First-Order Rewritability of Temporal Ontology-Mediated Queries

A. Artale[1], **R. Kontchakov**[3], **A. Kovtunova**[1], **V. Ryzhikov**[1], **F. Wolter**[2] and **M. Zakharyaschev**[3]

[1]Faculty of Computer Science, Free University of Bozen-Bolzano, Italy, {botoeva,ryzhikov}@inf.unibz.it

[2]Dept. of Computer Science, University of Liverpool, UK, wolter@liverpool.ac.uk

[3]Dept. of Computer Science, Birkbeck, London, UK, {roman,michael}@dcs.bbk.ac.uk

Abstract

Aiming at ontology-based data access over temporal, in particular streaming data, we design a language of ontology-mediated queries by extending OWL 2 QL and SPARQL with temporal operators, and investigate rewritability of these queries into two-sorted first-order logic with $<$ and PLUS over time.

1 Introduction

Ontology-based data access (OBDA), one of the most promising applications of description logics, has recently been touted as a key ingredient in data stream management systems [Calbimonte *et al.*, 2012; Baader *et al.*, 2013; Özçep and Möller, 2014; Kharlamov *et al.*, 2014]. Its role is to facilitate querying data streams from heterogeneous sensor networks (measuring temperature, vibration, wind speed and direction, blood pressure, etc.) in order to detect, analyse and predict certain events or situations. For example, we may want to detect areas of the country that have been hit by blizzards—severe snowstorms with low temperatures and strong winds for at least three hours—using the weather stations' sensors streaming raw data such as $(\mathit{sensorId}, \mathit{temperature}, \mathit{timestamp})$ [Zhang *et al.*, 2012]. The OBDA component would give us an ontology, $\mathcal{O}$, that defines a high-level vocabulary—possibly including the relation 'area x was hit by a blizzard at time t'—which is linked to the raw data by means of (say, R2RML) mappings. Given a query in the vocabulary of $\mathcal{O}$ such as $\exists t\, \mathit{Blizzard}(x,t)$, the system would first rewrite the pair $(\mathcal{O}, \exists t\, \mathit{Blizzard}(x,t))$, called a *temporal ontology-mediated query* (TOMQ), into a query over the original timestamped data and then evaluate it using standard relational or stream data management systems.

The OBDA scenario outlined above is similar to the classical one [Calvanese *et al.*, 2007b]. An essential difference, however, is that now we query *temporal* data and therefore, require temporal constructs in the ontology and/or query languages, which, on the other hand, should not undermine the rewritability property of TOMQs. As temporal extensions of knowledge representation formalisms are notorious for their bad computational behaviour [Lutz *et al.*, 2008], a first natural step could be to keep standard atemporal OBDA languages (e.g., OWL 2 QL or *DL-Lite* logics), assuming that ontology axioms hold at all moments of time, but add temporal constructs to queries. This approach was taken by [Gutiérrez-Basulto and Klarman, 2012; Baader *et al.*, 2013; Borgwardt *et al.*, 2013; Özcep *et al.*, 2013; Klarman and Meyer, 2014] and shown to preserve query rewritability. Note, however, that the inability to define temporal predicates such as $\mathit{Blizzard}(x,t)$ in ontologies leaves the burden of encoding them within queries to the user, which goes against the OBDA paradigm. Moreover, natural queries such as 'check if a weather station has been serviced every 24 hours' are not expressible in these languages.

The first attempt to introduce linear-time temporal logic (*LTL*) operators to the TOMQs' ontologies was made by Artale *et al.* [2013b], who showed that the operators $\Diamond_F$ and $\Diamond_P$ (sometime in the future and past) on the left-hand side of *DL-Lite* axioms allow rewritings of conjunctive queries into the two-sorted first-order language FO($<$) with variables of sorts 'object' and 'time' and an explicit temporal precedence relation $<$. On the other hand, it was observed that the next- and previous-time operators $\bigcirc_F$ and $\bigcirc_P$—perhaps the most powerful and versatile temporal modelling constructs—do not support FO($<$)-rewritability.

