Query Inseparability for Description Logic Knowledge Bases

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Abstract

We investigate conjunctive query inseparability of description logic (DL) knowledge bases (KBs) with respect to a given signature, a fundamental problem for KB versioning, module extraction, forgetting and knowledge exchange. We study the data and combined complexity of deciding KB query inseparability for fragments of *Horn-*$\mathcal{ALCHI}$, including the DLs underpinning *OWL 2 QL* and *OWL 2 EL*. While all of these DLs are P-complete for data complexity, the combined complexity ranges from P to ExpTime and 2ExpTime. We also resolve two major open problems for *OWL 2 QL* by showing that TBox query inseparability and the membership problem for universal UCQ-solutions in knowledge exchange are both ExpTime-complete for combined complexity.

Introduction

A description logic (DL) knowledge base (KB) consists of a terminological box (TBox), storing conceptual knowledge, and an assertion box (ABox), storing data. Typical applications of KBs involve answering queries over incomplete data sources (ABoxes) augmented by ontologies (TBoxes) that provide additional information about the domain of interest as well as a convenient vocabulary for user queries. The standard query language in such applications, which balances expressiveness and computational complexity, is the language of conjunctives queries (CQs).

With typically large data, often tangled ontologies, and the hard problem of answering CQs over ontologies, various transformation and comparison tasks are becoming indispensable for KB engineering and maintenance. For example, to make answering certain CQs more efficient, one may want to extract from a given KB a smaller module returning the same answers to those CQs as the original KB; to provide the user with a more convenient query vocabulary, one may want to reformulate the KB in a new language. These tasks are known as module extraction (Stuckenschmidt, Parent, and Spaccapietra 2009) and knowledge exchange (Arenas et al. 2012); other relevant tasks include versioning, revision and forgetting (Jiménez-Ruiz et al. 2011; Wang, Wang, and Topor 2010; Lin and Reiter 1994).

In this paper, we investigate the following relationship between KBs which is fundamental for all such tasks. Let Σ be a signature consisting of concept and role names. We call KBs $\mathcal{K}_1$ and $\mathcal{K}_2$ Σ-*query inseparable* and write $\mathcal{K}_1 \equiv_\Sigma \mathcal{K}_2$ if any CQ formulated in Σ has the same answers over $\mathcal{K}_1$ and $\mathcal{K}_2$. Note that even for Σ containing all concept and role names, Σ-query inseparability does not necessarily imply logical equivalence. The relativisation to (smaller) signatures is crucial to support the tasks mentioned above:

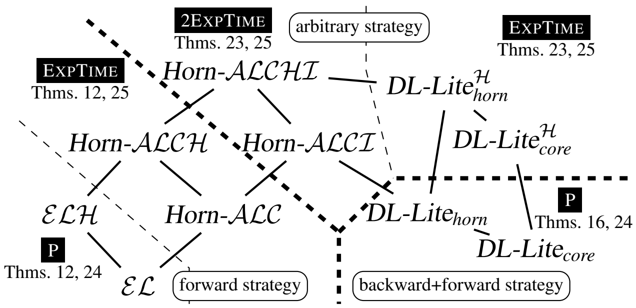
(versioning) When comparing two *versions* $\mathcal{K}_1$ and $\mathcal{K}_2$ of a KB with respect to their answers to CQs in a relevant signature Σ, the basic task is to check whether $\mathcal{K}_1 \equiv_\Sigma \mathcal{K}_2$.

(modularisation) A Σ-*module of* a KB $\mathcal{K}$ is a KB $\mathcal{K}' \subseteq \mathcal{K}$ such that $\mathcal{K}' \equiv_\Sigma \mathcal{K}$. If we are only interested in answering CQs in Σ over $\mathcal{K}$, then we can achieve our aim by querying any Σ-module of $\mathcal{K}$ instead of $\mathcal{K}$ itself.

(knowledge exchange) In knowledge exchange, we want to transform a KB $\mathcal{K}_1$ in a signature Σ_1 to a new KB $\mathcal{K}_2$ in a disjoint signature Σ_2 connected to Σ_1 via a declarative mapping specification given by a TBox $\mathcal{T}_{12}$. Thus, the target KB $\mathcal{K}_2$ should satisfy the condition $\mathcal{K}_1 \cup \mathcal{T}_{12} \equiv_{\Sigma_2} \mathcal{K}_2$, in which case it is called a *universal UCQ-solution* (CQ and UCQ inseparabilities coincide for Horn DLs).

(forgetting) A KB $\mathcal{K}'$ results from *forgetting* a signature Σ in a KB $\mathcal{K}$ if $\mathcal{K}' \equiv_{\mathsf{sig}(\mathcal{K}) \setminus \Sigma} \mathcal{K}$ and $\mathsf{sig}(\mathcal{K}') \subseteq \mathsf{sig}(\mathcal{K}) \setminus \Sigma$. Thus, the result of forgetting Σ does not use Σ and gives the same answers to CQs without symbols in Σ as $\mathcal{K}$.

We investigate the data and combined complexity of deciding Σ-query inseparability for KBs given in various fragments of the DL *Horn-*$\mathcal{ALCHI}$ (Krötzsch, Rudolph, and Hitzler 2013), which include *DL-Lite*$^{\mathcal{H}}_{core}$ (Calvanese et al. 2007) and $\mathcal{EL}$ (Baader, Brandt, and Lutz 2005) underlying the W3C profiles *OWL 2 QL* and *OWL 2 EL*. For all of these DLs, Σ-query inseparability turns out to be P-complete for data complexity, which matches the data complexity of CQ evaluation for all of our DLs lying outside the *DL-Lite* family. For combined complexity, the obtained tight complexity results are summarised in the diagram below. Most interesting are ExpTime-completeness of *DL-Lite*$^{\mathcal{H}}_{core}$ and 2ExpTime-completeness of *Horn-*$\mathcal{ALCI}$, which contrast with NP-completeness and ExpTime-completeness of CQ evaluation for those logics. For *DL-Lite* without role inclusions and $\mathcal{ELH}$, Σ-query inseparability is P-complete, while CQ evaluation is NP-complete. In general, it is the combined presence of inverse roles and qualified existential restrictions (or role inclusions) that makes Σ-query inseparability hard. To establish the upper complexity bounds, we develop a uniform game-theoretic technique for checking finite Σ-homomorphic embeddability between (possibly infinite) materialisations of KBs.



Σ-query inseparability for KBs has not been investigated systematically before. The polynomial upper bound for $\mathcal{EL}$ was established as a preliminary step to study TBox inseparability (Lutz and Wolter 2010), and this notion was also used to study forgetting for *DL-Lite*$^{\mathcal{N}}_{bool}$ (Wang et al. 2010).

We apply our results to resolve two important open problems. First, we show that the membership problem for universal UCQ-solutions in knowledge exchange for KBs in *DL-Lite*$^{\mathcal{H}}_{core}$ is ExpTime-complete for combined complexity, which settles an open question of (Arenas et al. 2013), where only PSpace-hardness was established. We also show that Σ-query inseparability of *DL-Lite*$^{\mathcal{H}}_{core}$ TBoxes is ExpTime-complete, which closes the PSpace–ExpTime gap that was left open by Konev et al. (2011).

Recall that TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are Σ-query inseparable if, for *all* Σ-ABoxes $\mathcal{A}$ (which only use concept and role names from Σ), the KBs $(\mathcal{T}_1, \mathcal{A})$ and $(\mathcal{T}_2, \mathcal{A})$ are Σ-query inseparable. TBox and KB inseparabilities have different applications. The former supports ontology engineering when data is not known or changes frequently: one can equivalently replace one TBox with another only if they return the same answers to queries for *every* Σ-ABox. In contrast, KB inseparability is useful in applications where data is stable such as knowledge exchange, module extraction or forgetting for a stable KB in order to re-use it in a new application or as a compilation step to make CQ answering more efficient. As we show below, TBox and KB Σ-query inseparabilities also have different computational properties.

TBox Σ-query inseparability has been extensively studied (Kontchakov, Wolter, and Zakharyaschev 2010; Lutz and Wolter 2010; Konev et al. 2012). For work on different notions of TBox inseparability and the corresponding notions of modules and forgetting, we refer the reader to (Cuenca Grau et al. 2008; Konev, Walther, and Wolter 2009; Del Vescovo et al. 2011; Nikitina and Rudolph 2012; Nikitina and Glimm 2012; Lutz, Seylan, and Wolter 2012).

Omitted proofs can be found in the full version available at `www.dcs.bbk.ac.uk/~roman`.

Horn-$\mathcal{ALCHI}$ and its Fragments

All the DLs for which we investigate KB Σ-query inseparability are Horn fragments of $\mathcal{ALCHI}$. To define these DLs, we fix sequences of *individual names* a_i, *concept names* A_i, and *role names* P_i, where $i < \omega$. A *role* is either a role name P_i or an *inverse role* P_i^-; we assume that $(P_i^-)^- = P_i$. $\mathcal{ALCI}$*-concepts*, C, are defined by the grammar

$$C ::= A_i \mid \top \mid \bot \mid \neg C \mid C_1 \sqcap C_2 \mid C_1 \sqcup C_2 \mid \exists R.C \mid \forall R.C,$$

where R is a role. $\mathcal{ALC}$*-concepts* are $\mathcal{ALCI}$-concepts without inverse roles; $\mathcal{EL}$*-concepts* are $\mathcal{ALC}$-concepts without the constructs $\bot$, $\sqcup$, $\neg$ and $\forall R.C$. *DL-Lite*$_{horn}$*-concepts* are $\mathcal{ALCI}$-concepts without $\sqcup$, $\neg$ and $\forall R.C$, in which $C = \top$ in every occurrence of $\exists R.C$. Finally, *DL-Lite*$_{core}$*-concepts* are *DL-Lite*$_{horn}$-concepts without $\sqcap$; in other words, they are *basic concepts* of the form $\bot$, $\top$, A_i or $\exists R.\top$.

For a DL $\mathcal{L}$, an $\mathcal{L}$*-concept inclusion* (CI) takes the form $C \sqsubseteq D$, where C and D are $\mathcal{L}$-concepts. An $\mathcal{L}$*-TBox*, $\mathcal{T}$, contains a finite set of $\mathcal{L}$-CIs. An $\mathcal{ALCHI}$, *DL-Lite*$^{\mathcal{H}}_{horn}$ and *DL-Lite*$^{\mathcal{H}}_{core}$ TBox can also contain a finite set of *role inclusions* (RIs) $R_1 \sqsubseteq R_2$, where the R_i are roles. In $\mathcal{ELH}$ TBoxes, RIs do not have inverse roles. *DL-Lite* TBoxes may also contain *disjointness constraints* $B_1 \sqcap B_2 \sqsubseteq \bot$ and $R_1 \sqcap R_2 \sqsubseteq \bot$, for basic concepts B_i and roles R_i.

To introduce the Horn fragments of these DLs, we require the following (standard) recursive definition (Hustadt, Motik, and Sattler 2005; Kazakov 2009): a concept C occurs positively in C; if C occurs positively (respectively, negatively) in C' then C occurs positively (negatively) in $C' \sqcup D$, $C' \sqcap D$, $\exists R.C'$, $\forall R.C'$, $D \sqsubseteq C'$, and it occurs negatively (positively) in $\neg C'$ and $C' \sqsubseteq D$. Now, we call a TBox $\mathcal{T}$ *Horn* if no concept of the form $C \sqcup D$ occurs positively in $\mathcal{T}$, and no concept of the form $\neg C$ or $\forall R.C$ occurs negatively in $\mathcal{T}$. In the DL *Horn-*$\mathcal{L}$, where $\mathcal{L}$ is one of our DLs, only Horn $\mathcal{L}$ TBoxes are allowed. Clearly, the $\mathcal{EL}$ and *DL-Lite* TBoxes are Horn by definition.

An *ABox*, $\mathcal{A}$, is a finite set of *assertions* of the form $A_k(a_i)$ or $P_k(a_i, a_j)$. An $\mathcal{L}$-TBox $\mathcal{T}$ and an ABox $\mathcal{A}$ together form an $\mathcal{L}$ *knowledge base* (KB) $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. The set of individual names in $\mathcal{K}$ is denoted by $\mathsf{ind}(\mathcal{K})$.

The semantics for the DLs is defined in the usual way based on interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ that comply with the *unique name assumption*: $a_i^{\mathcal{I}} \neq a_j^{\mathcal{I}}$ for $i \neq j$ (Baader et al. 2003). We write $\mathcal{I} \models \alpha$ in case an inclusion or assertion α is true in $\mathcal{I}$. If $\mathcal{I} \models \alpha$, for all $\alpha \in \mathcal{T} \cup \mathcal{A}$, then $\mathcal{I}$ is a *model* of a KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$; in symbols: $\mathcal{I} \models \mathcal{K}$. $\mathcal{K}$ is *consistent* if it has a model. $\mathcal{K} \models \alpha$ means that $\mathcal{I} \models \alpha$ for all $\mathcal{I} \models \mathcal{K}$.

A *conjunctive query* (CQ) $\boldsymbol{q}(\vec{x})$ is a formula $\exists \vec{y}\, \varphi(\vec{x}, \vec{y})$, where φ is a conjunction of atoms of the form $A_k(z_1)$ or $P_k(z_1, z_2)$ with $z_i \in \vec{x} \cup \vec{y}$. A tuple $\vec{a} \subseteq \mathsf{ind}(\mathcal{K})$ (of the same length as $\vec{x}$) is a *certain answer* to $\boldsymbol{q}(\vec{x})$ over $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ if $\mathcal{I} \models \boldsymbol{q}(\vec{a})$ for all $\mathcal{I} \models \mathcal{K}$; in this case we write $\mathcal{K} \models \boldsymbol{q}(\vec{a})$. If $\vec{x} = \emptyset$, the answer to $\boldsymbol{q}$ is 'yes' if $\mathcal{K} \models \boldsymbol{q}$ and 'no' otherwise.

For combined complexity, the problem '$\mathcal{K} \models \boldsymbol{q}(\vec{a})$?' is NP-complete for the *DL-Lite* logics (Calvanese et al. 2007), $\mathcal{EL}$ and $\mathcal{ELH}$ (Rosati 2007), and ExpTime-complete for the remaining Horn DLs above (Eiter et al. 2008). For data complexity (with fixed $\mathcal{T}$ and $\boldsymbol{q}$), this problem is in AC^0 for the *DL-Lite* logics (Calvanese et al. 2007) and P-complete for the remaining DLs (Rosati 2007; Eiter et al. 2008).

A *signature*, Σ, is a set of concept and role names. By a Σ-concept, Σ-role, Σ-CQ, etc. we understand any concept, role, CQ, etc. constructed using the names from Σ.

Σ-Query Entailment and Inseparability

We define the central notions of this paper.

Definition 1 Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be KBs and Σ a signature.

- $\mathcal{K}_1$ Σ-*query entails* $\mathcal{K}_2$ if $\mathcal{K}_2 \models \boldsymbol{q}(\vec{a})$ implies $\mathcal{K}_1 \models \boldsymbol{q}(\vec{a})$ for *all* Σ-CQs $\boldsymbol{q}(\vec{x})$ and all $\vec{a} \subseteq \mathsf{ind}(\mathcal{K}_2)$.
- $\mathcal{K}_1$ and $\mathcal{K}_2$ are Σ-*query inseparable* if they Σ-query entail each other. In this case we write $\mathcal{K}_1 \equiv_\Sigma \mathcal{K}_2$.

Observe that Σ-query inseparability is weaker than logical equivalence even if $\Sigma = \mathsf{sig}(\mathcal{K}_1) \cup \mathsf{sig}(\mathcal{K}_2)$, where $\mathsf{sig}(\mathcal{K}_i)$ is the signature of $\mathcal{K}_i$. For example, $(\emptyset, \{A(a)\})$ is $\{A, B\}$-query inseparable from $(\{B \sqsubseteq A\}, \{A(a)\})$ but the two KBs are clearly not logically equivalent. Since checking Σ-query inseparability can be reduced to two Σ-query entailment checks, we can prove complexity upper bounds for entailment. Conversely, for most languages we have a semantically transparent reduction of Σ-query entailment to Σ-query inseparability:

Theorem 2 *Let $\mathcal{L}$ be any of our DLs containing $\mathcal{EL}$ or having role inclusions. Then Σ-query entailment for $\mathcal{L}$-KBs is* LOGSPACE-*reducible to Σ-query inseparability for $\mathcal{L}$-KBs.*

Proof sketch. Let $\mathcal{K}_i = (\mathcal{T}_i, \mathcal{A}_i)$, $i = 1, 2$, and Σ be given. We may assume that $\Sigma = \mathsf{sig}(\mathcal{K}_1) \cap \mathsf{sig}(\mathcal{K}_2)$. We also assume that $\mathcal{L}$ has role inclusions, $\mathcal{K}_1$ and $\mathcal{K}_2$ are consistent and the *trivial interpretation* $\mathcal{I}_\emptyset$ (with $|\Delta^{\mathcal{I}_\emptyset}| = 1$ and $S^{\mathcal{I}_\emptyset} = \emptyset$, for any S) is a model of the $\mathcal{T}_i$ (a proof without those assumptions is given in the full version). Let $\mathcal{K}_i'$ be a copy of $\mathcal{K}_i$ in which all symbols S are replaced by fresh S_i, and let $\mathcal{K}_i^\Sigma$ extend $\mathcal{K}_i'$ with $S_i \sqsubseteq S$, for $S \in \Sigma$. One can show that $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{K}_1 \equiv_\Sigma \mathcal{K}_1^\Sigma \cup \mathcal{K}_2^\Sigma$. ❑

That $\mathcal{I}_\emptyset \models \mathcal{K}_i$ is essential in the reduction above. Take $\mathcal{T}_1 = \{A \sqsubseteq B, A \sqsubseteq \exists R.C\}$, $\mathcal{T}_2 = \{\top \sqsubseteq B, C \sqcap B \sqsubseteq \bot\}$ and $\Sigma = \{A, B, R, C\}$. Then $\mathcal{K}_1 = (\mathcal{T}_1, \{A(a)\})$ Σ-query entails $\mathcal{K}_2 = (\mathcal{T}_2, \{A(a)\})$ but $\mathcal{K}_1 \not\equiv_\Sigma \mathcal{K}_1^\Sigma \cup \mathcal{K}_2^\Sigma$.

We now consider the relationship between inseparability and universal UCQ-solutions in knowledge exchange. Suppose $\mathcal{K}_1$ and $\mathcal{K}_2$ are KBs in disjoint signatures Σ_1 and Σ_2. Let $\mathcal{T}_{12}$ be a mapping consisting of inclusions of the form $S_1 \sqsubseteq S_2$, where the S_i are concept (or role) names in Σ_i. Then $\mathcal{K}_2$ is a universal UCQ-solution for $(\mathcal{K}_1, \mathcal{T}_{12}, \Sigma_2)$ if $\mathcal{K}_1 \cup \mathcal{T}_{12} \equiv_{\Sigma_2} \mathcal{K}_2$. Deciding the latter is called the *membership problem for universal UCQ-solutions*. For DLs $\mathcal{L}$ with role inclusions, the problem whether $\mathcal{K}_1 \cup \mathcal{T}_{12} \equiv_{\Sigma_2} \mathcal{K}_2$ is a Σ_2-query inseparability problem in $\mathcal{L}$. Conversely, we have:

Theorem 3 *Σ-query entailment for any of our DLs $\mathcal{L}$ is* LOGSPACE-*reducible to the membership problem for universal UCQ-solutions in $\mathcal{L}$.*

Proof sketch. We want do decide whether $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$. We again assume that $\mathcal{I}_\emptyset \models \mathcal{T}_i$ and use the proof of Theorem 2 (for the general case, see the full version). We may assume that $\Sigma = \mathsf{sig}(\mathcal{K}_1) \cap \mathsf{sig}(\mathcal{K}_2)$. Let $\Sigma_1 = \mathsf{sig}(\mathcal{K}_1)$. Then $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{K}_1$ Σ_1-query entails $\mathcal{K}_2$. By the proof of Theorem 2, the latter is the case iff $\mathcal{K}_1$ Σ_1-query entails $\mathcal{K}_1^{\Sigma_1} \cup \mathcal{K}_2^{\Sigma_1}$. Clearly, $\mathcal{K}_1^{\Sigma_1} \cup \mathcal{K}_2^{\Sigma_1}$ Σ_1-query entails $\mathcal{K}_1$, and so the two KBs are Σ_1-query inseparable. Then $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{K}_1$ is a universal UCQ-solution for $(\mathcal{K}_1' \cup \mathcal{K}_2', \mathcal{T}_{12}, \Sigma_1)$, where $\mathcal{T}_{12} = \{S_1 \sqsubseteq S, S_2 \sqsubseteq S \mid S \in \Sigma_1\}$. ❑

Semantic Characterisation

In this section, we give a semantic characterisation of KB Σ-query entailment based on an abstract notion of materialisation and finite homomorphisms between such models.

Let $\mathcal{K}$ be a KB. An interpretation $\mathcal{I}$ is called a *materialisation* of $\mathcal{K}$ if, for all CQs $\boldsymbol{q}(\vec{x})$ and tuples $\vec{a} \subseteq \mathsf{ind}(\mathcal{K})$,

$$\mathcal{K} \models \boldsymbol{q}(\vec{a}) \quad \text{iff} \quad \mathcal{I} \models \boldsymbol{q}(\vec{a}).$$

We say that $\mathcal{K}$ is *materialisable* if it has a materialisation.

Materialisations can be used to characterise KB Σ-query entailment by means of Σ-homomorphisms. For an interpretation $\mathcal{I}$ and a signature Σ, the Σ-*types* $\boldsymbol{t}_\Sigma^{\mathcal{I}}(x)$ and $\boldsymbol{r}_\Sigma^{\mathcal{I}}(x, y)$ of $x, y \in \Delta^{\mathcal{I}}$ are defined by taking:

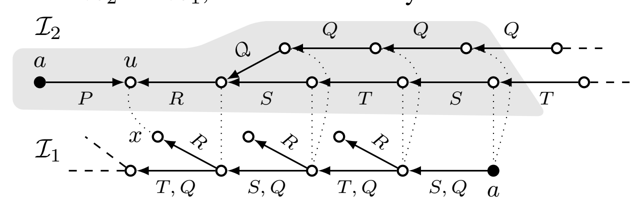
$$\boldsymbol{t}_\Sigma^{\mathcal{I}}(x) = \{\, \Sigma\text{-concept name } A \mid x \in A^{\mathcal{I}} \,\},$$
$$\boldsymbol{r}_\Sigma^{\mathcal{I}}(x, y) = \{\, \Sigma\text{-role } R \mid (x, y) \in R^{\mathcal{I}} \,\}.$$

Suppose $\mathcal{I}_i$ is a materialisation of $\mathcal{K}_i$, $i = 1, 2$. A function $h\colon \Delta^{\mathcal{I}_2} \to \Delta^{\mathcal{I}_1}$ is a Σ-*homomorphism* from $\mathcal{I}_2$ to $\mathcal{I}_1$ if, for any $a \in \mathsf{ind}(\mathcal{K}_2)$ and any $x, y \in \Delta^{\mathcal{I}_2}$,

- $h(a^{\mathcal{I}_2}) = a^{\mathcal{I}_1}$ whenever $\boldsymbol{t}_\Sigma^{\mathcal{I}_2}(a) \neq \emptyset$ or $\boldsymbol{r}_\Sigma^{\mathcal{I}_2}(a, y) \neq \emptyset$ for some $y \in \Delta^{\mathcal{I}_2}$, and
- $\boldsymbol{t}_\Sigma^{\mathcal{I}_2}(x) \subseteq \boldsymbol{t}_\Sigma^{\mathcal{I}_1}(h(x))$, $\boldsymbol{r}_\Sigma^{\mathcal{I}_2}(x, y) \subseteq \boldsymbol{r}_\Sigma^{\mathcal{I}_1}(h(x), h(y))$.

