct a family of 2UC graph pairs with the same degree sequence and a large number of common cards. Note that, we do not claim that this example is maximal with respect to the number of common cards (as we have shown for previous examples).

We recall from Section 1.7, that if $F$ is a connected graph of order $p$, then $S_q[F]$ denotes the graph of order $p(q+1)$ that consists of $F$ with $q$ leaves added to each of its vertices. For $F \cong K_p$, we let $S'_q[K_p]$ denote the graph $S_q[K_p]$ with a single leaf removed, and let $S''_q[K_p]$ denote the graph $S_q[K_p]$ with two leaves, adjacent to different vertices, removed. Then,

(a) $d_{p+q-1}(S_q[K_p]) = p$, $d_{p+q-2}(S_q[K_p]) = 0$, $d_1(S_q[K_p]) = pq$ and $d_i(S_q[K_p]) = 0$ for all $i \neq 1, p+q-2, p+q-1$;

(b) $d_{p+q-1}(S'_q[K_p]) = p-1$, $d_{p+q-2}(S'_q[K_p]) = 1$, $d_1(S'_q[K_p]) = pq-1$ and $d_i(S'_q[K_p]) = 0$ for all $i \neq 1, p+q-2, p+q-1$;

(c) $d_{p+q-1}(S''_q[K_p]) = p-2$, $d_{p+q-2}(S''_q[K_p]) = 2$, $d_1(S''_q[K_p]) = pq-2$ and $d_i(S''_q[K_p]) = 0$ for all $i \neq 1, p+q-2, p+q-1$.

By construction, for any leaf $v$ of $S_q[K_p]$, there is an isomorphism $\phi$ from $S_q[K_p] - v$ to $S'_q[K_p]$. Moreover, $S'_q[K_p] - \phi(w) \cong S''_q[K_p]$, for any leaf $w$ of $S_q[K_p]$, adjacent to a different vertex than $v$. This discussion leads to our example.

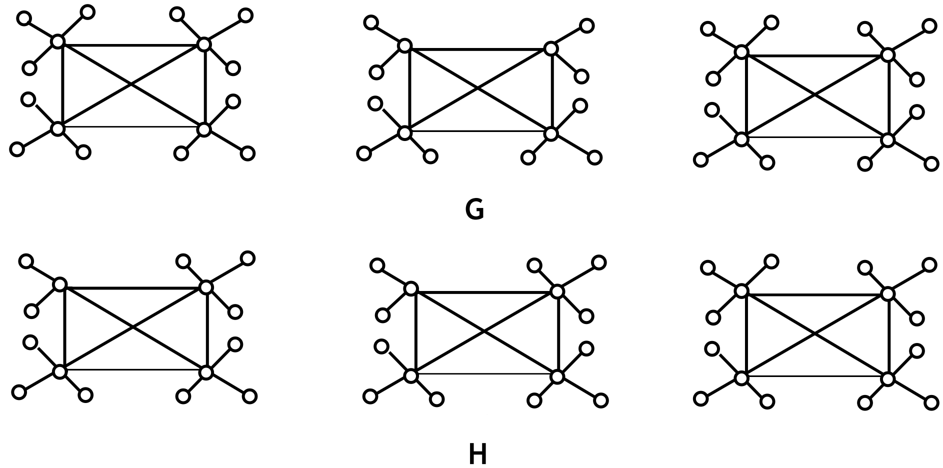


Figure 6.3: The pair of graphs in Example 6.2.13 of order 46 with 18 common cards.