The aim of this paper is to launch a systematic investigation of rewritability of TOMQs with arbitrary temporal operators. Apart from FO($<$), we consider two more target languages for rewritings: FO($<,+$), which complements FO($<$) with a ternary predicate PLUS for 'addition' and still ensures query evaluation in LogTime-uniform AC^0 for data complexity; and monadic second-order logic MSO($<$), which guarantees query evaluation over finite linear orders (flows of time) in $NC^1 \subseteq$ LogSpace for data complexity. Axioms of our ontologies are given in clausal normal form

$$\lambda_1 \sqcap \dots \sqcap \lambda_n \ \sqsubseteq\ \lambda_{n+1} \sqcup \dots \sqcup \lambda_{n+m}, \qquad (1)$$

where the λ_i are all either *DL-Lite* concepts or roles, possibly prefixed with the operators $\bigcirc_F$, $\bigcirc_P$, $\Box_F$ and $\Box_P$ (always in the future and the past). Let $\boldsymbol{o} \in \{\Box, \bigcirc, \Box\bigcirc\}$ and $\boldsymbol{c} \in \{\mathit{bool}, \mathit{horn}, \mathit{krom}, \mathit{core}\}$. We denote by $\mathit{DL\text{-}Lite}_{\boldsymbol{c}}^{\boldsymbol{o}}$ the *temporal description logic* with axioms of the form (1), where the λ_i can only use the (future and past) operators indicated in $\boldsymbol{o}$, and $m \le 1$ if $\boldsymbol{c} = \mathit{horn}$; $n+m \le 2$ if $\boldsymbol{c} = \mathit{krom}$; $n+m \le 2$ and $m \le 1$ if $\boldsymbol{c} = \mathit{core}$; and arbitrary n, m if $\boldsymbol{c} = \mathit{bool}$. For example, the $\mathit{DL\text{-}Lite}_{\mathit{horn}}^{\bigcirc}$ axioms

$$\bigsqcap_{0 \le i \le 2} \bigcirc_F^i \mathit{BlizzardCondition} \sqsubseteq \bigcirc_F^j \mathit{Blizzard}, \quad j = 0, 1, 2,$$

$$\mathit{SevereSnow} \sqcap \mathit{LowTemp} \sqcap \mathit{StrongWind} \sqsubseteq \mathit{BlizzardCondition},$$

where $\bigcirc_F^i$ is a sequence of i-many $\bigcirc_F$, define the temporal concept *Blizzard* (the concepts on the left-hand side of the second axiom are defined via mappings from the sensor data), while the $\textit{DL-Lite}^{\bigcirc}_{core}$ axiom $\textit{Scheduled} \sqsubseteq \bigcirc_F^{24}\textit{Scheduled}$ can be used to detect 'service every 24 hours' (see below).

The queries in our TOMQs range from atoms of the form $A(x, t)$ or $P(x, y, t)$, for A a concept and P a role name, in *atomic* TOMQs (or TOMAQs), to arbitrary positive temporal concepts and roles such as $(\textit{Blizzard}\,\mathcal{U}(\textit{Rain}\sqcup\textit{Flooding}))(x, t)$ in *instance* TOMQs (TOMIQs), where $\mathcal{U}$ is the 'until' operator, and further to two-sorted FO-formulas built from positive temporal concepts and roles. It is well known that, due to the open-world semantics of OBDA, negation in queries results in non-tractable (often undecidable) query evaluation even in the atemporal case. In classical OBDA, a way to retain all FO-constructs in queries was proposed by Calvanese *et al.* [2007a] who interpreted them under an epistemic semantics. We follow in their footsteps: for example, the query $\textit{Scheduled}(x, t) \land \neg\textit{Serviced}(x, t)$ returns weather stations x that are not *known* (or not *certain*) to have been serviced at scheduled times t. With this semantics, TOMQs cover all first-order features of SPARQL 1.1 under the OWL 2 QL entailment regime [Kontchakov *et al.*, 2014]. Although the variables in TOMQs range over the active domain, their positive temporal concepts can express (via qualified existential restrictions and role intersections) tree-shaped CQs with one answer variable.