As answers to Σ-CQs are preserved under Σ-homomorphisms, $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ if there is a Σ-homomorphism from $\mathcal{I}_2$ to $\mathcal{I}_1$. However, the converse does not hold:

Example 4 Suppose $\mathcal{I}_2$ and $\mathcal{I}_1$ below are materialisations of KBs $\mathcal{K}_2$ and $\mathcal{K}_1$, where a is the only ABox individual:



Let $\Sigma = \{Q, R, S, T\}$. Then there is no Σ-homomorphism from $\mathcal{I}_2$ to $\mathcal{I}_1$ (as $\boldsymbol{r}_\Sigma^{\mathcal{I}_2}(a, u) = \emptyset$, we can map u to, say, x but then only the shaded part of $\mathcal{I}_2$ can be mapped Σ-homomorphically to $\mathcal{I}_1$). However, for any Σ-query $\boldsymbol{q}(\vec{x})$, $\mathcal{I}_2 \models \boldsymbol{q}(\vec{a})$ implies $\mathcal{I}_1 \models \boldsymbol{q}(\vec{a})$ as any finite subinterpretation of $\mathcal{I}_2$ can be Σ-homomorphically mapped to $\mathcal{I}_1$.

We say that $\mathcal{I}_2$ is *finitely Σ-homomorphically embeddable into* $\mathcal{I}_1$ if, for every *finite* subinterpretation $\mathcal{I}_2'$ of $\mathcal{I}_2$, there exists a Σ-homomorphism from $\mathcal{I}_2'$ to $\mathcal{I}_1$.

To prove the following theorem, one can regard any finite subinterpretation of $\mathcal{I}_2$ as a CQ whose variables are elements of $\Delta^{\mathcal{I}_2}$, with the answer variables being in $\mathsf{ind}(\mathcal{K}_2)$.

Theorem 5 *Suppose $\mathcal{K}_i$ is a consistent KB with a materialisation $\mathcal{I}_i$, $i = 1, 2$. Then $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{I}_2$ is finitely Σ-homomorphically embeddable into $\mathcal{I}_1$.*

One problem with applying Theorem 5 is that materialisations are in general infinite for any of the DLs considered in this paper. We address this problem by introducing finite representations of materialisations. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a KB and let $\mathcal{G} = (\Delta^{\mathcal{G}}, \cdot^{\mathcal{G}}, \rightsquigarrow)$ be a finite structure such that $\Delta^{\mathcal{G}} = \mathsf{ind}(\mathcal{K}) \cup \Omega$, for $\mathsf{ind}(\mathcal{K}) \cap \Omega = \emptyset$, $\cdot^{\mathcal{G}}$ is an interpretation function on $\Delta^{\mathcal{G}}$ with $A_i^{\mathcal{G}} \subseteq \Delta^{\mathcal{G}}$, $P_i^{\mathcal{G}} \subseteq \mathsf{ind}(\mathcal{K}) \times \mathsf{ind}(\mathcal{K})$, and $(\Delta^{\mathcal{G}}, \rightsquigarrow)$ is a directed graph (containing loops) with nodes $\Delta^{\mathcal{G}}$ and edges $\rightsquigarrow \subseteq \Delta^{\mathcal{G}} \times \Omega$, in which every edge $u \rightsquigarrow v$ is labelled with a set $(u,v)^{\mathcal{G}} \neq \emptyset$ of roles satisfying the condition: if $u_1 \rightsquigarrow v$ and $u_2 \rightsquigarrow v$, then $(u_1,v)^{\mathcal{G}} = (u_2,v)^{\mathcal{G}}$. We call $\mathcal{G}$ a *generating structure for* $\mathcal{K}$ if the interpretation $\mathcal{M}$ defined below is a materialisation of $\mathcal{K}$.

A *path* in $\mathcal{G}$ is a sequence $\sigma = u_0 \dots u_n$ with $u_0 \in \mathsf{ind}(\mathcal{K})$ and $u_i \rightsquigarrow u_{i+1}$ for $i < n$. Let $\mathsf{tail}(\sigma) = u_n$ and let $\mathsf{path}(\mathcal{G})$ be the set of paths in $\mathcal{G}$. The materialisation $\mathcal{M}$ is given by:

$$\Delta^{\mathcal{M}} = \mathsf{path}(\mathcal{G}), \quad a^{\mathcal{M}} = a, \ \text{ for } a \in \mathsf{ind}(\mathcal{K}),$$
$$A^{\mathcal{M}} = \{\sigma \mid \mathsf{tail}(\sigma) \in A^{\mathcal{G}}\},$$
$$P^{\mathcal{M}} = P^{\mathcal{G}} \cup \{(\sigma, \sigma u) \mid \mathsf{tail}(\sigma) \rightsquigarrow u, \ P \in (\mathsf{tail}(\sigma), u)^{\mathcal{G}}\} \cup \{(\sigma u, \sigma) \mid \mathsf{tail}(\sigma) \rightsquigarrow u, \ P^- \in (\mathsf{tail}(\sigma), u)^{\mathcal{G}}\}.$$

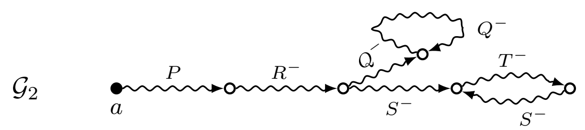
We say that a DL $\mathcal{L}$ has *finitely generated materialisations* if every $\mathcal{L}$-KB has a generating structure.

Theorem 6 *Horn-$\mathcal{ALCHI}$ and all of its fragments defined above have finitely generated materialisations. Moreover,*

– *for any $\mathcal{L} \in \{\mathcal{ALCHI}, \mathcal{ALCI}, \mathcal{ALCH}, \mathcal{ALC}\}$ and any Horn-$\mathcal{L}$ KB $(\mathcal{T}, \mathcal{A})$, a generating structure can be constructed in time $|\mathcal{A}| \cdot 2^{p(|\mathcal{T}|)}$, p a polynomial*;
– *for any $\mathcal{L}$ in the $\mathcal{EL}$ and DL-Lite families and any $\mathcal{L}$ KB $(\mathcal{T}, \mathcal{A})$, a generating structure can be constructed in time $|\mathcal{A}| \cdot p(|\mathcal{T}|)$, p a polynomial.*

Finite generating structures have been defined for $\mathcal{EL}$ (Lutz, Toman, and Wolter 2009), *DL-Lite* (Kontchakov et al. 2010) and more expressive Horn DLs (Eiter et al. 2008). With the exception of *DL-Lite*, however, the relation $\rightsquigarrow$ guiding the construction of materialisations was implicit. We show how the existing constructions can be converted to generating structures in the full version.

Example 7 The materialisation $\mathcal{I}_2$ from Example 4 can be generated by the structure $\mathcal{G}_2$ shown below:



For a generating structure $\mathcal{G}$ for $\mathcal{K}$ and a signature Σ, the Σ-*types* $\boldsymbol{t}_\Sigma^{\mathcal{G}}(u)$ and $\boldsymbol{r}_\Sigma^{\mathcal{G}}(u,v)$ of $u, v \in \Delta^{\mathcal{G}}$ are defined by:

$$\boldsymbol{t}_\Sigma^{\mathcal{G}}(u) = \{\, \Sigma\text{-concept name } A \mid u \in A^{\mathcal{G}} \,\},$$

$$\boldsymbol{r}_\Sigma^{\mathcal{G}}(u,v) = \begin{cases} \{\, \Sigma\text{-role } R \mid (u,v) \in R^{\mathcal{G}} \,\}, & \text{if } u, v \in \mathsf{ind}(\mathcal{K}),\\ \{\, \Sigma\text{-role } R \mid R \in (u,v)^{\mathcal{G}} \,\}, & \text{if } u \rightsquigarrow v,\\ \emptyset, & \text{otherwise,} \end{cases}$$

where $(P^-)^{\mathcal{G}}$ is the converse of $P^{\mathcal{G}}$. We also define $\bar{\boldsymbol{r}}_\Sigma^{\mathcal{G}}(u,v)$ to contain the inverses of the roles in $\boldsymbol{r}_\Sigma^{\mathcal{G}}(u,v)$; note that $\bar{\boldsymbol{r}}_\Sigma^{\mathcal{G}}(u,v)$ is not the same as $\boldsymbol{r}_\Sigma^{\mathcal{G}}(v,u)$; cf. the T^-, S^--cycle in Example 7. We write $u \rightsquigarrow^\Sigma v$ if $u \rightsquigarrow v$ and $\boldsymbol{r}_\Sigma^{\mathcal{G}}(u,v) \neq \emptyset$.

In the next section, we show that, for a DL $\mathcal{L}$ having finitely generated materialisations, the problem of checking Σ-query entailment between $\mathcal{L}$-KBs can be reduced to the problem of finding a winning strategy in a game played on the generating structures for these KBs.

Σ-Query Entailment by Games

Suppose a DL $\mathcal{L}$ has finitely generated materialisations, $\mathcal{K}_i$ is a consistent $\mathcal{L}$-KB, for $i = 1, 2$, and Σ a signature. Let $\mathcal{G}_i = (\Delta^{\mathcal{G}_i}, \cdot^{\mathcal{G}_i}, \rightsquigarrow_i)$ be a generating structure for $\mathcal{K}_i$ and let $\mathcal{M}_i$ be its materialisation; $\mathcal{G}_i^\Sigma$ and $\mathcal{M}_i^\Sigma$ denote the restrictions of $\mathcal{G}_i$ and $\mathcal{M}_i$ to Σ.

We begin with a very simple game on the finite generating structure $\mathcal{G}_2^\Sigma$ and the possibly infinite materialisation $\mathcal{M}_1^\Sigma$.

Infinite game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$. This game is played by two players: player 2 and player 1. The *states* of the game are of the form $\mathfrak{s}_i = (u_i \mapsto \sigma_i)$, for $i \geq 0$, where $u_i \in \Delta^{\mathcal{G}_2}$ and $\sigma_i \in \Delta^{\mathcal{M}_1}$ satisfy the following condition:

($\mathbf{s}_1$) $\boldsymbol{t}_\Sigma^{\mathcal{G}_2}(u_i) \subseteq \boldsymbol{t}_\Sigma^{\mathcal{M}_1}(\sigma_i)$.

The game starts in a state $\mathfrak{s}_0 = (u_0 \mapsto \sigma_0)$ with $\sigma_0 = u_0$ in case $u_0 \in \mathsf{ind}(\mathcal{K}_2)$. In each round $i > 0$, player 2 challenges player 1 with some $u_i \in \Delta^{\mathcal{G}_2}$ such that $u_{i-1} \rightsquigarrow_2^\Sigma u_i$. Player 1 has to respond with a $\sigma_i \in \Delta^{\mathcal{M}_1}$ satisfying **($\mathbf{s}_1$)** and

($\mathbf{s}_2$) $\boldsymbol{r}_\Sigma^{\mathcal{G}_2}(u_{i-1}, u_i) \subseteq \boldsymbol{r}_\Sigma^{\mathcal{M}_1}(\sigma_{i-1}, \sigma_i)$.

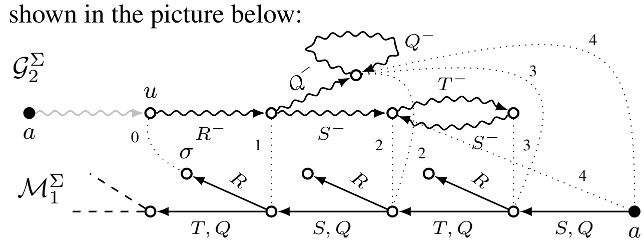
This gives the next state $\mathfrak{s}_i = (u_i \mapsto \sigma_i)$. Note that of all the u_i only u_0 may be an ABox individual; however, there is no such a restriction on the σ_i. A *play of length $n \geq 0$ starting from $\mathfrak{s}_0$* is any sequence $\mathfrak{s}_0, \dots, \mathfrak{s}_n$ of states obtained as described above. For an ordinal $\lambda \leq \omega$, we say that player 1 has a *λ-winning strategy in the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ starting from a state $\mathfrak{s}_0$* if, for any play of length $i < \lambda$, which starts from $\mathfrak{s}_0$ and conforms with this strategy, and any challenge of player 2 in round $i+1$, player 1 has a response.

The following theorem gives a game-theoretic flavour to the criterion of Theorem 5 (see the full paper for a proof).

Theorem 8 *$\mathcal{M}_2$ is finitely Σ-homomorphically embeddable into $\mathcal{M}_1$ iff the following conditions hold*:

(abox) $\boldsymbol{r}_\Sigma^{\mathcal{M}_2}(a,b) \subseteq \boldsymbol{r}_\Sigma^{\mathcal{M}_1}(a,b)$, *for any $a, b \in \mathsf{ind}(\mathcal{K}_2)$*;
(win) *for any $u_0 \in \Delta^{\mathcal{G}_2}$ and $n < \omega$, there exists $\sigma_0 \in \Delta^{\mathcal{M}_1}$ such that player 1 has an n-winning strategy in the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ starting from $(u_0 \mapsto \sigma_0)$.*

Example 9 Let $\Sigma = \{Q, R, S, T\}$. Consider $\mathcal{G}_2^\Sigma$ and $\mathcal{M}_2^\Sigma$ shown in the picture below:



For any $n < \omega$ and $u \in \Delta^{\mathcal{G}_2}$, player 1 has an n-winning strategy in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$. A 4-winning strategy starting from $(u \mapsto \sigma)$ is shown by dotted lines (in round 2, player 2 has two possible challenges). For a larger n, a suitable σ can be chosen further away from the root a of $\mathcal{M}_1$.

The criterion of Theorem 8 does not seem to be a big improvement on Theorem 5 as we still have to deal with an infinite materialisation. Our aim now is to show that condition **(win)** in the infinite game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ can be checked by analysing a more complex game on the *finite* generating structures $\mathcal{G}_2$ and $\mathcal{G}_1$. We consider four types of strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$. For each type, τ, we define a game $G^\tau_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ such that, for any $u_0 \in \Delta^{\mathcal{G}_2}$, the following conditions are equivalent:

($< \omega$) for every $n < \omega$, player 1 has an n-winning strategy of type τ in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ starting from some $(u_0 \mapsto \sigma^n_0)$;

(ω) player 1 has an ω-winning strategy in $G^\tau_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ starting from some state depending on u_0 and τ.

We start by considering 'forward' winning strategies that are sufficient for the DLs without inverse roles.

Forward strategy and game $G^f_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$. We say that a λ-strategy ($\lambda \le \omega$) for player 1 in the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ is *forward* if, for any play of length $i - 1 < \lambda$, which conforms with this strategy, and any challenge $u_{i-1} \rightsquigarrow^\Sigma_2 u_i$ by player 2, the response σ_i of player 1 is such that either $\sigma_{i-1}, \sigma_i \in \mathsf{ind}(\mathcal{K}_1)$ or $\sigma_i = \sigma_{i-1}v$, for some $v \in \Delta^{\mathcal{G}_1}$.

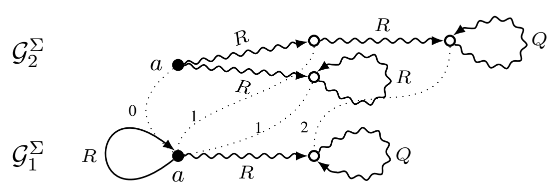
For example, if the $\mathcal{G}_i$, $i = 1, 2$, satisfy the condition

(f) the Σ-labels on $\rightsquigarrow_i$-edges contain no inverse roles,

then *every* strategy in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ is forward. This is clearly the case for *Horn-$\mathcal{ALCH}$*, *Horn-$\mathcal{ALC}$*, $\mathcal{ELH}$ and $\mathcal{EL}$, which by definition do not have inverse roles.

The existence of a forward λ-winning strategy for player 1 in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ is equivalent to the existence of such a strategy in the game $G^f_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$, which is defined similarly to $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ but with two modifications: (1) it is played on $\mathcal{G}_2$ and $\mathcal{G}_1$; and (2) the response $x_i \in \Delta^{\mathcal{G}_1}$ of player 1 to a challenge $u_{i-1} \rightsquigarrow^\Sigma_2 u_i$ must be such that either $x_{i-1}, x_i \in \mathsf{ind}(\mathcal{K}_1)$ or $x_{i-1} \rightsquigarrow_1 x_i$, and **(s$_1$)**–**(s$_2$)** hold (with $\mathcal{G}_1$ and x_i in place of $\mathcal{M}_1$ and σ_i).

Example 10 Let $\mathcal{G}_2$ and $\mathcal{G}_1$ be as shown below. Then, for any $u \in \Delta^{\mathcal{G}_2}$, there is $x \in \Delta^{\mathcal{G}_1}$ such that player 1 has an ω-winning strategy in $G^f_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ starting from $(u \mapsto x)$.



The next theorem follows from König's Lemma:

Lemma 11 *For $u_0 \in \Delta^{\mathcal{G}_2}$, condition ($< \omega$) holds for forward strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ iff (ω) holds in $G^f_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ for some state $(u_0 \mapsto x_0)$.*

$G^f_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ is a standard simulation or reachability game on finite graphs, where the existence of ω-winning strategies for player 1 follows from the existence of n-winning strategies for $n = O(|\mathcal{G}_2| \times |\mathcal{G}_1|)$, which can be checked in polynomial time (Mazala 2001; Baier and Katoen 2007). By Theorem 6 and **(f)**, we obtain:

Theorem 12 *For combined complexity, checking Σ-query entailment is in* P *for $\mathcal{EL}$ and $\mathcal{ELH}$ KBs, and in* ExpTime *for Horn-$\mathcal{ALC}$ and Horn-$\mathcal{ALCH}$ KBs. For data complexity, it is in* P *for all these DLs.*

In comparison to forward strategies, the winning strategies used in Example 9 can be described as 'backward.'

Backward strategy and game $G^b_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$. A λ-strategy for player 1 in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ is *backward* if, for any play of length $i - 1 < \lambda$, which conforms with this strategy, and any challenge $u_{i-1} \rightsquigarrow^\Sigma_2 u_i$ by player 2, the response σ_i of player 1 is the *immediate predecessor* of σ_{i-1} in $\mathcal{M}_1$ in the sense that $\sigma_{i-1} = \sigma_i w$, for some $w \in \Delta^{\mathcal{G}_1}$ (player 1 loses in case $\sigma_{i-1} \in \mathsf{ind}(\mathcal{K}_1)$). Note that, since $\mathcal{M}_1$ is tree-shaped, the response of player 1 to any different challenge $u_{i-1} \rightsquigarrow^\Sigma_2 u'_i$ must be the same σ_i; cf. Example 9.

That is why the states of the game $G^b_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ are of the form $\mathfrak{s}_i = (\Xi_i \mapsto x_i)$, where $\Xi_i \subseteq \Delta^{\mathcal{G}_2}$, $\Xi_i \neq \emptyset$, and $x_i \in \Delta^{\mathcal{G}_1}$ satisfy the following condition:

(s$'_1$) $\boldsymbol{t}^{\mathcal{G}_2}_\Sigma(u) \subseteq \boldsymbol{t}^{\mathcal{G}_1}_\Sigma(x_i)$, for all $u \in \Xi_i$.

The game starts in a state $\mathfrak{s}_0 = (\Xi_0 \mapsto x_0)$ such that

(s$'_0$) if $u \in \Xi_0 \cap \mathsf{ind}(\mathcal{K}_2)$, then $x_0 = u \in \mathsf{ind}(\mathcal{K}_1)$.

For each $i > 0$, player 2 always challenges player 1 with the set $\Xi_i = \Xi^\rightsquigarrow_{i-1}$, where

$$\Xi^\rightsquigarrow = \{v \in \Delta^{\mathcal{G}_2} \mid u \rightsquigarrow^\Sigma_2 v, \text{ for some } u \in \Xi\},$$

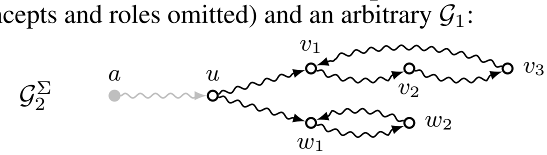
provided that it is not empty (otherwise, player 2 loses). Player 1 responds with $x_i \in \Delta^{\mathcal{G}_1}$ such that $x_i \rightsquigarrow_1 x_{i-1}$ and **(s$'_1$)** and the following condition hold:

(s$'_2$) $\boldsymbol{r}^{\mathcal{G}_2}_\Sigma(u, v) \subseteq \bar{\boldsymbol{r}}^{\mathcal{G}_1}_\Sigma(x_{i-1}, x_i)$, for all $u \in \Xi_{i-1}, v \in \Xi_i$.

Lemma 13 *For $u_0 \in \Delta^{\mathcal{G}_2}$, condition ($< \omega$) holds for backward strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ iff (ω) holds in $G^b_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ for some state $(\{u_0\} \mapsto x_0)$.*

Although Lemmas 11 and 13 look similar, the game $G^b_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ turns out to be more complex than $G^f_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$.

Example 14 To illustrate, consider $\mathcal{G}^\Sigma_2$ shown below (with concepts and roles omitted) and an arbitrary $\mathcal{G}_1$:



A play in $G^b_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ may proceed as: $(\{u\} \mapsto x_0)$, $(\{v_1, w_1\} \mapsto x_1)$, $(\{v_2, w_2\} \mapsto x_2)$, $(\{v_3, w_1\} \mapsto x_3)$, etc. This gives at least 6 different sets Ξ_i. But if $\mathcal{G}_2$ contained k cycles of lengths $p_1, \dots, p_k$, where p_i is the ith prime number, then the number of states in $G^b_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ could be exponential ($p_1 \times \cdots \times p_k$). In fact, we have the following:

Lemma 15 *Checking (ω) in Lemma 13 is* coNP*-hard.*

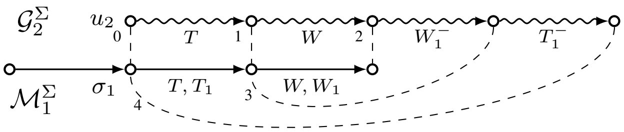
Observe that in the case of *DL-Lite$_{core}$* and *DL-Lite$_{horn}$* (which have inverse roles but no RIs), generating structures $\mathcal{G} = (\Delta^\mathcal{G}, \cdot^\mathcal{G}, \rightsquigarrow)$ can be defined so that, for any $u \in \Delta^\mathcal{G}$ and R, there is *at most one* v with $u \rightsquigarrow v$ and $R \in \boldsymbol{r}^\mathcal{G}(u, v)$ (Kontchakov et al. 2010). As a result, any n-winning strategy starting from $(u_0 \mapsto \sigma_0)$ in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ consists of a (possibly empty) backward part followed by a (possibly empty) forward part. Moreover, in the backward games for these DLs, the sets Ξ_i are always *singletons*. Thus, the number of states in the combined backward/forward games on the $\mathcal{G}_i$ is polynomial, and the existence of winning strategies can be checked in polynomial time.

Theorem 16 *Checking Σ-query entailment for DL-Lite$_{core}$ and DL-Lite$_{horn}$ KBs is in* P *for both combined and data complexity.*

An arbitrary strategy for player 1 in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ is a combination of a backward strategy and a number of start-bounded strategies to be defined next.

Start-bounded strategy and game $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$. A strategy for player 1 in the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ starting from a state $(u_0 \mapsto \sigma_0)$ is *start-bounded* if it never leads to $(u_i \mapsto \sigma_i)$ such that $\sigma_0 = \sigma_i v$, for some v and $i > 0$. In other words, player 1 cannot use those elements of $\mathcal{M}_1$ that are located closer to the ABox than σ_0; the ABox individuals in $\mathcal{M}_1$ can only be used if $\sigma_0 \in \mathsf{ind}(\mathcal{K}_1)$.

Example 17 The strategy starting from $(u_2 \mapsto \sigma_1)$ and shown below is start-bounded:



In the game $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$, player 1 will have to guess *all* the points of $\mathcal{G}_2$ that are mapped to the same point of $\mathcal{M}_1$.