**Example 6.2.13** For $n = 3p^2 - 2$, where $p \geq 3$, the following 2UC graph pair has the same degree sequence and $b(G,\, H) = 2(p-1)^2 = \frac{2}{3}(n + 5 - 2\sqrt{3n+6})$.

$$\begin{aligned} G &= (S_{p-1}[K_p] \oplus S''_{p-1}[K_p]) \oplus (S_{p-1}[K_p]) \\ H &= (S'_{p-1}[K_p] \oplus S'_{p-1}[K_p]) \oplus (S_{p-1}[K_p]). \end{aligned}$$

The removal of any leaf from component of $G$ isomorphic to $S_{p-1}[K_p]$ and an appropriate leaf from a component of $H$ isomorphic to $S'_{p-1}[K_p]$ gives isomorphic cards. So $b(G,\, H) = 2(p-1)^2 = \frac{2}{3}(n + 5 - 2\sqrt{3n+6})$. (a) to (c) above shows that they have the same degree sequence. Figure 6.3 shows these graphs for $p = 4$. □

By extending this example, we can find pairs of any order with the same degree sequence and a large number of common cards.

**Theorem 6.2.14** For $n \geq 10$, there exist 2UC graph pairs with the same degree sequence having at least $b(G,\, H) = 2\left\lfloor \sqrt{\frac{n+2}{3}} - 1 \right\rfloor^2 \geq \frac{2}{3}(n + 15 - 4\sqrt{3n+9})$.

*Proof* For $n = 3p^2 - 2$, where $p \geq 3$, the pair in Example 6.2.13 attains the bound. We can extend this example to all values of $n \geq 25$ by replacing $G$ and $H$ by $G \oplus \mathcal{F}$ and $H \oplus \mathcal{F}$, respectively, for some graph $\mathcal{F}$, where $1 \leq |V(\mathcal{F})| \leq 6p+2$ (in a similar manner to the extension of Example 5.5.12). This gives $b(G,\, H) = 2\left\lfloor \sqrt{\frac{n+2}{3}} - 1 \right\rfloor^2$, which has a minimum value of $\frac{2}{3}(n+15-4\sqrt{3n+9})$. For many values of $n \neq 3p^2-2$, we can usually increase the value of $b(G,\, H)$ by slightly changing the number of leaves adjacent to each of the vertices of the complete graphs $K_p$.

Finally, we note that when $p = 2$, the pair in Example 6.2.13 have
$b(G,\, H) = 4 > (p-1)^2$. So the theorem can therefore be extended to all values of $n \geq 10$. $\square$

We note that, instead of adding leaves to each vertex of each $K_p$ component in Example 6.2.13, we could add $p-1$ vertices, all adjacent to each other, as well as the vertex of $K_p$. These pair of graphs would clearly have the same degree sequence as each other and, in addition, would have the same number of common cards as the pair in the example.

The pair in Example 6.2.13, or the above variant, has a larger number of common cards than any pair with the same degree sequence yet published, for large $n$. It is easy to extend the example to find a pair of forests with the same degree sequence and a large number of common cards.

**Example 6.2.15** By replacing each of the complete graphs $K_p$ in Example 6.2.13 by the star $S_p^1$, and adjoining the sets of $p-1$ (or $p-2$) leaves only to the leaves of the stars, we can form a pair of forests with the same degree sequence that has the same number of common cards as the graphs in Example 6.2.13 and only three more vertices. We can extend this example to all values of $n \geq 10$ in the same way as in the proof of Theorem 6.2.14. So, for all $n \geq 10$, there exists pairs of such graphs with at least $\frac{2}{3}(n + 12 - 4\sqrt{3n})$ common cards. $\square$

## 6.3 Families of Connected Graph Pairs with a Large Number of Common Cards

In all of the examples of infinite families of graph pairs with a large number of common cards that we have presented in this thesis, at least one of the graphs has been disconnected. We can, however, easily modify these infinite families to obtain pairs of *connected* graphs that have approximately $\frac{2n}{3}$ common cards. The examples in the section illustrate this. We begin by complementing our examples.

**Theorem 6.3.1** For all $n \geq 4$, there exist non-isomorphic connected graphs $G$ and $H$ with $2\left\lfloor \frac{1}{3}(n-1) \right\rfloor$ common cards.

