Answers (tutorial on 28 November 2011)

1. (a) $(\neg B \sqcup \neg A)^{\mathcal{I}} = \{a, c, e, d\}$,
   $(\exists R.(\neg B \sqcup \neg A))^{\mathcal{I}} = \{a, b, c\}$,
   $(\forall R.(A \sqcap B))^{\mathcal{I}} = \{f, e, d\}$,
   $(\forall R.\forall R.(A \sqcap B))^{\mathcal{I}} = \{c, d, e, f\}$.

   (b) $(\exists R.B)^{\mathcal{I}} = \{2, 3, 5\}$,
   $(A \sqcup \exists R.B)^{\mathcal{I}} = \{2, 3, 4, 5\}$,
   $(\forall R.\neg A)^{\mathcal{I}} = \{1, 2, 4, 5\}$,
   $(B \sqcap \forall R.\neg A)^{\mathcal{I}} = \{1, 2\}$,
   $(\forall R.(B \sqcap \forall R.\neg A))^{\mathcal{I}} = \{1, 5\}$,
   $(\forall R.(B \sqcap \forall R.\neg A) \sqcup \neg B)^{\mathcal{I}} = \{1, 4, 5\}$.

2. Representations in negation normal form (NNF):

   (a) $\forall R.(A \sqcap \neg B) \sqcup \exists S.((\neg A \sqcup B) \sqcap \forall R.\neg A)$,

   (b) $\exists S.((\neg C \sqcup \forall R \exists S.\neg A) \sqcup \exists S(\neg A \sqcup \forall R.\forall R.\neg A))$,

   (c) $\neg B \sqcup (\forall R.\neg C \sqcap \exists S.\exists R.(\forall R.\neg A \sqcup \forall S.A))$;

3. Tableaux:

   (a) NNF $= \exists R.\neg A \sqcap \forall R.(A \sqcup B)$,
   $S_0 = \{x : \exists R.\neg A \sqcap \forall R.(A \sqcup B)\}$,
   $S_1 = S_0 \cup \{x : \exists R.\neg A, x : \forall R(A \sqcup B)\}$, ($\sqcap$-rule)
   $S_2 = S_1 \cup \{xRy, y : \neg A\}$, ($\exists$-rule)
   $S_3 = S_2 \cup \{y : A \sqcup B\}$, ($\forall$-rule applied to xRy and $x : \forall R(A \sqcup B)$)
   $S_4 = S_3 \cup \{y : A\}$;
   $S_{4'} = S_3 \cup \{y : B\}$;
   S_4 contains a clash $\{y : A, y : \neg A\}$, $S_{4'}$ gives an interpretation $\mathcal{I}$ with
   - $\Delta^{\mathcal{I}} = \{x, y\}$,
   - $A^{\mathcal{I}} = \emptyset$,
   - $B^{\mathcal{I}} = \{y\}$,
   - $R^{\mathcal{I}} = \{(x, y)\}$.

   (b) NNF $= \exists R.\exists R.\neg P \sqcap \forall R.P$,
   $S_0 = \{x : \exists R.\exists R.\neg P \sqcap \forall R.P\}$,
   $S_1 = S_0 \cup \{x : \exists R.\exists R.\neg P, x : \forall R.P\}$,
   $S_2 = S_1 \cup \{xRy, y : \exists R.\neg P\}$,
   $S_3 = S_2 \cup \{y : P\}$, ($\forall$-rule applied to xRy and $x : \forall R.P$)
   $S_4 = S_3 \cup \{yRz, z : \neg P\}$,
   S_4 gives an interpretation $\mathcal{I}$ with
   - $\Delta^{\mathcal{I}} = \{x, y, z\}$,
   - $P^{\mathcal{I}} = \{y\}$,
   - $R^{\mathcal{I}} = \{(x, y), (y, z)\}$.

(c) $S_0 = \{x : \exists R.(\forall S.C) \sqcap \forall R.(\exists S.\neg C)\}$,
$S_1 = S_0 \cup \{x : \exists R.\forall S.C, x : \forall R.\exists S.\neg C\}$,
$S_2 = S_1 \cup \{xRy, y : \forall S.C\}$,
$S_3 = S_1 \cup \{y : \exists S.\neg C\}$,
$S_4 = S_3 \cup \{yRz, z : \neg C\}$,
$S_5 = S_4 \cup \{z : C\}$,
S_5 contains a clash in z, and so the concept is not satisfiable.