The *states* of $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ are of the form $(\Gamma_i, \Xi_i \mapsto x_i)$, $i \geq 0$, where $\Gamma_i, \Xi_i \subseteq \Delta^{\mathcal{G}_2}$, $\Xi_i \neq \emptyset$, $x_i \in \Delta^{\mathcal{G}_1}$ and **(s$'_1$)** holds. The initial state is of the form $(\emptyset, \Xi_0 \mapsto x_0)$ such that **(s$'_0$)** holds. In each round $i > 0$, player 2 challenges player 1 with some $u \rightsquigarrow^\Sigma_2 v$ such that $u \in \Xi_{i-1}$ and

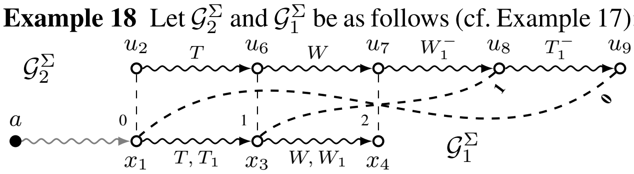
(nbk) if $v \in \Gamma_{i-1}$ then $\boldsymbol{r}^{\mathcal{G}_2}_\Sigma(u, v) \not\subseteq \bar{\boldsymbol{r}}^{\mathcal{G}_1}_\Sigma(x_{i-2}, x_{i-1})$.

Player 1 responds with either a state $(\Xi_{i-1}, \Xi_i \mapsto x_i)$ such that $x_{i-1} \rightsquigarrow_1 x_i$ (and so $x_i \notin \mathsf{ind}(\mathcal{K}_1)$) and **(s$''_2$)** holds, or a state $(\emptyset, \Xi_i \mapsto x_i)$ such that $x_{i-1}, x_i \in \mathsf{ind}(\mathcal{K}_1)$ and

(s$''_2$) $\boldsymbol{r}^{\mathcal{G}_2}_\Sigma(u, v) \subseteq \boldsymbol{r}^{\mathcal{G}_1}_\Sigma(x_{i-1}, x_i)$.

We make challenges $u \rightsquigarrow^\Sigma_2 v$, for which $u \in \Xi_{i-1}$ and **(nbk)** does not hold, 'illegitimate' because x_{i-2} can always be used as a response. Because of this, player 1 always moves 'forward' in $\mathcal{G}_1$, but has to guess appropriate sets Ξ_i in advance. Note that Γ_i is always uniquely determined by x_{i-1}, x_i and Ξ_{i-1} (and it is either Ξ_{i-1} or empty).

Example 18 Let $\mathcal{G}_2^\Sigma$ and $\mathcal{G}_1^\Sigma$ be as follows (cf. Example 17):



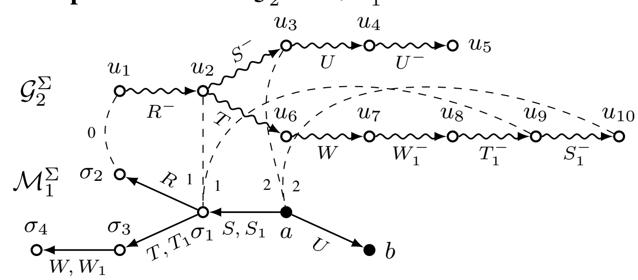
We show that player 1 has an ω-winning strategy in $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ starting from $(\emptyset, \{u_2, u_9\} \mapsto x_1)$. Player 2 challenges with $u_2 \rightsquigarrow^\Sigma_2 u_6$, and player 1 responds with $(\{u_2, u_9\}, \{u_6, u_8\} \mapsto x_3)$. Then player 2 picks $u_6 \rightsquigarrow^\Sigma_2 u_7$ and player 1 responds with $(\{u_6, u_8\}, \{u_7\} \mapsto x_4)$, where the game ends. Note the crucial guesses $\{u_2, u_9\} \mapsto x_1$ and $\{u_6, u_8\} \mapsto x_3$ made by player 1. If player 1 responded with $(\{u_2, u_9\}, \{u_6\} \mapsto x_3)$ (and failed to guess that u_8 must also be mapped to x_3), then after the challenge $u_6 \rightsquigarrow^\Sigma_2 u_7$ and response $(\{u_6, u_8\}, \{u_7\} \mapsto x_4)$), player 2 would pick $u_7 \rightsquigarrow^\Sigma_2 u_8$, to which player 1 could not respond.

Lemma 19 *For any $u_0 \in \Delta^{\mathcal{G}_2}$, condition $(< \omega)$ holds for start-bounded strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ iff (ω) holds in $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ for some state $(\emptyset, \Xi_0 \mapsto x_0)$ with $u_0 \in \Xi_0$.*

As we shall see in the next section, the problem of checking the conditions of this lemma is ExpTime-hard.

Arbitrary strategies and game $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$. An arbitrary winning strategy in the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ can be composed of one backward and a number of start-bounded strategies.

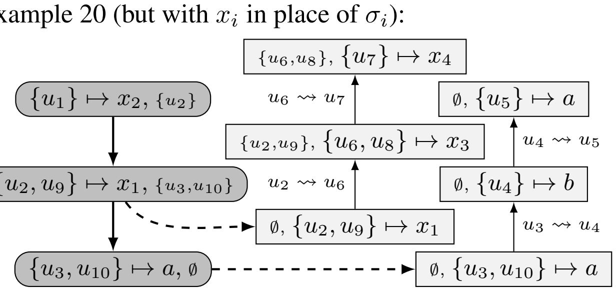
Example 20 Consider $\mathcal{G}_2^\Sigma$ and $\mathcal{M}_1^\Sigma$ shown below:



Starting from $(u_1 \mapsto \sigma_2)$, player 1 can respond to the challenges $u_1 \rightsquigarrow^\Sigma_2 u_2 \rightsquigarrow^\Sigma_2 u_3$ according to the backward strategy; the challenges $u_2 \rightsquigarrow^\Sigma_2 u_6 \rightsquigarrow^\Sigma_2 u_7 \rightsquigarrow^\Sigma_2 u_8 \rightsquigarrow^\Sigma_2 u_9$ according to the start-bounded strategy as in Example 17; the challenges $u_3 \rightsquigarrow^\Sigma_2 u_4 \rightsquigarrow^\Sigma_2 u_5$ also according to the obvious start-bounded strategy; finally, the challenge $u_9 \rightsquigarrow^\Sigma_2 u_{10}$ needs a response according to the backward strategy. We will combine the two backward strategies into a single one, but keep the start-bounded ones separate.

The game $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ begins as $G^b_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$, but with states of the form $(\Xi_i \mapsto x_i, \Psi_i)$, $i \geq 0$, where $\Xi_i \subseteq \Delta^{\mathcal{G}_2}$ and $x_i \in \Delta^{\mathcal{G}_1}$ satisfy **(s$'_1$)** and Ψ_i is a (possibly empty) subset of $\Xi^{\rightsquigarrow}_i$, which indicates initial challenges in start-bounded games. The initial state satisfies **(s$'_0$)**. In each round $i > 0$, if $x_{i-1} \in \mathsf{ind}(\mathcal{K}_1)$ then player 2 launches the start-bounded game $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ with the initial state $(\emptyset, \Xi_{i-1} \mapsto x_{i-1})$. Otherwise, if $x_{i-1} \notin \mathsf{ind}(\mathcal{K}_1)$, player 2 has two options. First, he can challenge player 1 with the set Ψ_{i-1} (that is, similar to the backward game but with a possibly smaller Ψ_{i-1} in place of $\Xi^{\rightsquigarrow}_{i-1}$); player 1 responds to this challenge with a state $(\Xi_i \mapsto x_i, \Psi_i)$ such that $\Psi_{i-1} \subseteq \Xi_i$, $x_i \rightsquigarrow_1 x_{i-1}$ and **(s$'_2$)** holds. Second, player 2 can launch the start-bounded game $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ with the initial state $(\emptyset, \Xi_{i-1} \mapsto x_{i-1})$, where the first challenge of player 2 must be picked from $\Phi_{i-1} = \Xi^{\rightsquigarrow}_{i-1} \setminus \Psi_{i-1}$.

Example 21 We illustrate the ω-winning strategy for player 1 in $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ starting from $(\{u_1\} \mapsto x_2, \{u_2\})$, where $\mathcal{G}_2^\Sigma$ is from Example 20 and $\mathcal{G}_1^\Sigma$ looks like $\mathcal{M}_1^\Sigma$ from Example 20 (but with x_i in place of σ_i):



Lemma 22 *For any $u_0 \in \Delta^{\mathcal{G}_2}$, condition $(<\omega)$ holds for arbitrary strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ iff (ω) holds in $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ for some state $(\Xi_0 \mapsto x_0, \Psi_0)$ with $u_0 \in \Xi_0$.*

Condition (ω) in the lemma above is checked in time $O(|\mathsf{ind}(\mathcal{K}_2)| \times 2^{|\Delta^{\mathcal{G}_2} \setminus \mathsf{ind}(\mathcal{K}_2)|} \times |\Delta^{\mathcal{G}_1}|)$, which can be readily seen by analysing the full game graph for $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ (similar to that in Example 21). By Theorem 6, we then obtain:

Theorem 23 *For combined complexity, Σ-query entailment is in* 2ExpTime *for Horn-$\mathcal{ALCHI}$ and Horn-$\mathcal{ALCI}$ KBs, and in* ExpTime *for DL-Lite$^{\mathcal{H}}_{horn}$ and DL-Lite$^{\mathcal{H}}_{core}$ KBs. For data complexity, these problems are all in* P.

Lower Bounds

We have shown that, for all of our DLs, Σ-query entailment and inseparability are in P for data complexity. The next theorem establishes a matching lower bound:

Theorem 24 *For data complexity, Σ-query entailment and inseparability are* P*-hard for DL-Lite$_{core}$ and $\mathcal{EL}$ KBs.*

Proof. The proof is by reduction of the P-complete entailment problem for *acyclic* Horn ternary clauses: given a conjunction φ of clauses of the form a_i and $a_i \wedge a_{i'} \to a_j$, $i, i' < j$, decide whether a_n is true in every model of φ. Consider the $\mathcal{EL}$ TBox $\mathcal{T} = \{V \sqsubseteq \exists P.(\exists R_1.V \sqcap \exists R_2.V)\}$ and an ABox $\mathcal{A}$ comprised of $F(a_n)$ and

$$P(a_i, a_i),\ R_1(a_i, a_i),\ R_2(a_i, a_i), \text{ for each clause } a_i \text{ in } \varphi,$$
$$P(a_j, c), R_1(c, a_i), R_2(c, a_{i'}), \text{ for } c = a_i \wedge a_{i'} \to a_j \text{ in } \varphi.$$

Set $\Sigma = \{F, P, R_1, R_2\}$, $\mathcal{K}_2 = (\mathcal{T}, \mathcal{A} \cup \{V(a_n)\})$ and $\mathcal{K}_1 = (\emptyset, \mathcal{A})$. Obviously, $\mathcal{K}_2$ Σ-query entails $\mathcal{K}_1$. On the other hand, the materialisation of $\mathcal{K}_2$ is (finitely) Σ-homomorphically embeddable in the materialisation of $\mathcal{K}_1$ iff φ derives a_n (see the full version for details). For *DL-Lite$_{core}$*, we take $\mathcal{T}$ to contain $V \sqsubseteq \exists P$, $\exists P^- \sqsubseteq \exists R_i$ and $\exists R_i^- \sqsubseteq V$, for $i = 1, 2$. ❑

For combined complexity, ExpTime-hardness of Σ-query inseparability for *Horn-$\mathcal{ALC}$* can be proved by reduction of the subsumption problem: we have $\mathcal{T} \models A \sqsubseteq B$ iff $(\mathcal{T}, \{A(a)\})$ and $(\mathcal{T} \cup \{A \sqsubseteq B\}, \{A(a)\})$ are $\{B\}$-query inseparable. We now establish matching lower bounds in the technically challenging cases.

Theorem 25 *For combined complexity, Σ-query entailment and inseparability are (i)* 2ExpTime*-hard for Horn-$\mathcal{ALCI}$ KBs and (ii)* ExpTime*-hard for DL-Lite$^{\mathcal{H}}_{core}$ KBs.*

Proof. The proof of (*i*) is by encoding alternating Turing machines (ATMs) with exponential tape and using the fact that AExpSpace = 2ExpTime; see, e.g. (Kozen 2006).

Let $M = (\Gamma, Q, q_0, q_1, \delta)$ be an ATM with a tape alphabet Γ, a set of states Q partitioned into existential $Q_\exists$ and universal $Q_\forall$ states, an initial state $q_0 \in Q_\exists$, an accepting state $q_1 \in Q$, and a transition function

$$\delta\colon (Q \setminus \{q_1\}) \times \Gamma \times \{1, 2\} \to Q \times \Gamma \times \{-1, 0, +1\},$$

which, for a state q and symbol a, gives two instructions, $\delta(q, a, 1)$ and $\delta(q, a, 2)$. We assume that existential and universal states strictly alternate: any transition from an existential state results in a universal state, and vice versa. We extend δ with the instructions $\delta(q_1, a, k) = (q_1, a, 0)$, for $a \in \Gamma$ and $k = 1, 2$, which go into an infinite loop if M reaches the accepting state q_1. Thus, assuming that M terminates on every input, it accepts $\vec{w}$ iff the modified ATM M' has a run on $\vec{w}$, all branches of which are infinite.

Our aim is to construct, given M and $\vec{w}$, TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ and a signature Σ such that M' has a run with only infinite branches iff the materialisation $\mathcal{M}_2$ of $(\mathcal{T}_2, \{A(c)\})$ is finitely Σ-homomorphically embeddable into the materialisation $\mathcal{M}_1$ of $(\mathcal{T}_1, \{A(c)\})$. Let f be a polynomial such that, on any input of length m, M uses at most $2^n - 2$ tape cells, with $n = f(m)$, which are numbered from 1 to $2^n - 2$, and the head stays to the right of cell 0, which contains the marker $\flat \in \Gamma$. The construction proceeds in five steps.

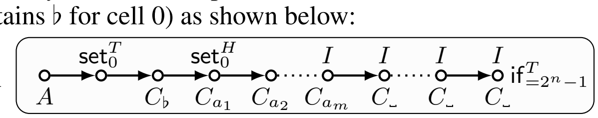
Step 0. We use tuples of $2n$ concepts to represent distances of up to 2^n between the cells on the tape in consecutive configurations. We refer to a tuple $Y_{n-1}, \overline{Y}_{n-1}, \dots, Y_0, \overline{Y}_0$ of concept names as Y and assume that the TBox contains the following CIs to encode an n-bit *R-counter on Y*:

$$\overline{Y}_k \sqcap Y_{k-1} \sqcap \dots \sqcap Y_0 \sqsubseteq \forall R.(Y_k \sqcap \overline{Y}_{k-1} \sqcap \dots \sqcap \overline{Y}_0), \quad n > k \geq 0,$$
$$\overline{Y}_i \sqcap \overline{Y}_k \sqsubseteq \forall R.\overline{Y}_i \text{ and } Y_i \sqcap \overline{Y}_k \sqsubseteq \forall R.Y_i, \quad n > i > k.$$

We use the expression $\mathsf{if}^Y_{=2^n-1}$ on the left-hand side of CIs to say that the Y-value is $2^n - 1$ (which is a shortcut for $Y_{n-1} \sqcap \dots \sqcap Y_0$); we also use $\mathsf{if}^Y_{<2^n-1}$ on the left-hand side of CIs for the complementary statement (which is a shortcut for n CIs with $\mathsf{if}^Y_{<2^n-1}$ replaced by each of $\overline{Y}_{n-1}, \dots, \overline{Y}_0$). Finally, we use set^Y_0 on the right-hand side of CIs for the reset command (which is equivalent to $\overline{Y}_{n-1} \sqcap \dots \sqcap \overline{Y}_0$). Note that the counter stops at $2^n - 1$: the R-successors of a domain element in $\mathsf{if}^Y_{=2^n-1}$ do not have to encode any value.

Step 1. First we encode configurations and transitions of M' using $\mathcal{T}_1$. We represent a configuration by a *block*, which is a sequence of $2^n + 1$ domain elements connected by a role P. The first element distinguishes the blocks for the two alternative transitions; using a P-counter on a tuple T, we assign indices from 0 to $2^n - 1$ to all other elements in each block. The element with index 0 is needed for padding. Each of the remaining $2^n - 1$ elements belongs to a concept C_a, for some $a \in \Gamma$: if the element with index $i + 1$ is in C_a, then the cell i is assumed to contain a in the configuration represented by the block (in particular, the element with index 1 contains $\flat$ for cell 0) as shown below:

$\mathcal{M}_1$



The first block represents the initial configuration: the input $\vec{w} = a_1 \dots a_m$ is followed by $2^n - m - 2$ blank symbols ␣ and the head is positioned over cell 1, which is indicated by the 0 value of the P-counter on a tuple H. This is achieved by the following CIs in the TBox $\mathcal{T}_1$:

$$A \sqsubseteq \exists P.(\mathsf{set}^T_0 \sqcap \exists P.(C_\flat \sqcap \exists P.(C_{a_1} \sqcap \mathsf{set}^H_0 \sqcap \exists P.(C_{a_2} \sqcap \exists P.(\dots \exists P.(C_{a_m} \sqcap I) \dots))))), \quad (\mathcal{T}_1\text{-}1)$$

$$\mathsf{if}^T_{<2^n-1} \sqcap I \sqsubseteq \exists P.(I \sqcap C_␣), \quad (\mathcal{T}_1\text{-}2)$$

$$\mathsf{if}^T_{=2^n-1} \sqcap I \sqsubseteq Z^0_{q_0 a_1}. \quad (\mathcal{T}_1\text{-}3)$$

Step 2. The contents of the tape and the head position in each configuration is encoded in a block of length $2^n + 1$; the current state $q \in Q$ is recorded in the concept Z^0_{qa} that contains the last element of the block ($a \in \Gamma$ specifies the contents of the active cell scanned by the head). At the end of the block, when the T-value reaches $2^n - 1$, we branch out one block for each of the two transitions, reset the P-counter on T, and propagate via Z^1_{qa} and Z^2_{qa} the current state and symbol in the active cell: for $q \in Q$ and $a \in \Gamma$, we add to $\mathcal{T}_1$ the CI

$$\mathsf{if}^T_{=2^n-1} \sqcap Z^0_{qa} \sqsubseteq \bigsqcap_{k=1,2} \exists P.(X_k \sqcap \exists P.(\mathsf{set}^T_0 \sqcap Z^k_{qa})), \quad (\mathcal{T}_1\text{-}4)$$

where X_1 and X_2 are two fresh concept names.

The acceptance condition for M' is enforced by means of $\mathcal{T}_2$, which uses a P-counter on a tuple T^0 for a block representing the initial configuration (a T^0-block):

$$A \sqsubseteq \exists P.\mathsf{set}^{T^0}_0, \qquad (\mathcal{T}_2\text{-}1)$$

$$\mathsf{if}^{T^0}_{<2^n-1} \sqsubseteq \exists P. \qquad (\mathcal{T}_2\text{-}2)$$

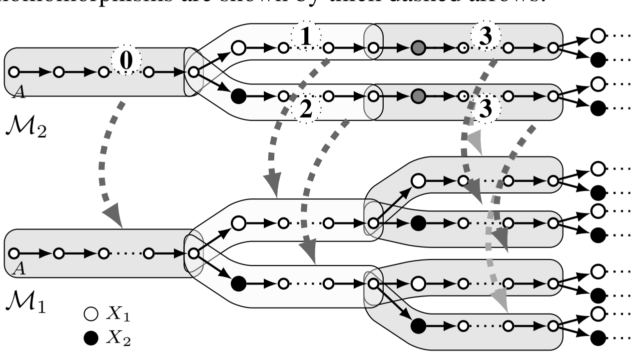
Two P-counters, on T^1 and T^2, are used for blocks representing configurations with universal sates (T^1- and T^2-blocks respectively) and one P-counter, on a tuple T^3, suffices for blocks representing configurations with existential states (T^3-blocks). These blocks are arranged into an infinite tree-like structure: the T^0-block is the root, from which a T^1- and a T^2-blocks branch out (successors of the initial state q_0 are universal). Each of them is followed by a T^3-block, which branches out a T^1- and a T^2-blocks, and so on. This is achieved by adding to $\mathcal{T}_2$ the following CIs:

$$\mathsf{if}^{T^k}_{=2^n-1} \sqsubseteq \bigsqcap_{j=1,2} \exists P.(X_j \sqcap \exists P.\mathsf{set}^{T^j}_0), \text{ for } k = 0, 3, \quad (\mathcal{T}_2\text{-}3)$$

$$\mathsf{if}^{T^k}_{<2^n-1} \sqsubseteq \exists P.G, \quad \text{for } k = 1, 2, 3, \qquad (\mathcal{T}_2\text{-}4)$$

$$\mathsf{if}^{T^k}_{=2^n-1} \sqsubseteq \exists P.\exists P.\mathsf{set}^{T^3}_0, \quad \text{for } k = 1, 2, \qquad (\mathcal{T}_2\text{-}5)$$

where G is a concept name. If $\Sigma = \{A, X_1, X_2, P\}$ then there is a unique Σ-homomorphism from the T^0-block in $\mathcal{M}_2$ to the block of the initial configuration in $\mathcal{M}_1$. Next, concepts X_1 and X_2 ensure that the T^1- and T^2-blocks are Σ-homomorphically mapped (in a unique way) into the respective blocks in $\mathcal{M}_1$, which reflects the acceptance condition of universal states. The following T^3-block, however, contains neither X_1 nor X_2 and can be mapped to either of the blocks in $\mathcal{M}_1$, which reflects the choice in existential states; see the picture below, where possible Σ-homomorphisms are shown by thick dashed arrows:



Step 3. Recall that the P-counter on H measures the distance from the head: if the active cell in the current configuration is k, then its H-value is 0 and the H-value of the cell $k - 2$ in a *successor* configuration is $2^n - 1$. So, until the H-counter reaches $2^n - 1$, the following CIs in $\mathcal{T}_1$ propagate the state and symbol in the active cell along the blocks: for $q \in Q$, $a \in \Gamma$ and $k = 0, 1, 2$,

$$\mathsf{if}^T_{<2^n-1} \sqcap \mathsf{if}^H_{<2^n-1} \sqcap Z^k_{qa} \sqsubseteq \bigsqcap_{b \in \Gamma} \exists P.(C_b \sqcap Z^k_{qa}) \qquad (\mathcal{T}_1\text{-}5)$$

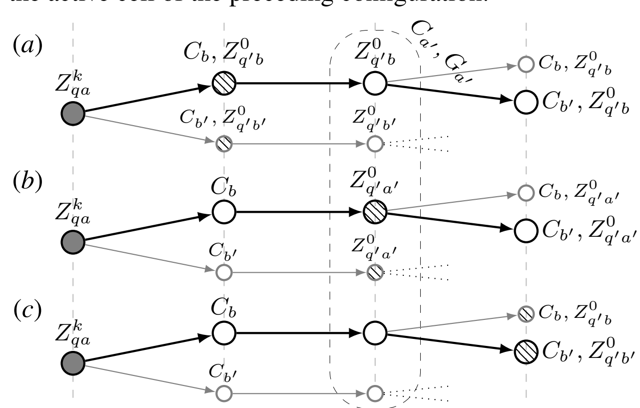
(for each $b \in \Gamma$, these CIs generate a branch in $\mathcal{M}_1$ to represent the same cell but with a different symbol, b, tentatively assigned to the cell—Step 4 will ensure that the correct branch and symbol are selected to match the cell contents in the preceding configuration). When the distance from the last head position is 2^n, the contents of the cell and the current state are changed according to δ:

$$\mathsf{if}^T_{<2^n-1} \sqcap \mathsf{if}^H_{=2^n-1} \sqcap Z^k_{qa} \sqsubseteq \bigsqcap_{b \in \Gamma} \exists P.(C_b \sqcap \Delta^k_{qa,b}), \quad (\mathcal{T}_1\text{-}6)$$

where $\delta(q, a, k) = (q', a', \sigma)$ and $\Delta^k_{qa,b}$ is the concept

$$\begin{array}{ll} \mathsf{set}^H_0 \sqcap Z^0_{q'b} \sqcap \exists P.(C_{a'} \sqcap G_{a'}), & \text{if } \sigma = -1, \\ \exists P.(C_{a'} \sqcap G_{a'} \sqcap \mathsf{set}^H_0 \sqcap Z^0_{q'a'}), & \text{if } \sigma = 0, \\ \exists P.(C_{a'} \sqcap G_{a'} \sqcap \bigsqcap_{b' \in \Gamma} \exists P.(C_{b'} \sqcap \mathsf{set}^H_0 \sqcap Z^0_{q'b'})), & \text{if } \sigma = +1 \end{array}$$

(the symbol in the active cell is changed according to the instruction, and the current state and symbol in the next active cell are then recorded in Z^0_{qa}). Since the head never visits cell 0, this happens over cells 0 to $2^n - 1$, that is, at least one element after the P-counter on T is reset to 0. These three situations are shown below, where grey and hatched nodes denote domain elements with H-values $2^n - 1$ and 0, respectively, and the domain elements in the dashed oval represent the active cell of the preceding configuration:



(Note that there is only one branch for the modified cell, which corresponds to the new symbol, a', in that cell; see explanations below.) Then, the current state and the symbol in the active cell are propagated along the tape using ($\mathcal{T}_1$-5).