*Proof* The complement of a disconnected graph is connected. So, for any 2UC graph pair $G$ and $H$ where both $G$ and $H$ contain at least two components, $G^C$ and $H^C$ must be a pair of connected graphs. By Lemma 6.2.4, $b(G^C, H^C) = b(G, H)$. Thus, by taking the complements of the graphs in Examples 5.5.12, 5.5.13 and 5.5.14, we obtain families of pairs of connected graphs with $2\left\lfloor \frac{1}{3}(n-1) \right\rfloor$ common cards. □

We now consider pairs of graphs of arbitrary connectivity. We first make the following observation. Let $A$ and $B$ be a pair of graphs of orders $a$ and $b$, respectively. Then the *join* of $A$ and $B$, denoted $A \vee B$, is the connected graph constructed from $A$ and $B$ by adding $ab$ new edges that join every vertex of $A$ to every vertex of $B$ (see [11]).

**Lemma 6.3.2** Let $G$, $H$ and $A$ be graphs. Then $b(G \vee A, H \vee A) \geq b(G, H)$.

*Proof* Let $v$ and $w$ be vertices of $G$ and $H$, respectively, such that $G - v \cong H - w$. Since $v$ is incident to every vertex of the subgraph of $G \vee A$ induced by $V(A)$, and similarly for $w$ and $H \vee A$, it is easy to see that $(G \vee A) - v \cong (H \vee A) - w$. The result then follows. □

This observation leads to the following theorem.

**Theorem 6.3.3** For all $n \geq 4$ and all $\kappa \leq n-4$, there exist non-isomorphic graphs $G$ and $H$ of connectivity $\kappa$ with $2\left\lfloor \frac{1}{3}(n-\kappa-1) \right\rfloor$ common cards.

*Proof* For any $\kappa \leq n-4$, let $G^*$ and $H^*$ be (one of) the appropriate pair of graphs of order $n-\kappa$ in Example 5.5.12, 5.5.13 and 5.5.14. Let $G \cong G^* \vee K_\kappa$ and $H \cong H^* \vee K_\kappa$. Clearly, $G$ and $H$ have connectivity $\kappa$. Moreover, $b(G, H) \geq 2\left\lfloor \frac{1}{3}(n-\kappa-1) \right\rfloor$, by Lemma 6.3.2. □

We can also construct pairs of graphs with high connectivity that have many common cards. If we replace $G^*$ and $H^*$ in the proof of Theorem 6.3.3 by their complements, it is not difficult to show that the resulting $G$ and $H$ both have connectivity $\left\lceil \frac{1}{3}(2n+\kappa-2) \right\rceil$, and the same number of common cards as in the theorem. We can clearly form pairs of graphs with the same number of edges or degree sequence of arbitrary connectivity and a large number of common cards using the above construction on the pairs in Examples 6.2.1 and 6.2.13.

A similar construction can be applied to the other families in Section 5.5. The following is of particular interest since the pair has the largest number of common cards for a pair of trees yet published, for large $n$.

**Theorem 6.3.4** For all $n \geq 11$, there exists non-isomorphic trees $G$ and $H$ with $2\left\lfloor \frac{1}{3}(n-5) \right\rfloor$ common cards.

*Proof* For $p = \left\lfloor \frac{n-5}{3} \right\rfloor$, let $G^*$ and $H^*$ be the forests in (6.2.1) of order $3p+4$. Let $G$ be the tree constructed from $G^*$ by adding a new "central" vertex and three edges joining this vertex to the three cut-vertices of $G^*$. We construct the tree $H$ in a similar manner from $H^*$. This pair of trees has $b(G, H) = 2p$ and is of order $3p+5$. We can extend this to the cases when $n \equiv 0$ or $1 \pmod 3$ by adding one or two leaves to the new "central' vertices. Figure 6.4 shows these trees for $p = 4$. □