(d) $S_0 = \{x : (\exists S.C \sqcap \exists S.D) \sqcap \forall S.(\neg C \sqcup \neg D)\}$,
$S_1 = S_0 \cup \{x : \exists S.C, x : \exists S.D, x : \forall S.(\neg C \sqcup \neg D)\}$,
$S_2 = S_1 \cup \{xSy, y : C\}$,
$S_3 = S_2 \cup \{y : \neg C \sqcup \neg D\}$,
$S_4 = S_3 \cup \{y : \neg C\}$, clash!
$S_{4'} = S_3 \cup \{y : D\}$,
$S_5 = S_{4'} \cup \{xSz, z : D\}$,
$S_6 = S_5 \cup \{z : \neg C \sqcup \neg D\}$,
$S_7 = S_6 \cup \{z : \neg C\}$
S_7 gives an interpretation $\mathcal{I}$ with

- $\Delta^{\mathcal{I}} = \{x, y, z\}$,
- $C^{\mathcal{I}} = \{y\}$,
- $D^{\mathcal{I}} = \{z\}$,
- $S^{\mathcal{I}} = \{(x, y), (x, z)\}$.

4. (a) $\exists R.(C \sqcap \exists R.(A \sqcap B) \sqcap \exists R.(A \sqcap \neg B))$,
   (b) $C \sqcap \exists R.(\neg B \sqcap \exists R.A) \sqcap \exists R.(B \sqcap \exists R.(B \sqcap \neg A) \sqcap \exists R.(A \sqcap B))$;

5. Inclusions:

   (a) Inclusion $\forall R.(\neg A \sqcup B) \sqsubseteq \neg \forall R.A \sqcup \forall R.B$ holds iff
   the concept $\forall R(\neg A \sqcup B) \sqcap \neg(\neg \forall R.A \sqcup \forall R.B)$ is not satisfiable.

   NNF $= \forall R(\neg A \sqcup B) \sqcap \forall R.A \sqcap (\exists R.\neg B)$,
   $S_0 = \{x : \forall R(\neg A \sqcup B) \sqcap \forall R.A \sqcap (\exists R.\neg B)\}$,
   $S_1 = S_0 \cup \{x : \forall R.(\neg A \sqcup B), x : \forall R.A, x : \exists R.\neg B\}$,
   $S_2 = S_1 \cup \{xRy, y : \neg B\}$,
   $S_3 = S_2 \cup \{y : A\}$,
   $S_4 = S_3 \cup \{y : \neg A \sqcup B\}$,
   $S_5 = S_4 \cup \{y : \neg A\}$, clash in y
   $S_{5'} = S_4 \cup \{y : B\}$, clash in y

   The concept is not satisfiable, hence the inclusion holds.

   (b) The inclusion $\neg \forall R.A \sqcup \forall R.B \sqsubseteq \forall R.(\neg A \sqcup B)$ holds iff
   the concept $(\neg \forall R \sqcup \forall R.B) \sqcap \neg(\forall R.(\neg A \sqcup B))$ is not satisfiable.

   NNF $= (\exists R.\neg A \sqcup \forall R.B) \sqcap \exists R.(A \sqcap \neg B)$,
   $S_0 = \{x : (\exists R.\neg A \sqcup \forall R.B) \sqcap \exists R.(A \sqcap \neg B)\}$,

$S_1 = S_0 \cup \{x : \exists R.\neg A \sqcup \forall R.B, x : \exists R(A \sqcap \neg B)\}$,
$S_2 = S_1 \cup \{x : \exists R.\neg A\}$
$S_{2'} = S_1 \cup \{x : \forall R.\neg B\}$ consider this case later if necessary
$S_3 = S_2 \cup \{xRy, y : \neg A\}$,
$S_4 = S_3 \cup \{xRz, z : A \sqcap \neg B\}$,
$S_5 = S_4 \cup \{z : A, z : \neg B\}$,
S_5 gives rise to an interpretation $\mathcal{I}$ where

- $\Delta^{\mathcal{I}} = \{x, y, z\}$,
- $A^{\mathcal{I}} = \{z\}$,
- $B^{\mathcal{I}} = \emptyset$,
- $S^{\mathcal{I}} = \{(x, y), (x, z)\}$.

The concept is satisfiable, hence the inclusion does not hold.

(c) See problem 6 at `http://www.dcs.bbk.ac.uk/~michael/sw/slides/SW%20Exercise-ans.pdf`