

# Interpreting Topological Logics over Euclidean Spaces

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## Introduction

Topological logics are a family of languages for representing and reasoning about topological data. The non-logical primitives of these languages stand for various topological relations and operations, and their valid formulas encode our knowledge about those relations and operations. Consider, for example, the six relations illustrated in Fig. 1. By em-

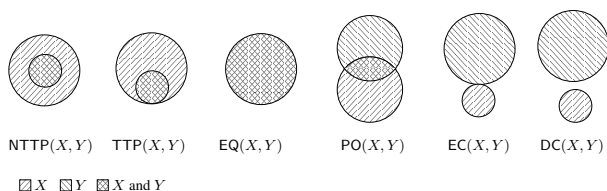


Figure 1:  $\mathcal{RCC8}$ -relations over disc-homeomorphs in  $\mathbb{R}^2$ .

ploying the binary predicates DC (disconnection), EC (external contact), PO (partial overlap), EQ (equality), TPP (tangential proper parthood) and NTPP (non-tangential proper parthood) to stand for these relations, the formula

$$\text{TPP}(r_1, r_2) \wedge \text{NTPP}(r_1, r_3) \rightarrow \text{PO}(r_2, r_3) \vee \text{TPP}(r_2, r_3) \vee \text{NTPP}(r_2, r_3) \quad (1)$$

makes the intuitively reasonable assertion that, if region  $r_1$  externally contacts region  $r_2$  and is a non-tangential proper part of region  $r_3$ , then  $r_2$  either partially overlaps, or else is a proper part (tangential or non-tangential) of  $r_3$ . This particular topological logic, known as  $\mathcal{RCC8}$ , has been intensively analysed in the literature on qualitative spatial reasoning; see e.g., (Egenhofer and Franzosa 1991; Randell, Cui, and Cohn 1992; Renz and Nebel 1999).

We referred to  $r_1$ ,  $r_2$  and  $r_3$  above as ‘regions,’ and depicted them as discs in the plane. But a moment’s thought shows that the set of valid formulas of any topological logic depends on the precise collection of regions we have in mind. For instance, there are  $\mathcal{RCC8}$ -formulas that are valid as long as regions are taken to be discs in the plane, but invalid when regions are allowed to be *disconnected* (i.e. to consist of more than one ‘piece’). Thus, an important semantic issue for a topological logic like  $\mathcal{RCC8}$  is to identify the intended models. In this paper, we show

how even relatively inexpressive topological logics are sensitive both to the spaces they are interpreted over and—more particularly—to the subsets of those spaces over which their variables are allowed to range. We identify the crucial notion of *tameness*, and chart the surprising patterns of sensitivity to the presence of non-tame regions exhibited by a range of topological logics in low-dimensional Euclidean spaces.

Historically,  $\mathcal{RCC8}$  has typically been interpreted over the *regular closed* sets of *arbitrary* topological spaces. (A set is *regular closed* if it is the topological closure of an open set.) This very general interpretation is *prima facie* surprising: after all, in qualitative spatial reasoning, it is the low-dimensional Euclidean spaces  $\mathbb{R}$ ,  $\mathbb{R}^2$  and  $\mathbb{R}^3$  that interest us—not the arbitrary topological spaces found in mathematics textbooks! The answer to this objection is that if an  $\mathcal{RCC8}$ -formula is valid over regular closed subsets of  $\mathbb{R}^n$  for some  $n \geq 1$ , then it is valid in all topological spaces whatsoever (Renz 1998). Put another way: the language  $\mathcal{RCC8}$  is almost totally *insensitive* to the underlying space.

However, this insensitivity disappears when we increase the expressive resources at our command. To illustrate, consider the effect of adding a unary predicate  $c$ , where  $c(r)$  means ‘ $r$  is connected.’ We call the resulting language  $\mathcal{RCC8}c$ . Thus, the  $\mathcal{RCC8}c$ -formula

$$\bigwedge_{1 \leq i \leq 3} c(r_i) \rightarrow \bigvee_{1 \leq i < j \leq 3} \neg \text{EC}(r_i, r_j) \quad (2)$$

states that no three connected regions  $r_1$ ,  $r_2$  and  $r_3$  can externally contact each other. In  $\mathbb{R}^2$  (or in  $\mathbb{R}^3$ ), this formula is clearly *invalid*. However, (2) is *valid* when interpreted over  $\mathbb{R}$ , since the non-empty connected regular closed subsets of  $\mathbb{R}$  are simply the (non-empty) closed intervals.

Actually, the standard notion of connectedness may be inappropriate for many applications of qualitative spatial reasoning, particularly in the context of geographical information systems (GIS). Consider, for example, the (closed) region formed by two triangles touching externally at a common vertex. Mathematically speaking, this set is connected; yet we are loath to take it to represent, say, a connected plot of land on a map. Accordingly, we introduce the unary predicate  $c^\circ$ , where  $c^\circ(r)$  means ‘the topological interior of  $r$  is connected,’ and denote by  $\mathcal{RCC8}c^\circ$  the result of adding  $c^\circ$  to  $\mathcal{RCC8}$ . Again, one can show that  $\mathcal{RCC8}c^\circ$ , like  $\mathcal{RCC8}c$ , is sensitive to the topological space in which it is interpreted.

| lang.         | $\mathbb{R}$ |        |        | $\mathbb{R}^2$           |                 |                          | $\mathbb{R}^3$    |                  |                | RC               |
|---------------|--------------|--------|--------|--------------------------|-----------------|--------------------------|-------------------|------------------|----------------|------------------|
|               | RCP          |        | RC     | RCP                      |                 | RC                       | RCP               |                  | RC             |                  |
| $RCC8c^\circ$ | NP           | $\neq$ | NP     | NP<br>Th. 6, 9           |                 |                          | NP                |                  |                |                  |
| $RCC8c$       |              | Th. 2  |        | NP<br>Th. 6, 9           |                 |                          | Th. 12            |                  |                |                  |
| $Bc^\circ$    | NP           |        |        | $\geq$ EXPTIME<br>Th. 11 | $\neq$<br>Th. 7 | ?                        | EXPTIME<br>Th. 15 | $\neq$<br>Th. 14 | ?              | ?                |
| $Bc$          | Th. 3        |        |        | $\geq$ EXPTIME<br>Th. 10 | $\neq$<br>Th. 8 | $\geq$ EXPTIME<br>Th. 10 | $\geq$ EXPTIME    | ?                | $\geq$ EXPTIME | ?                |
| $Cc^\circ$    | PSPACE       | $\neq$ | PSPACE | $\geq$ EXPTIME<br>Th. 10 | $\neq$<br>Th. 7 | $\geq$ EXPTIME<br>Th. 10 | $\geq$ EXPTIME    | $\neq$<br>Th. 14 | $\geq$ EXPTIME | $\neq$<br>Th. 13 |
| $Cc$          |              | Th. 2  |        | $\geq$ EXPTIME<br>Th. 10 | $\neq$<br>Th. 8 | $\geq$ EXPTIME<br>Th. 10 | $\geq$ EXPTIME    | ?                | $\geq$ EXPTIME | ?                |

Figure 2: Summary of the expressiveness and complexity results.

Another way to increase the expressive power of  $RCC8$  is to provide the means to talk about *combinations* of regions. Thus, in the language known as *Boolean RCC8* (Wolter and Zakharyashev 2000), we use  $r_1 + r_2$ ,  $r_1 \cdot r_2$  and  $-r$  for the regular closures of  $r_1 \cup r_2$ ,  $r_1 \cap r_2$  and the complement of  $r$ , respectively. We denote this extension of  $RCC8$  by  $\mathcal{C}$  (this nomenclature will be justified in the sequel). By extending  $\mathcal{C}$  with either of the predicates  $c$  or  $c^\circ$ , we obtain the languages  $Cc$  and  $Cc^\circ$ . For example, the  $Cc^\circ$ -formula

$$c^\circ(-r_1) \wedge c^\circ(-r_2) \wedge \neg c^\circ(-(r_1 + r_2)) \rightarrow \neg DC(r_1, r_2) \quad (3)$$

can be shown to be valid for regular closed sets in Euclidean space of any dimension (Theorem 13); yet it is invalid in other topological spaces—for example, the torus (Fig. 3).

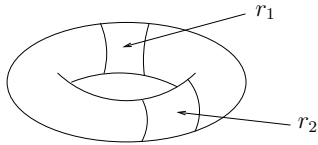


Figure 3: Invalidating (3) in the torus.

Once we have connectedness predicates at our disposal, two further topological languages suggest themselves. We denote by  $Bc$  the language featuring the Boolean function symbols  $+$ ,  $\cdot$  and  $-$ , together with the equality predicate  $=$  and the connectedness predicate  $c$ ; the language  $Bc^\circ$  is defined similarly, but with  $c$  replaced by  $c^\circ$ .

The language  $Bc^\circ$  nicely illustrates a subtle but important semantic issue which is often neglected in discussions of topological logics. Consider the  $Bc^\circ$ -formulas (for  $m \geq 3$ ):

$$\bigwedge_{1 \leq i \leq m} c^\circ(r_i) \wedge c^\circ\left(\sum_{1 \leq i \leq m} r_i\right) \rightarrow \bigvee_{2 \leq i \leq m} c^\circ(r_1 + r_i). \quad (4)$$

Interpreted over the regular closed subsets of  $\mathbb{R}^2$ , these formulas are *invalid*: Fig. 4 shows a counterexample (with  $m = 3$ ) in which the boundary between  $r_2$  and  $r_3$  is formed by the curve  $\sin(1/x)$  over the interval  $(0, 1]$ . Yet  $r_2$  and  $r_3$  are hardly plausible models of, for instance, regions occupied by physical objects resting on a surface, or plots of land in a cadastre. Crucially, it can be shown (Lemma 1) that (4) becomes valid as soon as we restrict attention to ‘well-behaved’ (as we shall say: *tame*) subsets of  $\mathbb{R}^n$ —in

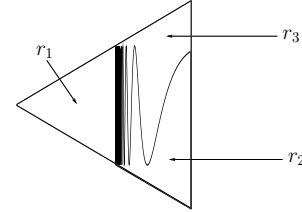


Figure 4: Three regular closed sets in  $\mathbb{R}^2$  satisfying (4).

particular, to polyhedra (or polygons). Therefore, in deciding how to interpret topological logics for spatial representation and reasoning, it is not sufficient merely to fix the topological space in question: we must also specify which subsets of that space we wish to count as *bona fide* regions.

This paper undertakes the kind of semantic analysis of topological logics we have just argued for. We consider the six languages introduced above, and interpret them in  $\mathbb{R}$ ,  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . We show that the dimensionality of the space is important for all of our languages, but that, in addition, these languages exhibit varying patterns of sensitivity to tameness in different dimensions. It turns out that only  $RCC8c$  and  $RCC8c^\circ$  are insensitive to tameness in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , while—surprisingly— $Bc$  and  $Bc^\circ$  do not feel it in  $\mathbb{R}$ ; in all these cases reasoning (satisfiability) proves to be NP-complete. Reasoning with  $Bc$ ,  $Bc^\circ$ ,  $Cc$  and  $Cc^\circ$  in  $\mathbb{R}^n$ , for  $n \geq 2$ , is shown to be generally EXPTIME-hard (apart from two cases that are still open). A matching upper bound is obtained for  $Bc^\circ$  over polyhedra in  $\mathbb{R}^n$ ,  $n \geq 3$ . The obtained results are collected in Fig. 2. The proofs of these results are occasionally intricate, and can only be sketched here. Full details can be found at <http://www.dcs.bbk.ac.uk/~roman>.

## Preliminaries

Let  $T$  be a topological space. We denote the closure of any  $X \subseteq T$  by  $X^-$ , its interior by  $X^\circ$  and its boundary by  $\delta X = X^- \setminus X^\circ$ . We call  $X$  *regular closed* if  $X = X^{\circ-}$ , and denote by  $RC(T)$  the set of all regular closed subsets of  $T$ . It is known that  $RC(T)$  forms a Boolean algebra under the operations  $r_1 + r_2 = r_1 \cup r_2$ ,  $r_1 \cdot r_2 = (r_1 \cup r_2)^{\circ-}$  and  $-r_1 = (T \setminus r_1)^-$ . A subset  $X \subseteq T$  is *connected* if it cannot be covered by the union of two non-empty and disjoint subsets which are open in the subspace topology on  $X$ . We say that  $X$  is *interior-connected* if  $X^\circ$  is connected.

By a *topological language* we mean a language featuring an infinite set of variables, a fixed non-logical signature of function symbols and predicates (with standard meanings as topological operations and relations), and the usual connectives of propositional logic. If  $\mathcal{L}$  is a topological language, a *frame* for  $\mathcal{L}$  is a collection  $\mathfrak{F}$  of subsets of some topological space  $T$ ; and a *model* for  $\mathcal{L}$  over  $\mathfrak{F}$  is a pair  $(\mathfrak{F}, \sigma)$ , where  $\sigma$  is a function from variables to elements of  $\mathfrak{F}$ . Thus, for any topological space  $T$ ,  $\text{RC}(T)$  is a frame for any of our topological languages; we denote by  $\text{RC}$  the class of all such frames. In this paper, we shall be concerned exclusively with frames which form subsets of  $\text{RC}(T)$  for some  $T$ . Restricting attention to regular closed sets is regarded as a convenient means of ignoring the boundaries of spatial regions.

Since the meanings of the non-logical primitives of  $\mathcal{L}$  are fixed, any model defines a notion of truth for  $\mathcal{L}$ -formulas in the obvious way. If  $\mathfrak{F}$  is a frame and  $\varphi$  an  $\mathcal{L}$ -formula, we say that  $\varphi$  is *satisfiable over*  $\mathfrak{F}$  if  $\varphi$  is true in some model over  $\mathfrak{F}$ . If  $\mathcal{K}$  is a class of frames, we say that  $\varphi$  is *satisfiable over*  $\mathcal{K}$  if  $\varphi$  is satisfiable over some frame in  $\mathcal{K}$ ; and we say that  $\varphi$  is *valid over*  $\mathcal{K}$  if  $\neg\varphi$  is not satisfiable over  $\mathcal{K}$ . Thus, satisfiability and validity are dual notions in the usual sense. A *topological logic* is a pair  $(\mathcal{L}, \mathcal{K})$  where  $\mathcal{L}$  is a topological language and  $\mathcal{K}$  a class of frames for  $\mathcal{L}$ . The *satisfiability problem* for  $(\mathcal{L}, \mathcal{K})$  is denoted by  $\text{Sat}(\mathcal{L}, \mathcal{K})$ .

For regular closed sets, the  $\mathcal{RCC8}$ -predicates are standardly interpreted as follows:

|                         |     |   |
|-------------------------|-----|---|
| $\text{DC}(r_1, r_2)$   | iff | $r_1 \cap r_2 = \emptyset$ ,  |
| $\text{EC}(r_1, r_2)$   | iff | $r_1 \cap r_2 \neq \emptyset$ but $r_1^\circ \cap r_2^\circ = \emptyset$ ,                  |
| $\text{PO}(r_1, r_2)$   | iff | $r_1^\circ \cap r_2^\circ, r_1^\circ \setminus r_2, r_2^\circ \setminus r_1 \neq \emptyset$ |
| $\text{EQ}(r_1, r_2)$   | iff | $r_1 = r_2$ ,   |
| $\text{TPP}(r_1, r_2)$  | iff | $r_1 \subseteq r_2$ but $r_1 \not\subseteq r_2^\circ$ and $r_2 \not\subseteq r_1$ ,         |
| $\text{NTPP}(r_1, r_2)$ | iff | $r_1 \subseteq r_2^\circ$ but $r_2 \not\subseteq r_1$ .                                     |

The unary predicates  $c$  and  $c^\circ$  are interpreted as the properties of connectedness and interior-connectedness, respectively. The function symbols  $+$ ,  $\cdot$ ,  $-$  and constants  $0$  and  $1$  are interpreted as the corresponding operations and elements in  $\text{RC}(T)$ . The *contact* predicate  $C$  holds between regions  $r_1$  and  $r_2$  if and only if  $r_1 \cap r_2 \neq \emptyset$ . Thus,  $C(r_1, r_2)$  is equivalent to  $\neg\text{DC}(r_1, r_2)$ . In the presence of the Boolean functions, all the  $\mathcal{RCC8}$ -predicates can be expressed in terms of  $C$  and  $=$  (i.e.,  $\text{EQ}$ ), and *vice versa* (Kontchakov et al. 2009); hence the name  $\mathcal{C}$  for the Boolean extension of  $\mathcal{RCC8}$ .

