Sets

A **set** is a collection of objects, called **elements** or **members**

- The set, call it A, whose elements are *Shakespeare, Tasmania, and Monday*
- The set, call it B, of all even positive numbers

Notation

$A = \{$*Shakespeare*, *Tasmania*, *Monday*$\}$ — list all elements and enclose with $\{\ \}$

$B = \{2, 4, 6, 8, \dots\}$ — **not entirely clear!**

$B = \{x \mid x \text{ is an even positive number}\}$ — specify a common **property** of elements

Write $a \in S$ to say that a is an element of S (from Greek ε)
Write $a \notin S$ to say that a is not an element of S

$$8 \in B \qquad 7 \notin B \qquad C = \{x \mid x \in \mathbb{N} \text{ and } x \notin B\}$$

where $\mathbb{N}$ is the set of all natural numbers. What is C?

Three fundamental features of sets

- **A set must be distinguished from its description**

  For instance, the following descriptions define the same set:

$$\{2,3,4\} \quad \{3,2,4\} \quad \{2,2,3,4,4\} \quad \{x \in \mathbb{N} \mid 2 \leq x \leq 4\} \quad \{y \in \mathbb{N} \mid 1 < y < 5\}$$

- **All elements of a set are distinct**

  In other words, no element may 'occur' more than once in a set

  We do not distinguish between $\{3,2,4\}$ and $\{2,2,3,4,4\}$

- **The elements of a set are not ordered in any way**

  We do not distinguish between $\{3,2,4\}$ and $\{2,3,4\}$

- A set can be an element of another set

  For example, $\{0,\{0\}\}$ has two elements: 0 and $\{0\}$

Subsets

A set B is a **subset** of a set A if every element of B is an element of A

Notation: $B \subseteq A$. Also say: B is **included** in A

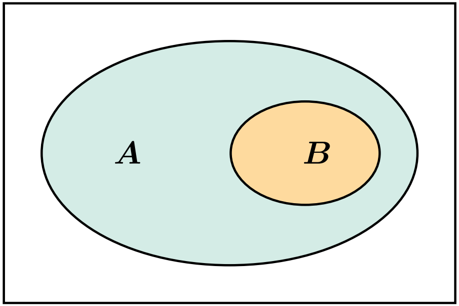


Figure 1: **Venn diagram** of $B \subseteq A$.

John Venn was a 19th-century British philosopher and mathematician who introduced the Venn diagram in 1881

$\{3,4,5\} \subseteq \{1,5,4,2,1,3\}$ $\{3,3,5\} \subseteq \{3,5\}$ $\{5,3\} \subseteq \{3,5\}$

Equal sets, proper subsets

Two sets A and B are **equal** if they have exactly the same elements

Notation: $A = B$

$$A = B \quad \text{iff} \quad A \subseteq B \quad \text{and} \quad B \subseteq A$$

$\{1\} = \{1, 1, 1\}$ $\qquad$ $\{1\} \neq \{\{1\}\}$ $\qquad$ $\{0, 2, 8\} = \{\sqrt{4}, 0/5, 2^3\}$

B is a **proper subset** of A if $B \subseteq A$ and $A \neq B$

Notation: $A \subset B$ or $A \subsetneqq B$

Also say: A is **properly included** in B

$\{1\} \subset \{1, 1, 2\}$ $\qquad$ $\{1\} \not\subseteq \{\{1\}\}$

Important sets

$\emptyset = \{ \}$	**empty set**, the set with no elements
$\mathbb{N} = \{0, 1, 2, 3, \dots\}$	natural numbers
$\mathbb{N}^{+} = \{1, 2, 3, \dots\}$	positive natural numbers
$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$	integer numbers
$\mathbb{Q} = \{x/y \mid x, y \in \mathbb{Z},\ y \neq 0\}$	rational numbers
$\mathbb{R} = \{\text{decimals}\}$	real numbers

Set operations: union

The **union** of sets A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$A \cup B$ is the set consisting of those elements that are in A or in B or both

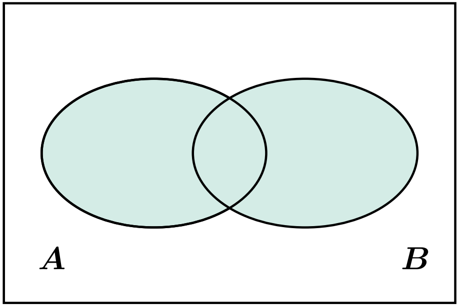


Figure 2: Venn diagram of $A \cup B$.