To only focus on rewritability, we assume that data instances are given as finite ABoxes with timestamped atoms of the form $A(a, n)$ and $P(a, b, n)$, $n \in \mathbb{Z}$ (e.g., generated from streamed data via mappings and the window operator [Calbimonte *et al.*, 2012]). We proceed in two steps. First, we investigate rewritability of TOMQs without roles, which can be regarded as *LTL* TOMQs. Using automata-theoretic techniques, we obtain the following rewritability classification:

<table>
<tr><th rowspan="2"></th><th colspan="3">TOMAQs</th><th colspan="3">TOMIQs / TOMQs</th></tr>
<tr><th>$\textit{LTL}^{\Box}_{\alpha}$</th><th>$\textit{LTL}^{\bigcirc}_{\alpha}$</th><th>$\textit{LTL}^{\Box\bigcirc}_{\alpha}$</th><th>$\textit{LTL}^{\Box}_{\alpha}$</th><th>$\textit{LTL}^{\bigcirc}_{\alpha}$</th><th>$\textit{LTL}^{\Box\bigcirc}_{\alpha}$</th></tr>
<tr><td>bool</td><td rowspan="4">FO(<)</td><td colspan="2" rowspan="2">MSO(<)</td><td>MSO(<)</td><td colspan="2" rowspan="3">MSO(<)</td></tr>
<tr><td>horn</td><td>FO(<)</td></tr>
<tr><td>krom</td><td rowspan="2">FO(<,+)</td><td rowspan="2">MSO(<)*</td><td>MSO(<)</td></tr>
<tr><td>core</td><td>FO(<)</td><td>FO(<,+)</td><td>MSO(<)*</td></tr>
</table>

*It is still open whether these can be improved to FO(<, +); all other results in the table are optimal.

In the second step, we reduce FO-rewritability of $\textit{DL-Lite}^{\Box\bigcirc}_{horn}$ TOMQs to FO-rewritability of certain $\textit{LTL}^{\Box\bigcirc}_{horn}$ TOMQs. In particular, we prove that all $\textit{DL-Lite}^{\Box}_{core}$ TOMQs are FO(<)-rewritable, while $\textit{DL-Lite}^{\bigcirc}_{core}$ TOMQs are FO(<, +)-rewritable. On the other hand, we show that some $\textit{DL-Lite}^{\Box}_{horn}$ TOMQs are NC^1-hard for data complexity, and so they cannot be FO-rewritable even using arbitrary numeric predicates.

2 Ontology-Mediated Queries

First we remind the reader of basic *DL-Lite* logics [Calvanese *et al.*, 2007b; Artale *et al.*, 2009]. Their language contains *object names* $a_0, a_1, \dots$, *concept names* $A_0, A_1, \dots$, and *role names* $P_0, P_1, \dots$. *Roles* R and *basic concepts* B are defined by the following grammar:

$$R \ ::= \ P_k \ \mid \ P_k^-, \qquad B \ ::= \ A_k \ \mid \ \exists R.$$

For $\boldsymbol{c} \in \{\textit{bool}, \textit{horn}, \textit{krom}, \textit{core}\}$, we denote by $\textit{DL-Lite}_{\boldsymbol{c}}$ the description logic with *concept* and *role inclusions* of the form (1) such that the λ_i are all either basic concepts or roles. As usual, we assume that the empty $\sqcap$ is $\top$ and the empty $\sqcup$ is $\bot$. Note that $\textit{DL-Lite}_{horn}$ and $\textit{DL-Lite}_{bool}$ contain role inclusions with monotone Boolean operators.

In temporal *DL-Lite*, we also allow applications of the operators $\bigcirc_F$, $\bigcirc_P$, $\Box_F$, $\Box_P$ to basic concepts and roles. For any $\boldsymbol{o} \in \{\Box, \bigcirc, \Box\bigcirc\}$, $\textit{DL-Lite}^{\boldsymbol{o}}_{\boldsymbol{c}}$ is the *temporal description logic* with concept and role inclusions of the form (1), where each λ_i is a basic concepts or roles are (possibly) prefixed by a string of (future or past) operators indicated in $\boldsymbol{o}$.