Fixing  $n \geq 1$ , any  $(n - 1)$ -dimensional hyperplane in  $\mathbb{R}^n$  bounds two regions in  $\text{RC}(\mathbb{R}^n)$ ; let us call these regions *half-spaces*. We denote by  $\text{RCP}(\mathbb{R}^n)$  the Boolean subalgebra of  $\text{RC}(\mathbb{R}^n)$  generated by the half-spaces. We call the elements of  $\text{RCP}(\mathbb{R}^n)$  *polyhedra* in  $\mathbb{R}^n$ , and the elements of  $\text{RCP}(\mathbb{R}^2)$  *polygons*. We have: (i) every polyhedron is the union of finitely many connected polyhedra; and (ii) every polyhedron satisfies the *curve-selection lemma* (Bochnak et al. 1998, p. 38). (The regions  $r_2$  and  $r_3$  of Fig. 4 lack curve-selection.) We call any collection of (regular closed) sets satisfying these two properties *tame*. Tame regions are regarded as well-behaved.

More generally, a subset of  $\mathbb{R}^n$  is *semi-algebraic* if it is definable by a formula with  $n$  free variables in the first-order language of fields; we denote the collection of regular closed

semi-algebraic subsets of  $\mathbb{R}^n$  by  $\text{RCS}(\mathbb{R}^n)$ . Semi-algebraic sets are certainly representationally adequate for all practical purposes, yet they are tame in the above sense. Since, however, even very expressive topological languages standardly cannot distinguish between semi-algebraic sets and polyhedra, we may as well, for the purpose of restricting attention to tame regions, focus on polyhedra; and that is what we do in the sequel. Note also that, in GISs, regions are usually represented as polygons.

## One-dimensional Euclidean space

First we consider the logics  $(\mathcal{L}, \text{RC}(\mathbb{R}))$  and  $(\mathcal{L}, \text{RCP}(\mathbb{R}))$ , where  $\mathcal{L}$  is any of topological languages introduced above. The one-dimensional case is simple to analyse, yet illustrates well the kinds of phenomena that will occupy us at greater length when we come to the 2D and 3D cases.

Over  $\mathbb{R}$ , the notions of connectedness and interior-connectedness coincide; hence, we have only the languages  $\mathcal{RCC8c}$ ,  $\mathcal{Bc}$  and  $\mathcal{Cc}$  to consider. We begin by observing that the dimensionality of the space is significant.

**Theorem 1.** For any  $\mathcal{L} \in \{\mathcal{RCC8c}, \mathcal{Bc}, \mathcal{Cc}\}$  and  $n \geq 2$ ,  $\text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R})) \subsetneq \text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^n))$ .

**Proof.** The inclusion holds because any tuple in  $\text{RC}(\mathbb{R})$  can easily be cylindrified to form a tuple in  $\text{RC}(\mathbb{R}^n)$ ,  $n \geq 2$ , satisfying the same  $\mathcal{L}$ -formulas. To see that it is proper, we observed above that (2) is invalid over  $\text{RC}(\mathbb{R}^n)$  for  $n \geq 2$ ; but it is valid over  $\text{RC}(\mathbb{R})$ . For  $\mathcal{Bc}$ , consider  $\bigwedge_{1 \leq i \leq 3} (c(r_i) \wedge (r_i \neq 0)) \rightarrow \bigvee_{1 \leq i < j \leq 3} \neg((r_i \cdot r_j = 0) \wedge c(r_i + r_j))$ .  $\square$

Next, we consider the issue of tameness. Over  $\mathbb{R}$ , the languages  $\mathcal{RCC8c}$  and  $\mathcal{Cc}$  are sensitive to the presence of non-tame regions:

**Theorem 2.**  $\text{Sat}(\mathcal{RCC8c}, \text{RCP}(\mathbb{R})) \subsetneq \text{Sat}(\mathcal{RCC8c}, \text{RC}(\mathbb{R}))$  and  $\text{Sat}(\mathcal{Cc}, \text{RCP}(\mathbb{R})) \subsetneq \text{Sat}(\mathcal{Cc}, \text{RC}(\mathbb{R}))$ .

**Proof.** The inclusions are trivial. To show that they are proper, the  $\mathcal{RCC8c}$ -formula  $c(r_1) \wedge \bigwedge_{1 \leq i < j \leq 4} \text{EC}(r_i, r_j)$  is satisfiable over  $\text{RC}(\mathbb{R})$ , but not over  $\text{RCP}(\mathbb{R})$ ; see Fig. 5.  $\square$

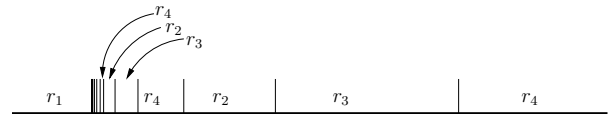


Figure 5: Subsets of  $\mathbb{R}$  used in the proof of Theorem 2.

The language  $\mathcal{Bc}$ , by contrast, is not sensitive to tameness:

**Theorem 3.**  $\text{Sat}(\mathcal{Bc}, \text{RC}(\mathbb{R})) = \text{Sat}(\mathcal{Bc}, \text{RCP}(\mathbb{R}))$ .

**Proof.** See <http://www.dcs.bbk.ac.uk/~roman>.  $\square$

Turning now to complexity, it is already known that  $\text{Sat}(\mathcal{Bc}, \text{RC}(\mathbb{R}))$  is NP-complete (Kontchakov et al. 2009), and  $\text{Sat}(\mathcal{Cc}, \text{RC}(\mathbb{R}))$  is PSPACE-complete (Kontchakov et al. 2008). The picture is completed by the following results:

**Theorem 4.** The problems  $\text{Sat}(\mathcal{RCC8c}, \text{RC}(\mathbb{R}))$  and  $\text{Sat}(\mathcal{RCC8c}, \text{RCP}(\mathbb{R}))$  are both NP-complete; the problem  $\text{Sat}(\mathcal{Cc}, \text{RCP}(\mathbb{R}))$  is PSPACE-complete.

**Proof.** See <http://www.dcs.bbk.ac.uk/~roman>.  $\square$

## Two-dimensional Euclidean space

We first observe that, for the languages we consider, confining attention to the space  $\mathbb{R}^2$  is significant for satisfiability.

**Theorem 5.** *For any of the languages  $\mathcal{L}$  considered in this paper, and any  $n \geq 3$ ,  $\text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^2)) \not\subseteq \text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^n))$ .*

**Proof.** Again, inclusion follows by cylindrification.

To show that it is proper, let  $r_i$  ( $1 \leq i \leq 5$ ) and  $r_{i,j}$  ( $1 \leq i < j \leq 5$ ) be variables, let  $\varphi$  be the  $\text{RCC8c}$ -formula

$$\bigwedge_{1 \leq i < j \leq 5} c(r_{i,j}) \wedge \bigwedge_{\{i,j\} \cap \{k,\ell\} = \emptyset} \text{DC}(r_{i,j}, r_{k,\ell}) \wedge \bigwedge_{i \in \{j,k\}} \text{TPP}(r_i, r_{j,k}),$$

and let  $\varphi^\circ$  be the  $\text{RCC8c}^\circ$ -formula obtained by replacing all occurrences of  $c$  by  $c^\circ$ . Thus,  $\varphi^\circ$  entails  $\varphi$ . A simple argument based on the non-planarity of the graph  $K_5$  shows that  $\varphi$  (and hence  $\varphi^\circ$ ) is not satisfiable over  $\text{RC}(\mathbb{R}^2)$ . On the other hand,  $\varphi^\circ$  (and hence  $\varphi$ ) is satisfiable over  $\text{RC}(\mathbb{R}^n)$  for  $n \geq 3$ . This deals with  $\text{RCC8c}$ ,  $\text{RCC8c}^\circ$ ,  $\mathcal{C}$  and  $\mathcal{C}^\circ$ . For  $\mathcal{Bc}$ , replace  $\text{DC}(r_{i,j}, r_{k,\ell})$  by  $\neg c(r_{i,j} + r_{k,\ell})$  and  $\text{TPP}(r_i, r_{j,k})$  by  $(r_i \cdot r_{j,k} = r_i) \wedge (r_i \neq 0)$ . For  $\mathcal{Bc}^\circ$ , use the formula  $\bigwedge_{1 \leq i \leq 5} (c^\circ(r_i) \wedge (r_i \neq 0)) \wedge \bigwedge_{1 \leq i < j \leq 5} (c^\circ(r_j + r_j) \wedge (r_i \cdot r_j = 0))$ .  $\square$

We now proceed to show (Theorems 6–8) that, over  $\mathbb{R}^2$ , our topological languages exhibit a different pattern of sensitivity to tameness to that which we observed over  $\mathbb{R}$ .

**Theorem 6.** *If an  $\text{RCC8c}$ - or  $\text{RCC8c}^\circ$ -formula is satisfiable over  $\text{RC}(\mathbb{R}^2)$ , then it can be satisfied over the frame of bounded regular closed polygons. In consequence:*

$$\text{Sat}(\text{RCC8c}, \text{RC}(\mathbb{R}^2)) = \text{Sat}(\text{RCC8c}, \text{RCP}(\mathbb{R}^2)),$$

$$\text{Sat}(\text{RCC8c}^\circ, \text{RC}(\mathbb{R}^2)) = \text{Sat}(\text{RCC8c}^\circ, \text{RCP}(\mathbb{R}^2)).$$

**Proof.** For the first statement, it suffices to construct, for any tuple  $r_1, \dots, r_n$  in  $\text{RC}(\mathbb{R}^2)$ , a corresponding tuple  $p_1, \dots, p_n$  in  $\text{RCP}(\mathbb{R}^2)$  satisfying exactly the same atomic  $\text{RCC8c}$ -formulas. We may assume that the  $r_i$  are distinct and non-empty. By reordering the variables if necessary, we can ensure that  $r_i \subseteq r_j$  implies  $i \leq j$ . For all  $i, j$  ( $1 \leq i < j \leq n$ ), let  $R_{ij} \in \{\text{DC}, \text{EC}, \text{PO}, \text{TPP}, \text{NTPP}\}$  be the unique relation such that  $R_{ij}(r_i, r_j)$ .

First, we construct regular closed sets  $r_1^+, \dots, r_n^+$  such that  $r_j \subseteq (r_j^+)^{\circ}$ ,  $(r_j^+)^{\circ}$  is connected whenever  $r_j$  is connected, and  $r_j^+ \cap r_{j'}^+ = \emptyset$  whenever  $r_j \cap r_{j'} = \emptyset$ , for all  $j, j'$ . This is possible because  $\mathbb{R}^2$  is a normal, locally connected topological space and the  $r_i$  are regular closed subsets of  $\mathbb{R}^2$ . For all  $i, j$  ( $1 \leq i < j \leq n$ ), pick points  $o, o', o''$  satisfying the conditions: (i) if  $R_{ij} = \text{EC}$ , then  $o \in \delta r_i \cap \delta r_j$ ; (ii) if  $R_{ij} = \text{PO}$ , then  $o \in r_i^{\circ} \cap r_j^{\circ}$ ,  $o' \in r_i^{\circ} \setminus r_j$ , and  $o'' \in r_j^{\circ} \setminus r_i$ ; (iii) if  $R_{ij} = \text{TPP}$ , then  $o \in \delta r_i \cap \delta r_j$  and  $o' \in r_j^{\circ} \setminus r_i$ ; (iv) if  $R_{ij} = \text{NTPP}$ , then  $o \in r_j^{\circ} \setminus r_i$ . And for all  $i$  ( $1 \leq i \leq n$ ), pick points  $o, o'$  satisfying the condition that, if  $r_i$  is not connected, then  $o$  and  $o'$  lie in different components of  $r_i$ . Enumerate the chosen (distinct) points as  $o_1, \dots, o_m$ : we call them *witness points*. We can draw disjoint closed disks  $d_1, \dots, d_m$ , centred on the respective witness points  $o_k$ , such that, for all  $j \leq n$  and  $k \leq m$ :  $o_k \in r_j^{\circ}$

implies  $d_k \subseteq r_j^{\circ}$ ; and  $o_k \in (r_j^+)^{\circ}$  implies  $d_k \subseteq (r_j^+)^{\circ}$ . Indeed, we can ensure that none of the sets  $(r_j^+)^{\circ}$  is disconnected by (simultaneous) removal of  $d_1, \dots, d_m$ .

We now begin the construction of the  $p_1, \dots, p_n$ . First, for each set  $r_j$  and each witness point  $o_k$ , we select a polygon  $w_{k,j}$  inside  $d_k$ . We refer to the  $w_{k,j}$  as *wedges*: for each  $j \leq n$ , and each  $k \leq m$  we will ensure below that  $w_{k,j} \subseteq p_j$ . Wedges are selected as follows. (i) If  $o_k \in \delta r_j$ , pick a point  $q_{k,j} \in \delta d_k \subseteq (r_j^+)^{\circ}$ , and let  $w_{k,j}$  be a lozenge with  $w_{k,j} \subseteq d_k$  and  $o_k, q_{k,j} \in \delta w_{k,j}$ ; see Fig. 6a. We may pick the  $q_{k,j}$  to be distinct, and construct the  $w_{k,j}$  so that no two such  $w_{k,j}$  have intersecting interiors. (ii) If  $o_k \in r_j^{\circ}$ , pick a point  $q_{k,j} \in \delta d_k \subseteq r_j^{\circ}$ , and let  $w_{k,j}$  be a lozenge such that  $w_{k,j} \subseteq d_k$ ,  $o_k \in (w_{k,j})^{\circ}$  and  $q_{k,j} \in \delta w_{k,j}$ . Again, we may pick the  $q_{k,j}$  to be distinct from each other and from the  $q_{k,j}$  selected in (i); see Fig. 6b. (iii) Otherwise, i.e., if  $o_k \notin r_j$ , let  $w_{k,j} = \emptyset$ . The wedges  $w_{k,j}$  will ensure that  $p_1, \dots, p_n$  contain certain witness points required for the satisfaction of the relevant atomic  $\text{RCC8c}$ -formulas. For example, if  $\text{PO}(r_i, r_j)$ , there will exist a witness point  $o_k \in r_i^{\circ} \cap r_j^{\circ}$ ; but then  $o_k \in w_{k,i}^{\circ} \cap w_{k,j}^{\circ}$ , whence  $o_k \in p_i^{\circ} \cap p_j^{\circ}$ , which is required to ensure that  $\text{PO}(p_i, p_j)$ .

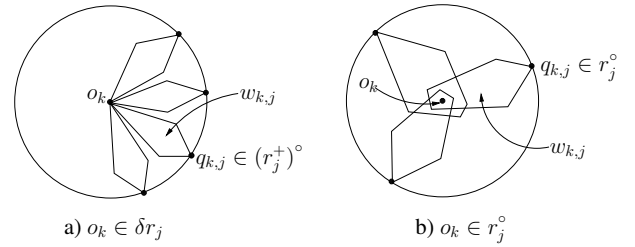


Figure 6: Wedges involving the witness point  $o_k$ .

Fix any  $j \leq n$ . If  $r_j$  is connected, we need to connect up the various wedges  $w_{k,j}$  so as to ensure that  $p_j$  is connected. Specifically, we construct a connected, regular closed polygon  $a_j \subseteq (r_j^+)^{\circ}$  such that  $a_j$  is externally connected to all the non-empty  $w_{k,j}$  (with  $k$  varying). The construction is quite elaborate, but the basic technique is illustrated in Fig. 7. The crucial point is that the  $a_j$  need only be connected—they need not have connected interiors; hence they may be crossed by regions  $a_{j'}$  ( $j' \neq j$ ) as long as we do not have  $\text{DC}(r_j, r_{j'})$ . The polygon  $a_j$  illustrated in Fig. 7 is crossed twice in this way. If  $r_j$  is not connected, set  $a_j = \emptyset$ . For all  $j \leq n$ , define  $b_j = a_j + \sum_{1 \leq k \leq m} w_{k,j}$ . Then  $b_j$  will be connected if and only if  $r_j$  is.

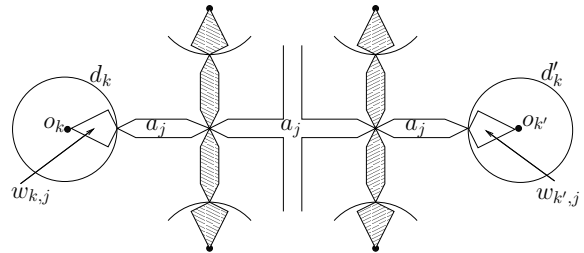


Figure 7: Connecting together wedges  $w_{k,j}$  and  $w_{k',j'}$ .