Suppose $A = \{4, 7, 8\}$ and $B = \{10, 4, 9\}$

Then $A \cup B = \{4, 7, 8, 9, 10\}$

Set operations: intersection

The **intersection** of sets A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

$A \cap B$ is the set consisting of all elements which are both in A and in B

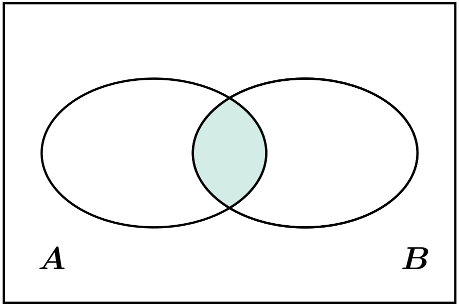


Figure 3: Venn diagram of $A \cap B$.

Suppose $A = \{4, 7, 8\}$ and $B = \{10, 4, 9\}$

Then $A \cap B = \{4\}$

If $A \cap B = \emptyset$ then A and B are called **disjoint**

Set operations: relative complement

The **complement** of a set B **relative** to a set A is the set

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

$A - B$ is also called the **difference** of A and B

$A - B$ is the set of all elements that belong to A but not to B

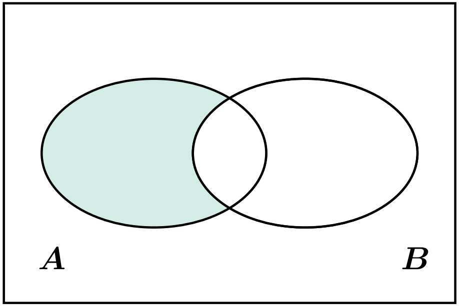


Figure 4: Venn diagram of $A - B$.

If $A = \{4, 7, 8\}$ and $B = \{10, 4, 9\}$ then $A - B = \{7, 8\}$

Set operations: (absolute) complement

In certain contexts we may regard all sets under consideration as being subsets of some given **universal set** U.

For instance, if we are investigating properties of the real numbers $\mathbb{R}$ (and subsets of $\mathbb{R}$), then we may take $\mathbb{R}$ as our universal set

Given a universal set U and $A \subseteq U$, the **complement** of A (in U) is the set

$$-A = U - A = \{x \in U \mid x \notin A\}$$

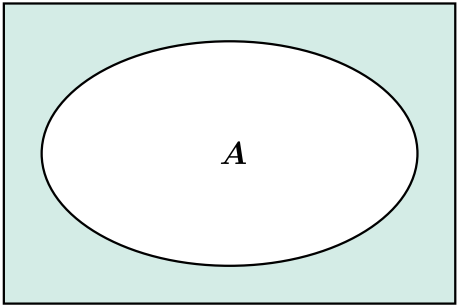


Figure 5: Venn diagram of $-A$

Set operations: powerset

The **powerset** of a set A is defined to be the set of **all** subsets of A

Notation: $\textsf{Pow}(A) = \{X \mid X \subseteq A\}$

Examples:

1. Let $A = \{2\}$. Then
$$\textsf{Pow}(A) = \{\emptyset, \{2\}\}$$
2. Let $B = \{1, 2, 3\}$. Then
$$\textsf{Pow}(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$$
3. Let $C = \emptyset$. Then
$$\textsf{Pow}(C) = \{\emptyset\} \qquad (\neq \emptyset)$$

More common notation: $\textsf{Pow}(A) = 2^A$

Why?

Because when A has n elements, then $\textsf{Pow}(A)$ has 2^n elements

Important equalities

- Associative laws:

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C$$

- Commutative laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

- Identity laws (where U is the universal set):

$$A \cup \emptyset = A, \quad A \cup U = U, \quad A \cap U = A, \quad A \cap \emptyset = \emptyset$$

- Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- Complement laws (where U is the universal set):

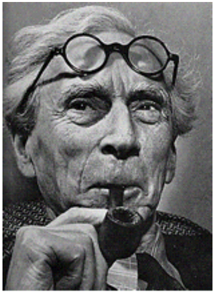
$$A \cup -A = U, \quad -U = \emptyset, \quad -(-A) = A, \quad A \cap -A = \emptyset, \quad -\emptyset = U$$

- De Morgan's laws:

$$-(A \cup B) = -A \cap -B, \quad -(A \cap B) = -A \cup -B$$

Russell's paradox (1901)

Russell's paradox shows that the 'object' $\{x \mid P(x)\}$ is not always meaningful.



Consider the set $A = \{B \mid B \notin B\}$

Give an example of an element of A

Problem: do we have $A \in A$?

For every set C, denote by $P(C)$ the statement $C \notin C$

Then $A = \{B \mid P(B)\}$.

- Suppose $A \in A$. Then not $P(A)$. Therefore, we must have $A \notin A$.
- But if $A \notin A$, then $P(A)$. Therefore, $A \in A$, which is a contradiction

Visit also http://plato.stanford.edu/entries/russell-paradox/

Popular version: the barber paradox

Suppose there is a town with just one male barber. According to law in this town, the barber shaves all and only those men in town who do not shave themselves. Who shaves the barber?