Step 4. The CIs ($\mathcal{T}_1$-5)–($\mathcal{T}_1$-6) generate a separate P-successor for each $b \in \Gamma$. The correct one is chosen by a finite Σ-homomorphism, h, from $\mathcal{M}_2$ to $\mathcal{M}_1$. To exclude wrong choices, we take

$$\Sigma \;=\; \{A, P, X_1, X_2\} \;\cup\; \{D_a \mid a \in \Gamma\}.$$

Recall that if $d_1 \in C_a^{\mathcal{M}_1}$, for some $a \in \Gamma$, then it represents a cell containing a. The following CIs in $\mathcal{T}_1$ ensure that, for each $b \in \Gamma$ different from a, there is a block of (2^n+1)-many P^--connected elements that ends in the concept D_b (called a *D_b-block* in the sequel):

$$C_a \sqsubseteq D_a \sqcap \bigsqcap\nolimits_{b \in \Gamma \setminus \{a\}} G_b, \qquad (\mathcal{T}_1\text{-}7)$$

$$G_b \sqsubseteq \exists P^-.(S_b \sqcap \mathsf{set}_0^B), \qquad (\mathcal{T}\text{-}1)$$

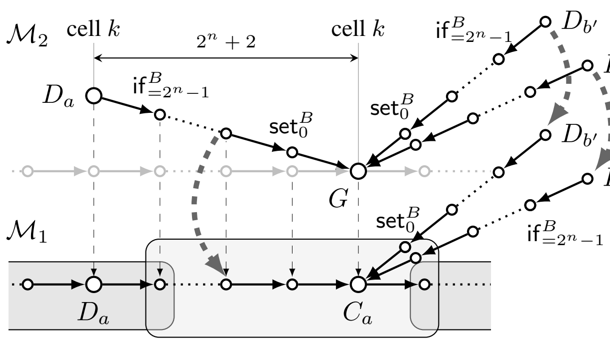
$$\mathsf{if}^B_{<2^n-1} \sqcap S_b \sqsubseteq \exists P^-.S_b, \qquad (\mathcal{T}\text{-}2)$$

$$\mathsf{if}^B_{=2^n-1} \sqcap S_b \sqsubseteq \exists P^-.D_b, \qquad (\mathcal{T}\text{-}3)$$

where we use a P^--counter on a tuple B (unlike P-counters in all other cases) and a concept S_b to propagate b along the whole block. Suppose $h(d_2) = d_1$ and d_2 belongs to G in $\mathcal{M}_2$ (it represents a cell in a non-initial configuration). Then the following CI and ($\mathcal{T}$-1)–($\mathcal{T}$-3), added to $\mathcal{T}_2$, generate a D_b-block, for *each* $b \in \Gamma$ (including a):

$$G \sqsubseteq \bigsqcap\nolimits_{b \in \Gamma} G_b. \qquad (\mathcal{T}_2\text{-}6)$$

Each of the D_b-blocks in $\mathcal{M}_2$, for $b \in \Gamma$ with $b \neq a$, can be mapped by h to the respective D_b-block in $\mathcal{M}_1$. By the choice of Σ, the only remaining D_a-block, in case a is tentatively contained in this cell, could be mapped (in the reverse order) along the branch in $\mathcal{M}_1$ *but only* if the cell contains a in the preceding configuration (that is, the element which is 2^n+1 steps closer to the root of $\mathcal{M}_1$ belongs to D_a):



Note (see $\Delta^k_{qa,b}$) that the cell whose content is changed generates the additional D_a-block in $\mathcal{M}_1$ to allow the respective D_a-block from $\mathcal{M}_2$ to be mapped there.

One can show that M' has a run with only infinite branches iff $(\mathcal{T}_1, \{A(c)\})$ Σ-query entails $(\mathcal{T}_2, \{A(c)\})$. It follows, by Theorem 2, that deciding Σ-query inseparability is 2ExpTime-hard.

(*ii*) A proof of ExpTime-hardness of Σ-query inseparability for *DL-Lite*$^{\mathcal{H}}_{core}$ KBs is given in the full paper. It uses the same idea of encoding computations of ATMs. One essential difference is that the expressive power of *DL-Lite*$^{\mathcal{H}}_{core}$ is not enough to represent n-bit counters in Step 0, and so we can only encode computations on polynomial tape. ❑

As a consequence of Theorems 3, 23 and 25 we obtain:

Theorem 26 *For combined complexity, the membership problem for universal UCQ-solutions is* 2ExpTime-*complete for Horn-$\mathcal{ALCHI}$ and Horn-$\mathcal{ALCI}$;* ExpTime-*complete for Horn-$\mathcal{ALCH}$, Horn-$\mathcal{ALC}$, DL-Lite$^{\mathcal{H}}_{horn}$ and DL-Lite$^{\mathcal{H}}_{core}$; and* P-*complete for $\mathcal{EL}$ and $\mathcal{ELH}$. For data complexity, all these problems are* P-*complete.*

In the case of *DL-Lite*$^{\mathcal{H}}_{core}$, we also obtain an ExpTime algorithm for checking the existence and computing universal UCQ-solutions. Indeed, given a KB $\mathcal{K}_1$, a target signature Σ_2 and a mapping $\mathcal{T}_{12}$, we first compute the Σ_2-ABox over $\mathsf{ind}(\mathcal{K}_1)$ that is implied by $\mathcal{K}_1$ and $\mathcal{T}_{12}$, and then check whether at least one KB $\mathcal{K}_2$ in Σ_2 with this ABox is a universal UCQ-solution (there are $\leq O(2^{|\Sigma_2|})$ such KBs). This gives an ExpTime upper bound for the non-emptiness problem for universal UCQ-solutions in *DL-Lite*$^{\mathcal{H}}_{core}$ (Arenas et al. 2013). Similarly, we can check in ExpTime whether the result of forgetting a signature in a *DL-Lite*$^{\mathcal{H}}_{core}$ KB exists.

Σ-query inseparability of *DL-Lite*$^{\mathcal{H}}_{core}$ TBoxes was known to sit between PSpace and ExpTime (Konev et al. 2011). Using the fact that witness ABoxes for *DL-Lite*$^{\mathcal{H}}_{core}$ TBox separability can always be chosen among the singleton ABoxes (Konev et al. 2011, Theorem 8), we can modify the proof of Theorem 25 to improve the PSpace lower bound:

Theorem 27 *Σ-query inseparability of DL-Lite$^{\mathcal{H}}_{core}$ TBoxes is* ExpTime-*complete.*

For more expressive DLs, TBox Σ-query inseparability is often harder than KB inseparability: for *DL-Lite$_{horn}$*, the space of relevant witness ABoxes for TBox separability is of exponential size and, in fact, TBox inseparability is NP-hard, while KB inseparability is in P. Similarly, Σ-query inseparability of $\mathcal{EL}$ KBs is tractable, while Σ-query inseparability of TBoxes is ExpTime-complete (Lutz and Wolter 2010). The complexity of TBox inseparability for Horn-DLs extending *Horn-$\mathcal{ALC}$* is not known.

Future Work

From a theoretical point of view, it would be of interest to investigate the complexity of Σ-query inseparability for KBs in more expressive Horn DLs (e.g., *Horn-$\mathcal{SHIQ}$*) and non-Horn DLs extending $\mathcal{ALC}$. We conjecture that the game technique developed in this paper can be extended to those DLs as well. Our games can also be used to define *efficient approximations* of Σ-query entailment and inseparability for KBs. The existence of a forward strategy, for example, provides a sufficient condition for Σ-query entailment for all of our DLs. Thus, one can extract a Σ-query module of a given KB $\mathcal{K}$ by exhaustively removing from $\mathcal{K}$ those inclusions and assertions α for which player 1 has a winning strategy in the game $G^f_\Sigma(\mathcal{G}_1, \mathcal{G}_2)$, where $\mathcal{G}_1$ is a generating structure for $\mathcal{K} \setminus \{\alpha\}$ and $\mathcal{G}_2$ for $\mathcal{K}$. The resulting modules are minimal for our DLs without inverse roles, and we conjecture that in practice they are often minimal for DLs with inverse roles as well; see (Konev et al. 2011) for experiments testing similar ideas for module extraction from TBoxes.

Finally, we plan to use the developed technique to investigate the complexity of the non-emptiness problem for universal UCQ-solutions in data exchange as well as algorithms for computing universal UCQ-solutions in various DLs.

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Appendix

The proofs of the theorems and lemmas from the paper are presented in the following order. We begin by proving Theorem 6, where we show how to construct finite generating structures for each of the languages and check that these structures give rise to materialisations. Theorem 6 is then used to prove Theorem 5, which characterises Σ-query entailment through finite Σ-homomorphisms. After that we prove Theorems 2 and 3 that establish a connection between query entailment, query inseparability and universal UCQ-solutions.

Next, we prove the results for our games: first, Theorem 8 relating finite Σ-homomorphisms with infinite games, and second, Lemma 22 saying that arbitrary games cover and admit arbitrary strategies. Proofs of Lemmas 11, 13 and 19 are obtained as corollaries of the proof of Lemma 22. Finally, we prove Lemma 19, establishing CONP-hardness of backward strategies.

We conclude with the proofs of lower bounds. First, we prove Theorem 24. Then, in the proof of Theorem 25 we show EXPTIME-hardness of Σ-query entailment in *DL-Lite*$^{\mathcal{H}}_{core}$.

Proof of Theorem 6: construction of generating structures

Theorem 6 *Horn-$\mathcal{ALCHI}$ and all of its fragments defined above have finitely generated materialisations. Moreover,*

– *for any $\mathcal{L} \in \{\mathcal{ALCHI}, \mathcal{ALCI}, \mathcal{ALCH}, \mathcal{ALC}\}$ and any Horn-$\mathcal{L}$ KB $(\mathcal{T}, \mathcal{A})$, a generating structure can be constructed in time $|\mathcal{A}| \cdot 2^{p(|\mathcal{T}|)}$, p a polynomial*;

– *for any $\mathcal{L}$ in the $\mathcal{EL}$ and DL-Lite families and any $\mathcal{L}$ KB $(\mathcal{T}, \mathcal{A})$, a generating structure can be constructed in time $|\mathcal{A}| \cdot p(|\mathcal{T}|)$, p a polynomial.*

We construct the generating structures first for *Horn-$\mathcal{ALCHI}$*, then for $\mathcal{ELH}$, and finally for *DL-Lite*$^{\mathcal{H}}_{horn}$. The construction for *Horn-$\mathcal{ALCI}$*, *Horn-$\mathcal{ALCH}$*, and *Horn-$\mathcal{ALC}$* is the same as for *Horn-$\mathcal{ALCHI}$*, the construction for $\mathcal{EL}$ is the same as for $\mathcal{ELH}$, and the construction for *DL-Lite*$_{core}$, *DL-Lite*$^{\mathcal{H}}_{core}$, and *DL-Lite*$_{horn}$ is the same as for *DL-Lite*$^{\mathcal{H}}_{horn}$ and therefore omitted.

Horn-$\mathcal{ALCHI}$

To construct the generating structure for *Horn-$\mathcal{ALCHI}$* TBoxes we first transform the TBox into normal form (Krötzsch, Rudolph, and Hitzler 2007). A *Horn-$\mathcal{ALCHI}$* TBox is in *normal form* if its concept inclusions are of the following form

$$\begin{array}{ll} A \sqsubseteq B, & A_1 \sqcap A_2 \sqsubseteq B, \\ A \sqsubseteq \bot, & \top \sqsubseteq B, \\ A \sqsubseteq \exists R.B, & A \sqsubseteq \forall R.B, \\ \exists R.A \sqsubseteq B, & R_1 \sqsubseteq R_2, \end{array}$$

where A, A_1, A_2, B are concept names and R, R_1, R_2 are roles. The following result is well-known (Krötzsch, Rudolph, and Hitzler 2007; Eiter et al. 2008):

Theorem A.28 *For every Horn-$\mathcal{ALCHI}$ TBox $\mathcal{T}$ one can construct in polynomial time a Horn-$\mathcal{ALCHI}$ TBox $\mathcal{T}'$ in normal form such that $\mathcal{T}$ and $\mathcal{T}'$ are Σ-query inseparable for the signature Σ of $\mathcal{T}$. Moreover,*

- *if $\mathcal{T}$ does not contain any role inclusions then $\mathcal{T}'$ does not contain any role inclusions*;
- *if $\mathcal{T}$ does not contain any inverse roles then $\mathcal{T}'$ does not contain any inverse roles.*

Now assume a consistent KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with a *Horn-$\mathcal{ALCHI}$* TBox $\mathcal{T}$ in normal form is given. By $\mathsf{sub}(\mathcal{T})$ we denote the set of subconcepts of concepts in $\mathcal{T}$. The $\mathcal{T}$-type of d in $\mathcal{I}$ is defined by taking

$$\boldsymbol{h}^{\mathcal{I}}(d) = \{C \in \mathsf{sub}(\mathcal{T}) \mid d \in C^{\mathcal{I}}\}.$$

We say that t is a *$\mathcal{T}$-type* if there exists a model $\mathcal{I}$ of $\mathcal{T}$ such that $t = \boldsymbol{h}^{\mathcal{I}}(d)$, for some $d \in \Delta^{\mathcal{I}}$. Denote by $\mathsf{type}(\mathcal{T})$ the set of all $\mathcal{T}$-types. It is well known that $\mathsf{type}(\mathcal{T})$ can be computed in exponential time in $|\mathcal{T}|$. We now construct the finitely generating structure $\mathcal{G} = (\Delta^{\mathcal{G}}, \cdot^{\mathcal{G}}, \rightsquigarrow)$ for $\mathcal{K}$, where $\Delta^{\mathcal{G}} = \mathsf{ind}(\mathcal{K}) \cup \Omega$.

For any role R in $\mathcal{T}$, we set

$$[R] = \{S \mid \mathcal{T} \models R \sqsubseteq S, \mathcal{T} \models S \sqsubseteq R\}.$$

We write $[R] \leq_{\mathcal{T}} [S]$ if $\mathcal{T} \models R \sqsubseteq S$; thus, $\leq_{\mathcal{T}}$ is a partial order on the set $\{[R] \mid R \text{ a role in } \mathcal{T}\}$. Ω will be a subset of the set of pairs $([R], t)$ with $t \in \mathsf{type}(\mathcal{T})$. First define $\rightsquigarrow$ as follows:

- $a \rightsquigarrow ([R], t)$ if $a \in \mathsf{ind}(\mathcal{K})$ and t is a maximal (with respect to set-inclusion) $\mathcal{T}$-type such that $\mathcal{K} \models \exists R.(\bigsqcap_{C \in t} C)(a)$ and $\mathcal{K} \not\models R(a, b)$ for any $b \in \mathsf{ind}(\mathcal{K})$ with $t \subseteq \{C \in \mathsf{sub}(\mathcal{T}) \mid \mathcal{K} \models C(b)\}$;
- $([R_1], t_1) \rightsquigarrow ([R_2], t_2)$ if t_2 is a maximal $\mathcal{T}$-type such that $\mathcal{T} \models (\bigsqcap_{C \in t_1} C) \sqsubseteq \exists R_2.(\bigsqcap_{C \in t_2} C)$.

Ω is defined as the set of all pairs $([R], t)$ such that there are $a \in \mathsf{ind}(\mathcal{K})$, $R_1, \dots, R_n = R$ and $t_1, \dots, t_n = t$ such that

$$a \rightsquigarrow ([R_1], t_1) \rightsquigarrow \dots \rightsquigarrow ([R_n], t_n).$$

We define the interpretation function $\cdot^{\mathcal{G}}$ by setting

$$A^{\mathcal{G}} = \{a \in \mathsf{ind}(\mathcal{K}) \mid \mathcal{K} \models A(a)\} \cup \{([R], t) \in \Omega \mid A \in t\},$$

$$P^{\mathcal{G}} = \{(a, b) \mid \text{there is } R(a, b) \in \mathcal{A} \text{ with } \mathcal{T} \models R \sqsubseteq P\},$$

and for every edge $u \rightsquigarrow v$ with $v = (R, t)$, we set

$$(u, v)^{\mathcal{G}} = \{S \mid [R] \leq_{\mathcal{T}} [S]\}.$$

It can now be proved that $\mathcal{G}$ is a generating structure for $\mathcal{K}$. Let $\mathcal{M}$ be the interpretation defined by unravelling $\mathcal{G}$.

Proposition A.29 *$\mathcal{M}$ is a model of $\mathcal{K}$.*

Proof. Clearly, $\mathcal{M}$ is a model of $\mathcal{A}$. We show $\mathcal{M} \models \mathcal{T}$ by verifying $\mathcal{M} \models \alpha$ for each $\alpha \in \mathcal{T}$.

$\alpha = A \sqsubseteq B$. Here we consider A to be either a concept name A, or $\top$. Let $x \in A^{\mathcal{M}}$. If $x = a \in \mathsf{ind}(\mathcal{K})$, then $\mathcal{K} \models A(a)$ by construction of $A^{\mathcal{M}}$. Since $A \sqsubseteq B \in \mathcal{T}$, it follows $\mathcal{K} \models B(a)$, hence $a \in B^{\mathcal{M}}$. If $x = x' \cdot ([R], t)$, then $A \in t$ and by construction of $\mathcal{G}$, t is a maximal $\mathcal{T}$-type such that $\mathcal{T} \models (\bigsqcap_{C \in t'} C) \sqsubseteq \exists R.(\bigsqcap_{C \in t} C)$, where $\mathsf{tail}(x') = ([R'], t')$. As $\mathcal{T} \models A \sqsubseteq B$, we have $B \in t$, so $x \in B^{\mathcal{M}}$.

$\alpha = A_1 \sqcap A_2 \sqsubseteq B$**.** The argument is analogous for $x \in A_1^{\mathcal{M}} \cap A_2^{\mathcal{M}}$.

$\alpha = A \sqsubseteq \exists R.B$**.** Let $x \in A^{\mathcal{M}}$. If $x = a \in \mathsf{ind}(\mathcal{K})$, then $\mathcal{K} \models A(a)$ by construction of $A^{\mathcal{M}}$. Since $A \sqsubseteq \exists R.B \in \mathcal{T}$, it follows $\mathcal{K} \models \exists R.B(a)$. Assume $\mathcal{K} \models \{R(a,b), B(b)\}$ for some $b \in \mathsf{ind}(\mathcal{K})$, then $(a,b) \in R^{\mathcal{M}}$ and $b \in B^{\mathcal{M}}$, so $\mathcal{M} \models \alpha$. Otherwise, take a maximal $\mathcal{T}$-type t with $B \in t$ such that $\mathcal{K} \models \exists R.(\bigsqcap_{C\in t} C)(a)$ and $\mathcal{K} \not\models R(a,b)$ for any $b \in \mathsf{ind}(\mathcal{K})$ with $t \subseteq \{C \in \mathsf{sub}(\mathcal{T}) \mid \mathcal{K} \models C(b)\}$. Then it holds that $a \rightsquigarrow ([R],t)$ and $B \in t$. From the construction of $R^{\mathcal{M}}$, we have that $(a, a\cdot([R],t)) \in R^{\mathcal{M}}$, and since $B \in t$, $([R],t) \in B^{\mathcal{M}}$. For the case $\mathsf{tail}(x) = ([R],t)$, the proof is similar.

$\alpha = A \sqsubseteq \forall R.B$**.** Let $x \in A^{\mathcal{M}}$, and assume $(x,y) \in R^{\mathcal{M}}$. We consider various cases of x and y:

- $x = a$, $y = b$ for $a, b \in \mathsf{ind}(\mathcal{K})$. Then $\mathcal{K} \models R(a,b)$ and $\mathcal{K} \models A(a)$, consequently $\mathcal{K} \models B(b)$, so $b \in B^{\mathcal{M}}$ by construction of $\mathcal{M}$.
- $x = a \in \mathsf{ind}(\mathcal{K})$, $y = a \cdot ([S],t)$ such that $\mathcal{T} \models S \sqsubseteq R$. Then t is a maximal $\mathcal{T}$-type such that $\mathcal{K} \models \exists S.(\bigsqcap_{C\in t} C)(a)$. Moreover, from $A \sqsubseteq \forall R.B \in \mathcal{T}$ it follows that $B \in t$. So, by definition of $B^{\mathcal{M}}$, $([S],t) \in B^{\mathcal{M}}$.
- $\mathsf{tail}(x) \notin \mathsf{ind}(\mathcal{K})$, $y = x\cdot([S],t)$ such that $\mathcal{T} \models S \sqsubseteq R$. The proof is as for the case above.
- $y = b \in \mathsf{ind}(\mathcal{K})$ and $x = y \cdot ([S],t)$ such that $\mathcal{T} \models S^- \sqsubseteq R$. Then $A \in t$ and so $\mathcal{K} \models \exists S.A(b)$, consequently $\mathcal{K} \models \exists R^-.A(b)$, and as $A \sqsubseteq \forall R.B$ is equivalent to $\exists R^-.A \sqsubseteq B$, finally, $\mathcal{K} \models B(b)$. Hence, $b \in B^{\mathcal{M}}$.
- $\mathsf{tail}(y) \notin \mathsf{ind}(\mathcal{K})$ and $x = y \cdot ([S],t)$ such that $\mathcal{T} \models S^- \sqsubseteq R$. Let $\mathsf{tail}(y) = ([Q],t')$, then t is a maximal $\mathcal{T}$-type such that $\mathcal{T} \models (\bigsqcap_{C\in t'} C) \sqsubseteq \exists S.(\bigsqcap_{C\in t} C)$, moreover $A \in t$. If follows $\mathcal{T} \models \exists S.(\bigsqcap_{C\in t} C) \sqsubseteq \exists R^-.A$, and as before, $\mathcal{T} \models \exists R^-.A \sqsubseteq B$. Therefore, $\mathcal{T} \models (\bigsqcap_{C\in t'} C) \sqsubseteq B$. t' is a maximal $\mathcal{T}$-type and so we can conclude that $B \in t'$. Hence, $y \in B^{\mathcal{M}}$.

$\alpha = \exists R.A \sqsubseteq B$**.** Observe that α is equivalent to the axiom $A \sqsubseteq \forall R^-.B$, whose satisfaction was shown above. That proof can be adjusted to the case of α.

$\alpha = A \sqsubseteq \bot$**.** Assume $A^{\mathcal{M}} \neq \emptyset$ and $x \in A^{\mathcal{M}}$. If $x = a \in \mathsf{ind}(\mathcal{K})$, then $\mathcal{K} \models A(a)$, and, since $\alpha \in \mathcal{T}$, we get a contradiction with $\mathcal{K}$ being consistent. If $\mathsf{tail}(x) = ([R],t)$ for some role R and $\mathcal{T}$-type t, then it follows $A \in t$, which contradicts the definition of a $\mathcal{T}$-type. Therefore, $A^{\mathcal{M}} = \emptyset$.

$\alpha = R_1 \sqsubseteq R_2$**.** Assume $(x,y) \in R_1^{\mathcal{M}}$. If $x = a$ and $y = b$ for $a,b \in \mathsf{ind}(\mathcal{K})$, it follows $\mathcal{K} \models R_1(a,b)$. From α we obtain that $\mathcal{K} \models R_2(a,b)$, therefore $(a,b) \in R_2^{\mathcal{M}}$. If $y = x \cdot ([R],t)$ for some R and t, by construction of $R_1^{\mathcal{M}}$, $\mathcal{T} \models R \sqsubseteq R_1$. Then because of α, $\mathcal{T} \models R \sqsubseteq R_2$, so finally, $(x,y) \in R_2^{\mathcal{M}}$. If $x = y \cdot ([R],t)$, then $\mathcal{T} \models R^- \sqsubseteq R_1$ and $\mathcal{T} \models R^- \sqsubseteq R_2$, so again $(x,y) \in R_2^{\mathcal{M}}$.