Using the polygons  $b_1, \dots, b_n$ , we construct the desired polygons  $p_1, \dots, p_n$ , relying on the fact that we earlier ensured that  $r_i \subseteq r_j$  implies  $i \leq j$ . Start by setting  $p_1 = b_1$ , and let  $p_1^+$  be a polygon containing  $p_1$  in its interior, but which is ‘close’ to  $p_1$  (specifically:  $p_1^+$  does not intersect any polygon involved in this construction that  $p_1$  does not intersect). For  $2 \leq j \leq n$ , set

$$p_j = b_j + \sum_{\text{TPP}(r_i, r_j)} p_i + \sum_{\text{NTPP}(r_i, r_j)} p_i^+$$

and again let  $p_j^+$  be a polygon containing  $p_j$  in its interior, and ‘close’ to  $p_j$ . This ensures that  $r_i \subseteq r_j$  implies  $p_i \subseteq p_j$ , and  $r_i \subseteq r_j^\circ$  implies  $p_i \subseteq p_j^\circ$ . We can then show that  $p_1, \dots, p_n$  satisfy exactly the same atomic  $\mathcal{RCC8c}$ -formulas as the  $r_1, \dots, r_n$ .

A similar (but not identical) construction can be carried out for the case of  $\mathcal{RCC8c}^\circ$ .  $\square$

An analogous result to Theorem 6 for a more expressive spatial logic can be found in (Davis et al. 1991, Sec. 8.1).

For the language  $\mathcal{Bc}$ , however, tameness does make a difference in two dimensions, both for connectedness and for interior-connectedness. The latter is easily dealt with:

**Lemma 1.** *The  $\mathcal{Bc}^\circ$ -formula (4) is valid in  $\text{RCP}(\mathbb{R}^n)$  for all  $m > n \geq 1$ .*

**Proof.** The case  $n = 1$  is trivial. See (Pratt-Hartmann 2007, p. 40) for the case  $n = 2$ ; the proof applies almost unaltered to higher dimensions.  $\square$

**Theorem 7.**  *$\text{Sat}(\mathcal{Bc}^\circ, \text{RCP}(\mathbb{R}^2)) \subsetneq \text{Sat}(\mathcal{Bc}^\circ, \text{RC}(\mathbb{R}^2))$  and  $\text{Sat}(\mathcal{Cc}^\circ, \text{RCP}(\mathbb{R}^2)) \subsetneq \text{Sat}(\mathcal{Cc}^\circ, \text{RC}(\mathbb{R}^2))$ .*

**Proof.** We need only show that the inclusions are proper. As observed above, (4) is invalid over  $\text{RC}(\mathbb{R}^2)$ ; but it is valid over  $\text{RC}(\mathbb{R}^2)$  by Lemma 1.  $\square$

For ordinary connectedness, much more work is required.

**Theorem 8.**  *$\text{Sat}(\mathcal{Cc}, \text{RCP}(\mathbb{R}^2)) \subsetneq \text{Sat}(\mathcal{Cc}, \text{RC}(\mathbb{R}^2))$ . In fact,  $\text{Sat}(\mathcal{Bc}, \text{RCP}(\mathbb{R}^2)) \subsetneq \text{Sat}(\mathcal{Bc}, \text{RC}(\mathbb{R}^2))$ .*

**Proof.** Again, we need only show that the inclusions are proper. We begin with the language  $\mathcal{Cc}$ ; the second statement of the theorem will follow by an easy adaptation. Let  $V$  be the set of variables  $\{v, h, s, t, t_0, r_0, r_1, r_2, r_3, r_4, r_5\}$  and, for any  $x \in V$ , let  $\hat{x}$  be a fresh variable. Consider the assignment of elements of  $\text{RC}(\mathbb{R}^2)$  to these variables shown in Fig. 8. Here, the regions  $v$  and  $h$  are unbounded, connected polygons, the regions  $s, t$  and  $t_0$  are bounded, connected polygons, and the regions  $r_i$  ( $0 \leq i < 6$ ) are all unbounded and have infinitely many components (and hence are not polygons). Also, for all  $x \in V$ , the region  $\hat{x}$  is a slightly ‘enlarged’ version of  $x$ , with  $x$  lying in the interior of  $\hat{x}$ . (In Fig. 8, we have drawn  $\hat{v}, \hat{t}_0$  and  $\hat{h}$  with dotted lines; the other regions  $\hat{x}$  are suppressed for clarity.)

Let  $\varphi_0$  be the conjunction of  $x \neq 0, x \cdot y = 0, \neg C(x, -\hat{x})$ , for distinct  $x, y \in V$ . The last of these ensures that  $\hat{x}$  represents a region whose interior contains  $x$ .

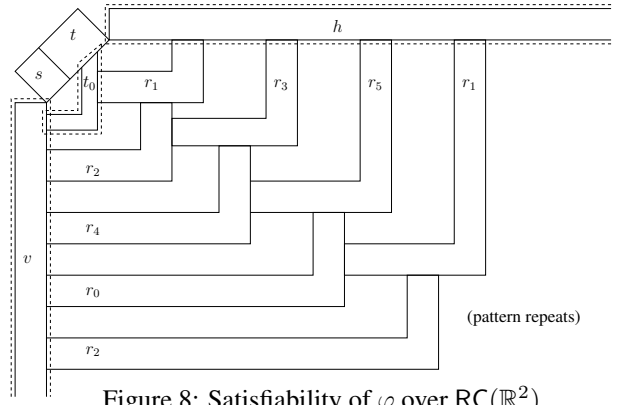


Figure 8: Satisfiability of  $\varphi$  over  $\text{RC}(\mathbb{R}^2)$ .

Let  $\varphi_1$  be the conjunction of the following formulas:

$$\begin{aligned} c(h) \quad c(v) \quad \neg C(v, h) \quad c(h + t + s + v) \\ \neg C(s, \hat{h}) \quad c((t + t_0) \cdot (-\hat{h}) + v) \\ \neg C(\hat{v}, \hat{t}) \quad c((t_0 + r_1) \cdot (-\hat{t}) \cdot (-\hat{v}) + h) \\ \neg C(\hat{t}_0, \hat{h}) \quad \neg C(t_0, s) \quad c((r_1 + r_2) \cdot (-\hat{t}_0) \cdot (-\hat{h}) + v). \end{aligned}$$

Let  $\varphi_2$  be the conjunction of the following formulas, for  $i = 0, 2, 4$ , and with arithmetic in the subscripts modulo 6:

$$\begin{aligned} \neg C(r_i, s) \quad \neg C(r_i, t) \quad \neg C(\hat{r}_i, \hat{h}) \\ c((r_i + r_{i+1}) \cdot (-\hat{r}_{i-1}) \cdot (-\hat{v}) + h). \end{aligned}$$

Let  $\varphi_3$  be the conjunction of the following formulas, for  $i = 1, 3, 5$ , with arithmetic in the subscripts modulo 6:

$$\begin{aligned} \neg C(r_i, s) \quad \neg C(r_i, t) \quad \neg C(\hat{r}_i, \hat{v}) \\ c((r_i + r_{i+1}) \cdot (-\hat{r}_{i-1}) \cdot (-\hat{h}) + v). \end{aligned}$$

Let  $\varphi_4$  be the conjunction of the following formulas:

$$\begin{aligned} \neg C(r_i, r_j), \quad (0 \leq i < j < 6, |i - j| > 1), \\ \neg C(t_0, r_j), \quad (2 \leq j < 6). \end{aligned}$$

Let  $\varphi = \bigwedge_{i=0}^4 \varphi_i$ . We claim that the model in Fig. 8 satisfies  $\varphi$ . The truth of  $\varphi_0$  and  $\varphi_1$  is easily checked. For  $\varphi_2$ , note that, for  $i$  even, each component of  $r_i$  contacts  $v$ , but not  $h$ ; on the other hand, the components of  $r_i$  together connect all the components of  $r_{i-1}$  to  $v$  (arithmetic modulo 6). The conjunct  $\varphi_3$  is handled analogously. For  $\varphi_4$ , observe that the components of  $r_0, \dots, r_5$  form a repeating pattern, with  $r_i \cap r_j = \emptyset$  whenever  $i$  and  $j$  differ by more than 1. Thus,  $\varphi$  is satisfiable over  $\text{RC}(\mathbb{R}^2)$ .

We outline the proof that  $\varphi$  is not satisfiable over  $\text{RCP}(\mathbb{R}^2)$ . Refer to Fig. 8, and pick any  $i$  ( $0 \leq i < 6$ ) and any component of  $r'_i$  of  $r_i$ . Let  $r'_{i+1}$  be the component of  $r_{i+1}$  which contacts  $r'_i$ . Draw a Jordan curve in  $r'_i + r'_{i+1} + v + s + t + h$  enclosing  $t_0$ . By doing this for all  $r'_i$  and  $r'_{i+1}$ , we obtain an infinite sequence  $\{\gamma_{i,j}\}$  of nested Jordan curves ( $0 \leq i < 6, 1 \leq j$ ), with each  $\gamma_{i,j}$  drawn in  $r_i + r_{i+1} + v + s + t + h$ . Suppose, then that  $\varphi$  is satisfied by any tuple of regions  $x, \hat{x}$ , for  $x \in V$ . On the assumption that these regions are in  $\text{RCP}(\mathbb{R}^2)$ , it can be shown that just such a sequence of Jordan curves  $\{\gamma_{i,j}\}$  must exist. But then each set  $\mathbb{R}^2 \setminus (r_i + r_{i+1} + v + s + t + h)$  has infinitely many components, contradicting the supposition that the satisfying tuple is in  $\text{RCP}(\mathbb{R}^2)$ .  $\square$

We turn now to complexity-theoretic issues, employing a surprising theorem on graph-drawing. Let  $\mathbf{D}$  be the frame consisting of all regular closed subsets of  $\mathbb{R}^2$  homeomorphic to closed discs. (It does not matter that  $\mathbf{D}$  is not a Boolean algebra.) Then  $Sat(\mathcal{RCC8}, \mathbf{D})$  is in NP (Schaefer, Sedgwick, and Štefankovič 2003). Using Theorem 6, we can show that  $Sat(\mathcal{RCC8c}, \mathcal{RC}(\mathbb{R}^2))$  is also in NP. The following lemma enables us to reduce  $Sat(\mathcal{RCC8c}, \mathcal{RCP}(\mathbb{R}^2))$  non-deterministically to  $Sat(\mathcal{RCC8}, \mathbf{D})$ .

**Lemma 2.** *Let  $\varphi$  be an  $\mathcal{RCC8c}$ -formula, and suppose  $\varphi$  is satisfied by bounded polygons  $r_1, \dots, r_n$ . Then  $\varphi$  is satisfied by bounded polygons  $r'_1, \dots, r'_n$  such that: for all  $i \leq n$ , (a) if  $u$  is a connected component of  $r_i^\circ$ , then  $u^- \in \mathbf{D}$ ; and (b)  $(r'_i)^\circ$  has at most  $O(n^3)$  components.*

**Theorem 9.** *The problems  $Sat(\mathcal{RCC8c}, \mathcal{RCP}(\mathbb{R}^2))$  and  $Sat(\mathcal{RCC8c}^\circ, \mathcal{RCP}(\mathbb{R}^2))$  are both NP-complete.*

**Proof.** Suppose  $\varphi$  is an  $\mathcal{RCC8c}$ -formula with  $n$  variables. We describe an NP procedure for determining whether  $\varphi$  is satisfiable over  $\mathcal{RCP}(\mathbb{R}^2)$ . For each  $i$  ( $1 \leq i \leq n$ ), take up to  $O(n^3)$  fresh variables  $t_{i,1}, \dots, t_{i,m_i}$ , and list all these variables as  $t_1, \dots, t_m$ . For all  $i, j$  ( $1 \leq i < j \leq m$ ), guess an  $\mathcal{RCC8}$ -relation  $R_{ij}$ , and let  $\psi$  be the conjunction of all the formulas  $R_{ij}(t_i, t_j)$ . By the result of (Schaefer, Sedgwick, and Štefankovič 2003), we check, in NP, that  $\psi$  is satisfiable over  $\mathbf{D}$ . Finally, we check, in deterministic polynomial time, that, if  $\psi$  is satisfied by the  $t_1, \dots, t_m$ , then  $\varphi$  is satisfied by the regions  $r_1, \dots, r_n$ , where  $r_i = t_{i,1} + \dots + t_{i,m_i}$ . If both of these tests succeed, we report that  $\psi$  is satisfiable. It follows from Theorem 8 and Lemma 2 this procedure has a successful run if and only if  $\varphi$  is satisfied over  $\mathcal{RCP}(\mathbb{R}^2)$  (and  $\mathcal{RC}(\mathbb{R}^2)$ ). The case of  $\mathcal{RCC8c}^\circ$  is handled similarly.  $\square$

The precise computational complexity of the languages  $\mathcal{Bc}$ ,  $\mathcal{Cc}$  and  $\mathcal{Cc}^\circ$  is not known; we have only the following:

**Theorem 10.**  *$Sat(\mathcal{L}, \mathcal{RC}(\mathbb{R}^2))$  and  $Sat(\mathcal{L}, \mathcal{RCP}(\mathbb{R}^2))$  are all EXPTIME-hard, for  $\mathcal{L} \in \{\mathcal{Bc}, \mathcal{Cc}, \mathcal{Cc}^\circ\}$ .*

**Proof.** The proof employs a technique developed in (Kontchakov et al. 2009, Theorem 5.9) For details, see <http://www.dcs.bbk.ac.uk/~roman>.  $\square$

**Theorem 11.**  *$Sat(\mathcal{Bc}^\circ, \mathcal{RCP}(\mathbb{R}^2))$  is EXPTIME-hard.*

**Proof.** See the proof of Theorem 15.  $\square$

At the time of writing, no non-trivial lower complexity bound is known for  $Sat(\mathcal{Bc}^\circ, \mathcal{RC}(\mathbb{R}^2))$ . Moreover, no upper bound at all is known for these problems. Thus, we do not know whether  $Sat(\mathcal{Bc}, \mathcal{RC}(\mathbb{R}^2))$ ,  $Sat(\mathcal{Bc}, \mathcal{RCP}(\mathbb{R}^2))$ , etc. are even decidable.

### Three-dimensional Euclidean space

Languages based on  $\mathcal{RCC8}$  cannot distinguish between Euclidean spaces of more than 3 dimensions. Indeed, they are even insensitive to the tameness of sets, and to the distinction between connectedness and interior-connectedness.

**Theorem 12.** *The problems  $Sat(\mathcal{RCC8c}, \mathcal{K})$  are identical, where  $\mathcal{K}$  is any of  $\mathcal{RC}$ ,  $\mathcal{RC}(\mathbb{R}^n)$  or  $\mathcal{RCP}(\mathbb{R}^n)$  for any  $n \geq 3$ .*

Further, for an  $\mathcal{RCC8c}$ -formula  $\varphi$ , let  $\varphi^\circ$  be the result of replacing all occurrences of  $c$  with  $c^\circ$ . Then  $\varphi$  is satisfiable over  $\mathcal{K}$  if and only if  $\varphi^\circ$  is.

**Proof.** Follows from the observation (Renz 1998) that any satisfiable  $\mathcal{RCC8}$ -formula is satisfied by (interior-) connected polyhedra in  $\mathbb{R}^n$ , for  $n \geq 3$ .  $\square$

It is open whether  $Sat(\mathcal{L}, \mathcal{RC}(\mathbb{R}^n)) = Sat(\mathcal{L}, \mathcal{RC}(\mathbb{R}^3))$ , for  $n > 3$ , where  $\mathcal{L}$  is any of  $\mathcal{Bc}$ ,  $\mathcal{Cc}$  or  $\mathcal{Cc}^\circ$ . The best result we have is:

**Theorem 13.**  *$Sat(\mathcal{Cc}^\circ, \mathcal{RC}(\mathbb{R}^n)) \subsetneq Sat(\mathcal{Cc}^\circ, \mathcal{RC})$  for  $n \geq 1$ .*

**Proof.** Recall that (3) can be invalidated in the torus (Fig. 3). To show it is valid over Euclidean spaces, we use the following fact (Newman 1951, p. 137): if  $r_1, r_2 \in \mathcal{RC}(\mathbb{R}^n)$  are non-intersecting, and points  $p_1$  and  $p_2$  lie in the same component of  $\mathbb{R}^n \setminus r_i = (-r_i)^\circ$  for  $i = 1, 2$ , then  $p_1$  and  $p_2$  lie in the same component of  $\mathbb{R}^n \setminus (r_1 \cup r_2) = (-(r_1 + r_2))^\circ$ . Thus, (3) is valid over  $\mathcal{RC}(\mathbb{R}^n)$ .  $\square$

In the case  $\mathcal{L} = \mathcal{Bc}^\circ$ , however, we can give an answer. A *connected partition* in  $\mathcal{RCP}(\mathbb{R}^n)$  is a tuple of non-empty, pairwise disjoint elements of  $\mathcal{RCP}(\mathbb{R}^n)$ , having connected interiors, which sum to the entire space. If  $r_1, \dots, r_n$  is a connected partition, its *neighbourhood graph* is the graph  $(V, E)$  with vertices  $V = \{r_1, \dots, r_n\}$  and edges  $E = \{(r_i, r_j) \mid i \neq j \text{ and } (r_i + r_j)^\circ \text{ is connected}\}$ .