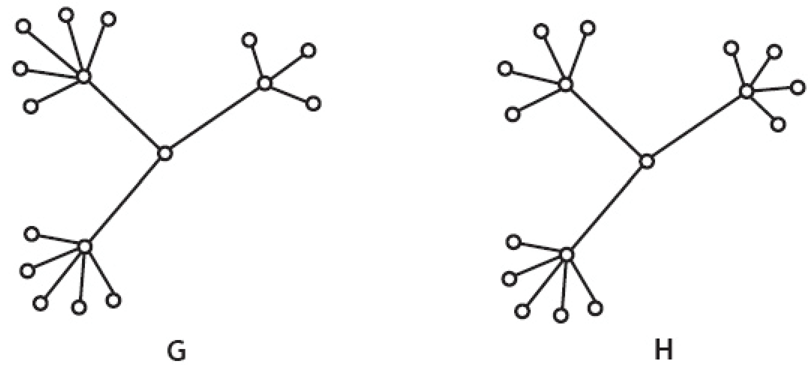


Figure 6.4: The pair of trees in Theorem 6.3.4 of order 17 with 8 common cards.

We can use a similar construction by adjoining a cycle of length 4 or 5 to the central vertices of these trees, and form pairs of unicyclic graphs having only slightly fewer common cards. For $n \equiv 1 \pmod 3$, however, we may simply add three edges joining the three cut-vertices of Example 6.2.1 to obtain a pair of unicyclic graphs with $b(G,\, H) = \frac{2}{3}(n-4)$. Figure 6.5 shows these graphs for $p = 4$.

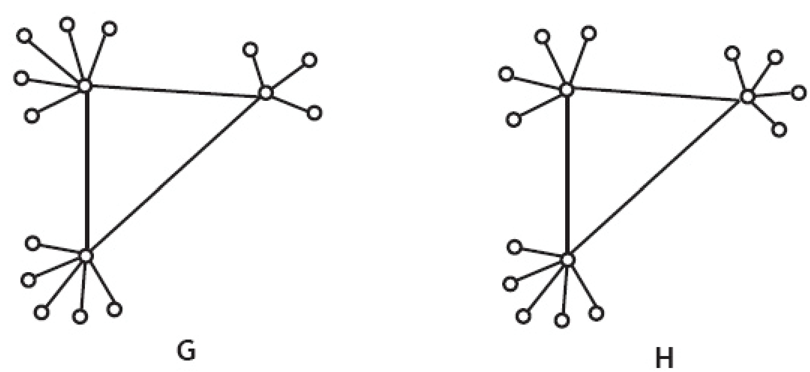


Figure 6.5: A pair of unicyclic graphs of order 16 with 8 common cards.

Finally, by using the tree construction described in Theorem 6.3.4 on the pair of forests in Example 6.2.15, we can form an infinite family of pairs of trees having the same degree sequence and at least $\frac{2}{3}(n + 11 - 4\sqrt{3n - 3})$ common cards. We can similarly construct pairs of unicyclic graphs with the same degree sequence and at least $\frac{2}{3}(n + 7 - 4\sqrt{3n - 7})$ common cards.

By employing methods similar to which we have used above, we can find families of graph pairs with a large number of common cards that belong to various different classes of graphs, both connected and disconnected. More importantly, however, as far as we are able to ascertain, no family of graph pairs with a higher number of common cards has yet been published. We therefore conjecture that no pair of graphs can have more than $2\left\lfloor \frac{1}{3}(n-1) \right\rfloor$ common cards.

**Conjecture 6.3.5** For large enough $n$, every finite simple undirected graph is determined, up to isomorphism, by any $2\left\lfloor \frac{1}{3}(n-1) \right\rfloor + 1$ of its vertex-deleted subgraphs.

Additionally we conjecture, that for large $n$, we can construct pairs of graphs from many classes that approach this bound.

We have shown that for these values of $n$, the only family of 2UC graph pairs that attain the bound of Conjecture 6.3.5 are those given in Examples 5.5.12, 5.5.13 and 5.5.14. Moreover, the only examples in this chapter that attain the bound are those formed by complementing the aforementioned examples, that is those given in Theorem 6.3.1.

