Problem: Design a DFA $A$ such that $L(A)$ consists of all strings of $\blacksquare$ and $\Diamond$ which are of length $3n$, for $n = 0, 1, 2, \ldots$

(1) Determine what to remember about the input string.
   Assign a state to each of the possibilities: $q_{3n}$, $q_{3n+1}$, $q_{3n+2}$.

(2) Add the transitions telling how the possibilities rearrange,
   select the initial state and the favourable states.

(3) Test the automaton.
**Problem:** Design a DFA \( A \) such that \( L(A) \) consists of all strings of \( a \)'s and \( b \)'s which contain \( aab \) as a substring.

There are four possibilities to remember:

- haven’t seen any symbols of the pattern: (initial) state \( s \)
- have just seen an \( a \): state \( q_a \)
- have just seen \( aa \): state \( q_{aa} \) or
- have just seen the entire pattern \( aab \): (favourable) state \( q_{aab} \).
DFA as a theoretical model for programming

- unusual, different from conventional programming languages
- major concepts of imperative programming (e.g., sequences, branching, loops) can be simulated
- elegant, simple to use, has a thoroughly developed theory
- all pattern matching (e.g., WWW search engines) can be described
- limited expressive power (the automaton is not capable of increasing any of its resources during computation)

What about ‘subprogram calls’?
Combining two automata

**Problem:** Design an automaton that accepts all the strings that have

either two consecutive a’s or two consecutive b’s

- On page 24 (4), there is a DFA $A_4$ accepting all the strings that have two consecutive a’s. This automaton can easily be transformed into an automaton $A'_4$ that accepts all the strings that have two consecutive b’s.

- Since we already have ‘programs’ that solve the problem for aa’s and, respectively, bb’s, an obvious idea would be to design a ‘program’ that would just ‘call’ the given programs as ‘subprograms.’ A possible solution would be to ‘combine’ the initial states of $A_4$ and $A'_4$. 

Combining two automata (cont.)

**Given:**

\[ L(A_4) = \text{all words having two consecutive } a \text{'s} \]

\[ L(A'_4) = \text{all words having two consecutive } b \text{'s} \]

**Needed:** automaton \( A \) such that

\[ L(A) = L(A_4) \cup L(A'_4) \]
Observe that the state transition diagram on the previous slide no longer represents a DFA: there is more than one arrow coming out of the initial state $s$ with identical labels (say, $a$).

We say that this automaton is **nondeterministic** because, being in state $s$ and reading $a$ from the input tape, the device now has a choice: it can choose to move either to state $q_1$ or to state $s$.

What can affect this choice?

We may imagine that the automaton just tries to ‘guess’ which way will lead it finally to a favourable state.
But real computers cannot ‘guess’...

This argument can hardly be refuted.

However, one can justify this variant of finite automata by the following:

- Nondeterministic finite automata are much easier to design than deterministic ones.
  They allow us to model how ‘programs’ are built from ‘subprograms’.

- Whatever can be done by a nondeterministic finite automaton, it can also be done (though usually in a bit more complicated way) by a deterministic one.

In other words, nondeterminism does not increase the computational power of finite automata.
More nondeterminism

There are other features of nondeterministic finite automata (NFA):

- Just as from some states with some input symbols there may be several different possible next states, so with other combinations of states and input symbols there may be **no** possible moves at all (like in the example below, there is no $a$-arrow coming out of $s$).

- We also allow the machine to ‘jump’ from a state to another state, without consuming any input symbol. These ‘jumps’ will be indicated on the state transition diagrams by arrows labelled with the empty word $\varepsilon$.

  (Transition tables are no longer adequate to describe these automata.)

![State transition diagram](image_url)
Acceptance of words by NFAs

A word $w$ is accepted by an NFA $A$ if there is a computation of $A$ on input $w$ ending up with a configuration $(q, \varepsilon)$ for some favourable state $q$.

Otherwise, $A$ rejects the input word.

So an NFA can reject an input word because every computation on this input is

- either ‘stuck’
- or ends up with a configuration $(q', \varepsilon)$, but $q'$ is not a favourable state.