**Lemma 3.** *Let  $G$  be a connected graph. Then  $G$  is (isomorphic to) the neighbourhood graph of some connected partition in  $\mathbb{R}^n$ ,  $n \geq 3$ . If  $G$  is also planar, it is the neighbourhood graph of some connected partition in  $\mathbb{R}^2$ .*

**Proof.** To prove the second statement, take a plane embedding  $H$  of  $G$ , and let  $H^*$  be its geometric dual (which we may draw with piecewise linear edges). The faces of  $H^*$  then form a connected partition in  $\mathcal{RCP}(\mathbb{R}^2)$ , and the geometric dual  $H^{**}$  of  $H^*$  is a drawing of the neighbourhood graph of this connected partition. Since  $H$  is connected,  $H^{**}$  is isomorphic to  $H$ , and hence to  $G$ . For the first statement, we proceed by induction on the number  $k$  of vertices of  $G$ . The case  $k = 1$  is trivial. If  $k > 1$ , let  $G' = G/e$  be the minor of  $G$  formed by collapsing some edge  $e$  of  $G$  into a single node. By inductive hypothesis, let  $r_1, \dots, r_{k-2}, r'_{k-1}$  be a connected partition in  $\mathbb{R}^n$  whose neighbourhood graph is  $G'$ , with the interior-connected polyhedron  $r'_{k-1}$  corresponding to the node  $e$ . It is routine to decompose  $r'_{k-1}$  into two interior-connected polyhedra  $r_{k-1}$  and  $r_k$  so that the neighbourhood graph of  $r_1, \dots, r_{k-1}, r_k$  is  $G$ .  $\square$

A *graph model* is a pair  $\mathfrak{G} = (G, \sigma)$ , where  $G = (V, E)$  is a graph and  $\sigma$  is a function mapping any variable of  $\mathcal{Bc}^\circ$  to a subset of  $V$ . The function symbols  $+$ ,  $\cdot$  and  $-$  are interpreted, respectively, as union, intersection and complement in the power-set algebra on  $V$ , and  $c^\circ$  is interpreted as the property of graph-theoretic connectedness.

**Lemma 4.** (i) *A  $\mathcal{Bc}^\circ$ -formula  $\varphi$  is satisfiable over  $\mathcal{RCP}(\mathbb{R}^2)$  if and only if it is true in a connected planar graph model.*

(ii) *A  $\mathcal{Bc}^\circ$ -formula  $\varphi$  is satisfiable over  $\mathcal{RCP}(\mathbb{R}^n)$ ,  $n \geq 3$ , if and only if it is true in a connected graph model.*

**Proof.** We prove (ii); the proof of (i) is similar. Suppose  $\varphi$  is satisfiable over  $\text{RCP}(\mathbb{R}^n)$ ,  $n \geq 3$ . Let  $\bar{s} = s_1, \dots, s_k$  be a tuple of polyhedra satisfying  $\varphi$ , and  $\bar{t}$  a connected partition in  $\text{RCP}(\mathbb{R}^2)$  such that, for all  $i \leq k$ , there exists a subset  $R_i \subseteq \bar{t}$  of these elements such that  $s_i = \sum R_i$ . Let  $G$  be the neighbourhood graph of  $\bar{t}$ ; and make  $\bar{G}$  into a graph model  $\mathfrak{G}$  by assigning to each variable  $x_i$  the set of nodes  $R_i$ . (Such a  $\bar{t}$  always exists as long as the  $\bar{s}$  are polyhedra.) Using Lemma 1, we can show that  $\varphi$  is true in  $\mathfrak{G}$ . Conversely, suppose  $\varphi$  is true in a connected graph model  $\mathfrak{G} = (G, \sigma)$ . By Lemma 3, let  $\bar{r}$  be a connected partition in  $\text{RCP}(\mathbb{R}^n)$  whose neighbourhood graph is isomorphic to  $G$ . Taking the nodes of  $G$  to be the elements  $\bar{r}$ , we define a model over  $\text{RCP}(\mathbb{R}^n)$  as follows: if  $\mathfrak{G}$  maps a variable  $x$  to the set of elements  $R \subseteq \bar{r}$ , interpret  $x$  as  $\sum R$ . One can check that  $\varphi$  is true in this model.  $\square$

Turning next to tameness, Theorem 12 has already shown that  $\mathcal{RCC8c}$  and  $\mathcal{RCC8c}^\circ$  are not sensitive to the difference between  $\text{RC}(\mathbb{R}^n)$  and  $\text{RCP}(\mathbb{R}^n)$  for  $n \geq 3$ . By contrast:

**Theorem 14.**  $\text{Sat}(\mathcal{Bc}^\circ, \text{RCP}(\mathbb{R}^n)) \subsetneq \text{Sat}(\mathcal{Bc}^\circ, \text{RC}(\mathbb{R}^n))$  and  $\text{Sat}(\mathcal{Cc}^\circ, \text{RCP}(\mathbb{R}^n)) \subsetneq \text{Sat}(\mathcal{Cc}^\circ, \text{RC}(\mathbb{R}^n))$ , for all  $n \geq 3$ .

**Proof.** By cylindrication of Fig. 4 and Lemma 1.  $\square$

For  $n \geq 3$ , the question of whether  $\text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^n)) = \text{Sat}(\mathcal{L}, \text{RCP}(\mathbb{R}^n))$ , where  $\mathcal{L}$  is either  $\mathcal{Bc}$  or  $\mathcal{Cc}$ , is open.

Finally, we address the complexity of satisfiability. As well as settling the insensitivity of  $\mathcal{Bc}^\circ$  to dimension  $\geq 3$  in Euclidean spaces, Lemma 4 gives us some complexity-theoretic information. Using (Kontchakov et al. 2009, Theorems 5.3 and 5.19), we can show

**Lemma 5.** *The problem of determining whether a  $\mathcal{Bc}^\circ$ -formula has a graph-model is EXPTIME-complete. It is EXPTIME-hard to decide whether a  $\mathcal{Bc}^\circ$ -formula has a planar graph-model.*

**Theorem 15.**  $\text{Sat}(\mathcal{Bc}^\circ, \text{RCP}(\mathbb{R}^3)) = \text{Sat}(\mathcal{Bc}^\circ, \text{RCP}(\mathbb{R}^n))$ , for  $n > 3$  and the problem is EXPTIME-complete.  $\text{Sat}(\mathcal{Bc}^\circ, \text{RCP}(\mathbb{R}^2))$  is EXPTIME-hard.

**Proof.** Follows from Lemmas 4 and 5.  $\square$

This result is, however, in stark contrast to the following:

**Theorem 16.**  $\text{Sat}(\mathcal{Bc}^\circ, \text{RC})$  is NP-complete.

**Proof.** See <http://www.dcs.bbk.ac.uk/~roman>.  $\square$

As shown in (Kontchakov et al. 2009),  $\text{Sat}(\mathcal{L}, \text{RC}(\mathbb{R}^n))$  and  $\text{Sat}(\mathcal{L}, \text{RCP}(\mathbb{R}^n))$  are EXPTIME-hard, for  $\mathcal{L}$  any of  $\mathcal{Bc}$ ,  $\mathcal{Cc}$  or  $\mathcal{Cc}^\circ$  and  $n \geq 3$ . The upper complexity bounds for these problems are open.

## Conclusions

We investigated the six languages  $\mathcal{RCC8c}$ ,  $\mathcal{Bc}$ ,  $\mathcal{Cc}$ ,  $\mathcal{RCC8c}^\circ$ ,  $\mathcal{Bc}^\circ$  and  $\mathcal{Cc}^\circ$  obtained by extending  $\mathcal{RCC8}$  with the connectedness and interior connectedness predicates  $c$  and  $c^\circ$ , as well as the Boolean function-symbols  $+$ ,  $\cdot$  and  $-$ , paying particular regard to issues that arise when interpreting these languages over low-dimensional Euclidean spaces. We

showed that—in contrast to the less expressive  $\mathcal{RCC8}$ —the dimensionality of the space is important for all of our languages, and that, in addition, these languages exhibit varying patterns of sensitivity to tameness in different dimensions. Thus,  $\mathcal{RCC8c}$  and  $\mathcal{RCC8c}^\circ$  both distinguish between  $\text{RC}(\mathbb{R}^n)$  and  $\text{RCP}(\mathbb{R}^n)$  for  $n = 1$ , but do not for  $n \geq 2$ ;  $\mathcal{Bc}$  does for  $n = 2$ , but not for  $n = 1$ ;  $\mathcal{Bc}^\circ$  does for  $n = 2$ , but not for  $n = 1$  or  $n \geq 3$ ;  $\mathcal{Cc}$  does for  $n = 1$  and 2;  $\mathcal{Cc}^\circ$  does in all dimensions. We also obtained results on the complexity of reasoning in these logics. For example, the satisfiability problems for  $\mathcal{RCC8c}$  and  $\mathcal{RCC8c}^\circ$ , under all the interpretations considered here, are NP-complete, whereas the corresponding problems for  $\mathcal{Bc}$ ,  $\mathcal{Bc}^\circ$ ,  $\mathcal{Cc}$  and  $\mathcal{Cc}^\circ$  in  $\mathbb{R}^n$ , for  $n \geq 2$ , are generally EXPTIME-hard. (Two cases are still open). A matching EXPTIME upper bound was proved for  $\mathcal{Bc}^\circ$  over polyhedra in  $\mathbb{R}^n$ ,  $n \geq 3$ . The expressiveness and complexity problems that still remain open are indicated in Fig. 2.

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## References

- Bochnak, J.; Coste, M.; and Roy, M.-F. 1998. *Real Algebraic Geometry*. Springer.
- Davis, E.; Gotts, N.; and Cohn A. 1999. Constraint networks of topological relations and convexity. *Constraints* 4:241–280.
- Egenhofer, M., and Franzosa, R. 1991. Point-set topological spatial relations. *International J. of Geographical Information Systems* 5:161–174.
- Kontchakov, R.; Pratt-Hartmann, I.; Wolter, F.; and Zakharyashev, M. 2008. Topology, connectedness and modal logic. In *AiML* 7, 151–176. College Publications.
- Kontchakov, R.; Pratt-Hartmann, I.; Wolter, F.; and Zakharyashev, M. 2009. Spatial logics with connectedness predicate. Submitted; available at <http://www.dcs.bbk.ac.uk/~michael/connect-5.pdf>
- Newman, M. 1951. *Elements of the Topology of Plane Sets of Points*. 2nd edition. Cambridge.
- Pratt-Hartmann, I. 2007. First-order mereotopology. In Aiello, M.; Pratt-Hartmann, I.; and van Benthem, J., eds., *Handbook of Spatial Logics*. Springer. 13–97.
- Randell, D.; Cui, Z.; and Cohn, A. 1992. A spatial logic based on regions and connection. In *Proc. of KR*, 165–176.
- Renz, J., and Nebel, B. 1999. On the complexity of qualitative spatial reasoning. *Artificial Intelligence* 108:69–123.
- Renz, J. 1998. A canonical model of the region connection calculus. In *Proc. of KR*, 330–341.
- Schaefer, M.; Sedgwick, E.; and Štefankovič, D. 2003. Recognizing string graphs in NP. *J. of Computer and System Sciences* 67:365–380.
- Wolter, F., and Zakharyashev, M. 2000. Spatial reasoning in RCC-8 with Boolean region terms. In *Proc. of ECAI*, 244–248.

### Proof of Theorem 3

**Theorem 3.**  $Sat(\mathcal{B}c, RC(\mathbb{R})) = Sat(\mathcal{B}c, RCP(\mathbb{R}))$ .

**Proof.** Let a  $\mathcal{B}c$ -formula  $\varphi$  be given. We may assume without loss of generality that  $\varphi$  is of the form

$$(\rho = 0) \wedge \bigwedge_{1 \leq j \leq m} (\sigma_j \neq 0) \wedge \bigwedge_{1 \leq i \leq n} (c(\pi_i) \wedge (\pi_i \neq 0)) \wedge \bigwedge_{1 \leq k \leq p} \neg c(\tau_k),$$

since, given any  $\mathcal{B}c$ -formula  $\psi$ , we may easily guess such a  $\varphi$  and show in polynomial time that  $\varphi$  and  $\psi$  are satisfiable over the same domains.

We describe a non-deterministic procedure which, given a formula  $\varphi$  of the form above, terminates with either success or failure in time bounded by a polynomial function of  $|\varphi|$ . We show that if the procedure has a successful run, then  $\varphi$  is satisfiable over  $RCP(\mathbb{R})$  and if  $\varphi$  is satisfiable over  $RC(\mathbb{R})$  then the procedure has a successful run.

Let  $E = 2(m + n + 3p)$ . Denote by  $\Xi$  the set of regular closed intervals  $(-\infty, 0], [0, 1], \dots, [E-1, E], [E, +\infty)$  and by  $\Delta$  the set of integers in the interval  $[0, E]$ . In what follows we construct a function  $\lambda$ , which maps  $\Xi$  to the power set of the set of subterms of  $\varphi$ . An interval  $[a, b]$  with  $a, b \in \Delta$  is regular closed if  $b - a \geq 1$ . We start off with  $\lambda(I) = \emptyset$  for every  $I \in \Xi$ .

1. For every  $j$  ( $1 \leq j \leq m$ ), choose a regular closed interval  $[a, b]$ ,  $a, b \in \Delta$ , and add  $\sigma_j$  to  $\lambda(I)$ , for each  $I \in \Xi$  with  $I \subseteq [a, b]$ .
2. For every  $i$  ( $1 \leq i \leq n$ ), choose a regular closed interval  $[a, b]$ ,  $a, b \in \Delta$ , and add  $\pi_i$  to  $\lambda(I)$ , for each  $I \in \Xi$  with  $I \subseteq [a, b]$  and add  $-\pi_i$  to  $\lambda(I)$ , for each  $I \in \Xi$  with  $I \subseteq [0, a] \cup [b, E]$ ; if  $a > 0$  add  $-\pi_i$  to  $\lambda((-\infty, 0])$ , otherwise, add either  $\pi_i$  or  $-\pi_i$  to  $\lambda((-\infty, 0])$ ; if  $b < E$  add  $-\pi_i$  to  $\lambda([E, +\infty))$ , otherwise, add either  $\pi_i$  or  $-\pi_i$  to  $\lambda([E, +\infty))$ ;
3. For every  $k$  ( $1 \leq k \leq p$ ), choose a pair of regular closed intervals  $[a_1, b_1]$  and  $[a_2, b_2]$  with  $a_1, b_1, a_2, b_2 \in \Delta$  such that  $a_2 - b_1 \geq 1$  and add  $\tau_k$  to  $\lambda(I)$ , for each  $I \in \Xi$  with  $I \subseteq [a_1, b_1] \cup [a_2, b_2]$ , and add  $-\tau_k$  to  $\lambda(I)$ , for each  $I \in \Xi$  with  $I \subseteq [b_1, a_2]$  (the latter is possible for at least one  $I \in \Xi$ ).
4. For every  $I \in \Xi$ , guess a  $\mathcal{B}c$ -term  $\xi_I$  of the form  $\prod_{1 \leq i \leq \ell} \pm r_i$ , where  $r_1, \dots, r_\ell$  are all the variables of  $\varphi$ , and fail if  $\xi_I \leq \rho$  or  $\xi_I \leq -\prod \lambda(I)$ . Succeed otherwise.

Suppose the procedure has a successful termination. Then we define an interpretation over  $RCP(\mathbb{R})$  by setting  $r$  to be the union of all the intervals  $I \in \Xi$  with  $\xi_I \leq r$ . Step 4 ensures that  $\rho = 0$  and that, for every  $I \in \Xi$ ,  $I \subseteq \prod \lambda(I)$ . Hence: Step 1 ensures that, for every  $j$  ( $1 \leq j \leq m$ ),  $\sigma_j$  is non-empty; Step 2 ensures that, for every  $i$  ( $1 \leq i \leq n$ ),  $\pi_i$  is connected and non-empty (note that  $\pi_i$  need not be bounded); Step 3 ensures that, for every  $k$  ( $1 \leq k \leq p$ ),  $\tau_j$  is not connected. Thus, the interpretation is as required.

Conversely, if  $\varphi$  is true in a model over  $RC(\mathbb{R})$ , it is easy to see how the intervals and the terms  $\xi_I$  may be selected so as to ensure successful termination of the procedure.