- if the barber does shave himself, then the barber (himself) must not shave himself
- if the barber does not shave himself, then the barber (himself) must shave himself

Sequences, tuples, and Cartesian products

A **sequence** of objects is a list of these objects **taken in a certain order**

The sequences $(1, 2, 3)$ $(2, 1, 3)$ $(3, 1, 2)$ are **different**

The sets $\{1, 2, 3\}$ $\{2, 1, 3\}$ $\{3, 1, 2\}$ are **the same**

Finite sequences are called **tuples**. A sequence with k elements is a **k-tuple**

A 2-tuple is also called a **pair**

The **Cartesian product** $A \times B$ of sets A and B is the *set* of all pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Example. Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

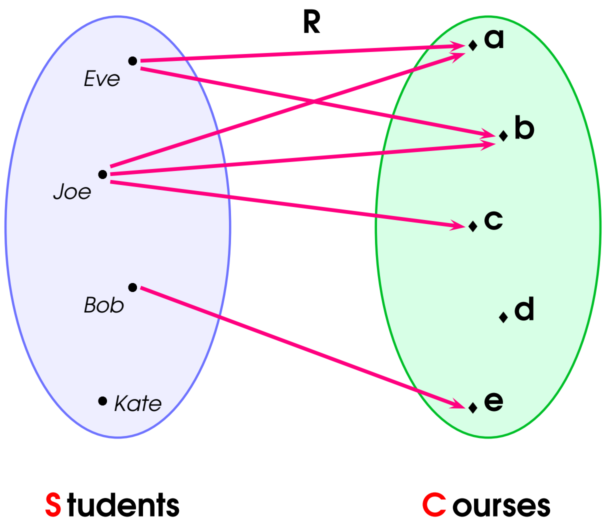
$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}$$

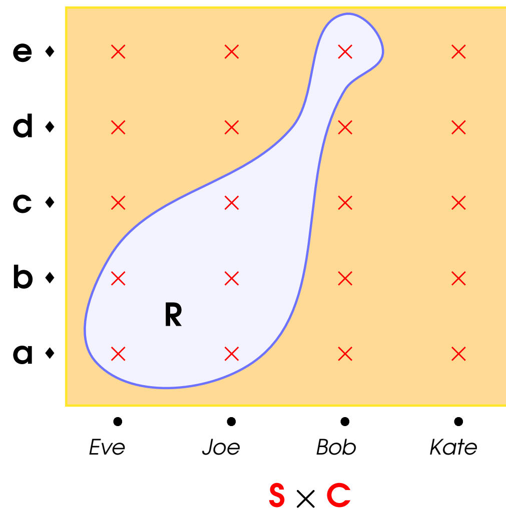
$\mathbb{R} \times \mathbb{R}$ is the **Euclidean plane**. What is $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$?

Binary relations

Let S be the set of students at Birkbeck, and C the set of available courses. The registration database or the relationship **‘registered for’** can be represented as the set

$$R = \{(s, c) \in S \times C \mid s \text{ registered for } c\}$$





Binary relations: definitions and examples

A **binary relation** between two sets A and B is a subset R of the Cartesian product $A \times B$. If $A = B$, then R is a **relation on** A

If $(x, y) \in R$ then we say that **x is R-related to y** and write xRy

- 'Smaller than' on $\mathbb{Z}$ $\quad$ $< \; = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x < y\}$

$$-1 < 0, \quad -1 < 1, \quad 0 < 2, \ldots \quad \text{but} \quad 1 \not< 0, \quad 1 \not< -1, \quad 2 \not< 0, \ldots$$

- 'Smaller than or equal to' on $\mathbb{Z}$ $\quad$ $\leq \; = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \leq y\}$

$$-1 \leq -1, \quad -1 \leq 1, \quad 0 \leq 0, \ldots \quad \text{but} \quad 1 \not\leq 0, \quad 1 \not\leq -1, \quad 2 \not\leq 0, \ldots$$

Note that $\quad < \; \subsetneq \; \leq$

- 'Input–output' relation IO_P for a given computer program P

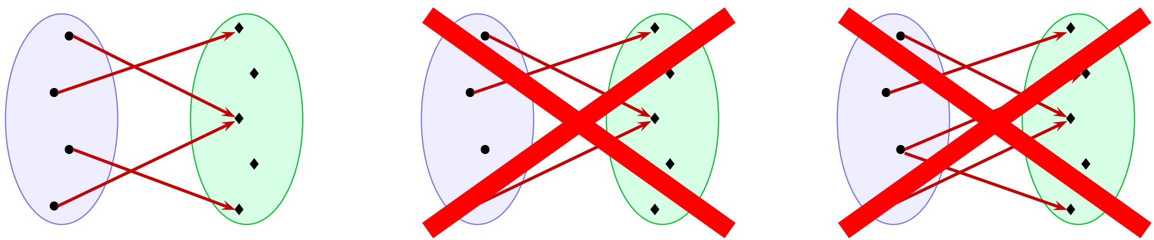
Let I be the set of possible inputs for P, and O the set of possible outputs