There can be more than one computation on an input word!!
Example: how an NFA works

- Computations on input \( baba \):
  
  \[
  (s, baba), (q, baba) \quad \text{stuck}
  \]
  
  \[
  (s, baba), (r, aba), (r, ba), (q, a), (s, \varepsilon) \quad \text{accepted}
  \]

- Computations on input \( aa \):
  
  \[
  (s, aa), (q, aa), (s, a), (q, a), (s, \varepsilon) \quad \text{accepted}
  \]

- Computations on input \( babba \):
  
  \[
  (s, babba), (q, babba) \quad \text{stuck}
  \]
  
  \[
  (s, babba), (r, abba), (r, bba), (q, ba) \quad \text{stuck}
  \]
  
  \[
  (s, babba), (r, abba), (q, bba) \quad \text{stuck, no more} \sim \text{rejected}
  \]
NFA: another example

- **Computations on input** 11:
  
  \[(s, 11), (q, 1)\] stuck
  
  \[(s, 11), (p, 11), (q, 1)\] stuck, no more \(\leadsto\) rejected

- **Computations on input** 00:
  
  \[(s, 00), (s, 0), (s, \varepsilon)\] stuck
  
  \[(s, 00), (s, 0), (p, \varepsilon)\] stuck
  
  \[(s, 00), (s, 0), (p, 0)\] stuck
  
  \[(s, 00), (p, 00)\] stuck, no more \(\leadsto\) rejected

- **Computations on input** 001:
  
  \[(s, 001), (s, 01), (s, 1), (q, \varepsilon)\] accepted
Nondeterministic Finite Automaton (NFA)

Formal definition:

A nondeterministic finite automaton (NFA) is a quintuple

\[ A = (Q, \Sigma, \Delta, s, F) \]

where

- \( Q \) is a finite set of states
- \( \Sigma \) is a finite set, the input alphabet
- \( s \in Q \) is the initial state
- \( F \subseteq Q \) is the set of favourable (or accepting) states
- \( \Delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q \) is a ternary relation, the transition relation.

(If \( A \), being in state \( q \in Q \), has read the symbol \( a \in \Sigma \), it enters one of the states \( q' \) for which \( (q, a, q') \) is in \( \Delta \). For any state \( q'' \) such that \( (q, \varepsilon, q'') \) is in \( \Delta \), \( A \) can `jump' from state \( q \) to state \( q'' \) without reading any input symbol.)
Configurations of NFAs

Let $A = (Q, \Sigma, \Delta, s, F)$ be an NFA.

- A **configuration** of $A$ is any pair of the form

  $$(\text{state in } Q, \text{word over } \Sigma)$$

- Configuration $(q, w)$ **yields** configuration $(q', w')$ in one step if
  
  - either $w = \sigma w'$ for some $\sigma$ in $\Sigma$ and the triple $(q, \sigma, q')$ is in $\Delta$
  
  - or $w = w'$ and the triple $(q, \varepsilon, q')$ is in $\Delta$ (‘jump’).

Observe that in general a configuration can yield **several** (or **no**) configurations in one step. In other words, a configuration does not determine uniquely the configuration it yields in one step.

  That is why the automaton $A$ is called **nondeterministic**
Computations of NFAs

A computation of $A$ on input word $w$ is any sequence of configurations

$$(q_1, w_1), (q_2, w_2), \ldots, (q_k, w_k)$$

such that

- $(q_1, w_1) = (s, w)$ where $s$ is the initial state of $A$
- every $(q_i, w_i)$ yields $(q_{i+1}, w_{i+1})$ in one step, and
- either $w_k$ is the empty word $\varepsilon$, or the configuration $(q_k, w_k)$ yields no configuration in one step (it gets stuck)

There can be more than one computation on an input word
NFAs and parallelism

Nondeterminism can be viewed as a kind of parallel computation wherein several ‘processes’ can be running concurrently.

- Suppose we are running an NFA on an input string and come to a state and read a symbol with multiple ways to proceed. Then the machine splits into multiple copies of itself and follows all the possibilities in parallel. If there are subsequent choices, the machine splits again.

- If the next input symbol does not appear on any arrows exiting the state occupied by a copy of the machine, that copy ‘dies,’ the branch of computation associated to it is ‘stuck.’

