# $\mathcal{ALC}$ TABLEAU

$$1.\ \neg(\forall R.A \sqcup \exists R.(\neg A \sqcap \neg B))$$

First of all we transform the concept to an equivalent concept in negation normal form:

$$= \neg(\forall R.A) \sqcap \neg(\exists R.(\neg A \sqcap \neg B)) =$$
$$= (\exists R.\neg A) \sqcap (\forall R.\neg(\neg A \sqcap \neg B))) =$$
$$= (\exists R.\neg A) \sqcap \forall R.(A \sqcup B)$$

Next we apply the $\mathcal{ALC}$ tableau algorithm to it:

$S_0 = \{x : \exists R.\neg A \sqcap \forall R.(A \sqcup B)\}$

$S_0 \rightarrow_{\sqcap} S_1 = S_0 \cup \{x : \exists R.\neg A,\ x : \forall R.(A \sqcup B)\}$

$S_1 \rightarrow_{\exists} S_2 = S_1 \cup \{(x, y) : R,\ y : \neg A\}$ $y$ is a fresh individual

$S_2 \rightarrow_{\forall} S_3 = S_2 \cup \{y : A \sqcup B\}$

$+S_3 \rightarrow_{\sqcup} S_{4.1} = S_3 \cup \{y : A\}$ CLASH with $S_2$

$+S_3 \rightarrow_{\sqcup} S_{4.2} = S_3 \cup \{y : B\}$

$S_{4.2}$ is a complete and clash-free constraint system.

It induces the model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where:

$\Delta^{\mathcal{I}} = \{x, y\}$

$R^{\mathcal{I}} = \{(x, y)\}$

$A^{\mathcal{I}} = \emptyset$

$B^{\mathcal{I}} = \{y\}$

$$2.\ \exists R.(\forall S.C) \sqcap \forall R.(\exists S.\neg C)$$

The concept is already expressed in negation normal form, we apply the $\mathcal{ALC}$ tableau algorithm to it:

$S_0 = \{x : \exists R.(\forall S.C) \sqcap \forall R.(\exists S.\neg C)\}$

$S_0 \rightarrow_{\sqcap} S_1 = S_0 \cup \{x : \exists R.(\forall S.C),\ x : \forall R.(\exists S.\neg C)\}$

$S_1 \rightarrow_\exists S_2 = S_1 \cup \{(x, y) : R,\ y : \forall S.C\}$ $y$ is a fresh individual

$S_2 \rightarrow_\forall S_3 = S_2 \cup \{y : \exists S.\neg C\}$

$S_3 \rightarrow_\exists S_4 = S_3 \cup \{(y, z) : S,\ z : \neg C\}$ $z$ is a fresh individual

$S_4 \rightarrow_\forall S_5 = S_4 \cup \{z : C\}$ CLASH with $S_4$

The tableau does not lead to a complete and clash-free constraint system, therefore the concept is not satisfiable.

3. $(\exists S.C \sqcap \exists S.D) \sqcap \forall S.(\neg C \sqcup \neg D)$

The concept is already expressed in negation normal form, we apply the $\mathcal{ALC}$ tableau algorithm to it:

$S_0 = \{x : (\exists S.C \sqcap \exists S.D) \sqcap \forall S.(\neg C \sqcup \neg D)\}$

$S_0 \rightarrow_\sqcap S_1 = S_0 \cup \{x : \exists S.C,\ x : \exists S.D,\ x : \forall S.(\neg C \sqcup \neg D)\}$

$S_1 \rightarrow_\exists S_2 = S_1 \cup \{(x, y) : S,\ y : C\}$ $y$ is a fresh individual

$S_2 \rightarrow_\exists S_3 = S_2 \cup \{(x, z) : S,\ z : D\}$ $z$ is a fresh individual

$S_3 \rightarrow_\forall S_4 = S_3 \cup \{y : \neg C \sqcup \neg D,\ z : \neg C \sqcup \neg D\}$

$+S_4 \rightarrow_\sqcup S_{5.1} = S_4 \cup \{y : \neg C\}$ CLASH with $S_2$

$+S_4 \rightarrow_\sqcup S_{5.2} = S_4 \cup \{y : \neg D\}$

$+S_{5.2} \rightarrow_\sqcup S_{6.2} = S_{5.2} \cup \{z : \neg C\}$

$+S_{5.2} \rightarrow_\sqcup S_{6.3} = S_{5.2} \cup \{z : \neg D\}$ CLASH with $S_3$

$S_{6.2}$ is a complete and clash-free constraint system.