A $\textit{DL-Lite}^{\boldsymbol{o}}_{\boldsymbol{c}}$ *TBox* $\mathcal{T}$ (*RBox* $\mathcal{R}$) is a finite set of $\textit{DL-Lite}^{\boldsymbol{o}}_{\boldsymbol{c}}$ concept (role) inclusions; $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ is a $\textit{DL-Lite}^{\boldsymbol{o}}_{\boldsymbol{c}}$ *ontology*. An *ABox* (or *data instance*), $\mathcal{A}$, is a finite set of atoms of the form $A(a, \ell)$ and $P(a, b, \ell)$, where a, b are object names and $\ell \in \mathbb{Z}$. We denote by $\mathsf{ind}(\mathcal{A})$ the set of object names in $\mathcal{A}$, by $\min \mathcal{A}$ and $\max \mathcal{A}$ the minimal and maximal numbers in $\mathcal{A}$, and set $\mathsf{tem}(\mathcal{A}) = \{n \in \mathbb{Z} \mid \min \mathcal{A} \le n \le \max \mathcal{A}\}$. Without loss of generality, we assume that $\min \mathcal{A} = 0$ and $\max \mathcal{A} \ge 1$. A $\textit{DL-Lite}^{\boldsymbol{o}}_{\boldsymbol{c}}$ *knowledge base* (KB) is a pair $(\mathcal{O}, \mathcal{A})$, where $\mathcal{O}$ is a $\textit{DL-Lite}^{\boldsymbol{o}}_{\boldsymbol{c}}$ ontology and $\mathcal{A}$ an ABox.

A *temporal interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}(n)})$, where $\Delta^{\mathcal{I}} \neq \emptyset$ and $\mathcal{I}(n) = (\Delta^{\mathcal{I}}, a_0^{\mathcal{I}}, \dots, A_0^{\mathcal{I}(n)}, \dots, P_0^{\mathcal{I}(n)}, \dots)$ is a standard DL interpretation for each time instant $n \in \mathbb{Z}$, that is, $a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, $A_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}}$ and $P_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Thus, we assume that the domain $\Delta^{\mathcal{I}}$ and the interpretations $a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ of the object names are the same for all $n \in \mathbb{Z}$. The DL and temporal constructs are interpreted in $\mathcal{I}(n)$ as follows:

$$(P_i^-)^{\mathcal{I}(n)} = \{(x, y) \mid (y, x) \in P_i^{\mathcal{I}(n)}\},$$
$$(\exists R)^{\mathcal{I}(n)} = \big\{x \mid (x, y) \in R^{\mathcal{I}(n)}, \text{ for some } y\big\},$$
$$(\Box_F \lambda)^{\mathcal{I}(n)} = \bigcap\nolimits_{k>n} \lambda^{\mathcal{I}(k)}, \qquad (\bigcirc_F \lambda)^{\mathcal{I}(n)} = \lambda^{\mathcal{I}(n+1)},$$

and symmetrically for $\Box_P$ and $\bigcirc_P$; as usual, $\bot$ is interpreted by $\emptyset$ and $\top$ by $\Delta^{\mathcal{I}}$ for concepts and by $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for roles. Concept and role inclusions are interpreted in $\mathcal{I}$ *globally* in the sense that (1) holds in $\mathcal{I}$ if $\bigcap \lambda_i^{\mathcal{I}(n)} \subseteq \bigcup \lambda_j^{\mathcal{I}(n)}$ for *all* $n \in \mathbb{Z}$. We call $\mathcal{I}$ a *model* of $(\mathcal{O}, \mathcal{A})$ and write $\mathcal{I} \models (\mathcal{O}, \mathcal{A})$ if all concept and role inclusions from $\mathcal{O}$ hold in $\mathcal{I}$, and $a^{\mathcal{I}} \in A^{\mathcal{I}(n)}$ for $A(a, n) \in \mathcal{A}$, and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in P^{\mathcal{I}(n)}$ for $P(a, b, n) \in \mathcal{A}$.