- If a state with some exiting $\varepsilon$-arrows is encountered then, without reading any input, the machine splits into multiple copies: one following each exiting $\varepsilon$-arrow, and one staying at the current state.
Differences between NFAs and DFAs

- There can be **more than one** arrow labelled by the same symbol coming out of a state (e.g., there are two \(a\)-arrows out of \(r\)).
- There can be **no** arrow labelled by some symbol coming out of a state (e.g., there is no \(b\)-arrow out of \(q\)).
- There can be arrows labelled by not symbols from the input alphabet but by the empty word \(\varepsilon\). These arrows represent ‘jumps’ (e.g. there is a jump from \(s\) to \(q\)).

Every DFA is an NFA as well!
Pattern matching revisited

**Problem:** Design a finite automaton \( A \) such that \( L(A) \) consists of all strings of \( a \)'s and \( b \)'s which contain \( aab \) as a substring.

- It is much easier to design an NFA than a DFA.
- We will show now that every NFA can be converted to a DFA accepting precisely the same words.
Two automata $A$ and $A'$ are said to be **equivalent** if they accept the same language.

**Claim:** For every NFA $A$, there is a DFA $A'$ equivalent to $A$.

Moreover, there is a ‘mechanical method,’ an **algorithm**, that carries out this conversion from a nondeterministic automaton to its deterministic equivalent.
The ‘subset construction’

Let $A = (Q, \Sigma, \Delta, s, F)$ be a NFA. We will turn $A$ to an equivalent DFA $A'$.

- The states of $A'$: all subsets of $Q$, that is, $\text{Pow}(Q)$

- The initial state of $A'$: the set containing the initial state $s$ of $A$ plus all those states from $Q$ that are reachable from $s$ by $\varepsilon$-jumps.

- The favourable states of $A'$: all those subsets of $Q$ that contain at least one favourable state of $A$.

In other words, $P \subseteq Q$ represents a favourable state of $A'$ if $P \cap F \neq \emptyset$. 
Example: the new states

NFA $A$:

$\begin{array}{c}
\text{∅} \\
\{s\} \\
\{r\} \\
\{s, r\} \\
\{q\} \\
\{s, q\} \\
\{q, r\} \\
\{s, q, r\} \\
\end{array}$

the $2^3 = 8$ new states of DFA $A'$:
Subset construction: the new transition function

Remember, being in a state and reading an input symbol, the transition function \( \delta \) tells the machine’s next state.

And the new states are sets of ‘old’ states now.

The transition function of \( A' \): for every subset \( P \) of \( Q \) and every symbol \( a \) in \( \Sigma \),

\[
\delta(P, a) = \text{the set of all states that are reachable from some state in } P \text{ by an } a\text{-arrow, followed by possibly one or more } \varepsilon\text{-jumps.}
\]
Computing the new transition function

\[ \delta(\{s, q\}, a) = \{s, q\} : \]

\[ \delta(\{s, q\}, b) = \{r\} : \]
The new transition function

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${s}$</td>
<td>$\emptyset$</td>
<td>${r}$</td>
</tr>
<tr>
<td>${q}$</td>
<td>${s, q}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>${r}$</td>
<td>${q, r}$</td>
<td>${q}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${s, q}$</td>
<td>${s, q}$</td>
<td>${r}$</td>
</tr>
<tr>
<td>${s, r}$</td>
<td>${q, r}$</td>
<td>${q, r}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${q, r}$</td>
<td>${s, q, r}$</td>
<td>${q}$</td>
</tr>
<tr>
<td>${s, q, r}$</td>
<td>${s, q, r}$</td>
<td>${q, r}$</td>
</tr>
</tbody>
</table>

![Transition diagram]
Removing the unreachable states

Observe that there is no computation going through the states \( \{s\} \) and \( \{s, r\} \), so we can safely remove them together with their outgoing arrows:

The initial state is never removable!
State minimisation problem

We saw that it is very easy to design nondeterministic finite automata for, say, pattern matching problems. However, an NFA cannot be directly implemented as a computer program. One must convert it into a DFA. But the number of states of the resulting DFA may increase significantly

\[ \text{number of states of the original NFA in the worst case} \]

An important field where finite automata are being used is hardware design: some components of hardware are based on simulation of DFA. An important objective here is to use finite automata that have a minimum possible number of states. There are methods for doing this. (The ‘subset construction’ is just the initial phase of design.)