This shows that the both problems are in NP.  $\square$

### Proof of Theorem 16

**Theorem 16.**  $Sat(\mathcal{B}c^\circ, RC)$  is NP-complete.

**Proof.** Let  $\varphi$  be a  $\mathcal{B}c^\circ$ -formula of the form

$$(\rho = 0) \wedge \bigwedge_{1 \leq j \leq m} (\sigma_j \neq 0) \wedge \bigwedge_{1 \leq i \leq n} (c^\circ(\pi_i) \wedge (\pi_i \neq 0)) \wedge \bigwedge_{1 \leq k \leq p} \neg c^\circ(\tau_k).$$

We show that (i) if  $\varphi$  is satisfiable over  $RC$  then it is satisfiable in a model  $\mathfrak{A}$  over an Aleksandrov space with at most  $2 \cdot 2^\ell + n$  points, where  $\ell$  is the number of variables in  $\varphi$ ; and (ii) how one can select a submodel  $\mathfrak{B}$  of  $\mathfrak{A}$ , which contains at most  $m + 2n + 2p + n \cdot p$  points and satisfies  $\varphi$ . Thus, we establish a polynomial finite model property for  $\mathcal{B}c^\circ$  over  $RC$ , which gives us NP membership (NP-hardness is trivial).

(i) By a *type*  $\xi$  for  $\varphi$  we mean any term of the form  $\prod_{1 \leq i \leq \ell} \pm r_i$ , where  $r_1, \dots, r_\ell$  are all the variables of  $\varphi$ . Let  $W_0$  contain a pair of distinct points  $x_\xi$  and  $x'_\xi$ , for each type  $\xi$  inconsistent with  $\rho$ , i.e.,  $\xi \not\models \rho$  (we need two points to make some regions disconnected). Let  $W_1$  contain a distinct point  $z_{\pi_i}$  for each positive  $c^\circ(\pi_i)$  in  $\varphi$ . Let  $R$  be the reflexive closure of

$$\{(z_{\pi_i}, x_\xi), (z_{\pi_i}, x'_\xi) \mid \xi \models \pi_i\}.$$

Define a valuation  $\cdot^{\mathfrak{A}}$  by taking

$$r_i^{\mathfrak{A}} = \{x_\xi, x'_\xi \mid \xi \models r_i\} \cup \{z_{\pi_i} \mid \pi_i \models r_i\}.$$

It is readily seen that  $r_i^{\mathfrak{A}}$  is a regular closed set in the Aleksandrov topology induced by  $(W, R)$ .

We show now that  $\varphi$  is true in  $\mathfrak{A}$ . It is trivial for the first three conjuncts of  $\varphi$ . So, it remains to show that  $\mathfrak{A} \models c^\circ(\tau_k)$ , for each  $k$  ( $1 \leq k \leq p$ ). Suppose to the contrary that  $\tau_k$  is interior-connected in  $\mathfrak{A}$ . Then there is a sequence  $x_1, z_1, x_2, z_2, \dots, x_s$  such that  $z_i R x_i, z_i R x_{i+1}$  and the  $x_i$  and the  $z_i$  are all the points in  $\tau_k^{\mathfrak{A}}$ . Then, by the definition of  $\mathfrak{A}$ , there are  $\pi_{i_1}, \dots, \pi_{i_{s-1}}$  such that  $x_j, z_j, x_{j+1} \in \pi_{i_j}^{\mathfrak{A}} \cap \tau_k^{\mathfrak{A}}$ , for all  $j$  ( $1 \leq j < s$ ). It follows then that  $z_{i_j} \in (\pi_{i_j}^\circ)^{\mathfrak{A}} \cap (\tau_k^\circ)^{\mathfrak{A}}$  and, as  $z_{i_j}$  is the  $R$ -predecessor of all points in  $\pi_{i_j}^{\mathfrak{A}}$ , we obtain  $\pi_{i_j}^{\mathfrak{A}} \leq \tau_k^{\mathfrak{A}}$ . So, we have  $(\sum_{1 \leq j < n} \pi_{i_j})^{\mathfrak{A}} \leq \tau_k^{\mathfrak{A}}$  and  $(\sum_{1 \leq j < n} \pi_{i_j})^{\mathfrak{A}}$  is interior-connected. On the other hand, as the path contains all points in  $\tau_k^{\mathfrak{A}}$ , we obtain  $\tau_k^{\mathfrak{A}} \leq (\sum_{1 \leq j < n} \pi_{i_j})^{\mathfrak{A}}$ . Therefore,  $\tau_k^{\mathfrak{A}}$  coincides with the sum of the  $\pi_{i_j}$  and so, is interior-connected contrary to  $\varphi$  being satisfiable.

(ii) We select the following points:

- for each  $j$  ( $1 \leq j \leq m$ ), pick  $x \in W_0 \cap \sigma_j^{\mathfrak{A}}$ ;
- for each  $i$  ( $1 \leq i \leq n$ ), pick  $x \in W_0 \cap \pi_i^{\mathfrak{A}}$  and  $z_{\pi_i} \in W_1$ ;
- for each  $k$  ( $1 \leq k \leq p$ ), pick 2 points  $x_{\tau_k}, x'_{\tau_k} \in W_0 \cap \tau_k^{\mathfrak{A}}$  form two distinct components of  $\tau_k^{\mathfrak{A}}$  and up to  $n$  points  $y_{\overline{\tau_k}, \pi_i} \in W_0 \cap \pi_i^{\mathfrak{A}} \cap (-\tau_k)^{\mathfrak{A}}$ , for  $1 \leq i \leq n$  (the point is picked if the set is not empty).

As  $\varphi$  is true in  $\mathfrak{A}$ , all the points mentioned above do necessarily exist (apart from the  $y_{\overline{\tau_k}, \pi_i}$ , some of which may not exist). Let  $V$  be the set of all these points and  $\mathfrak{B}$  the restriction of  $\mathfrak{A}$  onto  $V$ . We claim that  $\varphi$  is true in  $\mathfrak{B}$ . Indeed, this

is clearly the case for the first three conjuncts of  $\varphi$ . So, it remains to show that  $\mathfrak{B} \not\models c^\circ(\tau_k)$ , for each  $k$  ( $1 \leq k \leq p$ ). This fact follows from the observation that  $z_{\pi_i} \in (\tau_k^\circ)^\mathfrak{A}$  if and only if  $z_{\pi_i} \in (\tau_k^\circ)^\mathfrak{B}$ , for each  $i$  ( $1 \leq i \leq n$ ) and therefore,  $\tau_k$  is interior-connected in  $\mathfrak{B}$  if and only if it is interior-connected in  $\mathfrak{A}$ .

This completes the proof.  $\square$

### Proof of Theorem 8

**Theorem 8.**  $Sat(\mathcal{C}c, RCP(\mathbb{R}^2)) \subsetneq Sat(\mathcal{C}c, RC(\mathbb{R}^2))$ . In fact,  $Sat(\mathcal{B}c, RCP(\mathbb{R}^2)) \subsetneq Sat(\mathcal{B}c, RC(\mathbb{R}^2))$ .

**Proof.** Let  $V$  be the set of variables  $\{v, h, s, t, t_0, r_0, r_1, r_2, r_3, r_4, r_5\}$  and, for any  $x \in V$ , let  $\hat{x}$  be a fresh variable. For all  $x \in V$ , the region  $\hat{x}$  is a slightly ‘enlarged’ version of  $x$ , with  $x$  lying in the interior of  $\hat{x}$ .

The formula  $\varphi$  we are about to construct contains conjuncts  $-C(x, -\hat{x})$ , for  $x \in V$ , which ensure that  $\hat{x}$  represents a region whose interior contains  $x$ . The other conjuncts of  $\varphi$  will be presented as they are required in the proof. The claim that  $\varphi$  is not satisfiable over  $RCP(\mathbb{R}^2)$  is established by showing that any satisfying assignment in  $RC(\mathbb{R}^2)$  has the property that the interiors of the regions  $r_i$  ( $0 \leq i < 6$ ) have infinitely many components. Suppose we have such a satisfying assignment.

$$(v \neq 0) \wedge (h \neq 0) \quad (5)$$

$$c(v) \wedge c(h) \quad (6)$$

$$c(h + t + s + v) \quad (7)$$

$$-C(v, h) \quad (8)$$

$$-C(t, v) \quad (9)$$

$$-C(s, \hat{h}) \quad (10)$$

$$c((t + t_0) \cdot (-\hat{h}) + v) \quad (11)$$

$$-C(t_0, s) \quad (12)$$

**First stage:** By (5), choose a point in  $v$  and a point in  $h$  and, by (7), connect the first to the second by an arc  $\alpha_0^*$  in  $v + s + t + h$ . Let  $q_0$  be the first point of  $\alpha_0^*$  in  $h$ . By (8),  $q_0 \notin v$  and  $q_0$  is not the first point of  $\alpha_0^*$ . Let  $p_0$  be the last point of  $\alpha_0^*$  in  $v$  and strictly before  $q_0$ . Let  $\alpha_0$  be the segment of  $\alpha_0^*$  from  $p_0$  to  $q_0$ . Hence, no interior point of  $\alpha_0$  lies in  $v$  or  $h$ , and so all points of  $\alpha_0$  lie in  $s + t$ .

By (9), some initial segment of  $\alpha_0$  lies in  $s$ , whence, by (10), there exists a point  $p_1^*$  on  $\alpha_0$  such that  $p_1^* \notin s$  and  $p_1^* \notin \hat{h}$ . It follows that  $p_1^* \in t \cdot (-\hat{h})$ . By (11), draw an arc  $\alpha_1^*$  from  $p_1^*$  to  $p_0 \in v$ , and lying in  $(t + t_0) \cdot (-\hat{h}) + v$ . Since  $p_1^* \notin v$ , let  $q_1$  be the first point of  $v$  on  $\alpha_1^*$  after  $p_1^*$ .

This arc has a last point of contact with  $\alpha_0$  strictly before  $q_1$ . For, all points of  $\alpha_1^*$  before  $q_1$  are in  $t + t_0$ , whence, by (9), some segment of  $\alpha_1^*$  leading to  $q_1 \in v$  must lie entirely in  $t_0$ . But then, by (12), this segment does not touch some initial segment of  $\alpha_0$ . Therefore, let  $p_1$  be the last point of  $\alpha_1^*$  before  $q_1$  lying on  $\alpha_0$ . Let  $\alpha_1$  be the segment of  $\alpha_1^*$  between  $p_1$  and  $q_1$ . Thus, no point of  $\alpha_1$  lies in  $h$ ,

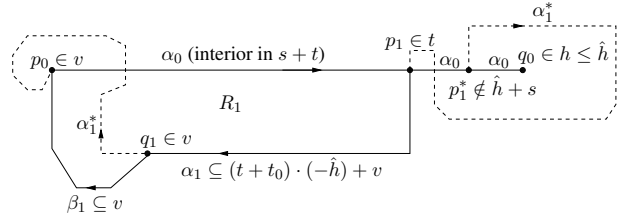


Figure 9: The first stage: the arc  $\alpha_1$  (solid line) is a sub-arc of  $\alpha_1^*$ . The Jordan curve (thick lines) is denoted by  $\gamma_1$ ;  $R_1$  is the open set it bounds not containing  $q_0$ .

and the only point of  $\alpha_1$  lying in  $v$  is  $q_1$ . By (6), let  $\beta_1$  be an arc in  $v$  from  $q_1$  to  $p_0$ . Thus,  $\beta_1 \cap \alpha_1 = \{q_1\}$  and  $\beta_1 \cap \alpha_0 = \{p_0\}$ . The situation is shown in Fig. 9. Let  $\gamma_1$  be the Jordan curve formed by  $\alpha_1$ ,  $\beta_1$  and the segment of  $\alpha_0$  lying between  $p_0$  and  $p_1$ , shown in thick lines. We denote the open set bounded by  $\gamma_1$  and not containing the point  $q_0$  by  $R_1$ . By drawing the configuration on the closed plane (with all Jordan arcs avoiding the point at infinity), we may without loss of generality regard  $R_1$  as the ‘inside’ of  $\gamma_1$ .

$$-C(\hat{t}, \hat{v}) \quad (13)$$

$$c((t_0 + r_1) \cdot (-\hat{t}) \cdot (-\hat{v}) + h) \quad (14)$$

$$-C(t_0 + r_1, s) \quad (15)$$

**Second stage:** Since  $p_1 \in t$  and  $q_1 \in v$ , by (13), there exists a point  $p_2^*$  on  $\alpha_1$  such that  $p_2^* \notin t$  and  $p_2^* \notin \hat{v}$ . It follows that  $p_2^* \in t_0 \cdot (-\hat{t}) \cdot (-\hat{v})$ . By (14), let  $\alpha_2^*$  be an arc from  $p_2^*$  to  $q_0$  lying in  $(t_0 + r_1) \cdot (-\hat{t}) \cdot (-\hat{v}) + h$ . Let  $p_2$  be the last point of  $\alpha_2^*$  lying on  $\alpha_1$ ; hence,  $p_2 \in t_0$ . Also  $p_2 \notin h$ , since it lies on  $\alpha_1$ . Let  $q_2$  be the first point of  $\alpha_2^*$  after  $p_2$  lying in  $h$ ; and let  $\alpha_2$  be the segment of  $\alpha_2^*$  from  $p_2$  to  $q_2$ . Thus, no point of  $\alpha_2$  other than  $q_2$  lies in  $h$ , and certainly, no point of  $\alpha_2$  lies in  $v$ . By (6), let  $\beta_2$  be an arc from  $q_2$  to  $q_0$  lying entirely in  $h$ . Notice that  $\beta_2$  cannot intersect  $\alpha_0$ ,  $\alpha_1$  or  $\alpha_2$  except at the endpoints  $q_2$  and  $q_0$ , because no other points of these arcs lie in  $h$ . We further claim that  $\alpha_2$  cannot enter region  $R_1$ , for it is impossible that any point in  $(\alpha_2 \cup \beta_2) \setminus \{p_2\}$  lies on  $\gamma_1$ . To see this, note that: (i) by construction, no point of  $\alpha_2$  apart from  $p_2$  lies on  $\alpha_1$ , (ii) by (15) and the fact that  $\alpha_2 \subseteq (t_0 + r_1) \cdot (-\hat{t})$ , no point on  $\alpha_2$  apart from  $q_2$  can lie in  $s + t \supseteq \alpha_0$ ; (iii) no point of  $\alpha_2$  lies in  $v \supseteq \beta_1$ ; (iv) no point of  $\beta_2$  lies on  $\gamma_1$ , since no point on  $\gamma_1$  lies in  $h$ . It follows that  $\alpha_2$  and  $\beta_2$  lie on the ‘outside’ of  $\gamma_1$  (since  $q_0$  does). The situation is shown in Fig. 10. Thus,  $\alpha_0$ ,  $\alpha_2$  and  $\beta_2$  divide the outside of  $\gamma_1$  into two residual domains; denote that residual domain which does not contain  $p_0$  by  $R_2$ . In addition, let  $\gamma_2$  be the Jordan curve formed by  $\alpha_2$ ,  $\beta_2$ ,  $\alpha_0$ ,  $\beta_1$  and  $\alpha_1$  from  $q_1$  to  $p_2$  (shown in thick lines). By drawing the configuration on the closed plane (with all Jordan arcs avoiding the point at infinity), we may without loss of generality regard  $R_2$  as lying ‘inside’  $\gamma_2$ . Note that  $S_2 = (R_1 \cup R_2)^{\circ}$  is the bounded open set having  $\gamma_2$  as its boundary.

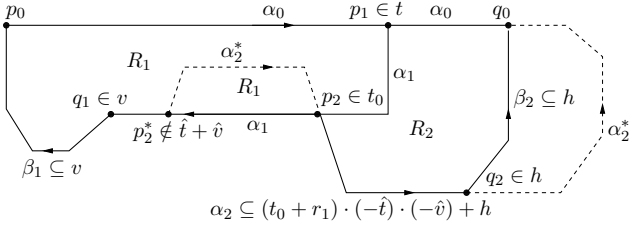


Figure 10: The second stage: the arc  $\alpha_2$  (solid line) is a sub-arc of  $\alpha_2^*$ . The Jordan curve (thick lines) is denoted by  $\gamma_2$ . The region  $S_2 = (R_1 \cup R_2)^{\circ-}$  is the bounded open set whose boundary is  $\gamma_2$ .

$$-C(\hat{t}_0, \hat{h}) \quad (16)$$

$$c((r_1 + r_2) \cdot (-\hat{t}_0) \cdot (-\hat{h}) + v) \quad (17)$$

$$-C(r_1 + r_2, t) \quad (18)$$

$$-C(r_1 + r_2, s + t) \quad (19)$$

**Third stage:** Since  $p_2 \in t_0$  and  $q_2 \in h$ , by (16), there exists a point  $p_3^*$  on  $\alpha_2$  such that  $p_3^* \notin \hat{t}_0$  and  $p_3^* \notin \hat{h}$ . It follows that  $p_3^* \in r_1 \cdot (-\hat{t}_0) \cdot (-\hat{h})$ . By (17), let  $\alpha_3^*$  be an arc from  $p_3^*$  to  $q_1$  lying in  $(r_1 + r_2) \cdot (-\hat{t}_0) \cdot (-\hat{h}) + v$ . Let  $p_3$  be the last point of  $\alpha_3^*$  lying on  $\alpha_2$ ; hence,  $p_3$  lies in  $r_1$ . Let  $q_3$  be the first point of  $\alpha_3^*$  after  $p_3$  lying in  $v$ ; and let  $\alpha_3$  be the segment of  $\alpha_3^*$  from  $p_3$  to  $q_3$ . Thus, no point of  $\alpha_3$  other than  $q_3$  lies in  $v$ , and certainly, no point of  $\alpha_3$  lies in  $h$ . By (6), let  $\beta_3^*$  be an arc from  $q_3$  to  $q_1$  lying entirely in  $v$ . Let  $q_3^*$  be the first point of  $\beta_3^*$  lying on  $\gamma_2 \cap v$  (i.e. lying on  $\beta_1$ ). Let  $\beta_3$  be the segment of  $\beta_3^*$  between  $q_3$  and  $q_3^*$ . The situation is shown in Fig. 11.