We can also construct families that attain the bound by adding either one or two isolated vertices, or a component isomorphic to $K_2$, to the example in Theorem 6.3.1 when $n \equiv 1 \pmod 3$. We can similarly construct a family that attains the bound by adding a single isolated vertex to the family from this theorem with $n \equiv 2 \pmod 3$. Finally, we could construct a family that attains the bound by taking the example in Theorem 6.3.1 when $n \equiv 1 \pmod 3$, adding an isolated vertex, complementing both whole graphs, and then adding another isolated vertex.

We finish this thesis by conjecturing that the examples in Examples 5.5.12, 5.5.13 and 5.5.14, these extra families, plus all of their complements, are, up to isomorphism, the only other ones that attain the bound, for large enough $n$. It is easy to see there are, up to isomorphism, 18 distinct families of pairs of graphs constructed in this way. Again, we know of no counter-example for $n \geq 22$.

**Conjecture 6.3.6** For large enough $n$, the only pairs of graphs that attain the bound in Conjecture 6.3.5 are, up to isomorphism, the 18 families of pairs of graphs that can be constructed from Example 5.5.12, by any combination of complementing, and adding up to two isolated vertices or a component isomorphic to $K_2$.

# Bibliography


[1] K.J. Asciak, On Certain Classes of Graphs with Large Reconstruction Numbers, MSc thesis, University of Malta, 1998.

[2] K.J. Asciak and J. Lauri, “On disconnected graphs with a large reconstruction number”, Ars Combinatoria, Vol 62 (2002), 173-181.

[3] J. Baldwin, Graph Reconstruction Numbers, MSc dissertation, Rochester Institute of Technology, 2004.

[4] B. Bollobás, “Almost every graph has reconstruction number three”, Journal of Graph Theory Vol 14, No 1 (1990), 1-4.

[5] J.A. Bondy, Some Uniqueness Theorems in Graph Theory, PhD thesis, University of Oxford, 1968.

[6] J.A. Bondy, “A graph reconstructor’s manual”, Surveys in Combinatorics, London Mathematics Society Lecture Notes Series, Vol 166 (1991), 221-252.

[7] J.A. Bondy, “On Ulam’s conjecture for separate graphs”, Pacific Journal of Mathematics, Vol 31 (1969), 281-288.

[8] J.A. Bondy, personal correspondence with W. Myrvold.

[9] J.A. Bondy and R.L Hemminger, “Graph reconstruction - a survey”, Journal of Graph Theory, Vol 1 (1977), 227-268.

[10] J.A. Bondy and U.S.R Murty, Graph Theory with Applications, Elsevier Amsterdam, 1976.

[11] J.A. Bondy and U.S.R Murty, Graph Theory, Graduate Texts in Mathematics, Springer, 2008.

[12] A. Bowler, The Graph Reconstruction Problem, MSc dissertation, University of Nottingham, 1988.

[13] M. Francalanza, Adversary-Reconstruction of Trees: The Case of Caterpillars and Sunshine Graphs, MSc thesis, University of Malta, 1999.

[14] D.L. Greenwell and R.L Hemminger, "Reconstructing graphs", The Many Facets of Graph Theory, Lecture Notes in Mathematics, Vol 110 (1968), 91-114.

[15] D.L. Greenwell, "Reconstructing graphs", Proceedings of the American Mathematical Society, Vol 30, No 3 (1971), 431-433.

[16] W. Giles, "Reconstructing trees from two point deleted subtrees", Discrete Mathematics, North-Holland Publishing Company, Vol 15 (1976), 325-332.

[17] P. Hall, "On representatives of subsets", Journal of the London Mathematics Society, Vol 10 (1935), 26-30.

[18] F. Harary, "On the reconstruction of a graph from a collection of subgraphs", in Theory of Graphs and its Applications (Proceedings of the Symposium Smolenice, 1963), Publishing House of the Czechoslovakian Academy of Science, Prague (1964), 47-52.