It induces the model $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where:

$\Delta^\mathcal{I} = \{x, y, z\}$

$S^\mathcal{I} = \{(x, y), (x, z)\}$

$C^\mathcal{I} = \{y\}$

$D^\mathcal{I} = \{z\}$

4. $\exists S.(C \sqcap D) \sqcap (\forall S.\neg C \sqcup \exists S.\neg D)$

The concept is already expressed in negation normal form, we apply the $\mathcal{ALC}$ tableau algorithm to it:

$S_0 = \{x : \exists S.(C \sqcap D) \sqcap (\forall S.\neg C \sqcup \exists S.\neg D)\}$

$S_0 \rightarrow_\sqcap S_1 = S_0 \cup \{x : \exists S.(C \sqcap D),\ x : (\forall S.\neg C \sqcup \exists S.\neg D)\}$

$S_1 \rightarrow_\exists S_2 = S_1 \cup \{(x, y) : S,\ y : C \sqcap D\}$ $y$ is a fresh individual

$S_2 \rightarrow_\sqcap S_3 = S_2 \cup \{y : C,\ y : D\}$

$+S_3 \rightarrow_\sqcup S_{4.1} = S_3 \cup \{x : \forall S.\neg C\}$

$S_{4.1} \rightarrow_\forall S_{5.1} = S_{4.1} \cup \{y : \neg C\}$ CLASH with $S_3$

$+S_3 \rightarrow_\sqcup S_{4.2} = S_3 \cup \{x : \exists S.\neg D\}$

$S_{4.2} \rightarrow_\exists S_{5.2} = S_{4.2} \cup \{(x, z) : S,\ z : \neg D\}$ $z$ is a fresh individual

$S_{5.2}$ is a complete and clash-free constraint system.

It induces the model $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where:

$\Delta^\mathcal{I} = \{x, y, z\}$

$S^\mathcal{I} = \{(x, y), (x, z)\}$

$C^\mathcal{I} = \{y\}$

$D^\mathcal{I} = \{y\}$

5. $C \sqcap \exists R.A \sqcap \exists R.B \sqcap \neg\exists R.(A \sqcap B)$

First of all we transform the concept to an equivalent concept in negation normal form:

$$C \sqcap \exists R.A \sqcap \exists R.B \sqcap \neg\exists R.(A \sqcap B)$$
$$C \sqcap \exists R.A \sqcap \exists R.B \sqcap \forall R.\neg(A \sqcap B)$$
$$C \sqcap \exists R.A \sqcap \exists R.B \sqcap \forall R.(\neg A \sqcup \neg B)$$

Next we apply the $\mathcal{ALC}$ tableau algorithm to it:

$S_0 = \{x : C \sqcap \exists R.A \sqcap \exists R.B \sqcap \forall R.(\neg A \sqcup \neg B)\}$

$S_0 \rightarrow_\sqcap S_1 = S_0 \cup \{x : C,\ x : \exists R.A,\ x : \exists R.B,\ x : \forall R.(\neg A \sqcup \neg B)\}$

$S_1 \rightarrow_\exists S_2 = S_1 \cup \{(x, y) : R,\ y : A\}$ $y$ is a fresh individual

$S_2 \rightarrow_\exists S_3 = S_2 \cup \{(x, z) : R,\ z : B\}$ $z$ is a fresh individual

$S_4 \rightarrow_\forall S_4 = S_3 \cup \{y : \neg A \sqcup \neg B,\ z : \neg A \sqcup \neg B\}$

$+S_4 \rightarrow_\sqcup S_{5.1} = S_4 \cup \{y : \neg A\}$ CLASH with $S_2$

$+S_4 \rightarrow_{\sqcup} S_{5.2} = S_4 \cup \{y : \neg B\}$

$+S_{5.2} \rightarrow_{\sqcup} S_{6.2} = S_{5.2} \cup \{z : \neg A\}$

$+S_{5.2} \rightarrow_{\sqcup} S_{6.3} = S_{5.2} \cup \{z : \neg B\}$ CLASH with $S_3$

$S_{6.2}$ is a complete and clash-free constraint system.