We need to show that the way in which  $\alpha_3$  and  $\beta_3$  have been drawn is sufficiently general. Recall that that  $R_1$  is bounded by the Jordan curve  $\gamma_1$ , and that  $S_2 = (R_1 \cup R_2)^{\circ-}$  is bounded by the Jordan curve  $\gamma_2$ . We first establish that  $\alpha_3$  cannot enter  $R_1$ . For: (i) all points of  $\alpha_1$  are in  $t_0 + t$ , and so by (18) cannot coincide with any point in  $(r_1 + r_2) \cdot (-\hat{t}_0)$ ; (ii) all points of  $\beta_3$  are in  $v$ , and  $q_1$  is the only point of  $\alpha_1$  in  $v$ ; (iii)  $\alpha_3 \setminus \{q_3\}$  has no points in  $v$ , and hence none on  $\beta_1$ ; (iv) by construction,  $\beta_3$  stops as soon as it touches  $\gamma_2$ ; (v) by (19), no points of  $(r_1 + r_2)$  can coincide with any points of  $s + t \geq \alpha_0$ ; (vi) no points of  $\alpha_0 \setminus \{p_0\}$  lie in  $v$  and hence none lie on  $\beta_3$ . We next establish that  $\alpha_3$  cannot enter  $R_2$ , either. For, (i) by construction, no point of  $\alpha_3$  apart from the first, can intersect  $\alpha_2$ , (ii) by (19) and (18), no points of  $(r_1 + r_2)$  can coincide with any points of  $s + t \geq \alpha_0$ ; (iii) no point of  $\alpha_3$  lies in  $h$ , and hence none lies on  $\beta_2$ . Thus,  $\alpha_3$  cannot enter  $S_2$ ; and  $\alpha_3$  and  $\beta_3$  divide the exterior of  $\gamma_2$  into two regions, forming a larger Jordan curve  $\gamma_3$ , shown by the thick lines in Fig. 11. By inspection, exactly one of these two regions will contain points in  $v$ . Drawing the configuration on the closed plane as before, we may without loss of generality regard the region containing points of  $v$  as the ‘outside’ of  $\gamma_3$ . The region  $S_3 = (S_2 \cup R_3)^{\circ-}$  is thus the interior of  $\gamma_3$ .

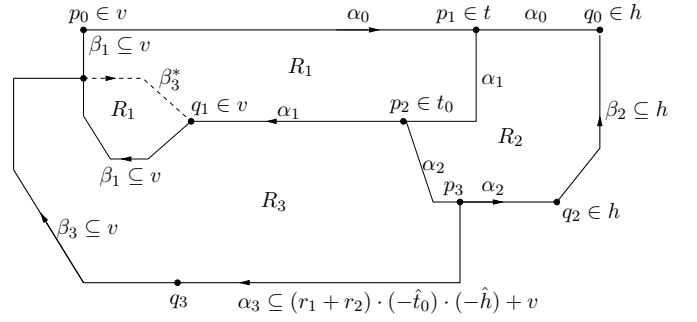


Figure 11: The third stage: the point  $p_3^*$  and arc  $\alpha_3^*$  are not shown, for clarity; the arc  $\beta_3$  (solid line) is a sub-arc of  $\beta_3^*$ . The Jordan curve (thick lines) is denoted by  $\gamma_3$ . The region  $S_3 = (S_2 \cup R_3)^{\circ-}$  is the bounded open set whose boundary is  $\gamma_3$ .

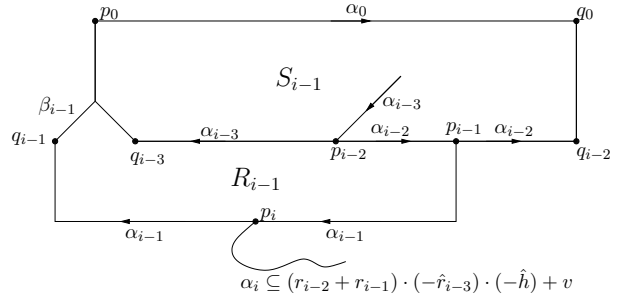


Figure 12: The general case: the arc  $\alpha_i$  ( $i$  even).

$$-C(\hat{r}_{i-3}, \hat{v}) \quad (i \text{ even}) \quad (20)$$

$$-C(\hat{r}_{i-3}, \hat{h}) \quad (i \text{ odd}) \quad (21)$$

$$c((r_{i-2} + r_{i-1}) \cdot (-\hat{r}_{i-3}) \cdot (-\hat{v}) + h) \quad (i \text{ even}) \quad (22)$$

$$c((r_{i-2} + r_{i-1}) \cdot (-\hat{r}_{i-3}) \cdot (-\hat{h}) + v) \quad (i \text{ odd}) \quad (23)$$

$$-C(r_{i-2} + r_{i-1}, r_{i-4}) \quad (24)$$

$$-C(r_{i-2} + r_{i-1}, r_{i-4} + r_{i-5}) \quad (25)$$

$$-C(r_{i-2} + r_{i-1}, t + t_0) \quad (i = 4) \quad (26)$$

$$-C(r_{i-2} + r_{i-1}, s + t) \quad (27)$$

**General stage  $i$ :** After this point, the process repeats itself through infinitely many stages; at each new stage, the numerical indices in the variables  $r_i$  are incremented (modulo 6) and  $h$  and  $v$  are transposed. The general situation (for  $i$  even), is illustrated in Fig. 12. In this stage,  $\alpha_i$  and  $\beta_i$  are about to be constructed. The arc  $\alpha_i \subseteq (r_{i-1} + r_i) \cdot (-\hat{r}_{i-2}) \cdot (-\hat{v}) + h$  will run from a point  $p_i$  on  $\alpha_{i-1}$  to a point  $q_i$  in  $h$  (the starting point  $p_i^*$  exists by (20) and the arc by (22)); and the arc  $\beta_i \subseteq h$ , will run from  $q_i$  to some point on  $\gamma_{i-1} \cap h = \gamma_{i-2} \cap h$ . The Jordan curve  $\gamma_{i-2}$ , enclosing  $S_{i-2}$ , is shown in thick lines.

The key observation is that neither  $\alpha_i$  nor  $\beta_i$  can enter  $S_{i-1} = (S_{i-2} \cup R_{i-1})^{\circ-}$ . We first show that  $\alpha_i$  cannot enter  $S_{i-2}$ . To see this, we note that: (i) since  $\alpha_i \subseteq (r_{i-2} + r_{i-1}) \cdot (-\hat{r}_{i-3})$  and  $\alpha_{i-2} \subseteq (r_{i-4} + r_{i-3})$ , by (24),  $\alpha_i \cap \alpha_{i-2} = \emptyset$ ; (ii) since  $\alpha_i \subseteq (r_{i-1} + r_{i-2})$  and  $\alpha_{i-3} \subseteq (r_{i-5} +$

$r_{i-4}$ ), by (25),  $\alpha_i \cap \alpha_{i-3} = \emptyset$ ; (iii) since  $\alpha_i \subseteq -\hat{v}$ ,  $\alpha_i$  does not intersect the segment of  $\gamma_{i-2}$  from  $q_{i-3}$  (clockwise) to  $p_0$ ; (iv) since  $\alpha_i \subseteq (r_{i-2} + r_{i-1})$  and  $\alpha_0 \subseteq s + t$ , by (27),  $\alpha_{i+1} \cap \alpha_0 = \emptyset$ ; (v) only the end-point  $q_i$  of  $\alpha_i$  is in  $h$  and so  $\alpha_i$  cannot intersect the segment of  $\gamma_{i-2}$  from  $q_0$  (clockwise) to  $q_{i-2}$ . We note here that, for  $i = 4$ , we use (26) instead of (24) and (25) to show (i) and (ii).

We next show that  $\alpha_i$  cannot enter  $R_{i-1}$ . To see this, we note that: (i) by construction,  $\alpha_i \cap \alpha_{i-1} = \{p_i\}$ ; (ii)  $\alpha_i$  cannot enter  $S_{i-2}$ , as we have just argued, and  $\beta_i \subseteq h$  certainly cannot cross the boundary between  $R_{i-1}$  and  $S_{i-2}$ , none of whose points are in  $h$  by construction of  $\alpha_{i-2}$ ,  $\alpha_{i-3}$ , and by (8); (iii)  $\alpha_i \subseteq -\hat{v}$  and so cannot intersect the segment of the boundary of  $R_{i-1}$  from  $q_{i-1}$  (clockwise) to the point where  $\beta_{i-1}$  reaches  $\gamma_{i-2}$ ;  $\beta_i$  cannot intersect this segment either, by (8). But this means that it is impossible for  $\alpha_i$  and  $\beta_i$  to connect a point of  $R_{i-1}$  to a point of  $\gamma_{i-1} \cap h$ ; hence  $\alpha_i$  cannot enter  $R_{i-1}$  at all, and so must be drawn (in the closed plane) as shown.

To complete the proof, observe that, for all  $k > 0$ ,  $p_{6k+2} \in r_0$ ; and since  $r_0$  is regular closed, there exist points in the interior of  $r_0$  arbitrarily close to  $p_{6k+2}$ . But  $p_{6k+2}$  and  $p_{6(k+1)+2}$  are separated by (for example)  $\gamma_{6k+6}$ , which lies entirely in the set  $s + t + h + v + r_3 + r_4 + r_5$ , and hence contains no interior points of  $r_0$ . Therefore,  $r_0^\circ$  has infinitely many components, as required.

For the second statement of the theorem, we replace every literal  $\neg C(x, y)$  in  $\varphi$  with a conjunction

$$(x \subseteq x^\dagger) \wedge (y \subseteq y^\dagger) \wedge c(x^\dagger) \wedge c(y^\dagger) \wedge \neg c(x^\dagger + y^\dagger),$$

where the variables  $x^\dagger$  and  $y^\dagger$  are chosen afresh for each replaced literal. Let the resulting  $\mathcal{B}c$ -formula be  $\psi$ . Trivially,  $\text{RCP}(\mathbb{R}^2) \models \psi \rightarrow \varphi$ , so that  $\psi$  is not satisfiable over  $\text{RCP}(\mathbb{R}^2)$ . That  $\psi$  is satisfiable over  $\text{RC}(\mathbb{R}^2)$  is almost immediate by inspection of Fig. 8.  $\square$

### Proof of Theorem 6

**Theorem 6.** *If an  $\text{RCC8}c$ - or  $\text{RCC8}c^\circ$ -formula is satisfiable over  $\text{RC}(\mathbb{R}^2)$ , then it can be satisfied over the frame of bounded regular closed polygons. In consequence:*

$$\text{Sat}(\text{RCC8}c, \text{RC}(\mathbb{R}^2)) = \text{Sat}(\text{RCC8}c, \text{RCP}(\mathbb{R}^2)),$$

$$\text{Sat}(\text{RCC8}c^\circ, \text{RC}(\mathbb{R}^2)) = \text{Sat}(\text{RCC8}c^\circ, \text{RCP}(\mathbb{R}^2)).$$

**Proof.** For the first statement, it suffices to construct, for any tuple  $r_1, \dots, r_n$  in  $\text{RC}(\mathbb{R}^2)$ , a corresponding tuple  $p_1, \dots, p_n$  in  $\text{RCP}(\mathbb{R}^2)$  satisfying exactly the same atomic  $\text{RCC8}c$ -formulas. We may assume that the  $r_i$  are distinct and non-empty. By reordering the  $r_i$  if necessary, we can ensure that  $r_i \subseteq r_j$  implies  $i \leq j$ . For all  $i, j$  ( $1 \leq i < j \leq n$ ), let  $R_{ij} \in \{\text{DC}, \text{EC}, \text{PO}, \text{TPP}, \text{NTPP}\}$  be the unique relation such that  $R_{ij}(r_i, r_j)$ .

**Step 1:** We construct regular closed sets  $r_1^+, \dots, r_n^+$  such that, for all  $j$  ( $1 \leq j \leq n$ ),  $r_j \subseteq (r_j^+)^\circ$ ,

$$\text{if } r_j \text{ is connected then } (r_j^+)^\circ \text{ is connected,} \quad (28)$$

$$\text{if } r_j \cap r_{j'} = \emptyset \text{ then } r_j^+ \cap r_{j'}^+ = \emptyset, \quad \text{for all } j, j'. \quad (29)$$

**Lemma 6.** *There are  $r_1^+, \dots, r_n^+$  in  $\text{RC}(\mathbb{R}^2)$  with (28)–(29).*

**Proof.** By the normality of  $\mathbb{R}^2$ , let  $s_1, \dots, s_n$  be closed sets such that  $r_j \subseteq s_j^\circ$ , for all  $j \leq n$ , and if  $r_j \cap r_{j'} = \emptyset$  then  $s_j \cap s_{j'} = \emptyset$ , for all  $j, j' \leq n$ . Fix any  $r_j$ . For all  $u \in r_j$ , let  $d_u$  be a connected and regular closed subset of  $s_j$  with  $u \in d_u^\circ$  (which is possible because  $\mathbb{R}^2$  is locally connected). Let  $r_j^+ = \sum_{u \in r_j} d_u$ . By construction,  $r_j \subseteq (r_j^+)^\circ$  and (29). To show (28), consider  $t_j = \bigcup_{u \in r_j} d_u^\circ$ . Since  $\bigcup_{u \in r_j} (d_u^\circ \cup r_j) = t_j$ , the set  $t_j$  is connected whenever  $r_j$  is. Clearly,  $t_j \subseteq (r_j^+)^\circ$ . On the other hand,  $t_j^-$  is regular closed and  $t_j^- \supseteq \sum_{u \in r_j} d_u^{\circ-} = r_j^+$ . Thus,  $t_j \subseteq (r_j^+)^\circ \subseteq r_j^+ \subseteq t_j^-$ , which means that if  $r_j$  is connected then  $(r_j^+)^\circ$  is sandwiched between a connected set  $t_j$  and its closure  $t_j^-$ , and hence is itself connected; cf. (28).  $\square$

**Step 2:** For all  $i, j$  ( $1 \leq i < j \leq n$ ), pick points  $o, o', o''$  satisfying the conditions:

- if  $R_{ij} = \text{EC}$ , then  $o \in \delta r_i \cap \delta r_j$ ;
- if  $R_{ij} = \text{PO}$ , then  $o \in r_i^\circ \cap r_j^\circ$ ,  $o' \in r_i^\circ \setminus r_j$ ,  $o'' \in r_j^\circ \setminus r_i$ ;
- if  $R_{ij} = \text{TPP}$ , then  $o \in \delta r_i \cap \delta r_j$  and  $o' \in r_j^\circ \setminus r_i$ ;
- if  $R_{ij} = \text{NTPP}$ , then  $o \in r_j^\circ \setminus r_i$ .

And for all  $j$  ( $1 \leq j \leq n$ ), pick points  $o, o'$  satisfying the condition that, if  $r_j$  is not connected, then  $o$  and  $o'$  lie in different components of  $r_j$ . Enumerate the chosen (distinct) points as  $o_1, \dots, o_m$ : we call them *witness points*. We can draw disjoint closed disks  $d_1, \dots, d_m$ , centred on the respective witness points  $o_k$ , such that, for all  $j \leq n$  and  $k \leq m$ :

$$\text{if } o_k \in r_j^\circ \text{ then } d_k \subseteq r_j^\circ, \quad (30)$$

$$\text{if } o_k \in (r_j^+)^\circ \text{ then } d_k \subseteq (r_j^+)^\circ, \quad (31)$$

$$\text{if } (r_j^+)^\circ \text{ is connected then so is } (r_j^+)^\circ \setminus d, \quad (32)$$

where

$$d = \bigcup_{k=1}^m d_k.$$

This can be done because, if  $s$  is a connected, open subset of  $\mathbb{R}^2$  and  $u \in s$ , then there exists a closed disc  $d$  such that  $u \in d \subseteq s$  and  $s \setminus d$  is connected.