[19] F. Harary and B. Manvel, "The reconstruction conjecture for labelled graphs", in (Proceedings of the Calgary international conference on combinatorial structures and their structures and their applications editors: R. K. Guy, H. Hanani, N.Sauer and J. Schonheim, Calgary, Alberta, 1969), Gordon and Breach, New York (1970), 131-146.

[20] P.J. Kelly, On Isometric Transformations, PhD thesis, University of Wisconsin, 1942.

[21] P.J. Kelly, "A congruence theorem for trees", Pacific Journal of Maths, Vol 7 (1957), 961-968.

[22] W.L. Kocay, "On reconstructing spanning subgraphs", Ars Combinatoria, Vol 11 (1981), 301-313.

[23] J. Lauri and R. Scapellato, Topics in Graph Automorphisms and Reconstruction, LMS Student Texts 54, 2003.

[24] J. Lauri, "Vertex-deleted and edge-deleted Subgraphs", in (A collection of papers by members of the University of Malta on the occasion of its quartercentenary celebrations editors: R. Ellul-Micallef and S. Fiorini), Malta (1992).

[25] B. Manvel, "Some basic observations on Kelly's conjecture for graphs", Discrete Mathematics Vol. 8 (1971), 181-185.

[26] B. Manvel, "On reconstruction of graphs", The Many Facets of Graph Theory, Lecture Notes in Mathematics, Vol 110 (1968), 91-114.

[27] B. Manvel, "On recontructing graphs from their subgraphs", Journal of Combinatorial Theory, Series B Vol 21 (1976), 156-165.

[28] B. Manvel, On Reconstruction of Graphs, PhD thesis, University of Michigan, 1970.

[29] B.D. McKay, "Small graphs are reconstructible", Australasian Journal of Combinatorics (1997), 123-126.

[30] B. McMullen, Reconstruction numbers, MSc thesis, Rochester Institute of Technology, 2005.

[31] R. Molina, "Correction of a proof on the ally reconstruction number of a disconnected graph", Ars Combinatoria, Vol 40 (1995), 59-64.

[32] V. Müller, "Probabilistic reconstruction from subgraphs", Commentary Mathematics of the University of Carolina, Vol 17 (1976), 109-719.

[33] W. Myrvold, Ally and Adversary Reconstruction Problems, PhD thesis, University of Waterloo, 1988.

[34] W. Myrvold, "The ally reconstruction number of of a disconnected graph", Ars Combinatoria, Vol 28 (1989), 123-127.

[35] W. Myrvold, “The degree sequence is reconstructible from n-1 Cards”, Discrete Mathematics Vol 102 (1990), 187-196.

[36] W. Myrvold, “The ally reconstruction number of a tree with five or more vertices is three”, Journal of Graph Theory 14 Vol 2 (1990), 149-166.

[37] P.K. Stockmeyer, “New counterexamples to the digraph reconstruction conjecture”, Notices American Mathematical Society, Vol 23 (1976a), A-654.

[38] D. Rivshin, A Computational Investigation of Graph Reconstruction, MSc thesis, Rochester Institute of Technology, 2007.

[39] W.T. Tutte, “All the King’s horses. A guide to reconstruction, in graph theory and related topics” in (Proceedings of the conference held in honour of W.T. Tutte on the occasion of his 60th birthday, J.A. Bondy and U.S.R. Murty), Academic Press, New York (1979), 15-33.

[40] S.M. Ulam, A Collection of Mathematical Problems, Wiley, New York, 1960.

[41] R. Wilson, Introduction to Graph Theory, Fourth Edition, Pearson Education Limited, Harlow, 1996.

[42] Y. Yongski, “The reconstruction conjecture is true if all 2-connected graphs are reconstructible”, Journal of Graph Theory Vol 12, No 2 (1988), 237-243.