Remark 1. Note that the *LTL* operators $\Diamond_F$, $\Diamond_P$, $\mathcal{U}$ and $\mathcal{S}$ ('since') can be expressed in $\textit{DL-Lite}^{\Box\bigcirc}_{bool}$ [Fisher *et al.*, 2001; Artale *et al.*, 2013a]. $\textit{DL-Lite}^{\Box}_{horn}$ extends the ontology language *TQL* of Artale *et al.* [2013b] as $\Diamond_P A \sqsubseteq B$ is equivalent to $A \sqsubseteq \Box_F B$. Note also that each of our ontology languages can say that a role or basic concept λ is *expanding* (using $\lambda \sqsubseteq \bigcirc_F \lambda$ in $\textit{DL-Lite}^{\bigcirc}_{core}$ and $\lambda \sqsubseteq \Box_F \lambda$ in $\textit{DL-Lite}^{\Box}_{core}$) or *rigid* (using, in addition, $\lambda \sqsubseteq \bigcirc_P \lambda$ and $\lambda \sqsubseteq \Box_P \lambda$).

To query temporal ontologies with data, we suggest the following language inspired by the recently standardised SPARQL 1.1 entailment regimes (`www.w3.org/TR/sparql11-entailment`); cf. also [Motik, 2012; Gutierrez *et al.*, 2007].

Positive temporal concepts $\varkappa$ and *positive temporal roles* ϱ

are defined by the grammars

$$\varkappa ::= \top \mid A_k \mid \exists R.\varkappa \mid \varkappa_1 \sqcap \varkappa_2 \mid \varkappa_1 \sqcup \varkappa_2 \mid \boldsymbol{op}\,\varkappa \mid \varkappa_1\, \boldsymbol{op}'\varkappa_2,$$
$$\varrho ::= R \mid \varrho_1 \sqcap \varrho_2 \mid \varrho_1 \sqcup \varrho_2 \mid \boldsymbol{op}\,\varrho \mid \varrho_1\, \boldsymbol{op}'\varrho_2,$$

where $\boldsymbol{op}$ ranges over $\bigcirc_F$, $\Diamond_F$, $\Box_F$, $\bigcirc_P$, $\Diamond_P$, $\Box_P$ and $\boldsymbol{op}'$ over $\mathcal{U}$, $\mathcal{S}$. The extensions of $\varkappa$ and ϱ in a temporal interpretation $\mathcal{I}$ are computed using the definition above and the following clauses (given only for $\varkappa$ and the future-time operators):

$$(\exists R.\varkappa)^{\mathcal{I}(n)} = \{x \mid (x, y) \in R^{\mathcal{I}(n)}, \text{ for some } y \in \varkappa^{\mathcal{I}(n)}\},$$
$$(\varkappa_1 \sqcap \varkappa_2)^{\mathcal{I}(n)} = \varkappa_1^{\mathcal{I}(n)} \cap \varkappa_2^{\mathcal{I}(n)},$$
$$(\varkappa_1 \sqcup \varkappa_2)^{\mathcal{I}(n)} = \varkappa_1^{\mathcal{I}(n)} \cup \varkappa_2^{\mathcal{I}(n)},$$
$$(\Diamond_F \varkappa)^{\mathcal{I}(n)} = \bigcup_{k>n} \varkappa^{\mathcal{I}(k)},$$
$$(\varkappa_1\, \mathcal{U}\, \varkappa_2)^{\mathcal{I}(n)} = \bigcup_{k>n} \big(\varkappa_2^{\mathcal{I}(k)} \cap \bigcap_{n<m<k} \varkappa_1^{\mathcal{I}(m)}\big).$$