**Step 3:** We now begin the construction of the  $p_1, \dots, p_n$ . First, for each set  $r_j$  and each witness point  $o_k$ , we select a polygon  $w_{k,j}$  such that

$$w_{k,j} \subseteq d_k. \quad (33)$$

We refer to the  $w_{k,j}$  as *wedges*: for each  $j \leq n$ , and each  $k \leq m$  we will make  $w_{k,j}$  part of  $p_j$ . Wedges are selected as follows.

- (i) If  $o_k \in \delta r_j$ , pick a point  $q_{k,j} \in \delta d_k \subseteq (r_j^+)^\circ$  and let  $w_{k,j}$  be a lozenge within  $d_k$  such that  $o_k, q_{k,j} \in \delta w_{k,j}$ ; see Fig. 6a. We may pick the  $q_{k,j}$  to be distinct, and construct the  $w_{k,j}$  so that no two such  $w_{k,j}$  have intersecting interiors.

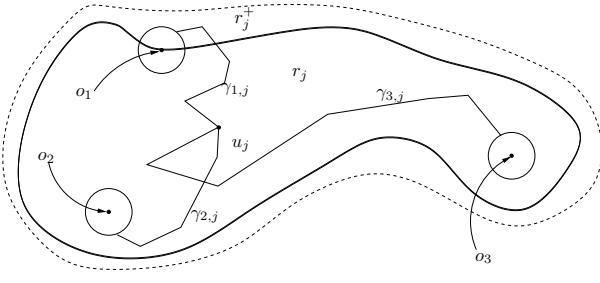


Figure 13: Disposition of the various arcs involving a particular connected element  $r_j$ . The outline of  $r_j$  is indicated with thickened lines, and its including region  $r_j^+$  with a dotted line. In this example,  $r_j$  involves three witness points:  $o_1$  (lying on  $\delta r_j$ ) and  $o_2, o_3$  (lying in  $r_j^\circ$ ). Notice that the arcs  $\gamma_{k,j}$  are contained within the larger set  $r_j^+$ , and not necessarily within  $r_j$ .

(ii) If  $o_k \in r_j^\circ$ , pick a point  $q_{k,j} \in \delta d_k \subseteq r_j^\circ$  and let  $w_{k,j}$  be a lozenge within  $d_k$  such that  $o_k \in (w_{k,j})^\circ$ ,  $q_{k,j} \in \delta w_{k,j}$ . Again, we may pick the  $q_{k,j}$  to be distinct from each other and from the  $q_{k,j}$  selected in (i); see Fig. 6b.

(iii) Otherwise, i.e., if  $o_k \notin r_j$ , let  $w_{k,j} = \emptyset$ .

The wedges  $w_{k,j}$  will ensure that  $p_1, \dots, p_n$  contain certain witness points required for the satisfaction of the relevant atomic RCC8c-formulas. For example, if  $\text{PO}(r_i, r_j)$ , there will exist a witness point  $o_k \in r_i^\circ \cap r_j^\circ$ ; but then  $o_k \in w_{k,i} \cap w_{k,j}$ , whence  $o_k \in p_i^\circ \cap p_j^\circ$ , which is required to ensure that  $\text{PO}(p_i, p_j)$ .

It is also obvious that the  $w_{k,j}$  may be constructed so that distinct  $w_{k,j}$  and  $w_{k',j'}$  share no bounding line segments. This condition is important in view of the following simple observation about sums of regular closed polygons:

**Lemma 7.** *Suppose that  $s_1, \dots, s_k$  are elements of  $\text{RCP}(\mathbb{R}^2)$  no two of which have any line segment common to their boundaries. Then  $(\sum_{1 \leq i \leq k} s_i)^\circ = \bigcup_{1 \leq i \leq k} s_i^\circ$ .*

**Step 4:** We must take steps now to connect up the wedges corresponding to the connected regions. For each connected set  $r_j$  ( $1 \leq j \leq n$ ), we do the following:

- Pick a point  $u_j \in (r_j^+)^\circ \setminus d$  and, for every  $k \leq m$ , select a piecewise-linear arc  $\gamma_{k,j} \subseteq (r_j^+)^\circ \setminus d$  that connects  $q_{k,j}$  with  $u_j$ .

This is possible because, by (28) and (32),  $(r_j^+)^\circ \setminus d$  is connected and  $q_{k,j} \in (r_j^+)^\circ \cap \delta d$ . Fig. 13 illustrates the disposition of the various arcs  $\gamma_{k,j}$  for a particular connected element  $r_j$ . We may evidently assume without loss of generality that, if  $\gamma_{k,j}$  and  $\gamma_{k',j'}$  are defined and distinct, then  $\gamma_{k,j}$  and  $\gamma_{k',j'}$  intersect at a finite number of points—i.e., do not have any line segments in common.

**Step 5:** For all  $j$  ( $1 \leq j \leq n$ ), denote

$$s_j = \bigcup_{\substack{j' < j \\ \text{TPP}(r_{j'}, r_j) \text{ or } \text{NTPP}(r_{j'}, r_j)}} \bigcup_{1 \leq k \leq m} (w_{k,j'} \cup \gamma_{k,j'}).$$

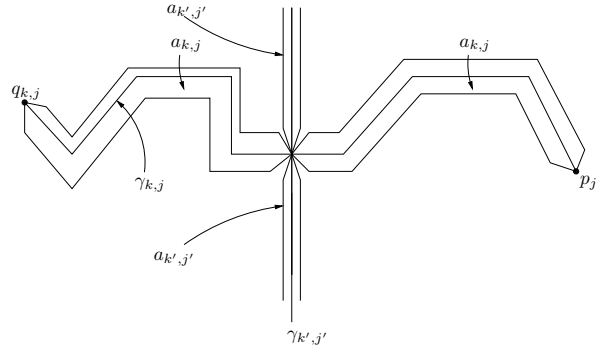


Figure 14: Illustration of the region  $a_{k,j}$  surrounding the arc  $\gamma_{k,j}$ , where  $\gamma_{k,j}$  is intersected once by another arc  $\gamma_{k',j'}$ .

This set is closed and semi-linear. So, therefore, are all of its finitely many components. For each of the connected components  $s_j^i$  of  $s_j$ , we may construct a connected, regular closed polygon  $z_j^i$  such that  $s_j^i \subseteq (z_j^i)^\circ$  and

the  $z_j^i$  are pairwise disjoint, for fixed  $j$ .

Denote  $z_j = \sum_i z_j^i$ .

**Step 6:** It is easy to see (Fig. 14) that, for every  $\gamma_{k,j}$  defined above, we can construct a regular closed polygon  $a_{k,j}$ , such that the following conditions are satisfied:

$$\gamma_{k,j} \subseteq a_{k,j} \subseteq \mathbb{R}^2 \setminus d^\circ, \quad (34)$$

$$\text{every } p \in \gamma_{k,j} \cap \delta a_{k,j} \text{ is either an endpoint of } \gamma_{k,j} \quad (35)$$

$$\text{or one of the (finitely many) points in } \gamma_{k,j} \cap \gamma_{k',j'}, \quad (36)$$

$$\text{if } \gamma_{k,j} \cap r_i = \emptyset \text{ then } a_{k,j} \cap r_i = \emptyset, \quad \text{for all } i \leq n, \quad (37)$$

$$\text{if } \gamma_{k,j} \subseteq (r_i^+)^\circ \text{ then } a_{k,j} \subseteq (r_i^+)^\circ, \quad \text{for all } i \leq n, \quad (37)$$

$$\text{if } \gamma_{k,j} \subseteq z_{j'}^\circ \text{ then } a_{k,j} \subseteq z_{j'}^\circ, \quad \text{for all } j' \leq j, \quad (38)$$

$$\text{if } \gamma_{k,j} \neq \gamma_{k',j'} \text{ then } (a_{k,j})^\circ \cap (a_{k',j'})^\circ = \emptyset. \quad (39)$$

If the arc  $\gamma_{k,j}$  is not defined, set  $a_{k,j} = \emptyset$ . The  $a_{k,j}$  can also be selected so that,  $w_{k,j} + a_{k,j}$  and  $w_{k',j'} + a_{k',j'}$  share no line segments on their boundaries, for distinct pairs  $(k, j)$  and  $(k', j')$ . This fact is significant in view of Lemma 7.

**Step 7:** Now define, inductively, the sequences  $p_1, \dots, p_n$  and  $p_1^+, \dots, p_n^+$  as follows. Suppose that  $p_1, \dots, p_{j-1}$  and  $p_1^+, \dots, p_{j-1}^+$  have been defined. Let

$$p_j = b_j + \sum_{\substack{j' < j \\ \text{TPP}(r_{j'}, r_j)}} p_{j'} + \sum_{\substack{j' < j \\ \text{NTPP}(r_{j'}, r_j)}} p_{j'}^+, \quad (40)$$

where

$$b_j = \sum_{1 \leq k \leq m} (w_{k,j} + a_{k,j}), \quad (41)$$

and let  $p_j^+$  be a regular closed polygon such that  $p_j \subseteq (p_j^+)^\circ$

and

every component of  $p_j^+$  includes a component of  $p_j$ , (42)

if  $p_j \cap p_{j'}^+ = \emptyset$  then  $p_j^+ \cap p_{j'}^+ = \emptyset$ , for  $j' < j$ , (43)

if  $p_j \cap b_i = \emptyset$  then  $p_j^+ \cap b_i = \emptyset$ , for  $j < i \leq n$ , (44)

if  $o_k \notin p_j$  then  $o_k \notin p_j^+$ , for  $k \leq m$ , (45)

if  $p_j \cap \delta z_{j'} = \emptyset$  then  $p_j^+ \cap \delta z_{j'} = \emptyset$ , for  $j' \leq j$ . (46)

The polygon  $p_j^+$  can also be chosen so that it shares no line segment with the boundary of any of the sets  $p_1, \dots, p_j, p_1^+, \dots, p_{j-1}^+$ . Again, this fact is significant in view of Lemma 7. Since all the regions concerned are bounded polygons, the existence of the  $p_j^+$  is unproblematic.

This concludes the construction of the bounded regular closed polygons  $p_1, \dots, p_n$ . We now proceed to show that  $p_1, \dots, p_n$  satisfy the same atomic  $\mathcal{RCC8c}$ -formulas as the given formula. First, we establish a series of lemmas.

**Lemma 8.** For all  $j$  and  $k$  ( $1 \leq j \leq n, 1 \leq k \leq m$ ),

(i)  $o_k \in r_j$  if and only if  $o_k \in p_j$  and

(ii)  $o_k \in r_j^\circ$  if and only if  $o_k \in p_j^\circ$ .

**Proof.** (i) If  $o_k \in r_j$ , then, by construction of  $w_{k,j}$ , we have  $o_k \in w_{k,j} \subseteq b_j \subseteq p_j$ . For the converse direction, we first observe that

$$\text{if } o_k \notin r_j \text{ then } o_k \notin b_j, \text{ for all } j \leq n. \quad (47)$$

Indeed, by (34),  $o_k \notin a_{k',j}$ , for each  $k'$ . By construction,  $o_k \notin r_j$  implies  $w_{k,j} = \emptyset$  and, for each  $k' \neq k$ , we have  $o_k \notin d_{k'}$ , whence, by (33),  $o_k \notin w_{k',j}$ .

Then we proceed to show (i) by induction on  $j$ . The case  $j = 1$  is immediate from (47). Let  $j > 1$  and  $o_k \notin r_j$ . By (47),  $o_k \notin b_j$  and, for all  $j' < j$  with  $\text{TPP}(r_{j'}, r_j)$  or  $\text{NTPP}(r_{j'}, r_j)$ , we have  $o_k \notin r_{j'}$ , whence, by IH,  $o_k \notin p_{j'}$  and, by (45),  $o_k \notin p_j^+$ . By (40),  $o_k \notin p_j$ .

(ii) If  $o_k \in r_j^\circ$ , then, by construction of  $w_{k,j}$ , we obtain  $o_k \in (w_{k,j})^\circ \subseteq b_j^\circ \subseteq p_j^\circ$ . For the converse direction, we first show that

$$\text{if } o_k \notin r_j^\circ \text{ then } o_k \notin b_j^\circ, \text{ for all } j \leq n. \quad (48)$$

In view of Lemma 7, it suffices to prove that  $o_k \notin (w_{k',j})^\circ$  and  $o_k \notin (a_{k',j})^\circ$ , for each  $k'$ . The latter is immediate from (34). The former holds because  $o_k \notin r_j^\circ$  implies  $o_k \notin (w_{k,j})^\circ$  and, for each  $k' \neq k$ , we have  $o_k \notin d_{k'}$ , whence, by (33),  $o_k \notin (w_{k',j})^\circ$ .

Then we proceed to show (ii) by induction on  $j$ . The case  $j = 1$  is immediate from (48). Let  $j > 1$  and  $o_k \notin r_j^\circ$ . By (48),  $o_k \notin b_j^\circ$ . For all  $j' < j$  with  $\text{TPP}(r_{j'}, r_j)$ , we have  $o_k \notin r_{j'}^\circ$ , whence, by IH,  $o_k \notin p_{j'}^\circ$ . For all  $j' < j$  with  $\text{NTPP}(r_{j'}, r_j)$ , we have  $o_k \notin r_{j'}$ , whence, by (i),  $o_k \notin p_j$  and, by (45),  $o_k \notin (p_j^+)^{\circ}$ . It follows from (40) and Lemma 7 that  $o_k \notin p_j$ .  $\square$

**Lemma 9.** For all  $i, j$  ( $1 \leq i, j \leq n$ ), if  $r_i \cap r_j = \emptyset$  then  $p_i^+ \cap p_j^+ = \emptyset$ .

**Proof.** First, we need to prove the following two statements:

$$\text{if } r_i \cap r_j = \emptyset \text{ then } b_i \cap b_j = \emptyset, \text{ for } i, j \leq n; \quad (49)$$

$$\text{if } r_j \cap r_i = \emptyset \text{ then } p_j^+ \cap b_i = \emptyset, \text{ for } j < i \leq n. \quad (50)$$

Observe that  $a_{k,j} + w_{k,j} \subseteq (r_j^+)^{\circ}$ , for all  $k \leq m$ : indeed, if  $a_{k,j} \neq \emptyset$  then  $\gamma_{k,j}$  is defined and  $\gamma_{k,j} \subseteq (r_j^+)^{\circ}$ , whence, by (37),  $a_{k,j} \subseteq (r_j^+)^{\circ}$ ; further, if  $w_{k,j} \neq \emptyset$  then  $o_k \in r_j$  whence, by (31),  $w_{k,j} \subseteq d_k \subseteq (r_j^+)^{\circ}$ . Now, if  $r_i \cap r_j = \emptyset$  then, by (29),  $r_i^+ \cap r_j^+ = \emptyset$  and thus (49).

For each  $i$ , we prove (50) by induction on  $j$ . The basis of induction,  $j = 1$ , follows from (49) and (44) as  $p_1 = b_1$ . For the induction step, let  $j > 1$  and  $r_j \cap r_i = \emptyset$ . By (49),  $b_j \cap b_i = \emptyset$ . For each  $j' < j$  with  $\text{TPP}(r_{j'}, r_j)$  or  $\text{NTPP}(r_{j'}, r_j)$ , we have  $j' < j < i$  and thus  $r_{j'} \cap r_i = \emptyset$ , whence, by IH,  $p_{j'}^+ \cap b_i = \emptyset$ . Therefore, by (40),  $p_j \cap b_i = \emptyset$  and, by (44),  $p_j^+ \cap b_i = \emptyset$ .

Finally, we prove the statement of the lemma: we show by induction on  $i$  that for all  $j, j' \leq i$ , if  $r_{j'} \cap r_j = \emptyset$  then  $p_{j'}^+ \cap p_j^+ = \emptyset$ . The case  $i = 1$  is trivial. Let  $i > 1$  and  $r_{j'} \cap r_j = \emptyset$ . We have to consider only the case  $j' < j = i$  (other cases are either immediate from IH or mirror image of this case). By (50),  $p_{j'}^+ \cap b_i = \emptyset$ . For each  $j'' < i$  with  $\text{TPP}(r_{j''}, r_i)$  or  $\text{NTPP}(r_{j''}, r_i)$ , we have  $r_{j'} \cap r_{j''} = \emptyset$ , whence, by IH,  $p_{j'}^+ \cap p_{j''}^+ = \emptyset$ . By (40),  $p_{j'}^+ \cap p_i = \emptyset$ , and thus, by (43),  $p_{j'}^+ \cap p_i^+ = \emptyset$ .  $\square$

**Lemma 10.** For all  $i, j$  ( $1 \leq i, j \leq n$ ), if  $r_i^\circ \cap r_j^\circ = \emptyset$  then  $p_i^\circ \cap p_j^\circ = \emptyset$ .