This finishes the proof. ❑

The proof above also shows that $\mathcal{T}$-types of a node in $\mathcal{M}$ coincide with the $\mathcal{T}$-types used in the construction of the node.

Lemma A.30 *Let $\mathcal{M}$ be the unravelling of $\mathcal{G}$. Then*

1. *for all $a \in \mathsf{ind}(\mathcal{K})$, $\boldsymbol{h}^{\mathcal{M}}(a) = \{C \in \mathsf{sub}(\mathcal{T}) \mid \mathcal{K} \models C(a)\}$;*
2. *For all $\sigma \cdot ([R],t) \in \Delta^{\mathcal{M}}$, $\boldsymbol{h}^{\mathcal{M}}(\sigma \cdot ([R],t)) = t$.*

Proposition A.31 *If $\mathcal{I}$ is a model of $\mathcal{K}$, then there exists a homomorphism from $\mathcal{M}$ to $\mathcal{I}$.*

Proof. We define a function $h : \Delta^{\mathcal{M}} \to \Delta^{\mathcal{I}}$ for each $\sigma \in \Delta^{\mathcal{M}}$ by induction on the length of σ, and simultaneously show it is a homomorphism, i.e.,

(1) $h(a^{\mathcal{M}}) = a^{\mathcal{I}}$ for $a \in \mathsf{ind}(\mathcal{K})$,

(2) $\boldsymbol{h}^{\mathcal{M}}(\sigma) \subseteq \boldsymbol{h}^{\mathcal{I}}(h(\sigma))$ for $\sigma \in \Delta^{\mathcal{M}}$,

(3) $\boldsymbol{r}^{\mathcal{M}}(\sigma,\sigma') \subseteq \boldsymbol{r}^{\mathcal{I}}(h(\sigma),h(\sigma'))$ for $\sigma,\sigma' \in \Delta^{\mathcal{M}}$.

First, for each $a \in \mathsf{ind}(\mathcal{K})$, we set $h(a^{\mathcal{M}}) = a^{\mathcal{I}}$. This ensures (1). Conditions 2 and 3 follow for $\sigma,\sigma' \in \mathsf{ind}(\mathcal{K})$ from Lemma A.30, the condition that $\mathcal{I}$ is a model of $\mathcal{K}$, and the construction of $\mathcal{M}$.

Let $\sigma \cdot ([S],t) \in \Delta^{\mathcal{M}}$ such that $h(\sigma)$ is defined. By construction of $\mathcal{M}$, it follows $\mathcal{K} \models \exists S.(\bigsqcap_{C\in t} C)(a)$ if $\sigma = a$, or $\mathcal{T} \models (\bigsqcap_{C\in t'} C) \sqsubseteq \exists S.(\bigsqcap_{C\in t} C)$ if $\mathsf{tail}(\sigma) = ([Q],t')$. By the condition that $\mathcal{I}$ is a model of $\mathcal{K}$, by Lemma A.30, and by the induction hypothesis $\boldsymbol{h}^{\mathcal{M}}(\sigma) \subseteq \boldsymbol{h}^{\mathcal{I}}(h(\sigma))$, it follows that there exists $z \in \Delta^{\mathcal{I}}$ such that $S \in \boldsymbol{r}^{\mathcal{I}}(h(\sigma),z)$ and $t \subseteq \boldsymbol{h}^{\mathcal{I}}(z)$. We set $h(\sigma \cdot ([S],t)) = z$ and show that (2) and (3) hold. (2) follows immediately from the fact that $\boldsymbol{h}^{\mathcal{M}}(\sigma \cdot ([S],t)) = t$ (by Lemma A.30). For (3), from $R \in \boldsymbol{r}^{\mathcal{M}}(\sigma, \sigma \cdot ([S],t))$ it follows $\mathcal{T} \models S \sqsubseteq R$, and since $\mathcal{I}$ is a model of $\mathcal{T}$, we get $R \in \boldsymbol{r}^{\mathcal{I}}(h(\sigma),z)$. ❑

$\mathcal{ELH}$

We now construct generating structures for $\mathcal{ELH}$. Again we first transform the TBox into normal form (Baader, Brandt, and Lutz 2005). An $\mathcal{ELH}$ TBox is in *normal form* if its concept inclusions are of the following form:

$$\begin{array}{ll} A \sqsubseteq B, & A_1 \sqcap A_2 \sqsubseteq B, \\ \top \sqsubseteq B, & \\ A \sqsubseteq \exists P.B, & \\ \exists P.A \sqsubseteq B, & P_1 \sqsubseteq P_2, \end{array}$$

where A, A_1, A_2, B are concept names and P, P_1, P_2 are role names. The following result is well known (Baader, Brandt, and Lutz 2005):

Theorem A.32 *For every $\mathcal{ELH}$ TBox $\mathcal{T}$ one can construct in polynomial time an $\mathcal{ELH}$ TBox $\mathcal{T}'$ in normal form such that $\mathcal{T}$ and $\mathcal{T}'$ are Σ-query inseparable for the signature Σ of $\mathcal{T}$.*

Assume $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with $\mathcal{T}$ an $\mathcal{ELH}$ TBox in normal form is given. We construct the generating structure $\mathcal{G} = (\Delta^{\mathcal{G}}, \cdot^{\mathcal{G}}, \rightsquigarrow)$ for $\mathcal{K}$ as follows, where $\Delta^{\mathcal{G}} = \mathsf{ind}(\mathcal{K}) \cup \Omega$ and Ω is a subset of the set of pairs $([P],A)$, for A and P, concept and role names in $\mathcal{T}$, respectively. (The class $[P]$ is defined in the construction for *Horn-$\mathcal{ALCHI}$*.) Define $\rightsquigarrow$ as follows:

- $a \rightsquigarrow ([P], A)$ if $a \in \mathsf{ind}(\mathcal{K})$ and A is a concept name in $\mathcal{T}$ such that $\mathcal{K} \models \exists P.A(a)$ and $\mathcal{K} \not\models P(a, b)$ for any $b \in \mathsf{ind}(\mathcal{K})$ with $\mathcal{K} \models A(b)\}$;
- $([P_1], A_1) \rightsquigarrow ([P_2], A_2)$ if $\mathcal{T} \models A_1 \sqsubseteq \exists P_2.A_2$.

Ω is defined as the set of all pairs $([P], A)$ such that there are $a \in \mathsf{ind}(\mathcal{K})$, $P_1, \ldots, P_n = P$ and $A_1, \ldots, A_n = A$ such that

$$a \rightsquigarrow ([P_1], A_1) \rightsquigarrow \cdots \rightsquigarrow ([P_n], A_n).$$

We define the interpretation function $\cdot^{\mathcal{G}}$ by setting

$$\begin{aligned} A^{\mathcal{G}} = &\ \{a \in \mathsf{ind}(\mathcal{K}) \mid \mathcal{K} \models A(a)\} \cup \\ &\ \{([P], B) \in \Omega \mid \mathcal{T} \models B \sqsubseteq A\}, \\ P^{\mathcal{G}} = &\ \{(a, b) \mid \text{there is } P'(a, b) \in \mathcal{A} \text{ with } \mathcal{T} \models P' \sqsubseteq P\} \end{aligned}$$

and for every edge $u \rightsquigarrow v$ with $v = ([P], A)$, we set

$$(u, v)^{\mathcal{G}} = \{P' \mid [P] \leq_{\mathcal{T}} [P']\}.$$

It can be shown that $\mathcal{G}$ is a generating structure for $\mathcal{K}$. Let $\mathcal{M}$ be the interpretation defined by unravelling $\mathcal{G}$.

Proposition A.33 *$\mathcal{M}$ is a model of $\mathcal{K}$.*

Proof. Clearly, $\mathcal{M}$ is a model of $\mathcal{A}$. We show $\mathcal{M} \models \mathcal{T}$ by verifying $\mathcal{M} \models \alpha$ for each $\alpha \in \mathcal{T}$.

$\alpha = A \sqsubseteq B$**.** Here we consider A to be either a concept name A, or $\top$. Let $x \in A^{\mathcal{M}}$. If $x = a \in \mathsf{ind}(\mathcal{K})$, then $\mathcal{K} \models A(a)$ by construction of $A^{\mathcal{M}}$. Since $A \sqsubseteq B \in \mathcal{T}$, it follows $\mathcal{K} \models B(a)$, hence $a \in B^{\mathcal{M}}$. If $x = x' \cdot ([R], C)$, then $\mathcal{T} \models C \sqsubseteq A$. Because of α, we have that $\mathcal{T} \models C \sqsubseteq B$, therefore $x \in B^{\mathcal{M}}$.

$\alpha = A_1 \sqcap A_2 \sqsubseteq B$**.** The argument is analogous for $x \in A_1^{\mathcal{M}} \cap A_2^{\mathcal{M}}$.

$\alpha = A \sqsubseteq \exists P.B$**.** Let $x \in A^{\mathcal{M}}$. If $x = a \in \mathsf{ind}(\mathcal{K})$, then $\mathcal{K} \models A(a)$ by construction of $A^{\mathcal{M}}$. Since $\alpha \in \mathcal{T}$, it follows $\mathcal{K} \models \exists P.B(a)$. Assume $\mathcal{K} \models \{P(a, b), B(b)\}$ for some $b \in \mathsf{ind}(\mathcal{K})$, then $(a, b) \in P^{\mathcal{M}}$ and $b \in B^{\mathcal{M}}$, so $\mathcal{M} \models \alpha$. Otherwise, we have that $a \rightsquigarrow ([P], B)$. From the construction of $\mathcal{M}$, we have that $(a, a \cdot ([P], B)) \in P^{\mathcal{M}}$, and $([P], B) \in B^{\mathcal{M}}$. For the case $\mathsf{tail}(x) = ([R], t)$, the proof is similar.

$\alpha = \exists P.A \sqsubseteq B$**.** Assume $(x, y) \in P^{\mathcal{M}}$ and $y \in A^{\mathcal{M}}$. We consider various cases of x and y:

- $x = b$, $y = a$ for $a, b \in \mathsf{ind}(\mathcal{K})$. Then $\mathcal{K} \models P(b, a)$ and $\mathcal{K} \models A(a)$, consequently $\mathcal{K} \models B(b)$, so $b \in B^{\mathcal{M}}$ by construction of $\mathcal{M}$.
- $x = b \in \mathsf{ind}(\mathcal{K})$, $y = b \cdot ([S], C)$. Then by construction of $\mathcal{M}$, $\mathcal{K} \models \exists S.C(b)$, moreover $\mathcal{T} \models \{S \sqsubseteq P, C \sqsubseteq A\}$. Next, $\mathcal{K} \models \exists P.A(b)$, and finally $\mathcal{K} \models B(b)$, so $b \in B^{\mathcal{M}}$.
- $\mathsf{tail}(x) = ([Q], D)$, $y = x \cdot ([S], C)$. Then by construction of $\mathcal{M}$, $\mathcal{T} \models D \sqsubseteq \exists S.C$, moreover $\mathcal{T} \models \{S \sqsubseteq P, C \sqsubseteq A\}$. It follows $\mathcal{T} \models D \sqsubseteq \exists P.A$, and because of α, $\mathcal{T} \models D \sqsubseteq B$. Hence, by construction of $B^{\mathcal{M}}$, $x \in B^{\mathcal{M}}$.

Observe that the case $x = y \cdot ([S], C)$ for some S and C is not possible as it would required that $\mathcal{T} \models S^- \sqsubseteq R$, which is not possible in $\mathcal{ELH}$.

$\alpha = P_1 \sqsubseteq P_2$**.** Assume $(x, y) \in P_1^{\mathcal{M}}$. If $x = a$ and $y = b$ for $a, b \in \mathsf{ind}(\mathcal{K})$, it follows $\mathcal{K} \models P_1(a, b)$. From α we obtain that $\mathcal{K} \models P_2(a, b)$, therefore $(a, b) \in P_2^{\mathcal{M}}$. If $y = x \cdot ([P], t)$ for some P and t, by construction of $P_1^{\mathcal{M}}$, $\mathcal{T} \models P \sqsubseteq P_1$. Then because of α, $\mathcal{T} \models P \sqsubseteq P_2$, so finally, $(x, y) \in R_2^{\mathcal{M}}$.

❑

Proposition A.34 *If $\mathcal{I}$ is a model of $\mathcal{K}$, then there exists a homomorphism from $\mathcal{M}$ to $\mathcal{I}$.*

Proof. Analogous to the proof of Proposition A.31. ❑

DL-Lite$^{\mathcal{H}}_{horn}$

Finally, assume $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with a *DL-Lite$^{\mathcal{H}}_{horn}$* TBox $\mathcal{T}$ is given. We construct the finitely generating structure $\mathcal{G} = (\Delta^{\mathcal{G}}, \cdot^{\mathcal{G}}, \rightsquigarrow)$ for $\mathcal{K}$ as follows, where $\Delta^{\mathcal{G}} = \mathsf{ind}(\mathcal{K}) \cup \Omega$. For each $[R]$ with R are role in $\mathcal{T}$, we introduce a *witness* $w_{[R]}$. Ω will be a subset of the set of all $w_{[R]}$. First define $\rightsquigarrow$ as follows:

- $a \rightsquigarrow w_{[R]}$ if $a \in \mathsf{ind}(\mathcal{K})$ and $[R]$ is $\leq_{\mathcal{T}}$-minimal such that $\mathcal{K} \models \exists R(a)$ and $\mathcal{K} \not\models R(a, b)$ for any $b \in \mathsf{ind}(\mathcal{K})$;
- $w_{[S]} \rightsquigarrow w_{[R]}$ if $[R]$ is $\leq_{\mathcal{T}}$-minimal with $\mathcal{T} \models \exists S^- \sqsubseteq \exists R$ and $[S^-] \neq [R]$.

Ω is defined as the set of all $w_{[R]}$ such that there are $a \in \mathsf{ind}(\mathcal{K})$ and $R_1, \ldots, R_n = R$ such that

$$a \rightsquigarrow w_{[R_1]} \rightsquigarrow \cdots \rightsquigarrow w_{[R_n]}.$$

We define the interpretation function $\cdot^{\mathcal{G}}$ by setting

$$\begin{aligned} A^{\mathcal{G}} = &\ \{a \in \mathsf{ind}(\mathcal{K}) \mid \mathcal{K} \models A(a)\} \cup \\ &\ \{w_{[R]} \in \Omega \mid \mathcal{T} \models \exists R^- \sqsubseteq A\}, \\ P^{\mathcal{G}} = &\ \{(a, b) \mid \text{there is } R(a, b) \in \mathcal{A} \text{ with } \mathcal{T} \models R \sqsubseteq P\}, \end{aligned}$$

and for every edge $u \rightsquigarrow v$ with $v = w_{[R]}$, we set

$$(u, v)^{\mathcal{G}} = \{R' \mid [R] \leq_{\mathcal{T}} [R']\}.$$

One can show that $\mathcal{G}$ is a generating structure for $\mathcal{K}$. Let $\mathcal{M}$ be the interpretation defined by unravelling $\mathcal{G}$. The following two propositions can be proved by analogy with Proposition 17 and Lemma 18 in the full version of (Konev et al. 2011).

Proposition A.35 *$\mathcal{M}$ is a model of $\mathcal{K}$.*

Proposition A.36 *If $\mathcal{I}$ is a model of $\mathcal{K}$, then there exists a homomorphism from $\mathcal{M}$ to $\mathcal{I}$.*

Finally, we show that $\mathcal{M}$ is indeed a materialisation.

Theorem A.37 *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a consistent $\mathcal{L}$-KB, and $\mathcal{M}$ the interpretation defined by unravelling the generating structure for $\mathcal{K}$. Then $\mathcal{M}$ is a materialisation of $\mathcal{K}$.*

Proof. We show that $\mathcal{K} \models \boldsymbol{q}[\vec{a}]$ iff $\mathcal{M} \models \boldsymbol{q}[\vec{a}]$, for each CQ $\boldsymbol{q}(\vec{x})$ and each tuple of constants $\vec{a} \subseteq \mathsf{ind}(\mathcal{K})$.

($\Rightarrow$) Assume $\mathcal{K} \models \boldsymbol{q}[\vec{a}]$. Then for each model $\mathcal{I}$ of $\mathcal{K}$, we have $\mathcal{I} \models \boldsymbol{q}[\vec{a}]$. Since $\mathcal{M}$ is a model of $\mathcal{K}$, we obtain $\mathcal{M} \models \boldsymbol{q}[\vec{a}]$.

($\Leftarrow$) Let $\mathcal{M} \models q[\vec{a}]$, moreover assume $\vec{a} = (a_1, \dots, a_k)$ for $a_i \in \mathsf{ind}(\mathcal{K})$, and

$$\boldsymbol{q}(\vec{x}) = \exists y_1 \dots \exists y_m \, \varphi(x_1, \dots, x_k, y_1, \dots, y_m).$$

Then there exist $\sigma_1, \dots, \sigma_m \in \Delta^{\mathcal{M}}$ such that $\mathcal{M} \models \varphi[a_1, \dots, a_k, \sigma_1, \dots, \sigma_m]$.

Let $\mathcal{I}$ be a model of $\mathcal{K}$, we show that $\mathcal{I} \models \boldsymbol{q}[\vec{a}]$. By Propositions A.31, A.34, A.36, there exists a homomorphism h from $\mathcal{M}$ to $\mathcal{I}$. Then it is easy to see that

$$\mathcal{I} \models \varphi[a_1, \dots, a_k, h(\sigma_1), \dots, h(\sigma_m)].$$

As $\mathcal{I}$ was an arbitrary model of $\mathcal{K}$, it follows that $\mathcal{K} \models \boldsymbol{q}[\vec{a}]$. ❑

Proof of Theorem 5

Theorem 5 *Suppose $\mathcal{K}_i$ is a consistent KB with a materialisation $\mathcal{M}_i$, $i = 1, 2$. Then $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{M}_2$ is finitely Σ-homomorphically embeddable into $\mathcal{M}_1$.*

Proof. ($\Rightarrow$) Assume $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$. Let Δ be a finite subset of $\Delta^{\mathcal{M}_2}$ such that $\Delta = \{a_1, \dots, a_k, \sigma_1, \dots, \sigma_m\}$ with $a_i \in \mathsf{Ind}(\mathcal{K}_2)$. Consider a query $\boldsymbol{q} = \exists y_1 \dots \exists y_m \, \varphi$, where for $1 \le i, i' \le k$ and $1 \le j, j' \le m$,

$$\varphi = \bigwedge_{A \in \boldsymbol{t}_\Sigma^{\mathcal{M}_2}(a_i)} A(a_i) \wedge \bigwedge_{R \in \boldsymbol{r}_\Sigma^{\mathcal{M}_2}(a_i, a_{i'})} R(a_i, a_{i'}) \wedge \bigwedge_{R \in \boldsymbol{r}_\Sigma^{\mathcal{M}_2}(a_i, \sigma_j)} R(a_i, y_j) \wedge \bigwedge_{A \in \boldsymbol{t}_\Sigma^{\mathcal{M}_2}(\sigma_j)} A(y_j) \wedge \bigwedge_{R \in \boldsymbol{r}_\Sigma^{\mathcal{M}_2}(\sigma_j, \sigma_{j'})} R(y_j, y_{j'})$$

Clearly, $\mathcal{M}_2 \models \boldsymbol{q}$, as $\mathcal{M}_2 \models \varphi[\sigma_1, \dots, \sigma_m]$. By Theorem A.37, $\mathcal{M}_1 \models \boldsymbol{q}$, and thus, $\mathcal{M}_1 \models \varphi[\sigma'_1, \dots, \sigma'_m]$, for some $\sigma'_1, \dots, \sigma'_m \in \Delta^{\mathcal{M}_1}$. We define $h \colon \Delta \to \Delta^{\mathcal{M}_1}$ by taking $h(a_i) = (a_i)$ and $h(\sigma_i) = \sigma'_i$. This function is a homomorphism: it maps every constant to itself, and from $\mathcal{M}_1 \models \varphi[\sigma'_1, \dots, \sigma'_m]$ it follows that for each $d, d' \in \Delta$, $\boldsymbol{t}_\Sigma^{\mathcal{M}_2}(d) \subseteq \boldsymbol{t}_\Sigma^{\mathcal{M}_1}(h(d))$ and $\boldsymbol{r}_\Sigma^{\mathcal{M}_2}(d, d') \subseteq \boldsymbol{r}_\Sigma^{\mathcal{M}_1}(d, d')$

($\Leftarrow$) Assume $\mathcal{M}_2$ is finitely Σ-homomorphically embeddable into $\mathcal{M}_1$. Let $\mathcal{K}_2 = (\mathcal{T}_2, \mathcal{A}_2)$, $\boldsymbol{q}$ be a Σ-query and $(a_1, \dots, a_k) \subseteq \mathsf{ind}(\mathcal{K}_2)$, $k \ge 0$, such that $\mathcal{K}_2 \models \boldsymbol{q}(a_1, \dots, a_k)$ and $\boldsymbol{q}(x_1, \dots, x_k) = \exists y_1, \dots, \exists y_m \, \varphi(\vec{x}, \vec{y})$, where $\varphi(\vec{x}, \vec{y})$ is a conjunction of atoms over variables $x_1, \dots, x_k$ and $y_1, \dots, y_m$. By Theorem A.37, $\mathcal{M}_2 \models \varphi[a_1, \dots, a_k, \sigma_1, \dots, \sigma_m]$ for some $\sigma_j \in \Delta^{\mathcal{M}_2}$. Let Δ be $\{a_1, \dots, a_k, \sigma_1, \dots, \sigma_m\}$ and h a Σ-homomorphism from $\mathcal{M}_2|_\Delta$, the restriction of $\mathcal{M}_2$ to Δ, to $\mathcal{M}_1$ with $h(a_i) = a_i$ for $1 \le i \le k$. By definition of homomorphism, we have $h(d) \in A^{\mathcal{M}_1}$ if $d \in A^{\mathcal{M}_2}$, for each concept A over Σ and $d \in \Delta$, and $(h(d), h(d')) \in R^{\mathcal{M}_1}$ if $(d, d') \in R^{\mathcal{M}_2}$, for each role R over Σ and $d, d' \in \Delta$. Which in turns implies that $\mathcal{M}_1 \models \varphi[a_1, \dots, a_k, h(\sigma_1), \dots, h(\sigma_m)]$, hence, $\mathcal{K}_1 \models \boldsymbol{q}(a_1, \dots, a_k)$. ❑

Proof of Theorem 2

Theorem 2 *Let $\mathcal{L}$ be any of our DLs containing $\mathcal{EL}$ or having role inclusions. Then Σ-query entailment for $\mathcal{L}$-KBs is* LOGSPACE*-reducible to Σ-query inseparability for $\mathcal{L}$-KBs.*

Proof. We complete the proof given in the paper by showing

Claim 1. $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{K}_1$ and $\mathcal{K}_1^\Sigma \cup \mathcal{K}_2^\Sigma$ are Σ-query inseparable.

The interesting direction is to show that if $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$, then $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_1^\Sigma \cup \mathcal{K}_2^\Sigma$. Assume that $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$. Consider materialisations $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{D}_1$ of $\mathcal{K}_1^\Sigma$, $\mathcal{K}_2^\Sigma$, and $\mathcal{K}_1$, respectively. The construction of the materialisations above shows that we may assume that

– $\mathcal{C}_i$ is a model of $\mathcal{K}_i^\Sigma$, for $i = 1, 2$;

– $a^{\mathcal{C}_1} = a^{\mathcal{C}_2}$ for all $a \in \mathsf{ind}(\mathcal{K}_1) \cap \mathsf{ind}(\mathcal{K}_2)$;

– $d \in \Delta^{\mathcal{C}_1} \cap \Delta^{\mathcal{C}_2}$ iff $d = a^{\mathcal{C}_1}$ for some $a \in \mathsf{ind}(\mathcal{K}_1) \cap \mathsf{ind}(\mathcal{K}_2)$.