It induces the model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where:

$\Delta^{\mathcal{I}} = \{x, y, z\}$

$R^{\mathcal{I}} = \{(x, y), (x, z)\}$

$A^{\mathcal{I}} = \{y\}$

$B^{\mathcal{I}} = \{z\}$

$C^{\mathcal{I}} = \{x\}$

6. $\neg\forall R.A \sqcap \forall R((\forall R.B) \sqcup A) \sqsubseteq \forall R.\neg(\exists R.A) \sqcap \exists R.(\exists R.B)$

We have to check whether this subsumption is true in all interpretations. This is *not* the case if, and only if, the intersection of the concept in the left-hand side and the negation of the right-hand side concept is satisfiable.

Therefore we check satisfiability of the concept

$$\neg\forall R.A \sqcap \forall R((\forall R.B) \sqcup A) \sqcap \neg(\forall R.\neg(\exists R.A) \sqcap \exists R.(\exists R.B))$$

First of all we transform the concept to an equivalent concept in negation normal form:

$$\exists R.\neg A \sqcap \forall R((\forall R.B) \sqcup A) \sqcap (\neg(\forall R.\neg(\exists R.A)) \sqcup \neg(\exists R.(\exists R.B)))$$

$$\exists R.\neg A \sqcap \forall R((\forall R.B) \sqcup A) \sqcap (\exists R.(\exists R.A) \sqcup \forall R.\neg(\exists R.B))$$

$$\exists R.\neg A \sqcap \forall R((\forall R.B) \sqcup A) \sqcap (\exists R.(\exists R.A) \sqcup \forall R.(\forall R.\neg B))$$

No we apply the $\mathcal{ALC}$ tableau algorithm:

$S_0 = \{x : \exists R.\neg A \sqcap \forall R((\forall R.B) \sqcup A) \sqcap (\exists R.(\exists R.A) \sqcup \forall R.(\forall R.\neg B))\}$

$S_0 \rightarrow_{\sqcap} S_1 = S_0 \cup = \{x : \exists R.\neg A,\ x : \forall R(\forall R.B \sqcup A),\ x : \exists R.(\exists R.A) \sqcup \forall R.(\forall R.\neg B)\}$

$S_1 \rightarrow_{\exists} S_2 = S_1 \cup \{(x, y) : R,\ y : \neg A\}$ $y$ is a fresh individual

$S_2 \rightarrow_{\forall} S_3 = S_2 \cup \{y : \forall R.B \sqcup A\}$

$+S_3 \rightarrow_{\sqcup} S_{4.1} = S_3 \cup \{y : \forall R.B\}$

$+S_3 \rightarrow_{\sqcup} S_{4.2} = S_3 \cup \{y : A\}$ CLASH with $S_2$

$+S_{4.1} \rightarrow_{\sqcup} S_{5.1.1} = S_{4.1} \cup \{x : \exists R.(\exists R.A)\}$

$S_{5.1.1} \rightarrow_{\exists} S_{6.1.1} = S_{5.1.1} \cup \{(x, z) : R,\ z : \exists R.A\}$ $z$ is a fresh individual

$S_{6.1.1} \rightarrow_{\exists} S_{7.1.1} = S_{6.1.1} \cup \{(z, w) : R,\ w : A\}$ $w$ is a fresh individual

$+S_{4.1} \rightarrow_{\sqcup} S_{5.1.2} = S_{4.1} \cup \{x : \forall R.(\forall R.\neg B)\}$

$S_{5.1.2} \rightarrow_{\forall} S_{6.1.2} = S_{5.1.2} \cup \{y : \forall R.\neg B\}$

Both of the branches lead to a clash free constraint system in $S_{7.1.1}$ and in $S_{6.1.2}$ (in this second case only if there is no R-relation arrow starting from y, so that both constraints $\forall R.B$ and $\forall R.\neg B$ can be satisfied).

$S_{7.1.1}$ induces the model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where:

$\Delta^{\mathcal{I}} = \{x, y, z, w\}$

$R^{\mathcal{I}} = \{(x, y), (x, z), (z, w)\}$

$A^{\mathcal{I}} = \{w\}$

$B^{\mathcal{I}} = \emptyset$

$S_{6.1.2}$ induces the model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where:

$\Delta^{\mathcal{I}} = \{x, y\}$

$R^{\mathcal{I}} = \{(x, y)\}$

$A^{\mathcal{I}} = \emptyset$

$B^{\mathcal{I}} = \emptyset$

Finally, since we have found two interpretations which can satisfy our concept, then the subsumption is not true in all interpretations.

These solutions were obtained and typeset by Fabrizio Prosperi