**Proof.** First, we need to prove the following two statements:

$$\text{if } r_i^\circ \cap r_j^\circ = \emptyset \text{ then } b_i^\circ \cap b_j^\circ = \emptyset, \text{ for } i, j \leq n; \quad (51)$$

$$\text{if } r_j^\circ \cap r_i^\circ = \emptyset \text{ then } p_j^\circ \cap b_i^\circ = \emptyset, \text{ for } j < i \leq n. \quad (52)$$

To show (51), suppose otherwise, that is,  $b_i^\circ \cap b_j^\circ \neq \emptyset$ . Then there exists  $k \leq m$  such that one of the following holds:  $o_k \in w_{k,i} \cap (w_{k,j})^\circ$  or  $o_k \in (w_{k,i})^\circ \cap w_{k,j}$  or  $o_k \in (w_{k,i})^\circ \cap (w_{k,j})^\circ$ . The first case is possible only if  $o_k \in r_i \cap r_j^\circ$ , the second only if  $o_k \in r_i^\circ \cap r_j$ , and the third only if  $o_k \in r_i^\circ \cap r_j^\circ$ . In all cases,  $r_i^\circ \cap r_j^\circ \neq \emptyset$ .

For each  $i$ , we prove (52) by induction on  $j$ . The basis of induction,  $j = 1$ , follows from (51) as  $p_1 = b_1$ . For the induction step, let  $j > 1$  and  $r_j^\circ \cap r_i^\circ = \emptyset$ . By (51),  $b_j^\circ \cap b_i^\circ = \emptyset$ . For all  $j' < j$  with  $\text{TPP}(r_{j'}, r_j)$ , we have  $r_{j'}^\circ \cap r_i^\circ = \emptyset$ , whence, by IH,  $p_{j'}^\circ \cap b_i^\circ = \emptyset$ ; for all  $j' < j$  with  $\text{NTPP}(r_{j'}, r_j)$ , we have  $r_{j'} \cap r_i = \emptyset$ , whence, by Lemma 9,  $p_{j'}^+ \cap b_i = \emptyset$  and  $(p_{j'}^+)^{\circ} \cap b_i^\circ = \emptyset$ . By (40) and Lemma 7,  $p_j^\circ \cap b_i^\circ = \emptyset$ .

Finally, we prove by induction on  $i$  that for all  $j, j' \leq i$ , if  $r_{j'}^\circ \cap r_j^\circ = \emptyset$  then  $p_{j'}^\circ \cap p_j^\circ = \emptyset$ . The case  $i = 1$  is trivial. Let  $i > 1$  and  $r_{j'}^\circ \cap r_j^\circ = \emptyset$ . We have to consider only the case  $j' < j = i$  (other cases are either immediate from IH or mirror image of this case). By (52),  $p_{j'}^\circ \cap b_i^\circ = \emptyset$ . For all  $j'' < i$  with  $\text{TPP}(r_{j''}, r_i)$ , we have  $r_{j'}^\circ \cap r_{j''}^\circ = \emptyset$ , whence by IH,  $p_{j'}^\circ \cap p_{j''}^\circ = \emptyset$ ; for all  $j'' < i$  with  $\text{NTPP}(r_{j''}, r_i)$ , we have  $r_{j'} \cap r_{j''} = \emptyset$ , whence, by Lemma 9,  $p_{j'}^+ \cap p_{j''}^+ = \emptyset$  and so  $p_{j'}^\circ \cap (p_{j''}^+)^{\circ} = \emptyset$ . By (40) and Lemma 7,  $p_{j'}^\circ \cap p_i^\circ = \emptyset$ .  $\square$

**Lemma 11.** For all  $j$  ( $1 \leq j \leq n$ ),  $r_j$  is connected if and only if  $p_j$  is connected.

**Proof.** ( $\Rightarrow$ ) By construction of the sets  $b_1, \dots, b_n$ , if  $r_j$  is connected then  $b_j$  is connected. We show by induction on  $j$  that every component of  $p_j$  includes a component of  $b_j$ . For  $j = 1$ , we have  $p_1 = b_1$ . Suppose,  $j > 1$  and  $p_j$  has a component not including (and hence not intersecting) any component of  $b_j$ . By construction, if  $r_{j'} \subseteq r_j$  then every component of  $b_{j'}$  intersects some component of  $b_j$ , since  $w_{k,j'} \neq \emptyset$  implies  $o_k \in r_{j'} \subseteq r_j$  and thus  $w_{k,j} \neq \emptyset$ . Then, by (40), there exists  $j' < j$  such that either  $\text{TPP}(r_{j'}, r_j)$  and some component  $e$  of  $p_{j'}$  does not intersect any component of  $b_j$  or  $\text{NTPP}(r_{j'}, r_j)$  and some component of  $p_{j'}^+$  does not intersect any component of  $b_j$ . In the former case, by IH,  $e$  includes some component of  $b_{j'}$  but since  $r_{j'} \subseteq r_j$ ,  $b_{j'}$  and hence  $e$  intersects some component of  $b_j$ , a contradiction. The latter case follows similarly, making use of (42).

( $\Leftarrow$ ) Now we show by induction on  $j$  that

$$\text{if } r_{j'} \subseteq r_j \text{ then } p_{j'} \cap \delta z_j = \emptyset, \text{ for all } j' \leq j. \quad (53)$$

As  $\delta z_j$  separates the components of  $s_{j'}$ , it follows that  $(w_{k,j'} \cup \gamma_{k,j'}) \cap \delta z_j = \emptyset$ , for all  $k \leq m$ , and so, by (38),  $b_{j'} \cap \delta z_j = \emptyset$ . Now, the basis of induction,  $j = 1$ , is trivial, since  $p_1 = b_1$ , and  $r_{j'} \subseteq r_1$  only if  $j' = 1$ . Let  $j > 1$ . We have  $b_{j'} \cap \delta z_j = \emptyset$ , and, for all  $j'' < j'$  with  $\text{TPP}(r_{j''}, r_{j'})$  or  $\text{NTPP}(r_{j''}, r_{j'})$ , we have  $r_{j''} \subseteq r_j$ , whence, by IH,  $p_{j''} \cap \delta z_j = \emptyset$  and, by (46),  $p_{j''}^+ \cap \delta z_j = \emptyset$ . By (40),  $p_{j'} \cap \delta z_j = \emptyset$ .

If  $r_j$  is not connected then there exist  $o_k$  and  $o_{k'}$  lying in separate components of  $r_j$ . We claim that  $o_k$  and  $o_{k'}$  lie in distinct components of  $z_j$ , which, by (53), implies that  $p_j$  is not connected. Suppose, to the contrary, that  $o_k$  and  $o_{k'}$  lie in the same component of  $z_j$ ; then they lie in the same component of  $s_j$ . But then, by the construction of the  $\gamma_{k,j}$  and  $w_{k,j}$ , there exists a sequence of arcs  $\gamma_{k_1,j_1}, \dots, \gamma_{k_\ell,j_\ell}$  along which it is possible to pass (possibly via wedges  $w_{k'',j'}$  with  $r_{j'} \subseteq r_j$ ) from  $o_k$  to  $o_{k'}$ . But in that case, we have, for all  $l \leq \ell$ : (i)  $r_{j_l}$  is connected; (ii)  $r_{j_l} \subseteq r_j$ ; and (iii)  $r_{j_l} \cap r_{j_{l+1}} \neq \emptyset$  if  $l < \ell$ . To see (iii), suppose  $r_{j_l} \cap r_{j_{l+1}} = \emptyset$ . By (29),  $r_{j_l}^+ \cap r_{j_{l+1}}^+ = \emptyset$ ; but  $\gamma_{k_l,j_l} \subseteq r_{j_l}^+$  and  $\gamma_{k_{l+1},j_{l+1}} \subseteq r_{j_{l+1}}^+$ . Thus,  $o_k$  and  $o_{k'}$  lie in the same component of  $r_j$  contrary to our assumption.  $\square$

Consider all the atomic formulas: (i) if  $\text{DC}(r_i, r_j)$  then, by Lemma 9,  $p_i \cap p_j = \emptyset$  and so  $\text{DC}(r_i, p_j)$ ; (ii) if  $\text{EC}(r_i, r_j)$  then, by Lemma 10,  $p_i^\circ \cap p_j^\circ = \emptyset$  and, by Lemma 8,  $\delta p_i \cap \delta p_j \neq \emptyset$  and so,  $\text{EC}(p_i, p_j)$ ; (iii) if  $\text{PO}(r_i, r_j)$  then, by Lemma 8,  $p_i^\circ \cap p_j^\circ \neq \emptyset$ ,  $p_i^\circ \setminus p_j^\circ \neq \emptyset$  and  $p_j^\circ \setminus p_i^\circ \neq \emptyset$  and so,  $\text{PO}(p_i, p_j)$ ; (iv) if  $\text{TPP}(r_i, r_j)$  then  $i < j$  and, by (40),  $p_i \subseteq p_j$ ; also, by Lemma 8,  $\delta p_i \cap \delta p_j \neq \emptyset$  and  $p_j^\circ \setminus p_i^\circ \neq \emptyset$  and so,  $\text{TPP}(p_i, p_j)$ ; (v) if  $\text{NTPP}(r_i, r_j)$  then  $i < j$ , whence, by (40),  $p_i \subseteq p_j^\circ$ , and so,  $\text{NTPP}(p_i, p_j)$ ; (vi) finally, by Lemma 11,  $r_i$  is connected if and only if  $p_i$  is connected.  $\square$

## Proof of Theorem 4

**Theorem 4.** The problems  $\text{Sat}(\text{RCC8c}, \text{RC}(\mathbb{R}))$  and  $\text{Sat}(\text{RCC8c}, \text{RCP}(\mathbb{R}))$  are both NP-complete; the problem  $\text{Sat}(\text{Cc}, \text{RCP}(\mathbb{R}))$  is PSPACE-complete.

**Proof.** (i) We show that an  $\text{RCC8c}$ -formula  $\varphi$  is satisfiable over  $\text{RCP}(\mathbb{R})$  if and only if it is satisfiable over the Aleksandrov space induced by a quasi-order of the form  $(\{x_0, \dots, x_n, z_0, \dots, z_{n-1}\}, R)$ , where  $R$  is the reflexive closure of  $\{(z_i, x_i), (z_i, x_{i+1}) \mid 1 \leq i < n\}$  and  $n \leq |\varphi|^2$ .

Without loss of generality we may assume that  $\varphi$  is a conjunction of atoms of the form:

- $(r \cdot r' \neq 0), (r \cdot (-r') \neq 0),$
- $C(r, r'), C(r, -r'),$
- $(r \cdot r' = 0), (r \leq r'),$
- $\neg C(r, r'), \neg C(r, -r'),$
- $c(r)$

(negative occurrences of  $c(r)$  can be equi-satisfiably replaced by  $(r' \leq r) \wedge (r'' \leq r) \wedge \neg C(r', r'')$ , where  $r'$  and  $r''$  are fresh variables).

Suppose  $\varphi$  is satisfiable in a model  $\mathfrak{M}$  over  $\text{RCP}(\mathbb{R})$ . We construct a model  $\mathfrak{B}$  over the Aleksandrov space induced by  $(W, R)$  in a number of steps.

**Step 1.** First we find points for  $W$  in the following way:

- (**sing**) if  $\mathfrak{M} \models (\tau \neq 0)$ , we pick a point  $x \in \tau^{\mathfrak{M}}$ ; and
- (**fork**) if  $\mathfrak{M} \models C(\tau, \tau')$ , we pick a pair of points  $x \in (\tau)^{\mathfrak{M}}$  and  $x' \in (\tau')^{\mathfrak{M}}$ .

Without loss of generality, we may assume that between no pair of points picked in (**fork**) lies another of the picked points (this can be done by selecting that pair close enough to the point of contact of  $\tau$  and  $\tau'$ ). We also assume that the same point may be picked twice. Denote by  $W_0$  the set of all the points picked above and let  $\prec$  be the strict linear order on  $W_0$  induced by their natural order in  $\mathfrak{M}$ . Let  $W = W_0$  and  $R = \emptyset$ . Next, for each pair  $x, x'$  of points picked in (**fork**), take a fresh point  $z$ , add it to  $W$  and  $(z, x), (z, x')$  to  $R$ . Note that  $(W, R)$  is a subgraph of the required quasi-order (i.e., each  $z$  has at most two successors and each  $x$  has at most two predecessors). Finally, we construct the model  $\mathfrak{B}$  based on the Aleksandrov space induced by  $(W, R)$  by setting  $x \in r^{\mathfrak{B}}$  iff  $x \in r^{\mathfrak{M}}$ , for each  $w \in W_0$  (the valuation in  $z \in W \setminus W_0$  needs no definition as if  $r^{\mathfrak{B}}$  are to be regular closed we must have  $z \in \mathfrak{B}$  iff  $x \in r^{\mathfrak{B}}$ , for some  $x \in W_0$  with  $zRx$ ). Note that  $\mathfrak{B} \models (\tau = 0)$  whenever  $\mathfrak{M} \models (\tau = 0)$  and  $\mathfrak{B} \models \neg C(\tau, \tau')$  whenever  $\mathfrak{M} \models \neg C(\tau, \tau')$ . If the space is connected and  $\mathfrak{B} \models c(r)$  whenever  $\mathfrak{M} \models c(r)$ , for all  $c(r)$ , we are done.

**Step 2.** So, suppose  $c(\tau_0)$ , where  $\tau_0$  is either  $r$  or  $1$ , is false in the model. Pick any two neighbouring (with respect to  $\prec$ ) points  $x, x' \in \tau_0^{\mathfrak{B}}$  without a common predecessor and take a fresh point  $y$ , add  $y$  to  $W$  with  $y \in r^{\mathfrak{B}}$  iff  $x, x' \in r^{\mathfrak{B}}$ . Clearly, adding this point to the model does not change the truth value of any subformula of the form  $(\tau \neq 0), C(\tau, \tau'), (\tau = 0)$  or  $\neg C(\tau, \tau')$ . So, it remains to connect  $y$  to both  $x$  and  $x'$ . We cannot, however, directly connect  $y$  to, say,  $x$  by

creating a common  $R$ -predecessor, i.e., a point of depth 1, because that might make one of the  $\neg C(\tau, \tau)$  subformulas false. Let  $r_{j_1}, \dots, r_{j_k}$  be a linear order on the regions containing  $x$  such that it is compatible with the subformulas of the form  $(r_i \leq r_j)$ , i.e., such that  $\mathfrak{M} \models (r_{j_i} \leq r_{j_{i'}})$  whenever  $j_i \leq j_{i'}$ . We proceed our construction in a step-by-step way. For step 0, let  $x_0 = y$ . For step  $i$ ,  $1 \leq i \leq k$ , take fresh points  $x_i$  and  $z_i$ , add them to  $W$ , add  $(z_i, x_{i-1})$  and  $(z_i, x_i)$  to  $R$ , let

$$x_i \in r^{\mathfrak{B}} \quad \text{iff} \quad x_{i-1} \in r^{\mathfrak{B}} \quad \text{or} \quad \mathfrak{M} \models r_{j_i} \leq r$$

and  $z_i \in r^{\mathfrak{B}}$  iff  $\{x_{i-1}, x_i\} \cap r^{\mathfrak{B}} \neq \emptyset$ . Clearly, all subformulas of the form  $(\tau \neq 0)$ ,  $C(\tau, \tau')$  and  $(\tau = 0)$  that are true in  $\mathfrak{M}$  are also true in  $\mathfrak{B}$ . It is also clear that the same holds for subformulas of the form  $\neg C(r, r')$ . So, it remains to show that the same holds for subformulas of the form  $\neg C(r, -r')$ . To this end we observe that, each triple  $x_{i-1}, z_i, x_i$  is either entirely in  $r^{\mathfrak{B}}$  or  $z_i$  is on the boundary of  $r^{\mathfrak{B}}$ , in which case  $x_{i-1} \notin r^{\mathfrak{B}}$  and  $x_i \in r^{\mathfrak{B}}$ . It also follows that in the latter case  $\mathfrak{M} \models r_{j_k} \leq r$  iff  $k \geq i$ , and so  $z_i$  cannot be a point of contact of  $r$  and any  $-r'$ . Finally, points  $x_k$  and  $x$  belong to precisely the same regions and can be identified. In the same way we connect  $y$  to  $x'$ .

Repeating **Step 2** for each pair of neighbouring (with respect to  $\prec$ ) points of depth 0 without a common predecessor, we construct the model as required.

(ii) For  $Sat(Cc, RCP(\mathbb{R}))$ , the model can be of exponential size (due to exponential paths required in **Step 2** to connect up disconnected regions) but a non-deterministic algorithm can guess such a model using only a polynomial number of cells on the tape of a Turing machine.  $\square$