Denote by $\mathcal{C}$ the union of $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by setting $\Delta^{\mathcal{C}} = \Delta^{\mathcal{C}_1} \cup \Delta^{\mathcal{C}_2}$ and $X^{\mathcal{C}} = X^{\mathcal{C}_1} \cup X^{\mathcal{C}_2}$ for all symbols X. We show that

(i) $\mathcal{C}$ is a model of $\mathcal{K}_1^\Sigma \cup \mathcal{K}_2^\Sigma$, and

(ii) $\mathcal{C}$ is finitely Σ-homomorphically embeddable into $\mathcal{D}_1$.

By (i) and (ii), $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_1^\Sigma \cup \mathcal{K}_2^\Sigma$. Item (i) follows from the assumption that the $\mathcal{C}_i$ are models of $\mathcal{K}_i^\Sigma$ and the assumption that the trivial interpretation is a model of $\mathcal{T}_i$.

For (ii), let $Y \subseteq \Delta^{\mathcal{C}}$ be finite. Since $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ and $\mathcal{K}_1$ trivially Σ-query entails $\mathcal{K}_1^\Sigma$ we have Σ-homomorphisms

– $f_1 : Y \cap \Delta^{\mathcal{C}_1} \to \Delta^{\mathcal{D}_1}$ from the Y-restriction of $\mathcal{C}_1$ to $\mathcal{D}_1$ and

– $f_2 : Y \cap \Delta^{\mathcal{C}_2} \to \Delta^{\mathcal{D}_1}$ from the Y-restriction of $\mathcal{C}_2$ to $\mathcal{D}_1$.

We may assume $f_1(a^{\mathcal{C}_1}) = f_2(a^{\mathcal{C}_2})$ for all $a \in \mathsf{Ind}(\mathcal{K}_1) \cap \mathsf{ind}(\mathcal{K}_2)$. Then $f_1 \cup f_2$ is a Σ-homomorphism from the Y-restriction of $\mathcal{C}$ to $\mathcal{D}_1$, as required.

We now consider the case when the trivial interpretation is not a model of $\mathcal{T}_i$. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are given. We construct $\mathcal{K}'_1$ and $\mathcal{K}'_2$ such that the trivial interpretation is a model of $\mathcal{T}'_1$ and $\mathcal{T}'_2$, respectively, and such that $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{K}'_1$ Σ-query entails $\mathcal{K}'_2$. The construction is by careful relativisation.

Let $A_\top^i$ be fresh concept names, for $i = 1, 2$. Set $\mathcal{A}'_i = \mathcal{A} \cup \{A_\top^i(a) \mid a \in \mathsf{ind}(\mathcal{K}_i)\}$.

Case 1. The $\mathcal{T}_i$ are *Horn-$\mathcal{ALCHI}$*-TBoxes. We assume they are in normal form. Now replace

– any inclusion $\top \sqsubseteq B$ by $A_\top^i \sqsubseteq B$;

– any inclusion $A \sqsubseteq \exists R.B$ by $A \sqsubseteq \exists R.(A_\top^i \sqcap B)$.

The remaining inclusions are not modified. Below we show that the $\mathcal{K}'_i = (\mathcal{T}'_i, \mathcal{A}'_i)$ are as required.

Note that the $\mathcal{K}'_i$ are consistent. Consider materialisations (and models) $\mathcal{C}_i$ and $\mathcal{D}_i$ of $\mathcal{K}_i$ and $\mathcal{K}'_i$ respectively. For a subset $S \subseteq \Delta^{\mathcal{D}_i}$, denote by $(S)^{\setminus A_\top^i}$ the set

$$\{av'_1 \cdots v'_n \mid av_1 \cdots v_n \in S, n \ge 0\},$$

where, if $v_j = ([R_j], t_j)$, then $v'_j = ([R_j], t_j \setminus \{A^i_\top\})$, and similarly for a set $S \subseteq \Delta^{\mathcal{D}_i} \times \Delta^{\mathcal{D}_i}$. We show $\Delta^{\mathcal{D}_i} = (A^i_\top)^{\mathcal{D}_i}$, $\Delta^{\mathcal{C}_i} = (\Delta^{\mathcal{D}_i})^{\setminus A^i_\top}$, $X^{\mathcal{C}_i} = (X^{\mathcal{D}_i})^{\setminus A^i_\top}$ for each symbol X distinct from $A^i_\top$, from which the required follows. First, by construction of $\mathcal{K}'_i$ and definition of the generating structure and materialisation, we have

$$(A^i_\top)^{\mathcal{D}_i} = \mathsf{ind}(\mathcal{K}_i) \cup \{av_1 \cdots v_n \mid a \in \mathsf{ind}(\mathcal{K}_i), v_j \in \Omega, a \rightsquigarrow v_1, v_j \rightsquigarrow v_{j+1}\},$$

thus $\Delta^{\mathcal{D}_i} = (A^i_\top)^{\mathcal{D}_i}$. On the other hand, clearly $\Delta^{\mathcal{C}_i} = \{av'_1 \cdots v'_n \mid av_1 \cdots v_n \in \Delta^{\mathcal{D}_i}\}$ for $v_j = ([R_j], t_j)$ and $v'_j = ([R_j], t_j \setminus \{A^i_\top\})$. Next, let $\mathcal{T}_i \models \top \sqsubseteq C$ for some concept C such that C is a concept name B, or a concept of the form $\exists R.B$ or $\exists R.\top$, since $\mathcal{T}_i$ is in normal form, it follows that $\top \sqsubseteq A \in \mathcal{T}_i$ for some concept name A and $\mathcal{T}_i \models A \sqsubseteq C$. Then, $\mathcal{T}'_i$ contains axiom $A^i_\top \sqsubseteq A$, and therefore $A^{\mathcal{D}_i} = B^{\mathcal{D}_i} = \Delta^{\mathcal{D}_i}$ if C is a concept name B and $R^{\mathcal{D}_i} = \{(x, x \cdot ([R], t) \mid x \in \Delta^{\mathcal{D}_i}, t$ is defined accordingly to $x\}$ if C is a concept of the form $\exists R.B$ or $\exists R.\top$. On the other hand, $B^{\mathcal{C}_i} = \Delta^{\mathcal{C}_i}$, or $R^{\mathcal{C}_i} = \{(x, x \cdot ([R], t)) \mid x \in \Delta^{\mathcal{C}_i}, t$ is defined accordingly to $x\}$. Now, if $\mathcal{T}_i \not\models \top \sqsubseteq C$, it is easy to see that the required holds as well.

Case 2. The $\mathcal{T}_i$ are $\mathcal{ELH}$-TBoxes. We assume they are in normal form. Now replace

– any inclusion $\top \sqsubseteq B$ by $A^i_\top \sqsubseteq B$;

– any inclusion $A \sqsubseteq \exists R.B$ by $A \sqsubseteq \exists R.(A^i_\top \sqcap B)$.

Thus, the construction is the same as above (we do not have to consider this case separately). One can show that the $\mathcal{K}'_i$ are as required.

Case 3. The $\mathcal{T}_i$ are *DL-Lite*$^{\mathcal{H}}_{core}$ or *DL-Lite*$^{\mathcal{H}}_{horn}$-TBoxes. Now replace

– any inclusion $\top \sqsubseteq B$ by $A^i_\top \sqsubseteq B$;

– any inclusion $B \sqsubseteq \exists R$ by $B \sqsubseteq \exists R$ and $\exists R^- \sqsubseteq A^i_\top$.

One can show that the $\mathcal{K}'_i$ are as required.

Next, we consider the case when $\mathcal{L}$ is a DL without role inclusions (with conjunction of concepts on the left-hand side). Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are given. We construct now $\mathcal{K}'_1$ and $\mathcal{K}'_2$ such that $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{K}'_1 \equiv_\Sigma \mathcal{K}'_1 \cup \mathcal{K}'_2$. Define $\mathcal{A}'_i$ as the union of

– $\mathcal{A}_i \cup \{A^i_\top(a) \mid a \in \mathsf{ind}(\mathcal{K}_i)\}$ and

– $\{A(a) \mid \mathcal{K}_i \models A(a)\} \cup \{P(a,b) \mid \mathcal{K}_i \models P(a,b)\}$.

Define $\mathcal{T}'_i$ as follows:

Case 1. If $\mathcal{T}_i$ are *Horn*-$\mathcal{ALCI}$-TBoxes in normal form then replace

– any inclusion $\top \sqsubseteq B$ by $A^i_\top \sqsubseteq B$;

– any inclusion $A \sqsubseteq B$ by $A \sqcap A^i_\top \sqsubseteq B$;

– any inclusion $A \sqsubseteq \exists R.B$ by $A \sqcap A^i_\top \sqsubseteq \exists R.(A^i_\top \sqcap B)$;

– any inclusion $A_1 \sqcap A_2 \sqsubseteq B$ by $A_1 \sqcap A_2 \sqcap A^i_\top \sqsubseteq B$;

– any inclusion $\exists R.A \sqsubseteq B$ by $A^i_\top \sqcap \exists R.(A \sqcap A^i_\top) \sqsubseteq B$;

– any inclusion $A \sqsubseteq \forall R.B$ by $A \sqcap A^i_\top \sqsubseteq \forall R.(\neg A^i_\top \sqcup B)$.

Note that we are not in normal form, but still in *Horn*-$\mathcal{ALCI}$.

Case 2. If the $\mathcal{T}_i$ are $\mathcal{EL}$-TBoxes in normal form then the construction is the same except that the final clause does not occur.

Observe that the trivial interpretation is a model of $\mathcal{T}_i$. We show that $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{K}'_1$ and $\mathcal{K}'_1 \cup \mathcal{K}'_2$ are Σ-query inseparable.

Consider materialisations (and models), $\mathcal{C}_i$ and $\mathcal{D}_i$, of $\mathcal{K}_i$ and $\mathcal{K}'_i$, respectively. As in the case with careful relativisation, one can show $\Delta^{\mathcal{D}_i} = (A^i_\top)^{\mathcal{D}_i}$, $\Delta^{\mathcal{C}_i} = (\Delta^{\mathcal{D}_i})^{\setminus A^i_\top}$ and $X^{\mathcal{C}_i} = (X^{\mathcal{D}_i})^{\setminus A^i_\top}$ for each symbol X distinct from $A^i_\top$. Note that in the case of *Horn*-$\mathcal{ALCHI}$, $(\Delta^{\mathcal{D}_1} \setminus \mathsf{ind}(\mathcal{K}_1)) \cap (\Delta^{\mathcal{D}_2} \setminus \mathsf{ind}(\mathcal{K}_2)) = \emptyset$ as for each $d \in \Delta^{\mathcal{D}_1} \setminus \mathsf{ind}(\mathcal{K}_1)$ such that $\mathsf{tail}(d) = ([S], t)$, we have $A^1_\top \in t$ and $A^2_\top \notin t$, and the other way around. And in the case of $\mathcal{EL}$, we can assume $(\Delta^{\mathcal{D}_1} \setminus \mathsf{ind}(\mathcal{K}_1)) \cap (\Delta^{\mathcal{D}_2} \setminus \mathsf{ind}(\mathcal{K}_2)) = \emptyset$ as we can rename the elements of $\Delta^{\mathcal{D}_i}$ to achieve that. Denote by $\mathcal{D}$ the union of $\mathcal{D}_1$ and $\mathcal{D}_2$ defined by setting $\Delta^{\mathcal{D}} = \Delta^{\mathcal{D}_1} \cup \Delta^{\mathcal{D}_2}$ and $X^{\mathcal{D}} = X^{\mathcal{D}_1} \cup X^{\mathcal{D}_2}$ for all symbols X. Then $\mathcal{D}$ is a model of $\mathcal{K}'_1 \cup \mathcal{K}'_2$.

Again, the interesting direction is to show that if $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$, then $\mathcal{K}'_1$ Σ-query entails $\mathcal{K}'_1 \cup \mathcal{K}'_2$. Assume that $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$, we show $\mathcal{D}$ is finitely Σ-homomorphically embeddable into $\mathcal{D}_1$. Let $Y \subseteq \Delta^{\mathcal{D}}$ be finite. Since $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ and $\mathcal{K}'_1$ trivially Σ-query entails $\mathcal{K}'_1$ we have Σ-homomorphisms

– $f_1 : Y \cap \Delta^{\mathcal{D}_1} \to \Delta^{\mathcal{D}_1}$ from the Y-restriction of $\mathcal{D}_1$ to $\mathcal{D}_1$ and

– $f_2 : Y \cap \Delta^{\mathcal{D}_2} \to \Delta^{\mathcal{D}_1}$ from the Y-restriction of $\mathcal{D}_2$ to $\mathcal{D}_1$.

We may assume $f_1(a^{\mathcal{D}_1}) = f_2(a^{\mathcal{D}_2})$ for all $a \in \mathsf{ind}(\mathcal{K}_1) \cap \mathsf{ind}(\mathcal{K}_2)$. Then $f_1 \cup f_2$ is a Σ-homomorphism from the Y-restriction of $\mathcal{D}$ to $\mathcal{D}_1$, as required.

Finally, we consider the case $\mathcal{K}_1$ is inconsistent. Let $\mathcal{A}'_1$ be the ABox extending $\mathcal{A}_1$ with

$$\{A(a) \mid a \in \mathsf{ind}(\mathcal{K}_2) \text{ and } \mathcal{K}_2 \not\models \boldsymbol{q}(a) \text{ for any } \Sigma\text{-query } \boldsymbol{q}\}$$

for some fresh concept name A. We show that $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{K}'_1 = (\mathcal{T}_1, \mathcal{A}'_1)$ and $\mathcal{K}_1 \cup \mathcal{K}_2$ are Σ-query inseparable. Note that $\mathcal{K}'_1$ and $\mathcal{K}_1 \cup \mathcal{K}_2$ are inconsistent. First, from $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ we obtain that $\mathsf{ind}(\mathcal{K}'_1) = \mathsf{ind}(\mathcal{K}_1) \cup \mathsf{ind}(\mathcal{K}_2)$, so the "only-if" direction follows immediately. Assume $\mathcal{K}'_1 \equiv_\Sigma \mathcal{K}_1 \cup \mathcal{K}_2$, then from their inconsistency, it follows $\mathsf{ind}(\mathcal{K}'_1) = \mathsf{ind}(\mathcal{K}_1) \cup \mathsf{ind}(\mathcal{K}_2)$. By construction of $\mathcal{A}'_1$, for each Σ-query $\boldsymbol{q}$ and tuple of constants $\vec{a} \subseteq \mathsf{ind}(\mathcal{K}_2)$ such that $\mathcal{K}_2 \models \boldsymbol{q}(\vec{a})$, we obtain $\vec{a} \subseteq \mathsf{ind}(\mathcal{K}_1)$. So we conclude $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$. ❑

Proof of Theorem 3

Theorem 3 *Σ-query entailment for any of our DLs $\mathcal{L}$ is* LOGSPACE*-reducible to the membership problem for universal UCQ-solutions in $\mathcal{L}$.*

Proof. We complete the proof given in the paper by considering the case when $\mathcal{I}_\emptyset$ is not a model of $\mathcal{T}_i$. As before, we may assume that $\Sigma = \mathsf{sig}(\mathcal{K}_1) \cap \mathsf{sig}(\mathcal{K}_2)$. Let $\Sigma_1 = \mathsf{sig}(\mathcal{K}_1)$. Then $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{K}_1$ Σ_1-query entails $\mathcal{K}_2$.

Define $\mathcal{K}_i' = (\mathcal{T}_i', \mathcal{A}_i')$ to be as in the case $\mathcal{L}$ is a DL without role inclusions in the proof of Theorem 2, where if $\mathcal{L}$ is *DL-Lite$_{core}$* or *DL-Lite$_{horn}$*, $\mathcal{T}_i'$ is defined by replacing in $\mathcal{T}_i$ any inclusion $\top \sqsubseteq B$ by $A^i_\top \sqsubseteq B$, and adding $\exists R^- \sqsubseteq A^i_\top$ for each role R. Moreover, define $\mathcal{K}_i''$ to be a copy of $\mathcal{K}_i'$ in which all symbols S except for $A^i_\top$ are replaced by fresh S_i. Then $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{K}_1$ is a universal UCQ-solution for $(\mathcal{K}_1'' \cup \mathcal{K}_2'', \mathcal{T}_{12}, \Sigma_1)$, where $\mathcal{T}_{12} = \{S_i \sqsubseteq S \mid S \in \Sigma_1,\ i = 1, 2\}$. ❑

Proof of Theorem 8

Theorem 8 *$\mathcal{M}_2$ is finitely Σ-homomorphically embeddable into $\mathcal{M}_1$ iff the following conditions hold*:

(abox) $\boldsymbol{r}_\Sigma^{\mathcal{M}_2}(a, b) \subseteq \boldsymbol{r}_\Sigma^{\mathcal{M}_1}(a, b)$, *for any* $a, b \in \mathsf{ind}(\mathcal{K}_2)$;

(win) *for any $u_0 \in \Delta^{\mathcal{G}_2}$ and $n < \omega$, there exists $\sigma_0 \in \Delta^{\mathcal{M}_1}$ such that player 1 has an n-winning strategy in the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ starting from $(u_0 \mapsto \sigma_0)$.*

Proof. ($\Rightarrow$) Suppose $\mathcal{M}_2$ is finitely Σ-homomorphically embeddable into $\mathcal{M}_1$. Then **(abox)** holds by the definition of Σ-homomorphism. To show that **(win)** holds, suppose $u_0 \in \Delta^{\mathcal{G}_2}$ and $n < \omega$ are given. Take the sub-interpretation $\mathcal{M}_2^{u_0, n}$ of $\mathcal{M}_2$ that contains σu_0, for some (say, the shortest) word σ, and all those elements of $\mathcal{M}_2$ whose distance from σu_0 does not exceed n. Let $h\colon \mathcal{M}_2^{u_0,n} \to \mathcal{M}_1$ be a Σ-homomorphism. Take $\sigma_0 = h(\sigma u_0)$. Clearly, u_0 and σ_0 satisfy **(s$_1$)** and **(s$_2$)**. We show that player 1 has an n-winning strategy in the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ starting from the state $(u_0 \mapsto \sigma_0)$. Suppose player 2 takes $u_0 \rightsquigarrow_2^\Sigma u_1$. Then $\sigma u_0 u_1$ is an element of $\mathcal{M}_2^{u_0,n}$, and player 1 responds with $\sigma_1 = h(\sigma u_0 u_1)$. Conditions **(s$_1$)** and **(s$_2$)** hold because h is a Σ-homomorphism. In the same way player 1 uses h to find responses to all challenges of player 2 in any round $k < n$ of the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$.

($\Leftarrow$) Let $\mathcal{M}_2'$ be a sub-interpretation of $\mathcal{M}_2$ containing n elements, or $\mathcal{M}_2$ itself if such a sub-interpretation does not exist. Consider first the case when $\mathcal{M}_2'$ is a tree with root σu_0 for some $u_0 \in \Delta^{\mathcal{G}_2}$. We define, by induction, a Σ-homomorphism $h\colon \mathcal{M}_2' \to \mathcal{M}_1$ as follows. Take an n-winning strategy for player 1 in the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ starting from a suitable state $(u_0 \mapsto \sigma_0)$ and set $h(\sigma u_0) = \sigma_0$. Suppose now that $\sigma u_0 \dots u_k$ is an element of $\mathcal{M}_2'$ such that whenever $u_i \rightsquigarrow_2^\Sigma \cdots \rightsquigarrow_2^\Sigma u_j$ for some $0 \le i \le j < k$ there is a play $(u_i \mapsto \sigma_i), \dots, (u_j \mapsto \sigma_j)$, which conforms with some n-winning strategy. Assume $u_{k-1} \rightsquigarrow_2^\Sigma u_k$, the current state is $(u_{k-1} \mapsto \sigma_{k-1})$ and the current n-winning strategy is $\mathcal{S}$, then $u_{k-1} \rightsquigarrow_2^\Sigma u_k$ is a valid challenge of player 2. Consider the reply $(u_k \mapsto \sigma_k)$ of player 1 according to $\mathcal{S}$. Then we set $h(\sigma u_0 \dots u_k) = \sigma_k$. Conditions **(s$_1$)** and **(s$_2$)** make sure that h is a Σ-homomorphism. If however it is not the case that $u_{k-1} \rightsquigarrow_2^\Sigma u$ for each $u \in \Delta^{\mathcal{G}_2}$, then $h(\sigma u_0 \dots u_k)$ can be defined equal to σ', where $\sigma' \in \Delta^{\mathcal{M}_1}$ is such that there is an n-winning strategy in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ from $(u_k \mapsto \sigma')$ (such σ' exists by **(win)**).

An arbitrary finite sub-interpretation of $\mathcal{M}_2$ can be represented as a union of finitely many maximal rooted ones in which ABox individuals can only be roots. Let h be the union of the corresponding Σ-homomorphisms for all these rooted sub-interpretations. In view of **(abox)**, h is a Σ-homomorphism. ❑

Proof of Lemma 22

Lemma 22 *For any $u_0 \in \Delta^{\mathcal{G}_2}$, condition $(< \omega)$ holds for arbitrary strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ iff (ω) holds in $G_\Sigma^a(\mathcal{G}_2, \mathcal{G}_1)$ for some state $(\Xi_0 \mapsto x_0, \Psi_0)$ with $u_0 \in \Xi_0$.*

Proof. ($\Rightarrow$) Let $\mathbb{S} = \{\mathcal{S}_n \mid n < \omega\}$ be the set of the given n-winning strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ and suppose that $\mathcal{S}_n$ begins with $(u_0 \mapsto \sigma_0^n)$, $n < \omega$.

We define a (possibly infinite) tree $\mathfrak{T}$ whose nodes are of the form $(u \mapsto z, k)$, where $u \in \Delta^{\mathcal{G}_2}$, z is a suffix of some element in $\Delta^{\mathcal{M}_1}$, $k < \omega$, whose edges are labelled with $u \rightsquigarrow_2^\Sigma u'$, and the following conditions hold:

(1) the root of $\mathfrak{T}$ is of the form $(u_0 \mapsto w, 0)$, $w \in \Delta^{\mathcal{G}_1}$;

(2) $\boldsymbol{t}_\Sigma^{\mathcal{G}_2}(u) \subseteq \boldsymbol{t}_\Sigma^{\mathcal{G}_1}(\mathsf{tail}(z))$;

(3) for any node $(u \mapsto z, k)$ in $\mathfrak{T}$ and any $u \rightsquigarrow_2^\Sigma u'$, there is exactly one $(u \rightsquigarrow_2^\Sigma u')$-successor of $(u \mapsto z, k)$ in $\mathfrak{T}$, which can be of the following forms:

- $(u' \mapsto w', k+1)$, if $z = w \in \Delta^{\mathcal{G}_1}$, $w' \rightsquigarrow_1^\Sigma w$ and $\boldsymbol{r}_\Sigma^{\mathcal{G}_2}(u, u') \subseteq \bar{\boldsymbol{r}}^{\mathcal{G}_1}(w', w)$;
- $(u' \mapsto z', k)$, if $z = z'w$, for $w \in \Delta^{\mathcal{G}_1}$, $w' = \mathsf{tail}(z')$, $w' \rightsquigarrow_1^\Sigma w$ and $\boldsymbol{r}_\Sigma^{\mathcal{G}_2}(u, u') \subseteq \bar{\boldsymbol{r}}_\Sigma^{\mathcal{G}_1}(w', w)$;
- $(u' \mapsto zw', k)$, if $z = z'w$, $w \rightsquigarrow_1^\Sigma w'$ and $\boldsymbol{r}_\Sigma^{\mathcal{G}_2}(u, u') \subseteq \boldsymbol{r}_\Sigma^{\mathcal{G}_1}(w, w')$;
- $(u' \mapsto b, -1)$, if $z = a \in \mathsf{ind}(\mathcal{K}_1)$, $b \in \mathsf{ind}(\mathcal{K}_1)$ and $\boldsymbol{r}_\Sigma^{\mathcal{G}_2}(u, u') \subseteq \boldsymbol{r}_\Sigma^{\mathcal{G}_1}(a, b)$;

(4) for any $k \ge 0$ and any nodes $(u \mapsto w, k)$, $(u' \mapsto w', k)$ in $\mathfrak{T}$ with $w, w' \in \Delta^{\mathcal{G}_1}$, it follows $w = w'$.

