# OWL 2: automated reasoning

## (with description logics)

# What are Description Logics?

- A **family** of logic-based **Knowledge Representation formalisms**
  - descendants of **semantic networks**
  - describe domain in terms of

    **concepts** (classes), **roles** (relationships) and **individuals**

Distinguished by:

- **formal semantics** (typically model theoretic)
  - decidable fragments of first-order logic
  - closely related to propositional modal and dynamic logics
- provision of **inference services**
  - sound and complete decision procedures for key problems
  - implemented systems (highly optimised)

# DL architecture

## Knowledge Base (KB)

**TBox** (terminological box, schema)

$$\mathsf{Man} \equiv \mathsf{Human} \sqcap \mathsf{Male}$$
$$\mathsf{Father} \equiv \mathsf{Man} \sqcap \exists \mathsf{hasChild}.\top$$
...

**ABox** (assertion box, data)

$$\mathsf{john} : \mathsf{Man}$$
$$(\mathsf{john}, \mathsf{mary}) : \mathsf{hasChild}$$
...

**Inference System**

**Interface**

# Description logics: $\mathcal{ALC}$ (syntax)

- Language:
  - concept names $A_0, A_1, \ldots$ (e.g., Person, Female, ...)
  - role names $R_0, R_1, \ldots$ (e.g., hasChild, loves, ...)
  - individuals $a_0, a_1, \ldots$ (e.g., john, mary, ...)
  - concept constructors: $\sqcap$, $\sqcup$, $\neg$, $\exists$, $\forall$
  - axiom constructors: $\equiv$, $\sqsubseteq$

- **$\mathcal{ALC}$ concepts:**
  - all concept names
  - $\top$, $\bot$, $\neg C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$, $\exists R.C$,

    where $C$, $D$ are concepts and $R$ a role name

**Examples:** Person $\sqcap$ Female, Person $\sqcap$ $\neg$Female,
Person $\sqcap$ $\exists$hasChild.$\top$, Person $\sqcap$ $\forall$hasChild.Male

# Description logics: $\mathcal{ALC}$ (semantics)

- (standard Tarski-style) **interpretation** is a structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$
  - $\Delta^{\mathcal{I}}$ is the **domain** (a non-empty set)
  - $\cdot^{\mathcal{I}}$ is an **interpretation function** that maps:
    * concept name $A_i \mapsto$ subset $A_i^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ $(A_i^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}})$
    * role name $R_i \mapsto$ binary relation $R_i^{\mathcal{I}}$ over $\Delta^{\mathcal{I}}$ $(R_i^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}})$
    * individual name $a_i \mapsto$ element $a_i^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ $(a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}})$

- interpretation of **complex concepts** in $\mathcal{I}$:

  ($A$ is a concept name, $C$, $D$ are concepts and $R$ a role name)

  - $(\top)^{\mathcal{I}} = \Delta^{\mathcal{I}}$ and $(\bot)^{\mathcal{I}} = \emptyset$
  - $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
  - $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$, $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$
  - $(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid$ there is $y \in C^{\mathcal{I}}$ such that $(x, y) \in R^{\mathcal{I}}\}$
  - $(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid$ for all $y \in \Delta^{\mathcal{I}}$, if $(x, y) \in R^{\mathcal{I}}$ then $y \in C^{\mathcal{I}}\}$

    $= \{x \in \Delta^{\mathcal{I}} \mid$ for all $y \in \Delta^{\mathcal{I}}$, either $(x, y) \notin R^{\mathcal{I}}$ or $y \in C^{\mathcal{I}}\}$

# Terminologies or TBoxes

statements about **how concepts are related to each other**

Terminology is a set $\mathcal{T}$ of **terminological axioms**:

- $C \sqsubseteq D$ — $C$ is subsumed by $D$ (or $D$ subsumes $C$)
- $C \equiv D$ — $C$ is equivalent to $D$

an interpretation $\mathcal{I}$ **satisfies** (models) an axiom

- $\mathcal{I} \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
- $\mathcal{I} \models C \equiv D$ iff $C^{\mathcal{I}} = D^{\mathcal{I}}$