$$\mathsf{IO}_P = \{(x, y) \in I \times O \mid P(x) = y\}$$

$x \, \mathsf{IO}_P \, y$ $\quad$ iff $\quad$ given x as an input, P returns y

Functions

A **function** from a set A to a set B is a binary relation $R \subseteq A \times B$ in which **every element** of A is R-related a **unique** element of B, or, in other words: for each $a \in A$ there is precisely one pair of the form (a, b) in R.

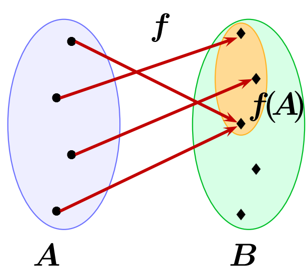


Which of the following relations are functions?

- $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = x - 1\}$
- $\{(x, y) \in \mathbb{Q} \times \mathbb{Q} \mid y = x - 1\}$
- $\{(x, y) \in \mathbb{N} \times \mathbb{R} \mid y^2 = x\}$

Notation

- Let f be a function from a set A to a set B. Since for each $x \in A$ there exists a uniquely determined $y \in B$ with $(x, y) \in f$, we write $y = f(x)$ and refer to $f(x)$ as the **image** of x under f
- We write $f : A \to B$ to indicate that f is a function from A to B
- A is called the **domain** of f. B is called the **codomain** of f
- The **range** of f is the set $f(A) = \{f(x) \mid x \in A\}$



Alphabets and words

An **alphabet** is a finite set Σ of symbols

Examples:
- $\Sigma_1 = \{a, b, c, \ldots, z\}$, the set of all lower-case letters
- $\Sigma_2 = \{0, 1\}$, the binary alphabet
- $\Sigma_3 = \{\Box, \Diamond, \heartsuit\}$

A **word** or **string** (over an alphabet Σ) is a finite sequence of symbols from Σ

Examples:
- *abracadabra*, *azwzax* (over Σ_1)
- 111111111110000000000, 000110 (over Σ_2)
- $\heartsuit\heartsuit\Box$, $\Box\Diamond\Box\Box\Diamond\heartsuit$ (over Σ_3)
- the **empty word** ε is a word over **any** alphabet Σ (but we may assume that ε is *NOT a symbol* of any of our alphabets)

Words (cont.)

- Σ^* is the set of all words over Σ (always contains ε)

- The **length** $|w|$ of a word w is the number of symbols in w

  $|w|$ = the number of occurrences of symbols in w

  e.g., $|azwza| = 5$, $|\heartsuit\heartsuit\Box| = 3$, $|\varepsilon| = 0$

- The **concatenation** of words x and y (notation: xy) is

  the word x followed by the word y

  - $w^n = \underbrace{ww\dots w}_{n}$ (e.g., $(\heartsuit\Box)^0 = \varepsilon$, $(01)^3 = 010101$, $a^4 = aaaa$)
  - $x\varepsilon = \varepsilon x = x$, for every word x

- if $w = xy$ then x is a **prefix** of w, and y a **suffix** of w

  e.g., *tor* is a prefix and *se* is a suffix of *tortoise*

Languages

A **language** over an alphabet Σ is a set of words over Σ,
that is, a **language** is a subset of Σ^*

Examples:

(1) $\Sigma = \{a, b, c, \dots, z\}$

- $L_1 =$ all English words
- $L_2 =$ all Latin words
- $L_3 = \{kdpekvg, leih, hkiiw, wowiszk\}$

(2) $\Sigma = \{0, 1\}$

- $L_4 = \{001, 101010, 111, 1001\}$
- $L_5 = \{0^n 1^m \mid n$ is an even, m is an odd number$\}$

What is 'iff' ?

If and only if (or **iff**) is a biconditional logical connective between statements: it is the standard conditional **if** combined with its reverse **only if**.

The truth of either one of the connected statements requires the truth of the other, i.e., either both statements are true, or both are false.

Alternative phrases to P 'iff Q':

- P is necessary and sufficient for Q
- P is equivalent to Q
- P precisely if Q
- P just in case Q

Example: Suppose $A \subseteq B$. Which of the following statements are true?

- For all x, $x \in A$ iff $x \in B$.
- For all x, $x \in A$ is sufficient for $x \in B$.
- For all x, $x \in B$ is sufficient for $x \in A$.
- For all x, $x \in A$ is necessary for $x \in B$.
- For all x, $x \in B$ is necessary for $x \in A$.