We call the tree $\mathfrak{T}$ *complete* if whenever a node $(u \mapsto z, k)$ is in $\mathfrak{T}$ and $u \rightsquigarrow_2^\Sigma u'$ then some node $(u' \mapsto z', k')$ is its $(u \rightsquigarrow_2^\Sigma u')$-successor in $\mathfrak{T}$. It will be shown later that given a complete tree $\mathfrak{T}$ we can construct an ω-winning strategy starting from some $(\Xi_0 \mapsto x_0, \Psi_0)$ in the game $G_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$. But first we show how to construct such a tree using $\mathbb{S}$.

For $\mathcal{S} \in \mathbb{S}$, we say that $\mathcal{S}$ *respects* $\mathfrak{T}$ if there exists a map

$$f_\mathcal{S}\colon \{(z, k) \mid (u \mapsto z, k) \in \mathfrak{T}\} \to \Delta^{\mathcal{M}_1}$$

such that:

1. $f_\mathcal{S}(z, k) = \delta z$, for some δ;
2. $(u \mapsto f_\mathcal{S}(z, k))$ is in $\mathcal{S}$, for any $(u \mapsto z, k)$ in $\mathfrak{T}$;
3. if $(u' \mapsto z', k')$ is a $(u \rightsquigarrow_2^\Sigma u')$-successor of $(u \mapsto z, k)$ in $\mathfrak{T}$, then, according to $\mathcal{S}$, player 1 responds to the challenge $u \rightsquigarrow_2^\Sigma u'$ of player 2 in the state $(u \mapsto f_\mathcal{S}(z, k))$ with $(u' \mapsto f_\mathcal{S}(z', k'))$.

The set $\mathbb{S}$ contains an n-winning strategy starting from $(u_0 \mapsto \sigma_0^n)$, for any $n < \omega$. As $\mathcal{G}_1$ is finite, we can find some x_0 such that $x_0 = \mathsf{tail}(\sigma_0^n)$ for infinitely many n. Denote by $\mathbb{S}_0$ the set of the corresponding strategies from $\mathbb{S}$. As an m-winning strategy is also an l-winning strategy for any $l \leq m$, $\mathbb{S}_0$ contains an n-winning strategy starting from some $(u_0 \mapsto \delta^n x_0)$, for any $n < \omega$. Define $\mathfrak{T}_0$ to be a tree with a single node $(u_0 \mapsto x_0, 0)$. For every $\mathcal{S} \in \mathbb{S}_0$, we set $f_{\mathcal{S}}(x_0, 0) = \delta_{\mathcal{S}} x_0$, where $\delta_{\mathcal{S}}$ is the corresponding δ^n. Thus, all the strategies in $\mathbb{S}_0$ respect $\mathfrak{T}_0$.

Suppose we have already constructed $\mathfrak{T}_i$ and $\mathbb{S}_i$ such that $\mathbb{S}_i$ contains an n-winning strategy for any $n < \omega$, and all of them respect $\mathfrak{T}_i$. If $\mathfrak{T}_i$ is incomplete then it contains a state $(u \mapsto z, k)$ without a $(u \rightsquigarrow_2^\Sigma u')$-successor, for some $u \rightsquigarrow_2^\Sigma u'$. (We always take such a state that is nearest to the root.) Suppose $f_{\mathcal{S}}(z, k) = \delta_{\mathcal{S}} w$. Consider the responses $u' \mapsto \sigma^n$ to the challenge $u \rightsquigarrow_2^\Sigma u'$ according to the n-winning strategies in $\mathbb{S}_i$, for $n < \omega$. Take some $w' \in \Delta^{\mathcal{G}_1}$ such that $w' = \mathsf{tail}(\sigma^n)$ for infinitely many n. Denote by $\mathbb{S}_{i+1}$ the set of the corresponding strategies from $\mathbb{S}_i$.

Suppose $w' \rightsquigarrow_1^\Sigma w$. If $z = w$ then we add the node $(u' \mapsto w', k+1)$ as a $(u \rightsquigarrow_2^\Sigma u')$-successor of $(u \mapsto z, k)$ to $\mathfrak{T}_i$, thus obtaining $\mathfrak{T}_{i+1}$. By the definition of the materialisations, we also have $\delta_{\mathcal{S}} = \delta'_{\mathcal{S}} w'$, for all $\mathcal{S} \in \mathbb{S}_{i+1}$. We then set $f_{\mathcal{S}}(w', k+1) = \delta_{\mathcal{S}}$. If $|z| > 1$ then $z = z'w'w$ and $z'w'$ is a suffix of $\delta_{\mathcal{S}}$. In this case, we add the node $(u' \mapsto z'w', k)$ as a $(u \rightsquigarrow_2^\Sigma u')$-successor of $(u \mapsto z, k)$ to $\mathfrak{T}_i$, thus obtaining $\mathfrak{T}_{i+1}$, and we set $f_{\mathcal{S}}(z'w', k) = \delta_{\mathcal{S}}$.

Suppose $w \rightsquigarrow_1^\Sigma w'$. In this case, we add $(u' \mapsto zw', k)$ as a $(u \rightsquigarrow_2^\Sigma u')$-successor of $(u \mapsto z, k)$ to $\mathfrak{T}_i$, thus obtaining $\mathfrak{T}_{i+1}$, and we set $f_{\mathcal{S}}(zw', k) = \delta_{\mathcal{S}} w w'$.

Suppose $w, w' \in \mathsf{ind}(\mathcal{K}_1)$ (hence, $\delta_{\mathcal{S}}$ is empty). In this case, we add $(u' \mapsto w', -1)$ as a $(u \rightsquigarrow_2^\Sigma u')$-successor of $(u \mapsto z, k)$ to $\mathfrak{T}_i$, thus obtaining $\mathfrak{T}_{i+1}$, and set $f_{\mathcal{S}}(w', -1) = w'$.

All $\mathcal{S} \in \mathbb{S}_{i+1}$ clearly respect $\mathfrak{T}_{i+1}$. It is easy to see that it satisfies **(4)**.

We proceed in the same way and construct a sequence of growing trees $\mathfrak{T}_0 \subseteq \mathfrak{T}_1 \subseteq \dots$ until we reach a complete finite tree $\mathfrak{T}_k$; otherwise we take $\mathfrak{T} = \bigcup_{n<\omega} \mathfrak{T}_n$, which is obviously complete.

Now we show that player 1 has an ω-winning strategy starting from some $(\Xi_0 \mapsto x_0, \Psi_0)$ in the game $G_\Sigma^a(\mathcal{G}_2, \mathcal{G}_1)$. Suppose that we have a complete tree $\mathfrak{T}$ with the root $(u_0 \mapsto x_0, 0)$. We then set:

$$\begin{aligned}
\Xi_0 &= \{u \mid (u \mapsto x_0, 0) \in \mathfrak{T}\},\\
\Phi_0 &= \{u' \mid u \rightsquigarrow_2^\Sigma u',\ u \in \Xi_0,\ (u' \mapsto x_0 w, 0) \in \mathfrak{T}\},\\
&\quad \cup \{u' \mid (u' \mapsto b, -1) \in \mathfrak{T} \text{ is a } (u \rightsquigarrow_2^\Sigma u')\text{-successor}\\
&\qquad\qquad \text{of } (u \mapsto x_0, 0) \in \mathfrak{T}\},\\
\Psi_0 &= \{u' \mid u \rightsquigarrow_2^\Sigma u',\ u \in \Xi_0,\ (u' \mapsto w, 1) \in \mathfrak{T}\}.
\end{aligned}$$

Note that, by **(4)**, if $(u \mapsto x, 0) \in \mathfrak{T}$ (and $|x| = 1$, that is, $x \in \Delta^{\mathcal{G}_1}$) then $x = x_0$. Moreover, if $x_0 \in \mathsf{ind}(\mathcal{K}_1)$, then $\Psi_0 = \emptyset$.

More generally, for any $i > 0$ such that $\mathfrak{T}$ contains some $(u \mapsto x, i)$, $|x| = 1$, and $x_{i-1} \notin \mathsf{ind}(\mathcal{K}_1)$, we set

$$\begin{aligned}
\Xi_i &= \{u \mid (u \mapsto x, i) \in \mathfrak{T}\},\\
\Phi_i &= \{u' \mid u \rightsquigarrow_2^\Sigma u',\ u \in \Xi_i,\ (u' \mapsto xw, i) \in \mathfrak{T}\}\\
&\quad \cup \{u' \mid (u' \mapsto b, -1) \in \mathfrak{T} \text{ is a } (u \rightsquigarrow_2^\Sigma u')\text{-successor}\\
&\qquad\qquad \text{of } (u \mapsto x, i) \in \mathfrak{T}\},\\
\Psi_i &= \{u' \mid u \rightsquigarrow_2^\Sigma u',\ u \in \Xi_i,\ (u' \mapsto w, i+1) \in \mathfrak{T}\}.
\end{aligned}$$

Note that, by **(4)**, all $(u \mapsto x, i) \in \mathfrak{T}$ with $x \in \Delta^{\mathcal{G}_1}$ share the same x, which we denote by x_i. And again, if $x_i \in \mathsf{ind}(\mathcal{K}_1)$, then $\Psi_i = \emptyset$.

By **(3)**, the states $\mathfrak{s}_i = (\Xi_i \mapsto x_i, \Psi_i)$ clearly define the backward part of an ω-winning strategy for player 1 in the game $G_\Sigma^a(\mathcal{G}_2, \mathcal{G}_1)$ starting from $\mathfrak{s}_0$.

Thus, it remains to define ω-winning strategies for the start-bounded game $G_\Sigma^s(\mathcal{G}_2, \mathcal{G}_1)$ starting from states of the form $(\emptyset, \Xi_k \mapsto x_k)$ and first-round challenges $u \rightsquigarrow_2^\Sigma v$ such that $u \in \Xi_k$ and $v \in \Phi_k$.

Let $k \geq 0$ be such that $\Phi_k \neq \emptyset$. We now transform $\mathfrak{T}$ into a tree $\mathfrak{W}_k$ representing an ω-winning strategy for player 1 in the game $G_\Sigma^s(\mathcal{G}_2, \mathcal{G}_1)$ starting from $(\emptyset, \Xi_k \mapsto x_k)$ and first-round challenges $u \rightsquigarrow_2^\Sigma v$ such that $u \in \Xi_k$ and $v \in \Phi_k$. Thus, $(\emptyset, \Xi_k \mapsto x_k)$ is the root of $\mathfrak{W}_k$ *associated with* x_k. Suppose that we have already defined a node $(\Gamma, \Xi \mapsto w)$ associated with a word δw. Let $u \in \Xi$ and $u \rightsquigarrow_2^\Sigma v$ be such that the node $(u \mapsto \delta w, k')$ in $\mathfrak{T}$, where k' equals to k or -1 has a $(u \rightsquigarrow_2^\Sigma v)$-successor of the form $(v \mapsto \delta w w', k')$ (if $(\Gamma, \Xi \mapsto w)$ is the root, we also require that $v \in \Phi_k$). Then we add to $\mathfrak{W}_k$ the node $(\Gamma', \Xi' \mapsto w')$, associated with $\delta w w'$, as a $(u \rightsquigarrow_2^\Sigma v)$-successor of $(\Gamma, \Xi \mapsto w)$, where

– $\Xi' = \{v' \mid (v' \mapsto \delta w w', k') \in \mathfrak{T}\}$,

– $\Gamma' = \Xi$.

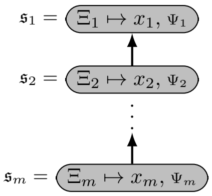
If $(\Gamma, \Xi \mapsto a)$ is associated with $a \in \mathsf{ind}(\mathcal{K}_1)$ and the node $(u \mapsto a, k')$, for $u \in \Xi$, where k' equals to k or -1, has a $(u \rightsquigarrow_2^\Sigma v)$-successor of the form $(v \mapsto b, -1)$ with $b \in \mathsf{ind}(\mathcal{K}_1)$ (note that if $(\Gamma, \Xi \mapsto a)$ is the root, then $\Phi_k = \Xi_k^{\rightsquigarrow}$), then we add to $\mathfrak{W}_k$ the node $(\emptyset, \Xi' \mapsto b)$, associated with b, as a $(u \rightsquigarrow_2^\Sigma v)$-successor of $(\Gamma, \Xi \mapsto w)$, where $\Xi' = \{v' \mid (v' \mapsto b, -1) \in \mathfrak{T}\}$.

We claim that $\mathfrak{W}_k$ thus constructed represents an ω-winning strategy for player 1 in the game $G_\Sigma^s(\mathcal{G}_2, \mathcal{G}_1)$ starting from $(\emptyset, \Xi_k \mapsto x_k)$ and first-round challenges $u \rightsquigarrow_2^\Sigma v$ such that $u \in \Xi_k$ and $v \in \Phi_k$.

($\Leftarrow$) Given $G_\Sigma^a(\mathcal{G}_2, \mathcal{G}_1)$ and $u_0 \in \Delta^{\mathcal{G}_2}$ suppose (ω) holds for some x_0, Ξ_0 and Ψ_0 such that $u_0 \in \Xi_0$. Let $n < \omega$, we are going to show there is $\sigma_0 \in \Delta^{\mathcal{M}_1}$ such that player 1 has an n-winning strategy starting from $(u_0 \mapsto \sigma_0)$ in the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$. To define σ_0 consider a N-winning strategy $\mathcal{S}$ of player 1 from $(\Xi_0 \mapsto x_0, \Psi_0)$ for $N = 2 \times |2^{\Omega_2}| \times |\Omega_1| + 1$, where Ω_i is such that $\Delta^{\mathcal{G}_i} = \mathsf{ind}(\mathcal{K}_i) \cup \Omega_i$, and a play

$$(\Xi_m \mapsto x_m, \Psi_m), \dots, (\Xi_1 \mapsto x_1, \Psi_1)$$

conforming with $\mathcal{S}$ such that $\Xi_m = \Xi_0$, $x_m = x_0$, and $\Psi_m = \Psi_0$. Denote by $\mathfrak{s}_i$ the state $(\Xi_i \mapsto x_i, \Psi_i)$ for $1 \leq i \leq m$.



Then, either $m < N$ and $\Psi_1 = \emptyset$, or $m = N$ and since the number of all possible states in $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ is less than N, there are integers c, r such that $m \geq c > c - r \geq 1$ and $\mathfrak{s}_c = \mathfrak{s}_{c-r}$.

Now, we set $\sigma_0 = \delta'\delta$, where δ and δ' are obtained as follows. In the fist case above, δ is equal to $x_1 \dots x_m$ and δ' is any (possibly empty) sequence such that $\delta'\delta \in \Delta^{\mathcal{M}_1}$ (such δ' obviously exists). In the second case δ is equal to the sequence of length $n+1$:

$$\delta = x_{c-o}x_{c-o+1} \cdots x_c \ \cdot \ \delta_{c,r} \ \cdots \ \delta_{c,r} \ \cdot \ x_{c+1} \cdots x_m$$

where $o = (n - (m - c)) \bmod r$, $\delta_{c,r} = x_{c-r+1}x_{c-r+2} \cdots x_c$, and δ' is obtained as before.

Let k be the length of δ, and y_i denote the i-th element of the sequence δ, $1 \leq i \leq k$. We define $\mu(i) \in \{1, \dots, m\}$ to be the number such that $y_i = x_{\mu(i)}$. In the first case above $\mu(i) = i$, whereas in the second case $\mu(i)$ equals to

$$\begin{cases} c - \big((o - i + 1) \bmod r\big), & \text{for } 1 \leq i \leq n - (m-c) + 1, \\ c + i - 1 - (n - (m - c)), & \text{for } n - (m - c) + 2 \leq i \leq n + 1, \end{cases}$$

which can be presented graphically:

	1	2		$o+1$	$o+2$		$o+r+1$
$\delta =$	x_{c-o}	x_{c-o-1}	$\cdots$	x_c	x_{c-r+1}	$\cdots$	x_c

$o+r+2$		$o+2r+1$		$n-m+c+2$		$n+1$
x_{c-r+1}	$\cdots$	x_c	$\cdots$	x_{c+1}	$\cdots$	x_m

Finally, it remains to produce an n-winning strategy $\mathcal{S}'$ of player 1 from $(u_0 \mapsto \sigma_0)$. For each challenge $u_{i-1} \rightsquigarrow^\Sigma_2 u_i$ from a state $(u_{i-1} \mapsto \sigma_{i-1})$ by player 2 in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$, we are going to define $\sigma_i \in \Delta^{\mathcal{M}_1}$ so that to set the response of player 1 according to $\mathcal{S}'$ to be $(u_i \mapsto \sigma_i)$. We will also define auxiliary f-values for these states $(u_i \mapsto \sigma_i)$ that relate them with the "original" states in $G^a_\Sigma(\mathcal{G}_1, \mathcal{G}_2)$.

We first set $f(u_0 \mapsto \sigma_0) = (\Xi_{\mu(k)} \mapsto x_{\mu(k)}, \Psi_{\mu(k)})$ and consider the challenge $u_0 \rightsquigarrow^\Sigma_2 u_1$ by player 2 in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$. If $u_1 \in \Psi_{\mu(k)}$ then $k > 1$ by the construction of δ. We set $\sigma_1 = \delta' y_1 \dots y_{k-1}$ and $f(u_1 \mapsto \sigma_1) = (\Xi_{\mu(k-1)} \mapsto x_{\mu(k-1)}, \Psi_{\mu(k-1)})$. If $u_1 \notin \Psi_{\mu(k)}$, then consider the start-bounded game $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ with the initial state $(\emptyset, \Xi_{\mu(k)} \mapsto x_{\mu(k)})$ and the first-round challenge $u_0 \rightsquigarrow^\Sigma_2 u_1$ (by the structure of the states in $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ this challenge is valid). Let $(\Gamma, \Xi \mapsto z)$ be the response of player 1 according to $\mathcal{S}$, for some $z \in \Delta^{\mathcal{G}_1}$. If $z \in \mathsf{ind}(\mathcal{K}_1)$, we set $\sigma_1 = z$ (note, in this case $\sigma_0 \in \mathsf{ind}(\mathcal{K}_1)$), otherwise we set $\sigma_1 = \sigma_0 z$. The f-value is defined as $f(u_1 \mapsto \sigma_1) = (\Gamma, \Xi \mapsto z)$.

Suppose now we defined $\mathcal{S}'$ for a number of steps $h < n$ and the response of player 1 to the challenge $u_{h-1} \rightsquigarrow^\Sigma_2 u_{h+1}$ from a state $(u_{h-1} \mapsto \sigma_{h-1})$ was defined as the state $(u_h \mapsto \sigma_h)$, moreover assume $\sigma_h = \delta' y_1 \dots y_{k'} z_1 \dots z_l$ for $0 \leq k' \leq k$ and $l \geq 0$ (we have also the value of f for this state). If now there is no valid challenge $u_h \rightsquigarrow^\Sigma_2 u_{h+1}$ then further moves of player 1 need not be defined. Otherwise consider the challenge $u_h \rightsquigarrow^\Sigma_2 u_{h+1}$ of player 2 in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$.

Suppose, first, $f(u_h \mapsto \sigma_h) = (\Gamma', \Xi' \mapsto x')$ where $x' = \mathsf{tail}(\sigma_h) = z_l$, and by induction hypothesis $u_h \in \Xi'$. Note that in this case, $l \geq 1$. If $u_h \rightsquigarrow^\Sigma_2 u_{h+1}$ is a challenge also from $(\Gamma', \Xi' \mapsto x')$ in $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$, consider the response $(\Gamma, \Xi \mapsto z)$ of player 1 to this challenge according to $\mathcal{S}$. If $z \in \mathsf{ind}(\mathcal{K}_1)$ we set $\sigma_{h+1} = z$, otherwise we set $\sigma_{h+1} = \sigma_h z$. The f-value is defined as $f(u_{h+1} \mapsto \sigma_{h+1}) = (\Gamma, \Xi \mapsto z)$. If $u_h \rightsquigarrow^\Sigma_2 u_{h+1}$ is *not* a valid challenge from $(\Gamma', \Xi' \mapsto x')$ in $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$, then **(nbk)** does not hold for this challenge, which means $x' \notin \mathsf{ind}(\mathcal{K}_1)$, $u_{h+1} \in \Gamma'$, and $\boldsymbol{r}^{\mathcal{G}_2}_\Sigma(u_h, u_{h+1}) \subseteq \bar{\boldsymbol{r}}^{\mathcal{G}_1}_\Sigma(z, x')$, where z is the element preceding z_l in σ_h. Two cases are possible:

- $l = 1$, therefore the predecessor of $(\Gamma', \Xi' \mapsto x')$ according to $\mathcal{S}$ is the starting state $(\emptyset, \Xi_{\mu(k')} \mapsto x_{\mu(k')})$ of the game $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$, which has been launched from $(\Xi_{\mu(k')} \mapsto x_{\mu(k')}, \Psi_{\mu(k')})$ in $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$. It follows, $\boldsymbol{r}^{\mathcal{G}_2}_\Sigma(u_h, u_{h+1}) \subseteq \bar{\boldsymbol{r}}^{\mathcal{G}_1}_\Sigma(y_{k'}, z_1)$, and as $\Gamma' = \Xi_{\mu(k')}$, we have $u_{h+1} \in \Xi_{\mu(k')}$. So we set $\sigma_{h+1} = \delta' y_1 \cdots y_{k'}$, and $f(u_{h+1} \mapsto \sigma_{h+1}) = (\Xi_{\mu(k')} \mapsto x_{\mu(k')}, \Psi_{\mu(k')})$.
- $l > 1$, we consider the predecessor $(\Gamma, \Xi \mapsto x)$ of $(\Gamma', \Xi' \mapsto x')$ in $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ according to $\mathcal{S}$, with $x = z_{l-1}$. We have $\Gamma' = \Xi$, hence $u_{h+1} \in \Xi$, so we set $\sigma_{h+1} = \delta' y_1 \dots y_{k'} z_1 \dots z_{l-1}$ and $f(u_{h+1} \mapsto \sigma_{h+1}) = (\Gamma, \Xi \mapsto x)$.

Alternatively, suppose $f(u_h \mapsto \sigma_h) = (\Xi' \mapsto x', \Psi')$, where $x' = \mathsf{tail}(\sigma_h)$, and by induction hypothesis $u_h \in \Xi'$. Then $l = 0$ and $(\Xi' \mapsto x', \Psi') = (\Xi_{\mu(k')} \mapsto x_{\mu(k')}, \Psi_{\mu(k')})$. We proceed here as in the base case. If $u_{h+1} \in \Psi_{\mu(k')}$, then $k' > 1$: indeed, by construction of δ, if $k = m$, then $\Psi_1 = \emptyset$, otherwise $k = n+1$, so provided that $h \leq n$, it cannot be the case $k' = 1$. We set $\sigma_{h+1} = \delta' y_1 \cdots y_{k'-1}$, and $f(u_{h+1} \mapsto \sigma_{h+1}) = (\Xi_{\mu(k'-1)} \mapsto x_{\mu(k'-1)}, \Psi_{\mu(k'-1)})$. If $u_{h+1} \notin \Psi_{\mu(k')}$, then consider the start-bounded game $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ with the initial state $(\emptyset, \Xi_{\mu(k')} \mapsto x_{\mu(k')})$ and the first-round challenge $u_h \rightsquigarrow^\Sigma_2 u_{h+1}$. Let $(\Gamma, \Xi \mapsto z)$ be the response of player 1 according to $\mathcal{S}$, for some $z \in \Delta^{\mathcal{G}_1}$, $u_{h+1} \in \Xi$. If $z \in \mathsf{Ind}(\mathcal{K}_1)$, we set $\sigma_{h+1} = z$, otherwise we set $\sigma_{h+1} = \sigma_h z$. The f-value is defined as $f(u_{h+1} \mapsto \sigma_{h+1}) = (\Gamma, \Xi \mapsto z)$.

We have constructed the strategy $\mathcal{S}'$ from $(u_0 \mapsto \sigma_0)$ in the game $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$. It can be straightforwardly verified that $\mathcal{S}'$ is n-winning. ❑

Proofs of Lemmas 11, 13 and 19

Lemma 11 *For $u_0 \in \Delta^{\mathcal{G}_2}$, condition $(< \omega)$ holds for forward strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ iff (ω) holds in $G^f_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ for some state $(u_0 \mapsto x_0)$.*

Proof. Can be obtained as a corollary of the proof of Lemma 22. Given $u_0 \in \Delta^{\mathcal{G}_2}$, it suffices to observe that condition $(< \omega)$ holds for forward strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$

iff (ω) holds in $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$, where all the states of the kinds $(\Xi_i \mapsto x_i, \Psi_i)$ and $(\Gamma_i, \Xi_i \mapsto x_i)$ are such that $\Xi_i = \{u\}$ for $u \in \Delta^{\mathcal{G}_2}$ and $\Psi_i = \Gamma_i = \emptyset$, for some state $(\{u_0\} \mapsto x_0, \emptyset)$. Such restricted $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ can be straightforwardly converted to $G^f_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$. ❑

Lemma 13 *For $u_0 \in \Delta^{\mathcal{G}_2}$, condition $(< \omega)$ holds for backward strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ iff (ω) holds in $G^b_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ for some state $(\{u_0\} \mapsto x_0)$.*

Proof. Can be obtained as a corollary of the proof of Lemma 22. Given $u_0 \in \Delta^{\mathcal{G}_2}$, it suffices to observe that condition $(< \omega)$ holds for backward strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ iff (ω) holds in $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$, where only the states of the kind $(\Xi_i \mapsto x_i, \Psi_i)$ occur and $\Psi_i = \Xi_i^{\rightsquigarrow}$, for some state $(\{u_0\} \mapsto x_0, \{u_0\}^{\rightsquigarrow})$. Such restricted $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ can be straightforwardly converted to $G^b_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$. ❑

Lemma 19 *For any $u_0 \in \Delta^{\mathcal{G}_2}$, condition $(< \omega)$ holds for start-bounded strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ iff (ω) holds in $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ for some state $(\emptyset, \Xi_0 \mapsto x_0)$ with $u_0 \in \Xi_0$.*

Proof. Can be obtained as a corollary of the proof of Lemma 22. Given $u_0 \in \Delta^{\mathcal{G}_2}$, it suffices to observe that condition $(< \omega)$ holds for start-bounded strategies in $G_\Sigma(\mathcal{G}_2, \mathcal{M}_1)$ iff (ω) holds in $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$, where all the states of the kind $(\Xi_i \mapsto x_i, \Psi_i)$ are such that $\Psi_i = \emptyset$, for some state $(\Xi_0 \mapsto x_0, \emptyset)$ with $u_0 \in \Xi_0$. Such restricted $G^a_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$ can be straightforwardly converted to $G^s_\Sigma(\mathcal{G}_2, \mathcal{G}_1)$. ❑

Proof of Lemma 15

Lemma 15 *Checking (ω) in Lemma 13 is* coNP*-hard.*

Proof. The proof is by reduction of the *unsatisfiability* problem for 3CNFs $\varphi = \bigwedge_{i=1}^m c_i$, where $c_i = l_{i1} \vee l_{i2} \vee l_{i3}$ and each l_{ij} is either one of the propositional variables $v_1, \dots v_k$ or a negation of such a variable.

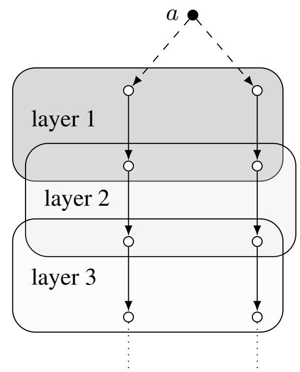
Let $p_1, \dots, p_k$ be the first k prime numbers (observe that $1 < p_j \leq k^2$, for all j). We take a role name R, a role name C_i, for each clause c_i in φ, and role names $S_{j\ell}$, for $1 \leq j \leq k$ and $1 \leq \ell \leq p_j$. Now we define a KB $\mathcal{K}_1 = (\mathcal{T}_2, \{\exists R(a)\})$, where $\mathcal{T}_2$ contains the following inclusions, for $1 \leq j \leq k$ and $1 \leq \ell < p_j$,

$$\exists R^- \sqsubseteq \exists S_{j1}, \quad \exists S^-_{j\ell} \sqsubseteq \exists S_{j\ell+1}, \quad \exists S^-_{jp_j} \sqsubseteq \exists S_{j1},$$

and the following inclusions, for $1 \leq j \leq k$ and $1 \leq i \leq m$:

$$S_{j1} \sqsubseteq C_i, \quad \text{if } v_j \text{ is a literal of } c_i,$$
$$S_{j2} \sqsubseteq C_i, \quad \text{if } \neg v_j \text{ is a literal of } c_i.$$

Intuitively, $\mathcal{M}_2$ is a tree with k branches with a common root edge R. The jth branch is obtained by unravelling the loop of p_j arrows $S_{j1}, \dots, S_{jp_j}$: the first arrow, S_{j1}, corresponds to v_j being true (in an assignment), while the second arrow, S_{j2}, to v_j being false. Therefore, $p_1 \times p_2 \times \cdots \times p_k$ layers (a layer i consists of all edges from points at the distance i from the root) contain representations of all possible assignments to $v_1, \dots, v_k$ (see figure below). The last two types of role inclusions make sure that roles $C_1, \dots, C_m$, which constitute the signature Σ, mark those assignments on which φ is true.



We define $\mathcal{K}_1 = (\mathcal{T}_1, \{A(a)\})$, where $\mathcal{T}_1$ consists of the following inclusions, for $1 \leq i, i' \leq m$,

$$A \sqsubseteq \exists T_i, \qquad \exists T_i^- \sqsubseteq \exists T_{i'},$$
$$T_i \sqsubseteq C^-_{i'}, \qquad \text{if } i' \neq i.$$

In $\mathcal{M}_1$, the path from each point to the root contains edges that are labelled by all of $C_1, \dots, C_m$ but one (note that the C_i edges point towards to root, in the opposite direction to the C_i edges of $\mathcal{M}_2$). Therefore, there is a finite Σ-homomorphism iff in each of the assignments one of the clauses is false (that is, iff φ is unsatisfiable).

The generating structure $\mathcal{G}_1$ is essentially a set of loops each of which is missing precisely one of the C_i. Therefore, the responses of player 1 correspond to choices of the missing C_i. Challenges by player 2, on the other hand, correspond to the subsets of $C_1, \dots, C_m$ in the layers of $\mathcal{M}_2$, the number of which may be exponential in k. Thus, player 2 can go through a sequence of exponentially many distinct challenges (assignments), to each of which player 1 will have to find a clause that is false under the assignment. The sequence, however, repeats itself after $p_1 \times p_2 \times \cdots \times p_k$ steps. ❑

Proof of Theorem 24

Theorem 24 *For data complexity, Σ-query entailment and inseparability are* P*-hard for DL-Lite$_{core}$ and $\mathcal{EL}$-KBs.*

Proof. The proof is by reduction of the P-complete entailment problem for *acyclic* Horn ternary clauses: given a conjunction φ of clauses of the form a_i and $a_i \wedge a_{i'} \to a_j$, $i, i' < j$, decide whether a_n is true in every model of φ. Consider a *DL-Lite$_{core}$* TBox $\mathcal{T}$ containing the CIs

$$V \sqsubseteq \exists P, \quad \exists P^- \sqsubseteq \exists R_i \text{ and } \exists R_i^- \sqsubseteq V, \text{ for } i = 1, 2,$$

and let an ABox $\mathcal{A}$ be comprised of $F(a_n)$ and

$$P(a_i, a_i),\ R_1(a_i, a_i),\ R_2(a_i, a_i), \text{ for each clause } a_i \text{ in } \varphi,$$
$$P(a_j, c), R_1(c, a_i), R_2(c, a_{i'}), \text{ for } c = a_i \wedge a_{i'} \to a_j \text{ in } \varphi.$$

Set $\Sigma = \{F, P, R_1, R_2\}$, $\mathcal{K}_2 = (\mathcal{T}, \mathcal{A} \cup \{V(a_n)\})$ and $\mathcal{K}_1 = (\emptyset, \mathcal{A})$. Obviously, $\mathcal{K}_2$ Σ-query entails $\mathcal{K}_1$. On the other hand, the materialisation of $\mathcal{K}_2$ is (finitely) Σ-homomorphically embeddable in the materialisation of $\mathcal{K}_1$

iff φ derives a_n. Indeed, the materialisation $\mathcal{M}_2$ of $\mathcal{K}_2$ is infinite, while finite materialisation $\mathcal{M}_1$ of $\mathcal{K}_1$ is finite. So, the only way to embed finite prefixes of $\mathcal{M}_2$ of arbitrary depth into $\mathcal{M}_1$ is by mapping subtrees of unbounded depth into the loops in $\mathcal{M}_1$ for unary clauses a_i in φ, which is only possible if there is a tree of rules of the form $a_i \wedge a_{i'} \to a_j$ with root a_n and leaves among the clauses a_i of φ (that is, if there is a derivation of a_n from φ).

For $\mathcal{EL}$, we can take $\mathcal{T} = \{V \sqsubseteq \exists P.(\exists R_1.V \sqcap \exists R_2.V)\}$. ❑

Proof of Theorem 25

Theorem 25 *For combined complexity, Σ-query entailment and inseparability are (i) 2*ExpTime*-hard for Horn-$\mathcal{ALCI}$ KBs and (ii)* ExpTime*-hard for DL-Lite$^{\mathcal{H}}_{core}$ KBs.*

Proof. The proof of (*ii*) is by encoding alternating Turing machines (ATMs) with polynomial tape and using the fact that APSpace = ExpTime.

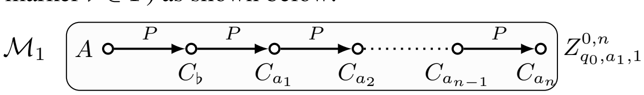
As in the proof of (*i*), let $M = (\Gamma, Q, q_0, q_1, \delta)$ be an ATM and let M' be the ATM obtained from M by extending it with two instructions that go into an infinite loop if M reaches the accepting state. Our aim is to construct, given M and an input $\vec{w}$, two TBoxes, $\mathcal{T}_1$ and $\mathcal{T}_2$, and a signature Σ such that M' has a run with only infinite branches iff the materialisation $\mathcal{M}_2$ of $(\mathcal{T}_2, \{A(c)\})$ is finitely Σ-homomorphically embeddable into the materialisation $\mathcal{C}_1$ of $(\mathcal{T}_1, \{A(c)\})$. Let f be a polynomial such that, on any input of length m, M' uses at most $n = f(m)$ cells.

The construction proceeds in four steps. In the definition of the TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$, we use concept inclusions of the form $B \sqsubseteq \exists R.(C_1 \sqcap \dots \sqcap C_k)$ as an abbreviation for

$$B \sqsubseteq \exists R_0, \quad R_0 \sqsubseteq R \quad \text{and} \quad \exists R_0^- \sqsubseteq C_i, \text{ for } 1 \leq i \leq k,$$

where R_0 is a fresh role name. If C_i is a complex concept then $\exists R_0^- \sqsubseteq C_i$ is also treated as an abbreviation for the respective concept and role inclusions.

Step 1. First we encode configurations and transitions of M' using $\mathcal{T}_1$. We represent a configuration (that is, the contents of every cell on the tape, the state and the position of the head) by a sequence of $(n + 2)$ domain elements connected by some role R, which will be called a *block*. More precisely, the first element in each block is used to distinguish the type of the block. Each of the remaining n elements is assigned an index from 0 to n. They encode the contents of the tape: if the element with index i belongs to C_a, for some $a \in \Gamma$, then the ith cell of the tape is assumed to contain a in the configuration defined by the block (cell 0 contains marker $\flat \in \Gamma$) as shown below:



The first block represents the initial configuration, that is, symbols $a_1, \dots, a_n$ written in the n cells of the tape (comprising the input $\vec{w}$ in the first m cells padded with the blanks) and the initial state q_0, which is achieved by the following inclusion in $\mathcal{T}_1$:

$$A \sqsubseteq \exists P.(C_\flat \sqcap \exists P.(C_{a_1} \sqcap \exists P.(C_{a_2} \sqcap \exists P.(\dots \exists P.(C_{a_n} \sqcap Z^{0,n}_{q_0,a_1,1}) \dots)))). \quad (\mathcal{T}_1\text{-}1)$$

Step 2. The contents of the tape and the head position in each configuration is encoded in a block of length $n + 2$; the current state $q \in Q$ and the position k of the head are recorded in the concept $Z^{0,n}_{q,a,k}$ that contains the last element of the block ($a \in \Gamma$ specifies the contents of the active cell scanned by the head). At the end of the block we branch out one block for each of the two transitions and propagate via the $Z^{1,i}_{q,a,k}$ and the $Z^{2,i}_{q,a,k}$ the current state, head position and symbol in the active cell: for $q \in Q$, $a \in \Gamma$ and $1 \leq k \leq n$, we add to $\mathcal{T}_1$ the inclusions

$$Z^{0,n}_{q,a,k} \sqsubseteq \prod_{j=1,2} \exists P.(X_j \sqcap Z^{j,-1}_{q,a,k}), \quad (\mathcal{T}_1\text{-}2)$$

where X_1 and X_2 are two fresh concept names (distinguishing the two branches).

The acceptance condition for M' is enforced by means of $\mathcal{T}_2$, which uses four types of blocks. The initial configuration is encoded by the following inclusion in $\mathcal{T}_2$:

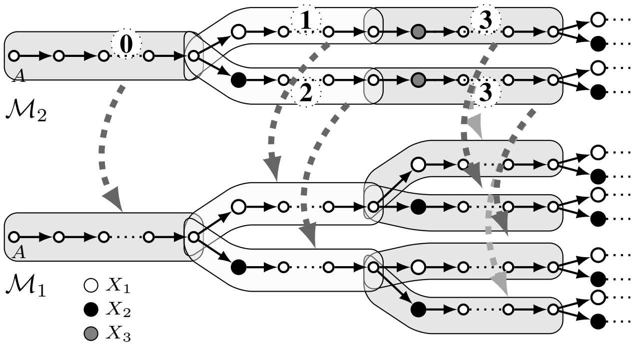
$$A \sqsubseteq \exists P.\underbrace{\exists P.\cdots\exists P.}_{n \text{ times}}(\exists P.X_1 \sqcap \exists P.X_2). \quad (\mathcal{T}_2\text{-}1)$$

Two types of blocks, starting with X_1 and X_2, respectively, represent configurations with universal sates; and one more type of blocks, starting with X_3, suffices for representing configurations with existential states. These blocks are arranged into an infinite tree-like structure: the block starting with A is the root, from which an X_1- and an X_2-blocks branch out (successors of the initial state q_0 are universal). Each of them is followed by an X_3-block (an existential state), which branches out an X_1- and an X_2-blocks, and so on. This is achieved by adding to $\mathcal{T}_2$ the following inclusions: for $j = 1, 2$,

$$X_j \sqsubseteq \exists P.\underbrace{\exists P.(G \sqcap \exists P.(\cdots \exists P.(G \sqcap}_{n \text{ times}} \exists P.X_3))), \quad (\mathcal{T}_2\text{-}2)$$

$$X_3 \sqsubseteq \exists P.\overbrace{\exists P.(G \sqcap \exists P.(\cdots \exists P.(G \sqcap}\prod_{j=1,2} \exists P.X_j))), \quad (\mathcal{T}_2\text{-}3)$$

where G is a concept name (containing all domain elements representing the tape). If $\Sigma = \{A, X_1, X_2, P\}$ then there is a unique Σ-homomorphism from the A-block in $\mathcal{M}_2$ to the block of the initial configuration in $\mathcal{M}_1$. Next, concepts X_1 and X_2 ensure that the following X_1- and X_2-blocks are Σ-homomorphically mapped (in a unique way) into the respective blocks in $\mathcal{M}_1$, which reflects the acceptance condition of universal states. The following block, however, begins with X_3, which is not in the signature, and thus can be mapped to either of the blocks in $\mathcal{M}_1$, which reflects the choice in existential states; see the picture below, where possible Σ-homomorphisms are shown by thick dashed arrows:



Step 3. Recall that the $Z^{j,i}_{q,a,k}$, for $-1 \le i \le n$, specify the position k of the head on the tape. Let the active cell in the current configuration be k; then until the cell $k-2$ is reached in a successive configuration, the following inclusions in $\mathcal{T}_1$ propagate the state ($q \in Q$), the symbol in the active cell ($a \in \Gamma$), the head position ($1 \le k \le n$) and the branch marker ($j = 0, 1, 2$) along the domain elements constituting blocks: for $-1 < i \le n$ with $i \neq k-1$,

$$Z^{j,i-1}_{q,a,k} \sqsubseteq \bigsqcap_{b \in \Gamma} \exists P.(C_b \sqcap Z^{j,i}_{q,a,k}) \qquad (\mathcal{T}_1\text{-}3)$$

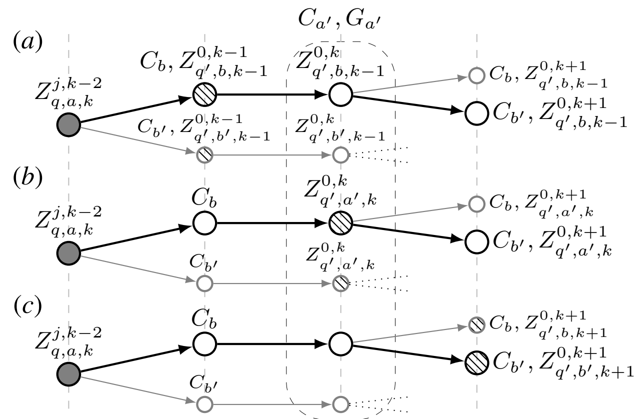
(for each $b \in \Gamma$, these inclusions generate a branch in $\mathcal{M}_1$ to represent the same cell but with a different symbol, b, tentatively assigned to the cell; Step 4 will ensure that the correct branch and symbol are selected to match the cell contents in the preceding configuration). We point out that, since the size of the tape is polynomial in the length of the input, we can use the subscripts of the $Z^{j,i}_{q,a,k}$ to specify the head position, k, and the cell number, i; in the proof of item (i), we had to use P-counters over H and the T^j, respectively. When the cell $k-2$ is reached, the contents of the active cell, the current state and the head position are changed according to δ:

$$Z^{j,k-2}_{q,a,k} \sqsubseteq \bigsqcap_{b \in \Gamma} \exists P.(C_b \sqcap \Delta^k_{qa,b}), \qquad (\mathcal{T}_1\text{-}4)$$

where $\delta(q, a, j) = (q', a', \sigma)$ and $\Delta^k_{qa,b}$ is the concept

$$\begin{array}{ll} \exists P.(C_{a'} \sqcap G_{a'} \sqcap Z^{0,k}_{q',b,k-1}), & \text{if } \sigma = -1, \\ \exists P.(C_{a'} \sqcap G_{a'} \sqcap Z^{0,k}_{q',a',k}), & \text{if } \sigma = 0, \\ \exists P.(C_{a'} \sqcap G_{a'} \sqcap \bigsqcap_{b' \in \Gamma} \exists P.(C_{b'} \sqcap Z^{0,k+1}_{q',b',k+1})), & \text{if } \sigma = +1 \end{array}$$

(the symbol in the active cell is changed according to the instruction, and the current state, symbol in the next active cell and the head position are then recorded in $Z^{0,i}_{q,a,k}$; note that the branch marker, $j = 1, 2$, is replaced by 0). These three situations are shown below, where hatched nodes denote domain elements for the active cell in a current configuration (where the symbol is recorded in the $Z^{0,i}_{q,a,k}$); the domain elements in the dashed oval represent the active cell of the preceding configuration and the grey nodes denote domain elements *two* cells before the active cell of that configuration (where inclusion ($\mathcal{T}_1$-4) becomes 'active'):



(Note that there is only one branch for the modified cell, which corresponds to the new symbol, a', in that cell; see explanations below.) Then the current state and the symbol in the active cell are further propagated along the tape using ($\mathcal{T}_1$-3) with $j = 0$ and $i > k-1$.

Step 4. The inclusons ($\mathcal{T}_1$-3)–($\mathcal{T}_1$-4) generate a separate P-successor for each $b \in \Gamma$. The correct one is chosen by a finite Σ-homomorphism, h, from $\mathcal{M}_2$ to $\mathcal{M}_1$. To exclude wrong choices, we take

$$\Sigma \quad = \quad \{A, P, X_1, X_2, P\} \quad \cup \quad \{D_a \quad | \quad a \quad \in \quad \Gamma\}.$$

Recall that if $d_1 \in C_a^{\mathcal{M}_1}$, for some $a \in \Gamma$, then it represents a cell containing a. The following inclusions in $\mathcal{T}_1$ ensure that, for each $b \in \Gamma$ different from a, there is a block of $n+2$-many P^--connected elements that ends in the concept D_b (called a D_b*-block* in the sequel):

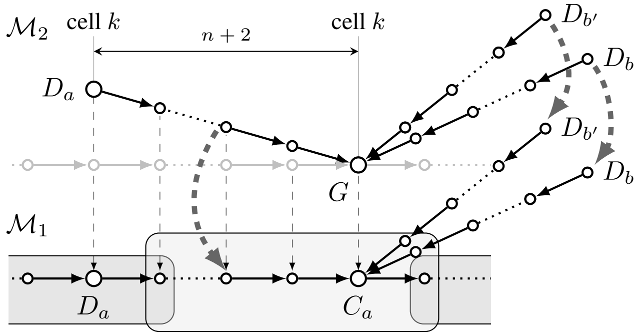
$$C_a \sqsubseteq D_a \sqcap \bigsqcap_{b \in \Gamma \setminus \{a\}} G_b, \qquad (\mathcal{T}_1\text{-}5)$$

$$G_b \sqsubseteq \exists P^-. \underbrace{\exists P^- . \cdots \exists P^-.}_{n \text{ times}} \exists P^-.D_b, \quad \text{for } b \in \Gamma. \quad (\mathcal{T}\text{-}1)$$

(Note that in this proof we do not need to use binary counters to reach the end of the block.) Suppose $h(d_2) = d_1$ and d_2 belongs to G in $\mathcal{M}_2$ (it represents a cell in a non-initial configuration). Then ($\mathcal{T}$-1) and the inclusions

$$G \sqsubseteq \bigsqcap_{b \in \Gamma} G_b \qquad (\mathcal{T}_2\text{-}4)$$

added to $\mathcal{T}_2$ generate a D_b-block, for *each* $b \in \Gamma$ (including a). Each of the D_b-blocks in $\mathcal{M}_2$, for $b \in \Gamma$ with $b \neq a$, can be mapped by h to the respective D_b-block in $\mathcal{M}_1$. By the choice of Σ, the only remaining D_a-block, in case a is tentatively contained in this cell, could be mapped (in the reverse order) along the branch in $\mathcal{M}_1$ *but only* if the cell contains a in the preceding configuration (that is, the element which is $n+2$ steps closer to the root of $\mathcal{M}_1$ belongs to D_a):



Note (see $\Delta^k_{qa,b}$) that the cell whose contents is changed generates the additional D_a-block in $\mathcal{M}_1$ to allow the respective D_a-block from $\mathcal{M}_2$ to be mapped there.

One can show now that $\mathcal{T}_1$ and $\mathcal{T}_2$ are as required: M' has a run with only infinite branches iff the materialisation $\mathcal{M}_2$ of $(\mathcal{T}_2, \{A(c)\})$ is finitely Σ-homomorphically embeddable into the materialisation $\mathcal{M}_1$ of $(\mathcal{T}_1, \{A(c)\})$, where Σ contains the concept and role names in $\mathcal{T}_2$. It remains to use Theorem 5 and the fact that APSPACE = EXPTIME. By Theorem 2, Σ-query inseparability is also EXPTIME-hard. ❑

References

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