An interpretation $\mathcal{I}$ is a **model** of a TBox $\mathcal{T}$ iff $\mathcal{I}$ satisfies **every axiom** of $\mathcal{T}$

# Example

Consider the TBox $\mathcal{T}$ with the following axioms:

$$\forall R.\neg B \sqsubseteq B$$

$$\exists R.(\exists R.C) \sqsubseteq \neg A \sqcup \neg B$$

and the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where

$$\Delta^{\mathcal{I}} = \{a, b, c, d, e, f\}$$

$$A^{\mathcal{I}} = \{a, c, e\}$$

$$B^{\mathcal{I}} = \{c, d, e, f\},$$

$$C^{\mathcal{I}} = \{e\}$$

$$R^{\mathcal{I}} = \{(a, f), (a, c), (b, d), (d, c), (c, e), (f, a)\}$$

- Is $\mathcal{I}$ a model of $\mathcal{T}$?

# TBox Inference Services

- **satisfiability of concepts**

  $C$ is satisfiable w.r.t. a TBox $\mathcal{T}$ iff there exists **some model** $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$

- **subsumption of concepts** $(C \sqsubseteq_{\mathcal{T}} D)$

  $C$ is subsumed by $D$ w.r.t. a TBox $\mathcal{T}$ iff for **every model** $\mathcal{I}$ of $\mathcal{T}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$

- **equivalence of concepts** $(C \equiv_{\mathcal{T}} D)$

  $C$ is equivalent to $D$ w.r.t. a TBox $\mathcal{T}$ iff for **every model** $\mathcal{I}$ of $\mathcal{T}$, $C^{\mathcal{I}} = D^{\mathcal{I}}$

- **disjointness of concepts**

  $C$ is disjoint with $D$ w.r.t. a TBox $\mathcal{T}$ iff for **every model** $\mathcal{I}$ of $\mathcal{T}$, $C^{\mathcal{I}} \cap D^{\mathcal{I}} = \emptyset$

# TBox Inference Services: example

$$\begin{aligned}
\text{Parent} &\equiv \text{Person} \sqcap \exists \text{hasChild.Person} \\
\text{Woman} &\equiv \text{Person} \sqcap \text{Female} \\
\text{Mother} &\equiv \text{Parent} \sqcap \text{Female}
\end{aligned}$$

the concepts Woman, Mother, Parent are satisfiable

however, the concept $\neg\text{Woman} \sqcap \text{Mother}$ is **unsatisfiable**:

We could unfold the definition of Woman and Mother:

$$\begin{aligned}
&\neg\text{Woman} \sqcap \text{Mother} \\
&\equiv \neg(\text{Person} \sqcap \text{Female}) \sqcap \text{Parent} \sqcap \text{Female} \\
&\equiv (\neg\text{Person} \sqcup \neg\text{Female}) \sqcap \text{Parent} \sqcap \text{Female} \\
&\equiv (\neg\text{Person} \sqcap \text{Parent} \sqcap \text{Female}) \sqcup (\neg\textbf{Female} \sqcap \text{Parent} \sqcap \textbf{Female}) \quad \text{(inconsistent)} \\
&\equiv \neg\text{Person} \sqcap \text{Parent} \sqcap \text{Female} \\
&\equiv \neg\textbf{Person} \sqcap (\textbf{Person} \sqcap \exists\text{hasChild.Person}) \sqcap \text{Female} \quad \text{(inconsistent)} \\
&\equiv \bot
\end{aligned}$$

Therefore, the concept $\neg\text{Woman} \sqcap \text{Mother}$ can never be satisfied

# Inference Services based on Satisfiability

All concept inference services can be reduced to concept (un)satisfiability:

- **subsumption of concepts**

  $C \sqsubseteq_{\mathcal{T}} D$ iff $C \sqcap \neg D$ is **unsatisfiable** w.r.t. $\mathcal{T}$

