Fundamentals of Computing

Tutorial 3 — Solutions

1. (a) $\text{succ} : \mathbb{N} \to \mathbb{N}$ is one-to-one because if $n + 1 = m + 1$ then $m = n$.
   $\text{succ} : \mathbb{N} \to \mathbb{N}$ is not onto: 0 cannot be obtained as a value. $\text{range}(\text{succ}) = \mathbb{N}^+$. 
(b) $+: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is not one-to-one: say, $2 + 8 = 3 + 7$ but $(2, 8) \neq (3, 7)$.
   $+: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is onto: every $n \in \mathbb{N}$ can be obtained as a sum, say, $n = n + 0$.

2. (a) $(f \circ f)(n) = f(f(n)) = f(2n + 1) = 2(2n + 1) + 1 = 4n + 3$.
(b) $(f \circ g)(n) = f(g(n)) = f(3n - 1) = 2(3n - 1) + 1 = 6n - 1$.
(c) $(g \circ f)(n) = g(f(n)) = g(2n + 1) = 3(2n + 1) - 1 = 6n + 2$.
(d) $(g \circ g)(n) = g(g(n)) = g(3n - 1) = 3(3n - 1) - 1 = 9n - 4$.

3. (a) $f \circ f = \begin{cases} n + 2, & 0 \leq n \leq 4; \\ n, & n = 5, 6; \\ n - 2, & n \geq 7. \end{cases}$
(b) $f \circ g = \begin{cases} 3n + 3, & n = 0, 1; \\ 3n + 1, & \text{otherwise}. \end{cases}$
(c) $g \circ f = \begin{cases} 3n - 1, & n > 5; \\ 3n + 5, & 0 \leq n \leq 5. \end{cases}$

4. (a) $\text{degree}(v_1) = \text{degree}(v_4) = 2$, $\text{degree}(v_2) = 5$, $\text{degree}(v_3) = 1$, $\text{degree}(v_5) = \text{degree}(v_7) = 3$, $\text{degree}(v_6) = 4$.
   Pendant: $v_3$.
   There are no isolated vertices.
(b) The sequence $(v_1, v_6, v_2, v_5, v_4)$ is not a path: A path is a sequence of vertices travelling along edges, but there is no edge between $v_2$ and $v_5$.
(c) The sequence $(v_6, v_7, v_5, v_6, v_1, v_6)$ is a simple cycle: It is a path that begins and ends with the same vertex (so it is a cycle), and it does not contain the same edge twice (so it is simple).
(d) There can be different solutions, depending on the order of listing of the vertices. This one belongs to the listing $v_1, v_2, v_3, v_4, v_5, v_6, v_7$:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

5. The function $f$ defined by taking $f(a) = 1$, $f(b) = 4$, $f(c) = 6$, $f(d) = 5$, $f(e) = 3$, $f(f) = 2$
   shows that these graphs are isomorphic (because $f$ is a bijection, and takes edges to edges, non-edges to non-edges). (There can be other ‘suitable’ functions as well.)

6. The automaton is shown in the picture below:
It accepts the first two words and rejects the third one. In general, it accepts those and only those words that do not have three consecutive 1.

7. (a) \((Q, \Sigma, \delta, s, F)\) where \(Q = \{s, q, r, p, t\}\), \(\Sigma = \{a, b\}\), \(F = \{s, q, r, p\}\) and the transition function \(\delta\) is as follows:

<table>
<thead>
<tr>
<th>(\delta)</th>
<th>(a)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s)</td>
<td>(q)</td>
<td>(t)</td>
</tr>
<tr>
<td>(q)</td>
<td>(q)</td>
<td>(r)</td>
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<tr>
<td>(r)</td>
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<td>(p)</td>
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<td>(t)</td>
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</tbody>
</table>

(b) \((s, aaabba), (q, aabba), (q, abba), (q, bba), (r, ba), (r, a), (p, \varepsilon)\); the string is accepted.

(c) \((s, aabaab), (q, abaab), (q, baab), (r, aab), (p, ab), (p, b), (t, \varepsilon)\); the string is not accepted.

(d) The language of this automaton consists of the empty string \(\varepsilon\) and all the strings of the form “a string of \(a\)s followed by a (possibly empty) string of \(b\)s followed by a (possibly empty) string of \(a\)s”.

8. (a) \((Q, \Sigma, \delta, s, F)\) where \(Q = \{s, q, r\}\), \(\Sigma = \{a, b\}\), \(F = \{r\}\) and the transition function \(\delta\) is as follows:

<table>
<thead>
<tr>
<th>(\delta)</th>
<th>(a)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s)</td>
<td>(q)</td>
<td>(s)</td>
</tr>
<tr>
<td>(q)</td>
<td>(r)</td>
<td>(s)</td>
</tr>
<tr>
<td>(r)</td>
<td>(r)</td>
<td>(q)</td>
</tr>
</tbody>
</table>

(b) \((s, aababa), (q, ababa), (r, aba), (q, a), (r, a), (p, \varepsilon)\); the string is accepted.

(c) \((s, bbaababa), (s, baababa), (s, aababa), (r, babaa), (q, abaa), (r, baaa), (q, aaaa), (r, aaaa), (r, a), (r, \varepsilon)\); the string is accepted.

No, it is not true, e.g. \(aba\) is not accepted.

(d) The language of this automaton is

\[
\{ waa(ba)^n a^m : w \text{ is any word of } a\text{s and } b\text{s, and } n, m \geq 0 \}.
\]

In other words, this language consists of all the strings of the following form: “take any word of \(a\)s and \(b\)s which is followed by \(aa\), then by any (possibly zero) number of repetitions of \(ba\), then by a (possibly empty) string of \(a\)s”.

9. The automaton accepts those non-empty words in which the number of \(b\)’s is divisible by 3.

10. The automaton looks as follows:
It accepts the words that have an odd number of sub-words of the form $ab$ and do not end with $a$.

11. The answers are given in the diagrams below: