# On the Decidability of Connectedness Constraints in 2D and 3D Euclidean Spaces

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## Abstract

We investigate (quantifier-free) spatial constraint languages with equality, contact and connectedness predicates, as well as Boolean operations on regions, interpreted over low-dimensional Euclidean spaces. We show that the complexity of reasoning varies dramatically depending on the dimension of the space and on the type of regions considered. For example, the logic with the interior-connectedness predicate (and without contact) is undecidable over polygons or regular closed sets in $\mathbb{R}^2$, EXPTIME-complete over polyhedra in $\mathbb{R}^3$, and NP-complete over regular closed sets in $\mathbb{R}^3$.

## 1 Introduction

A central task in Qualitative Spatial Reasoning is that of determining whether some described spatial configuration is geometrically realizable in 2D or 3D Euclidean space. Typically, such a description is given using a spatial logic—a formal language whose variables range over (typed) geometrical entities, and whose non-logical primitives represent geometrical relations and operations involving those entities. Where the geometrical primitives of the language are purely topological in character, we speak of a *topological logic*; and where the logical syntax is confined to that of propositional calculus, we speak of a *topological constraint language*.

Topological constraint languages have been intensively studied in Artificial Intelligence over the last two decades. The best-known of these, $\mathcal{RCC}8$ and $\mathcal{RCC}5$, employ variables ranging over regular closed sets in topological spaces, and a collection of eight (respectively, five) binary predicates standing for some basic topological relations between these sets [Egenhofer and Franzosa, 1991; Randell *et al.*, 1992; Bennett, 1994; Renz and Nebel, 2001]. An important extension of $\mathcal{RCC}8$, known as $\mathcal{BRCC}8$, additionally features standard Boolean operations on regular closed sets [Wolter and Zakharyaschev, 2000].

A remarkable characteristic of these languages is their *in*sensitivity to the underlying interpretation. To show that an $\mathcal{RCC}8$-formula is satisfiable in $n$-dimensional Euclidean space, it suffices to demonstrate its satisfiability in *any* topological space [Renz, 1998]; for $\mathcal{BRCC}8$-formulas, satisfiability in *any connected* space is enough. This inexpressiveness yields (relatively) low computational complexity: satisfiability of $\mathcal{BRCC}8$-, $\mathcal{RCC}8$- and $\mathcal{RCC}5$-formulas over arbitrary topological spaces is NP-complete; satisfiability of $\mathcal{BRCC}8$-formulas over connected spaces is PSPACE-complete.

However, satisfiability of spatial constraints by *arbitrary* regular closed sets by no means guarantees realizability by practically meaningful geometrical objects, where *connectedness* of regions is typically a minimal requirement [Borgo *et al.*, 1996; Cohn and Renz, 2008]. (A connected region is one which consists of a 'single piece.') It is easy to write constraints in $\mathcal{RCC}8$ that are satisfiable by connected regular closed sets over arbitrary topological spaces but not over $\mathbb{R}^2$; in $\mathcal{BRCC}8$ we can even write formulas satisfiable by connected regular closed sets over arbitrary spaces but not over $\mathbb{R}^n$ for any $n$. Worse still: there exist simple collections of spatial constraints (involving connectedness) that are satisfiable in the Euclidean plane, but only by 'pathological' sets that cannot plausibly represent the regions occupied by physical objects [Pratt-Hartmann, 2007]. Unfortunately, little is known about the complexity of topological constraint satisfaction by non-pathological objects in low-dimensional Euclidean spaces. One landmark result [Schaefer *et al.*, 2003] in this area shows that satisfiability of $\mathcal{RCC}8$-formulas by *disc homeomorphs* in $\mathbb{R}^2$ is still NP-complete (even though formulas can force arrangements that cut the plane into exponentially many regions). This paper investigates the computational properties of more general and flexible spatial logics with connectedness constraints interpreted over $\mathbb{R}^2$ and $\mathbb{R}^3$.

We consider two 'base' topological constraint languages. The language $\mathcal{B}$ features $=$ as its only predicate, but has function symbols $+$, $-$, $\cdot$ denoting the standard operations of fusion, complement and taking common parts defined for regular closed sets, as well as the constants $1$ and $0$ for the entire space and the empty set. Our second base language, $\mathcal{C}$, additionally features a binary predicate, $C$, denoting the 'contact' relation (two sets are in *contact* if they share at least one point). The language $\mathcal{C}$ is a notational variant of $\mathcal{BRCC}8$ (and thus an extension of $\mathcal{RCC}8$), while $\mathcal{B}$ is the analogous extension of $\mathcal{RCC}5$. We add to $\mathcal{B}$ and $\mathcal{C}$ one of two new unary predicates: $c$, representing the property of connectedness, and $c^\circ$, representing the (stronger) property of having a connected *interior*. We denote the resulting languages by $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$. We are interested in interpretations over (*i*) the regular closed sets of $\mathbb{R}^2$ and $\mathbb{R}^3$, and (*ii*) the regular closed *polyhe-*

*dra* in $\mathbb{R}^2$ and $\mathbb{R}^3$. (A set is polyhedral if it can be defined by finitely many bounding hyperplanes; see Sec. 2.) By restricting interpretations to polyhedra, we rule out pathological sets, and, in effect, use the same 'data structure' as in GISs.

When interpreted over *arbitrary* topological spaces, the complexity of reasoning with these languages is known: satisfiability of $\mathcal{B}c^\circ$-formulas is NP-complete, while for the other three languages, it is EXPTIME-complete. Likewise, the 1D Euclidean case is completely solved. For the spaces $\mathbb{R}^n$ ($n \geq 2$), however, most problems are still open. All four languages contain formulas satisfiable by regular closed sets in $\mathbb{R}^2$, but not by regular closed polygons; in $\mathbb{R}^3$, the analogous result is known only for $\mathcal{B}c^\circ$ and $\mathcal{C}c^\circ$. The satisfiability problem for $\mathcal{B}c$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ is EXPTIME-hard (in both polyhedral and unrestricted cases) for $\mathbb{R}^n$ ($n \geq 2$); however, the only known upper bound is that satisfiability of $\mathcal{B}c^\circ$-formulas by polyhedra in $\mathbb{R}^n$ ($n \geq 3$) is EXPTIME-complete. (See [Kontchakov *et al.*, 2010b] for a summary.)

This paper settles most of these open problems, revealing considerable differences between the computational properties of constraint languages with connectedness predicates when interpreted over $\mathbb{R}^2$ and over abstract topological spaces. Sec. 3 shows that $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ are all sensitive to restriction to polyhedra in $\mathbb{R}^n$ ($n \geq 2$). Sec. 4 establishes an unexpected result: all these languages are *undecidable* in $\mathbb{R}^2$, both in the polyhedral and unrestricted cases ([Dornheim, 1998] proves undecidability of the *first-order* versions of these languages). Sec. 5 resolves the open issue of the complexity of $\mathcal{B}c^\circ$ over regular closed sets (not just polyhedra) in $\mathbb{R}^3$ by establishing an NP upper bound. Thus, Qualitative Spatial Reasoning in Euclidean spaces proves much more challenging if connectedness of regions is to be taken into account. We discuss the obtained results in the context of spatial reasoning in Sec. 6. Omitted proofs can be found in [Kontchakov *et al.*, 2011].

## 2 Constraint Languages with Connectedness

Let $T$ be a topological space. We denote the closure of any $X \subseteq T$ by $X^-$, its interior by $X^\circ$ and its boundary by $\delta X = X^- \setminus X^\circ$. We call $X$ *regular closed* if $X = X^{\circ -}$, and denote by $\mathsf{RC}(T)$ the set of regular closed subsets of $T$. Where $T$ is clear from context, we refer to elements of $\mathsf{RC}(T)$ as *regions*. $\mathsf{RC}(T)$ forms a Boolean algebra under the operations $X + Y = X \cup Y$, $X \cdot Y = (X \cap Y)^{\circ -}$ and $-X = (T \setminus X)^-$. We write $X \leq Y$ for $X \cdot (-Y) = \emptyset$; thus $X \leq Y$ iff $X \subseteq Y$. A subset $X \subseteq T$ is *connected* if it cannot be decomposed into two disjoint, non-empty sets closed in the subspace topology; $X$ is *interior-connected* if $X^\circ$ is connected.

Any $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$, $n \geq 1$, bounds two elements of $\mathsf{RC}(\mathbb{R}^n)$ called *half-spaces*. We denote by $\mathsf{RCP}(\mathbb{R}^n)$ the Boolean subalgebra of $\mathsf{RC}(\mathbb{R}^n)$ generated by the half-spaces, and call the elements of $\mathsf{RCP}(\mathbb{R}^n)$ (regular closed) *polyhedra*. If $n = 2$, we speak of (regular closed) *polygons*. Polyhedra may be regarded as 'well-behaved' or, in topologists' parlance, '*tame*.' In particular, every polyhedron has finitely many connected components, a property which is not true of regular closed sets in general.

The topological constraint languages considered here all employ a countably infinite collection of variables $r_1, r_2, \ldots$ The language $\mathcal{C}$ features binary predicates $=$ and $C$, together with the individual constants 0, 1 and the function symbols $+, \cdot, -$. The *terms* $\tau$ and *formulas* $\varphi$ of $\mathcal{C}$ are given by:

$$\tau \quad ::= \quad r \;\mid\; \tau_1 + \tau_2 \;\mid\; \tau_1 \cdot \tau_2 \;\mid\; -\tau_1 \;\mid\; 1 \;\mid\; 0,$$
$$\varphi \quad ::= \quad \tau_1 = \tau_2 \;\mid\; C(\tau_1, \tau_2) \;\mid\; \varphi_1 \wedge \varphi_2 \;\mid\; \neg\varphi_1.$$

The language $\mathcal{B}$ is defined analogously, but without the predicate $C$. If $S \subseteq \mathsf{RC}(T)$ for some topological space $T$, an *interpretation over* $S$ is a function $\cdot^{\mathfrak{I}}$ mapping variables $r$ to elements $r^{\mathfrak{I}} \in S$. We extend $\cdot^{\mathfrak{I}}$ to terms $\tau$ by setting $0^{\mathfrak{I}} = \emptyset$, $1^{\mathfrak{I}} = T$, $(\tau_1 + \tau_2)^{\mathfrak{I}} = \tau_1^{\mathfrak{I}} + \tau_2^{\mathfrak{I}}$, etc. We write $\mathfrak{I} \models \tau_1 = \tau_2$ iff $\tau_1^{\mathfrak{I}} = \tau_2^{\mathfrak{I}}$, and $\mathfrak{I} \models C(\tau_1, \tau_2)$ iff $\tau_1^{\mathfrak{I}} \cap \tau_2^{\mathfrak{I}} \neq \emptyset$. We read $C(\tau_1, \tau_2)$ as '$\tau_1$ *contacts* $\tau_2$.' The relation $\models$ is extended to non-atomic formulas in the obvious way. A formula $\varphi$ is *satisfiable over* $S$ if $\mathfrak{I} \models \varphi$ for some interpretation $\mathfrak{I}$ over $S$.

Turning to languages with connectedness, we define $\mathcal{B}c$ and $\mathcal{C}c$ to be the extensions of $\mathcal{B}$ and $\mathcal{C}$ with the unary predicate $c$. We set $\mathfrak{I} \models c(\tau)$ iff $\tau^{\mathfrak{I}}$ is connected in the topological space under consideration. Similarly, we define $\mathcal{B}c^\circ$ and $\mathcal{C}c^\circ$ to be the extensions of $\mathcal{B}$ and $\mathcal{C}$ with the predicate $c^\circ$, setting $\mathfrak{I} \models c^\circ(\tau)$ iff $(\tau^{\mathfrak{I}})^\circ$ is connected. *Sat*$(\mathcal{L}, S)$ is the set of $\mathcal{L}$-formulas satisfiable over $S$, where $\mathcal{L}$ is one of $\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{B}c^\circ$ or $\mathcal{C}c^\circ$ (the topological space is implicit in this notation, but will always be clear from context). We shall be concerned with *Sat*$(\mathcal{L}, S)$, where $S$ is $\mathsf{RC}(\mathbb{R}^n)$ or $\mathsf{RCP}(\mathbb{R}^n)$ for $n = 2, 3$.

To illustrate, consider the $\mathcal{B}c^\circ$-formulas $\varphi_k$ given by

$$\bigwedge_{1 \leq i \leq k} \bigl(c^\circ(r_i) \wedge (r_i \neq 0)\bigr) \wedge \bigwedge_{i<j} \bigl(c^\circ(r_i + r_j) \wedge (r_i \cdot r_j = 0)\bigr). \quad (1)$$

One can show that $\varphi_3$ is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but not over $\mathsf{RC}(\mathbb{R})$, as no three intervals with non-empty, disjoint interiors can be in pairwise contact. Also, $\varphi_5$ is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, for $n \geq 3$, but not over $\mathsf{RC}(\mathbb{R}^2)$, as the graph $K_5$ is non-planar. Thus, $\mathcal{B}c^\circ$ is sensitive to the dimension of the space. Or again, consider the $\mathcal{B}c^\circ$-formula

$$\bigwedge_{1 \leq i \leq 3} c^\circ(r_i) \;\wedge\; c^\circ(r_1 + r_2 + r_3) \;\wedge\; \bigwedge_{2 \leq i \leq 3} \neg c^\circ(r_1 + r_i). \quad (2)$$

One can show that (2) is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, for any $n \geq 2$ (see, e.g., Fig. 1), but not over $\mathsf{RCP}(\mathbb{R}^n)$. Thus $\mathcal{B}c^\circ$ is sensitive to tameness in Euclidean spaces. It is known [Kontchakov *et al.*, 2010b] that, for the Euclidean *plane*, the same is true of $\mathcal{B}c$ and $\mathcal{C}c$: there is a $\mathcal{B}c$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, but not over $\mathsf{RCP}(\mathbb{R}^2)$. (The example required to show this is far more complicated than the $\mathcal{B}c^\circ$-formula (2).) In the next section, we prove that any of $\mathcal{B}c$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ contains formulas satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, for every $n \geq 2$, but only by regions with infinitely many components. Thus, all four of our languages are sensitive to tameness in all dimensions greater than one.

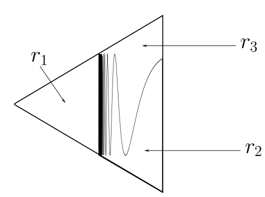


Figure 1: Three regions in $\mathsf{RC}(\mathbb{R}^2)$ satisfying (2).

## 3 Regions with Infinitely Many Components

Fix $n \geq 2$ and let $d_0, d_1, d_2, d_3$ be regions partitioning $\mathbb{R}^n$:

$$\left(\sum_{0\leq i\leq 3} d_i = 1\right) \quad \wedge \quad \bigwedge_{0\leq i<j\leq 3}(d_i \cdot d_j = 0). \qquad (3)$$

We construct formulas forcing the $d_i$ to have infinitely many connected components. To this end we require non-empty regions $a_i$ contained in $d_i$, and a non-empty region $t$:

$$\bigwedge_{0\leq i\leq 3}\big((a_i \neq 0) \wedge (a_i \leq d_i)\big) \quad \wedge \quad (t \neq 0). \qquad (4)$$

The configuration of regions we have in mind is depicted in Fig. 2, where components of the $d_i$ are arranged like the layers of an onion. The 'innermost' component of $d_0$ is surrounded by a component of $d_1$, which in turn is surrounded by a component of $d_2$, and so on. The region $t$ passes through every layer, but avoids the $a_i$. To enforce a configuration of this sort, we need the following three formulas, for $0 \leq i \leq 3$:

$$c(a_i + d_{\lfloor i+1\rfloor} + t), \qquad (5)$$

$$\neg C(a_i, d_{\lfloor i+1\rfloor} \cdot (-a_{\lfloor i+1\rfloor})) \quad \wedge \quad \neg C(a_i, t), \qquad (6)$$

$$\neg C(d_i, d_{\lfloor i+2\rfloor}), \qquad (7)$$

where $\lfloor k\rfloor = k \bmod 4$. Formulas (5) and (6) ensure that each component of $a_i$ is in contact with $a_{\lfloor i+1\rfloor}$, while (7) ensures that no component of $d_i$ can touch any component of $d_{\lfloor i+2\rfloor}$.

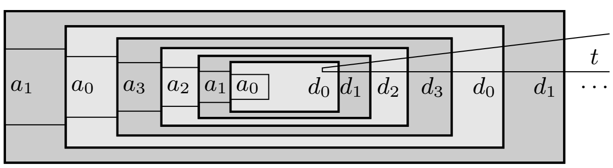


Figure 2: Regions satisfying $\varphi_\infty$.

Denote by $\varphi_\infty$ the conjunction of the above constraints. Fig. 2 shows how $\varphi_\infty$ can be satisfied over $\mathsf{RC}(\mathbb{R}^2)$. By cylindrification, it is also satisfiable over any $\mathsf{RC}(\mathbb{R}^n)$, for $n > 2$.

The arguments of this section are based on the following property of regular closed subsets of Euclidean spaces:

**Lemma 1** *If $X \in \mathsf{RC}(\mathbb{R}^n)$ is connected, then every component of $-X$ has a connected boundary.*

The proof of this lemma, which follows from [Newman, 1964], can be found in [Kontchakov *et al.*, 2011]. The result fails for other familiar spaces such as the torus.

**Theorem 2** *There is a $\mathcal{C}c$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but not by regions with finitely many components.*

**Proof.** Let $\varphi_\infty$ be as above. To simplify the presentation, we ignore the difference between variables and the regions they stand for, writing, for example, $a_i$ instead of $a_i^{\mathfrak{I}}$. We construct a sequence of disjoint components $X_i$ of $d_{\lfloor i\rfloor}$ and open sets $V_i$ connecting $X_i$ to $X_{i+1}$ (Fig. 3). By the first conjunct of (4), let $X_0$ be a component of $d_0$ containing points in $a_0$. Suppose $X_i$ has been constructed. By (5) and (6), $X_i$ is in contact with $a_{\lfloor i+1\rfloor}$. Using (7) and the fact that $\mathbb{R}^n$ is locally connected, one can find a component $X_{i+1}$ of $d_{\lfloor i+1\rfloor}$ which has points in $a_{i+1}$, and a connected open set $V_i$ such that $V_i \cap X_i$ and $V_i \cap X_{i+1}$ are non-empty, but $V_i \cap d_{\lfloor i+2\rfloor}$ is empty.

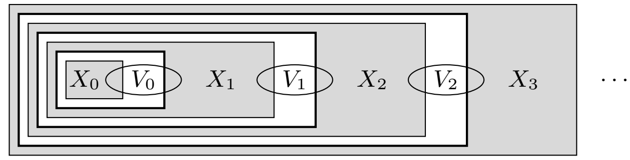


Figure 3: The sequence $\{X_i, V_i\}_{i\geq 0}$ generated by $\varphi_\infty$. ($S_{i+1}$ and $R_{i+1}$ are the 'holes' of $X_{i+1}$ containing $X_i$ and $X_{i+2}$.)

To see that the $X_i$ are distinct, let $S_{i+1}$ and $R_{i+1}$ be the components of $-X_{i+1}$ containing $X_i$ and $X_{i+2}$, respectively. It suffices to show $S_{i+1} \subseteq S_{i+2}^\circ$. Note that the connected set $V_i$ must intersect $\delta S_{i+1}$. Evidently, $\delta S_{i+1} \subseteq X_{i+1} \subseteq d_{\lfloor i+1\rfloor}$. Also, $\delta S_{i+1} \subseteq -X_{i+1}$; hence, by (3) and (7), $\delta S_{i+1} \subseteq d_i \cup d_{\lfloor i+2\rfloor}$. By Lemma 1, $\delta S_{i+1}$ is connected, and therefore, by (7), is entirely contained either in $d_{\lfloor i\rfloor}$ or in $d_{\lfloor i+2\rfloor}$. Since $V_i \cap \delta S_{i+1} \neq \emptyset$ and $V_i \cap d_{\lfloor i+2\rfloor} = \emptyset$, we have $\delta S_{i+1} \not\subseteq d_{\lfloor i+2\rfloor}$, so $\delta S_{i+1} \subseteq d_i$. Similarly, $\delta R_{i+1} \subseteq d_{i+2}$. By (7), then, $\delta S_{i+1} \cap \delta R_{i+1} = \emptyset$, and since $S_{i+1}$ and $R_{i+1}$ are components of the same set, they are disjoint. Hence, $S_{i+1} \subseteq (-R_{i+1})^\circ$, and since $X_{i+2} \subseteq R_{i+1}$, also $S_{i+1} \subseteq (-X_{i+2})^\circ$. So, $S_{i+1}$ lies in the interior of a component of $-X_{i+2}$, and since $\delta S_{i+1} \subseteq X_{i+1} \subseteq S_{i+2}$, that component must be $S_{i+2}$. ❑

Now we show how the $\mathcal{C}c$-formula $\varphi_\infty$ can be transformed to $\mathcal{C}c^\circ$- and $\mathcal{B}c$-formulas with similar properties. Note first that all occurrences of $c$ in $\varphi_\infty$ have positive polarity. Let $\varphi_\infty^\circ$ be the result of replacing them with the predicate $c^\circ$. In Fig. 2, the connected regions mentioned in (5) are in fact interior-connected; hence $\varphi_\infty^\circ$ is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$. Since interior-connectedness implies connectedness, $\varphi_\infty^\circ$ entails $\varphi_\infty$, and we obtain:

**Corollary 3** *There is a $\mathcal{C}c^\circ$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but not by regions with finitely many components.*

To construct a $\mathcal{B}c$-formula, we observe that all occurrences of $C$ in $\varphi_\infty$ are negative. We eliminate these using the predicate $c$. Consider, for example, the formula $\neg C(a_i, t)$ in (6). By inspection of Fig. 2, one can find regions $r_1$, $r_2$ satisfying

$$c(r_1) \wedge c(r_2) \wedge (a_i \leq r_1) \wedge (t \leq r_2) \wedge \neg c(r_1 + r_2). \qquad (8)$$

On the other hand, (8) entails $\neg C(a_i, t)$. By treating all other non-contact relations similarly, we obtain a $\mathcal{B}c$-formula $\psi_\infty$ that is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, and that entails $\varphi_\infty$. Thus:

**Corollary 4** *There is a $\mathcal{B}c$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but not by regions with finitely many components.*

Obtaining a $\mathcal{B}c^\circ$ analogue is complicated by the fact that we must enforce non-contact constraints using $c^\circ$ (rather than $c$). In the Euclidean plane, this can be done using *planarity constraints*; see [Kontchakov *et al.*, 2011].

**Theorem 5** *There is a $\mathcal{B}c^\circ$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, but not by regions with finitely many components.*

Theorem 2 and Corollary 4 entail that, if $\mathcal{L}$ is $\mathcal{B}c$ or $\mathcal{C}c$, then $\mathit{Sat}(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n)) \neq \mathit{Sat}(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$ for $n \geq 2$. Theorem 5 fails for $\mathsf{RC}(\mathbb{R}^n)$ with $n \geq 3$ (Sec. 5). However, we know from (2) that $\mathit{Sat}(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^n)) \neq \mathit{Sat}(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^n))$ for all $n \geq 2$. Theorem 2 fails in the 1D case; moreover, $\mathit{Sat}(\mathcal{L}, \mathsf{RC}(\mathbb{R})) = \mathit{Sat}(\mathcal{L}, \mathsf{RCP}(\mathbb{R}))$ only in the case $\mathcal{L} = \mathcal{B}c$ or $\mathcal{B}c^\circ$ [Kontchakov *et al.*, 2010b].

# 4 Undecidability in the Plane

Let $\mathcal{L}$ be any of $\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{B}c^\circ$ or $\mathcal{C}c^\circ$. In this section, we show, via a reduction of the *Post correspondence problem* (PCP), that *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ is r.e.-hard, and *Sat*$(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ is r.e.-complete. An *instance* of the PCP is a quadruple $\mathbf{w} = (S, T, \mathsf{w}_1, \mathsf{w}_2)$ where $S$ and $T$ are finite alphabets, and each $\mathsf{w}_i$ is a word morphism from $T^*$ to $S^*$. We may assume that $S = \{0, 1\}$ and $\mathsf{w}_i(t)$ is non-empty for any $t \in T$. The instance $\mathbf{w}$ is *positive* if there exists a non-empty $\tau \in T^*$ such that $\mathsf{w}_1(\tau) = \mathsf{w}_2(\tau)$. The set of positive PCP-instances is known to be r.e.-complete. The reduction can only be given in outline here: for details, see [Kontchakov *et al.*, 2011].

To deal with arbitrary regular closed subsets of $\mathsf{RC}(\mathbb{R}^2)$, we use the technique of 'wrapping' a region inside two bigger ones. Let us say that a *3-region* is a triple $\mathfrak{a} = (a, \dot{a}, \ddot{a})$ of elements of $\mathsf{RC}(\mathbb{R}^2)$ such that $0 \neq \ddot{a} \ll \dot{a} \ll a$, where $r \ll s$ abbreviates $\neg C(r, -s)$. It helps to think of $\mathfrak{a} = (a, \dot{a}, \ddot{a})$ as consisting of a kernel, $\ddot{a}$, encased in two protective layers of shell. As a simple example, consider the sequence of 3-regions $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ depicted in Fig. 4, where the innermost regions form a sequence of externally touching polygons. When describing arrangements of 3-regions, we use

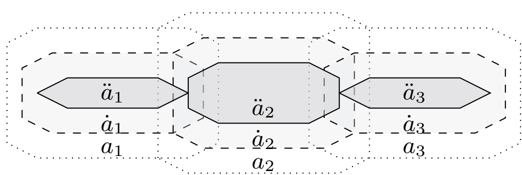


Figure 4: A chain of 3-regions satisfying $\mathsf{stack}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3)$.

the variable $\mathfrak{r}$ for the triple of variables $(r, \dot{r}, \ddot{r})$, taking the conjuncts $\ddot{r} \neq 0$, $\ddot{r} \ll \dot{r}$ and $\dot{r} \ll r$ to be implicit. As with ordinary variables, we often ignore the difference between 3-region variables and the 3-regions they stand for.

For $k \geq 3$, define the formula $\mathsf{stack}(\mathfrak{a}_1, \dots, \mathfrak{a}_k)$ by

$$\bigwedge_{1 \leq i \leq k} c(\dot{a}_i + \ddot{a}_{i+1} + \cdots + \ddot{a}_k) \quad \wedge \quad \bigwedge_{j-i>1} \neg C(a_i, a_j).$$

Thus, the triple of 3-regions in Fig. 4 satisfies $\mathsf{stack}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3)$. This formula plays a crucial role in our proof. If $\mathsf{stack}(\mathfrak{a}_1, \dots, \mathfrak{a}_k)$ holds, then any point $p_0$ in the inner shell $\dot{a}_1$ of $\mathfrak{a}_1$ can be connected to any point $p_k$ in the kernel $\ddot{a}_k$ of $\mathfrak{a}_k$ via a Jordan arc $\gamma_1 \cdots \gamma_k$ whose $i$th segment, $\gamma_i$, never leaves the outer shell $a_i$ of $\mathfrak{a}_i$. Moreover, each $\gamma_i$ intersects the inner shell $\dot{a}_{i+1}$ of $\mathfrak{a}_{i+1}$, for $1 \leq i < k$.

This technique allows us to write $\mathcal{C}c$-formulas whose satisfying regions are guaranteed to contain various networks of arcs, exhibiting almost any desired pattern of intersections. Now recall the construction of Sec. 3, where constraints on the variables $d_0, \dots, d_3$ were used to enforce 'cyclic' patterns of components. Using $\mathsf{stack}(\mathfrak{a}_1, \dots, \mathfrak{a}_k)$, we can write a formula with the property that the regions in any satisfying assignment are forced to contain the pattern of arcs having the form shown in Fig. 5. These arcs define a 'window,' containing a sequence $\{\zeta_i\}$ of 'horizontal' arcs ($1 \leq i \leq n$), each connected by a corresponding 'vertical arc,' $\eta_i$, to some point on the 'top edge.' We can ensure that each $\zeta_i$ is included in a

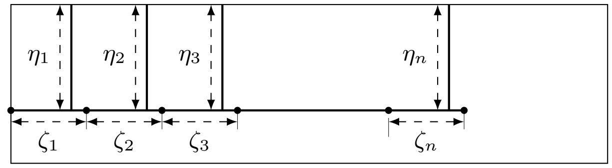


Figure 5: Encoding the PCP: Stage 1.

region $a_{\lfloor i \rfloor}$, and each $\eta_i$ ($1 \leq i \leq n$) in a region $b_{\lfloor i \rfloor}$, where $\lfloor i \rfloor$ now indicates $i$ mod 3. By repeating the construction, a second pair of arc-sequences, $\{\zeta'_i\}$ and $\{\eta'_i\}$ ($1 \leq i \leq n'$) can be established, but with each $\eta'_i$ connecting $\zeta'_i$ to the 'bottom edge.' Again, we can ensure each $\zeta'_i$ is included in a region $a'_{\lfloor i \rfloor}$ and each $\eta'_i$ in a region $b'_{\lfloor i \rfloor}$ ($1 \leq i \leq n'$). Further, we can ensure that the final horizontal arcs $\zeta_n$ and $\zeta'_{n'}$ (but no others) are joined by an arc $\zeta^*$ lying in a region $z^*$. The cru-

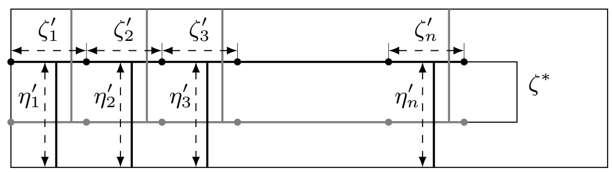


Figure 6: Encoding the PCP: Stage 2.

cial step is to match up these arc-sequences. To do so, we write $\neg C(a'_i, b_j) \wedge \neg C(a_i, b'_j) \wedge \neg C(b_i + b'_i, b_j + b'_j + z^*)$, for all $i$, $j$ ($0 \leq i, j < 3$, $i \neq j$). A simple argument based on planarity considerations then ensures that the upper and lower sequences of arcs must cross (essentially) as shown in Fig. 6. In particular, we are guaranteed that $n = n'$ (without specifying the value $n$), and that, for all $1 \leq i \leq n$, $\zeta_i$ is connected by $\eta_i$ (and also by $\eta'_i$) to $\zeta'_i$.

Having established the configuration of Fig. 6, we write $(b_i \leq l_0 + l_1) \wedge \neg C(b_i \cdot l_0, b_i \cdot l_1)$, for $0 \leq i < 3$, ensuring that each $\eta_i$ is included in exactly one of $l_0$, $l_1$. These inclusions naturally define a word $\sigma$ over the alphabet $\{0, 1\}$. Next, we write $\mathcal{C}c$-constraints which organize the sequences of arcs $\{\zeta_i\}$ and $\{\zeta'_i\}$ (independently) into consecutive blocks. These blocks of arcs can then be put in 1–1 correspondence using essentially the same construction used to put the individual arcs in 1–1 correspondence. Each pair of corresponding blocks can now be made to lie in exactly one region from a collection $t_1, \dots, t_\ell$. We think of the $t_j$ as representing the letters of the alphabet $T$, so that the labelling of the blocks with these elements defines a word $\tau \in T^*$. It is then straightforward to write non-contact constraints involving the arcs $\zeta_i$ ensuring that $\sigma = \mathsf{w}_1(\tau)$ and non-contact constraints involving the arcs $\zeta'_i$ ensuring that $\sigma = \mathsf{w}_2(\tau)$. Let $\varphi_{\mathbf{w}}$ be the conjunction of all the foregoing $\mathcal{C}c$-formulas. Thus, if $\varphi_{\mathbf{w}}$ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, then $\mathbf{w}$ is a positive instance of the PCP. On the other hand, if $\mathbf{w}$ is a positive instance of the PCP, then one can construct a tuple satisfying $\varphi_{\mathbf{w}}$ over $\mathsf{RCP}(\mathbb{R}^2)$ by 'thickening' the above collections of arcs into polygons in the obvious way. So, $\mathbf{w}$ is positive iff $\varphi_{\mathbf{w}}$ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$ iff $\varphi_{\mathbf{w}}$ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. This shows r.e.-hardness of *Sat*$(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^2))$ and *Sat*$(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}^2))$. Membership of the latter problem in r.e. is immediate because all polygons may be assumed to have vertices with rational coordinates, and so may be effectively enumerated. Using the techniques of Corollaries 3–4 and Theorem 5, we obtain:

**Theorem 6** *For $\mathcal{L} \in \{\mathcal{B}c^\circ, \mathcal{B}c, \mathcal{C}c^\circ, \mathcal{C}c\}$, Sat$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ is r.e.-hard, and Sat$(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ is r.e.-complete.*

The complexity of *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^3))$ remains open for the languages $\mathcal{L} \in \{\mathcal{B}c, \mathcal{C}c^\circ, \mathcal{C}c\}$. However, as we shall see in the next section, for $\mathcal{B}c^\circ$ it drops dramatically.

## 5 $\mathcal{B}c^\circ$ in 3D

In this section, we consider the complexity of satisfying $\mathcal{B}c^\circ$-constraints by polyhedra and regular closed sets in three-dimensional Euclidean space. Our analysis rests on an important connection between geometrical and graph-theoretic interpretations. We begin by briefly discussing the results of [Kontchakov *et al.*, 2010a] for the *polyhedral* case.

Recall that every partial order $(W, R)$, where $R$ is a transitive and reflexive relation on $W$, can be regarded as a topological space by taking $X \subseteq W$ to be open just in case $x \in X$ and $xRy$ imply $y \in X$. Such topologies are called *Aleksandrov spaces*. If $(W, R)$ contains no proper paths of length greater than 2, we call $(W, R)$ a *quasi-saw* (Fig. 8). If, in addition, no $x \in W$ has more than two proper $R$-successors, we call $(W, R)$ a 2-*quasi-saw*. The properties of 2-quasi-saws we need are as follows [Kontchakov *et al.*, 2010a]:

- satisfiability of $\mathcal{B}c$-formulas in arbitrary topological spaces coincides with satisfiability in 2-quasi-saws, and is ExpTime-complete;
- $X \subseteq W$ is connected in a 2-quasi-saw $(W, R)$ iff it is interior-connected in $(W, R)$.

The following construction lets us apply these results to the problem *Sat*$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^3))$. Say that a *connected partition* in $\mathsf{RCP}(\mathbb{R}^3)$ is a tuple $X_1, \ldots, X_k$ of non-empty polyhedra having connected and pairwise disjoint interiors, which sum to the entire space $\mathbb{R}^3$. The *neighbourhood graph* $(V, E)$ of this partition has vertices $V = \{X_1, \ldots, X_k\}$ and edges $E = \{\{X_i, X_j\} \mid i \neq j \text{ and } (X_i + X_j)^\circ \text{ is connected}\}$ (Fig. 7). One can show that *every* connected graph is the neighbourhood graph of some connected partition in $\mathsf{RCP}(\mathbb{R}^3)$. Furthermore, every neighbourhood graph $(V, E)$ gives rise to a 2-quasi-saw, namely, $(W_0 \cup W_1, R)$, where $W_0 = V$, $W_1 = \{z_{x,y} \mid \{x, y\} \in E\}$, and $R$ is the reflexive closure of $\{(z_{x,y}, x), (z_{x,y}, y) \mid \{x, y\} \in E\}$. From this, we see that (*i*) a $\mathcal{B}c^\circ$-formula $\varphi$ is satisfiable over $\mathsf{RCP}(\mathbb{R}^3)$ iff (*ii*) $\varphi$ is satisfiable over a connected 2-quasi-saw iff (*iii*) the $\mathcal{B}c$-formula $\varphi^\bullet$, obtained from $\varphi$ by replacing every occurrence of $c^\circ$ with $c$, is satisfiable over a connected 2-quasi-saw. Thus, *Sat*$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^3))$ is ExpTime-complete.

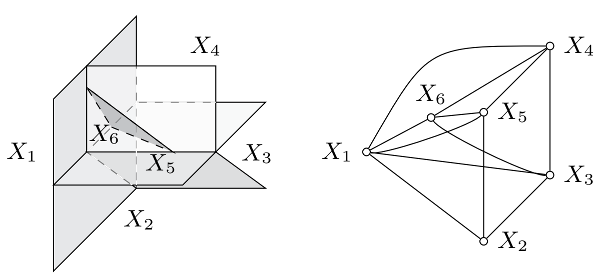


Figure 7: A connected partition and its neighbourhood graph.

The picture changes if we allow variables to range over $\mathsf{RC}(\mathbb{R}^3)$ rather than $\mathsf{RCP}(\mathbb{R}^3)$. Note first that the $\mathcal{B}c^\circ$-formula (2) is not satisfiable over 2-quasi-saws, but has a quasi-saw model as in Fig. 8. Some extra geometrical work will show

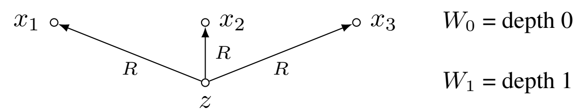


Figure 8: A quasi-saw model $\mathfrak{I}$ of (2): $r_i^{\mathfrak{I}} = \{x_i, z\}$.

now that (*iv*) a $\mathcal{B}c^\circ$-formula is satisfiable over $\mathsf{RC}(\mathbb{R}^3)$ iff (*v*) it is satisfiable over a connected quasi-saw. And as shown in [Kontchakov *et al.*, 2010a], satisfiability of $\mathcal{B}c^\circ$-formulas in connected spaces coincides with satisfiability over connected quasi-saws, and is NP-complete.

**Theorem 7** *The problem Sat$(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^3))$ is* NP*-complete.*

**Proof.** From the preceding discussion, it suffices to show that (*v*) implies (*iv*) for any $\mathcal{B}c^\circ$-formula $\varphi$. So suppose $\mathfrak{A} \models \varphi$, with $\mathfrak{A}$ based on a finite connected quasi-saw $(W_0 \cup W_1, R)$, where $W_i$ contains all points of depth $i \in \{0, 1\}$ (Fig. 8). Without loss of generality we will assume that there is a special point $z_0$ of depth 1 such that $z_0 R x$ for all $x$ of depth 0. We show how $\mathfrak{A}$ can be embedded into $\mathsf{RC}(\mathbb{R}^3)$.

Take pairwise disjoint *closed* balls $B_x^1$, for $x$ of depth 0, and pairwise disjoint *open* balls $D_z$, for all $z$ of depth 1 except $z_0$ (we assume the $D_z$ are disjoint from the $B_x^1$). Let $D_{z_0}$ be the closure of the complement of all $B_x^1$ and $D_z$. We expand the $B_x^1$ to sets $B_x$ forming a connected partition in $\mathsf{RC}(\mathbb{R}^3)$ (i.e. they sum to $\mathbb{R}^3$, and their interiors are non-empty, connected and pairwise disjoint). To construct the $B_x$, let $q_1, q_2, \ldots$ be an enumeration of all the points in the interiors of any of the $D_z$ with *rational* coordinates. For $x \in W_0$, we set $B_x$ to be $\left(\bigcup_{k \geq 1} B_x^k\right)^-$, where the regular closed sets $B_x^k$ are defined inductively as follows (Fig. 9). Suppose, for $k \geq 1$, $B_x^k$ has been defined for all $x \in W_0$. Let $q_i$ be the first point in the list $q_1, q_2, \ldots$ that is not in any $B_x^k$ yet. If $q_i$ is in the interior of some $D_z$, take a closed ball in the interior of $D_z$ centred on $q_i$ and disjoint from the $B_x^k$. Now pick some $x$ such that $zRx$, and expand $B_x^k$ by the closed ball around $q_i$ together with a closed 'rod' connecting it to $B_x^1$, in such a way that the rod is disjoint from the rest of the $B_x^k$; the result is denoted by $B_x^{k+1}$. Consider the function $f$ mapping regular closed sets $X \subseteq W$ to $\mathsf{RC}(\mathbb{R}^3)$, defined by $f(X) = \sum_{x \in X \cap W_0} B_x$. Since the $B_x$ form a partition, $f$ preserves $+, \cdot, -, 0$ and 1. And since, for all $z$, $\sum\{B_x \mid zRx\}$ is interior connected, $f$ preserves interior-connectedness. By carefully adding extra balls and rods in the construction of the $B_x^k$, we can further ensure that non-interior-connected elements of $\mathsf{RC}(W, R)$ are mapped to non-interior connected elements of $\mathsf{RC}(\mathbb{R}^3)$ (for details, see [Kontchakov *et al.*, 2011]). Defining an interpretation $\mathfrak{I}$ over $\mathsf{RC}(\mathbb{R}^3)$ by setting $r^{\mathfrak{I}} = f(r^{\mathfrak{A}})$ then secures $\mathfrak{I} \models \varphi$. ❑

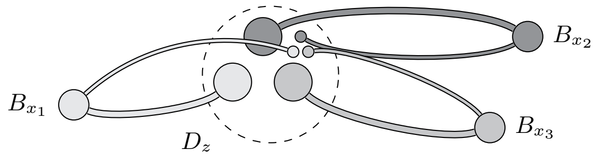

Figure 9: Filling $D_z$ with $B_{x_i}$, for $zRx_i$, $i = 1, 2, 3$.

The remarkably diverse computational behaviour of $\mathcal{B}c^\circ$ over $\mathsf{RC}(\mathbb{R}^3)$, $\mathsf{RCP}(\mathbb{R}^3)$ and $\mathsf{RCP}(\mathbb{R}^2)$ can be explained as follows. To satisfy a $\mathcal{B}c^\circ$-formula $\varphi$ in $\mathsf{RC}(\mathbb{R}^3)$, it suffices to find polynomially many points in the regions mentioned in $\varphi$ (witnessing non-emptiness or non-interior-connectedness constraints), and then to 'inflate' those points to (possibly interior-connected) regular closed sets using the technique of Fig. 9. By contrast, over $\mathsf{RCP}(\mathbb{R}^3)$, one can write a $\mathcal{B}c^\circ$-formula analogous to (8) stating that two interior-connected polyhedra do not share a 2D face. Such 'face-contact' constraints can be used to generate constellations of exponentially many polyhedra simulating runs of alternating Turing machines on polynomial tapes, leading to EXPTIME-hardness. Finally, over $\mathsf{RCP}(\mathbb{R}^2)$, planarity considerations endow $\mathcal{B}c^\circ$ with the extra expressive power required to enforce full non-contact constructs (not possible in higher dimensions), and thus to encode the PCP as sketched in Sec. 4.

## 6 Conclusion

This paper investigated topological constraint languages featuring connectedness predicates and Boolean operations on regions. Unlike their less expressive cousins, $\mathcal{RCC}8$ and $\mathcal{RCC}5$, such languages are highly sensitive to the spaces over which they are interpreted, and exhibit more challenging computational behaviour. Specifically, we demonstrated that the languages $\mathcal{C}c$, $\mathcal{C}c^\circ$ and $\mathcal{B}c$ contain formulas satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but only by regions with infinitely many components. Using a related construction, we proved that the satisfiability problem for any of $\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{B}c^\circ$ and $\mathcal{C}c^\circ$, interpreted either over $\mathsf{RC}(\mathbb{R}^2)$, or over its polygonal subalgebra, $\mathsf{RCP}(\mathbb{R}^2)$, is *undecidable*. Finally, we showed that the satisfiability problem for $\mathcal{B}c^\circ$, interpreted over $\mathsf{RC}(\mathbb{R}^3)$, is NP-complete, which contrasts with EXPTIME-completeness for $\mathsf{RCP}(\mathbb{R}^3)$. The complexity of satisfiability for $\mathcal{B}c$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ over $\mathsf{RC}(\mathbb{R}^n)$ or $\mathsf{RCP}(\mathbb{R}^n)$ for $n \geq 3$ remains open. The obtained results rely on certain distinctive topological properties of Euclidean spaces. Thus, for example, the argument of Sec. 3 is based on the property of Lemma 1, while Sec. 4 similarly relies on *planarity* considerations. In both cases, however, the moral is the same: the topological spaces of most interest for Qualitative Spatial Reasoning exhibit special characteristics which any topological constraint language able to express connectedness must take into account.

The results of Sec. 4 pose a challenge for Qualitative Spatial Reasoning in the Euclidean plane. On the one hand, the relatively low complexity of $\mathcal{RCC}8$ over disc-homeomorphs suggests the possibility of usefully extending the expressive power of $\mathcal{RCC}8$ without compromising computational properties. On the other hand, our results impose severe limits on any such extension. We observe, however, that the constructions used in the proofs depend on a strong interaction between the connectedness predicates and the Boolean operations on regular closed sets. We believe that by restricting this interaction one can obtain non-trivial constraint languages with more acceptable complexity. For example, the extension of $\mathcal{RCC}8$ with connectedness constraints is still in NP for both $\mathsf{RC}(\mathbb{R}^2)$ and $\mathsf{RCP}(\mathbb{R}^2)$ [Kontchakov *et al.*, 2010b].

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## References

[Bennett, 1994] B. Bennett. Spatial reasoning with propositional logic. In *Proc. of KR*, pages 51–62. 1994.

[Borgo *et al.*, 1996] S. Borgo, N. Guarino, and C. Masolo. A pointless theory of space based on strong connection and congruence. In *Proc. of KR*, pages 220–229. 1996.

[Cohn and Renz, 2008] A. Cohn and J. Renz. Qualitative spatial representation and reasoning. In *Handbook of Knowledge Representation*, pages 551–596. Elsevier, 2008.

[Dornheim, 1998] C. Dornheim. Undecidability of plane polygonal mereotopology. In *Proc. of KR*. 1998.

[Egenhofer and Franzosa, 1991] M. Egenhofer and R. Franzosa. Point-set topological spatial relations. *International J. of Geographical Information Systems*, 5:161–174, 1991.

[Kontchakov *et al.*, 2010a] R. Kontchakov, I. Pratt-Hartmann, F. Wolter, and M. Zakharyaschev. Spatial logics with connectedness predicates. *Logical Methods in Computer Science*, 6(3), 2010.

[Kontchakov *et al.*, 2010b] R. Kontchakov, I. Pratt-Hartmann, and M. Zakharyaschev. Interpreting topological logics over Euclidean spaces. In *Proc. of KR*. 2010.

[Kontchakov *et al.*, 2011] R. Kontchakov, Y. Nenov, I. Pratt-Hartmann, and M. Zakharyaschev. On the decidability of connectedness constraints in 2D and 3D Euclidean spaces. arXiv:1104.0219, 2011.

[Newman, 1964] M.H.A. Newman. *Elements of the Topology of Plane Sets of Points*. Cambridge, 1964.

[Pratt-Hartmann, 2007] I. Pratt-Hartmann. First-order mereotopology. In *Handbook of Spatial Logics*, pages 13–97. Springer, 2007.

[Randell *et al.*, 1992] D. Randell, Z. Cui, and A. Cohn. A spatial logic based on regions and connection. In *Proc. of KR*, pages 165–176. 1992.

[Renz and Nebel, 2001] J. Renz and B. Nebel. Efficient methods for qualitative spatial reasoning. *J. Artificial Intelligence Research*, 15:289-318, 2001.

[Renz, 1998] J. Renz. A canonical model of the region connection calculus. In *Proc. of KR*, pages 330–341. 1998.

[Schaefer *et al.*, 2003] M. Schaefer, E. Sedgwick, and D. Štefankovič. Recognizing string graphs in NP. *J. of Computer and System Sciences*, 67:365–380, 2003.

[Wolter and Zakharyaschev, 2000] F. Wolter and M. Zakharyaschev. Spatial reasoning in $\mathcal{RCC}8$ with Boolean region terms. In *Proc. of ECAI*, pages 244–248. 2000.