- **equivalence of concepts**

  $C \equiv_{\mathcal{T}} D$ iff $C \sqsubseteq_{\mathcal{T}} D$ and $D \sqsubseteq_{\mathcal{T}} C$
  iff $(C \sqcap \neg D) \sqcup (\neg C \sqcap D)$ is **unsatisfiable** w.r.t. $\mathcal{T}$

- **disjointness of concepts**

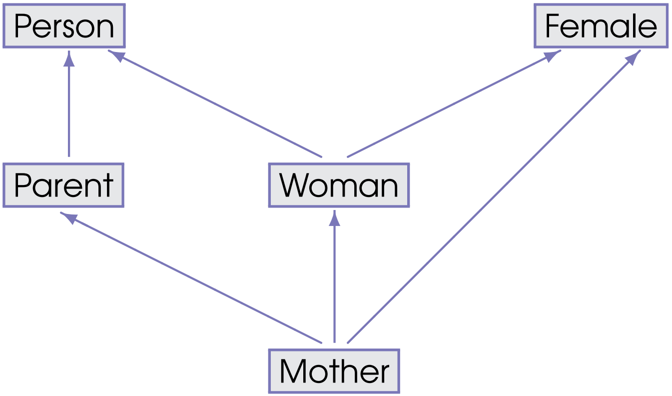
  $C$ is disjoint with $D$ w.r.t. $\mathcal{T}$ iff $C \sqcap D$ is **unsatisfiable** w.r.t. $\mathcal{T}$

# Inference Service: Concept Subsumption

Consider the question: is a mother always a woman?

Does the concept `Woman` subsume the concept `Mother`?

DL reasoners offer the computation of **subsumption hierarchy** (taxonomy) of all named concepts




yes, `Woman` subsumes `Mother` (see also slide 9.2)

# World Description of ABox

Asserts knowledge about **individuals**

ABox $\mathcal{A}$ is a set of **assertional axioms**

- $a : C$ concept assertion for an individual
- $(a, b) : R$ role assertion for a pair of individuals

an interpretation $\mathcal{I}$ **satisfies** (models) an assertion

- $\mathcal{I} \models a : C$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$
- $\mathcal{I} \models (a, b) : R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$

An interpretation $\mathcal{I}$ is a **model** of a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$ iff
$\mathcal{I}$ satisfies **every axiom** of $\mathcal{T}$ and $\mathcal{A}$

# ABox Inference Services

- **ABox consistency**:

  is the collection of assertions $\mathcal{A}$ satisfiable (w.r.t. a TBox $\mathcal{T}$)?

  $\mathcal{A}$ is consistent w.r.t. a TBox $\mathcal{T}$ iff there exists **some model** $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{A}$

- **instance checking**:

  is $a$ an instance of a concept $C$?

  $a$ is an instance of $C$ w.r.t. a TBox $\mathcal{T}$ and an Abox $\mathcal{A}$ iff

  $a^{\mathcal{I}} \in C^{\mathcal{I}}$, **for every model** $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{A}$

- **ABox realisation**:

  for all individuals in $\mathcal{A}$,

  compute their **most specific concept names** w.r.t. $\mathcal{T}$

# ABox and TBox Inference Services based on Consistency

All inference services can be reduced to ABox consistency:

- **instance checking**

  $a$ is an instance of $C$ (w.r.t. a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$) iff

  $\mathcal{A} \cup \{a\colon \neg C\}$ is **inconsistent** (w.r.t. $\mathcal{T}$)

- **concept satisfiability**

  $C$ is satisfiable w.r.t. a TBox $\mathcal{T}$ iff $\{a\colon C\}$ is **consistent** w.r.t. $\mathcal{T}$

  ($a$ does not occur in $\mathcal{T}$)

- **concept subsumption**

  $C$ is subsumed by $D$ w.r.t. a TBox $\mathcal{T}$ iff

  $\{a\colon C \sqcap \neg D\}$ is **inconsistent** w.r.t. $\mathcal{T}$

  ($a$ does not occur in $\mathcal{T}$)

# ABox Inference Services: example

Consider the ABox $\mathcal{A}$:

1. $(\mathsf{john}, \mathsf{susan}) : \mathsf{friend}$
2. $(\mathsf{john}, \mathsf{andrea}) : \mathsf{friend}$
3. $(\mathsf{susan}, \mathsf{andrea}) : \mathsf{loves}$
4. $(\mathsf{andrea}, \mathsf{bill}) : \mathsf{loves}$
5. $\mathsf{susan} : \mathsf{Female}$
6. $\mathsf{bill} : \neg\mathsf{Female}$

Represent the following query (to $\mathcal{A}$) as an inference service problem and find an answer:

> Does John have a female friend who is in love with a male (not female) person?

$$\mathsf{john} : \exists \mathsf{friend}.(\mathsf{Female} \sqcap \exists \mathsf{loves}.\neg\mathsf{Female})$$

# Multiple models vs. Single model

- **DL KB** doesn't define a single model, it is a set of constraints that define **a set of possible models**
  - no constraints (empty KB) means any model is possible
  - more constraints means fewer models
  - too many constraints may mean no possible model (inconsistent KB)

- In contrast, **DBs** (and frame/rule KR systems) make assumptions such that DB/KB defines **a single model**
  - **unique name assumption** (different names always interpreted as different individuals)
  - **closed world assumption** (domain consists only of individuals named in the DB/KB)
  - **minimal models** (extensions are as small as possible)

# Open World Assumption

$\mathcal{A} = \{$ harry : Male
(peter, harry) : hasChild $\}$

Is peter an instance of $\forall$hasChild.Male w.r.t. $\mathcal{A}$?

**No!** Although the ABox contains only knowledge about one male child,
it is always assumed that the represented information is **incomplete**

In order to prevent this, we could

- add peter : $\forall$hasChild.Male to the ABox $\mathcal{A}$
- or assert that information about a second child will not be added in the future, i.e., close a role for an individual

however, it is **not possible** in $\mathcal{ALC}$ since we need number restrictions:

peter : $\leq 1$ hasChild

# More DL Constructors

- **qualified number restrictions** $\leq n\, R.C, \quad \geq n\, R.C$
  - $(\leq n\, R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \; |\{y \in \Delta^{\mathcal{I}} \mid (x, y) \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\}| \leq n\; \}$
  - $(\geq n\, R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \; |\{y \in \Delta^{\mathcal{I}} \mid (x, y) \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\}| \geq n\; \}$

  **Example:** $\geq 3$ hasChild.Male (qualified) vs. $\geq 3$ hasChild (simple)

- **inverse roles** $R^{-}$
  - $(R^{-})^{\mathcal{I}} = \{(y, x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (x, y) \in R^{\mathcal{I}}\}$

- **transitive roles** *transitive*$(R)$
  - $\mathcal{I} \models$ *transitive*$(R)$ iff $R^{\mathcal{I}}$ is transitive, i.e.,
    for all $x, y, z \in \Delta^{\mathcal{I}}$ $\quad ((x, y) \in R^{\mathcal{I}} \land (y, z) \in R^{\mathcal{I}} \to (x, z) \in R^{\mathcal{I}})$

- **role hierarchies** $R \sqsubseteq S$
  - $\mathcal{I} \models R \sqsubseteq S$ iff $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$

  **Example:** hasSon $\sqsubseteq$ hasChild

# Exercise

Using the individuals john and cnets, concepts Course, Lecturer, MSc and BSc, and roles teaches and hasDegree, represent the following knowledge base as an $\mathcal{ALC}$ knowledge base $\mathcal{K}$:

- Everybody who teaches a course must either have an MSc degree or be a lecturer
- Every lecturer teaches some course
- Every lecturer has a BSc degree
- Everybody with an MSc degree has a BSc degree as well
- John teaches the Computer Networks course

- Is the statement "John has an MSc degree" a logical consequence of the knowledge base $\mathcal{K}$?

- Is the statement "Everybody who teaches a course must have a BSc degree" a logical consequence of the knowledge base $\mathcal{K}$?