A *DL-Lite*$_c^o$ *ontology-mediated instance query* (TOMIQ) is a pair of the form $(\mathcal{O}, \varkappa)$ or $(\mathcal{O}, \varrho)$, for a *DL-Lite*$_c^o$ ontology $\mathcal{O}$. A *certain answer* to $(\mathcal{O}, \varkappa)$ over an ABox $\mathcal{A}$ is a pair (a, ℓ) such that $a \in \mathsf{ind}(\mathcal{A})$, $\ell \in \mathsf{tem}(\mathcal{A})$ and $a^{\mathcal{I}} \in \varkappa^{\mathcal{I}(\ell)}$ for every $\mathcal{I} \models (\mathcal{O}, \mathcal{A})$. A *certain answer* to $(\mathcal{O}, \varrho)$ over $\mathcal{A}$ is a triple (a, b, ℓ) with $a, b \in \mathsf{ind}(\mathcal{A})$, $\ell \in \mathsf{tem}(\mathcal{A})$ and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in \varrho^{\mathcal{I}(\ell)}$ for every $\mathcal{I} \models (\mathcal{O}, \mathcal{A})$. Let $\mathsf{ans}(\mathcal{O}, \varkappa, \mathcal{A})$ denote the set of certain answers to $(\mathcal{O}, \varkappa)$ over $\mathcal{A}$, and similarly for $(\mathcal{O}, \varrho)$.

As a technical tool in our proofs, we also require 'certain answers' in which ℓ can range over the whole $\mathbb{Z}$ rather than only the *active domain* $\mathsf{tem}(\mathcal{A})$; we denote the set of such certain answers *over* $\mathcal{A}$ *and* $\mathbb{Z}$ by $\mathsf{ans}^{\mathbb{Z}}(\mathcal{O}, \varkappa, \mathcal{A})$ or $\mathsf{ans}^{\mathbb{Z}}(\mathcal{O}, \varrho, \mathcal{A})$.

Example 2. Let $\mathcal{O} = \{\bigcirc_P A \sqsubseteq B,\ \bigcirc_P B \sqsubseteq A\}$, $\mathcal{A} = \{A(a, 0)\}$ and $\varkappa = \bigcirc_F^2 B$. Then $a^{\mathcal{I}} \in B^{\mathcal{I}(2n+1)}$, for any $n \geq 0$ and $\mathcal{I} \models (\mathcal{O}, \mathcal{A})$. So, $\mathsf{ans}^{\mathbb{Z}}(\mathcal{O}, \varkappa, \mathcal{A}) = \{(a, 2n + 1) \mid n \geq 0\}$ but $\mathsf{ans}(\mathcal{O}, \varkappa, \mathcal{A}) = \{(a, 1)\}$ since $\mathsf{tem}(\mathcal{A}) = \{0, 1\}$.

Given an ABox $\mathcal{A}$ and convex $D \supseteq \mathsf{tem}(\mathcal{A})$, denote by $\mathfrak{S}_{\mathcal{A}}^D$ the two-sorted structure with object domain $\mathsf{ind}(\mathcal{A})$ and temporal domain D such that $\mathfrak{S}_{\mathcal{A}}^D \models A(a, \ell)$ iff $A(a, \ell) \in \mathcal{A}$ and $\mathfrak{S}_{\mathcal{A}}^D \models P(a, b, \ell)$ iff $P(a, b, \ell) \in \mathcal{A}$, for concept and role names A and P. Let $\boldsymbol{q} = (\mathcal{O}, \varkappa)$ be a TOMIQ and $\Phi(x, t)$ a constant-free FO-formula whose signature is the concept and role names in $\boldsymbol{q}$, $<$ and $=$. We call $\Phi(x, t)$ an FO($<$)-*rewriting of* $\boldsymbol{q}$ if, for any ABox $\mathcal{A}$, $a \in \mathsf{ind}(\mathcal{A})$ and $\ell \in \mathsf{tem}(\mathcal{A})$, we have $(a, \ell) \in \mathsf{ans}(\boldsymbol{q}, \mathcal{A})$ iff $\mathfrak{S}_{\mathcal{A}}^{\mathsf{tem}(\mathcal{A})} \models \Phi(a, \ell)$. If $\Phi(x, t)$ also uses the predicate PLUS(k, n_1, n_2) (for $k = n_1 + n_2$) then it is called an FO($<, +$)-*rewriting of* $\boldsymbol{q}$. FO-rewritings of TOMIQs $(\mathcal{O}, \varrho)$ are defined analogously. Note that FO-formulas using $<$ and PLUS as built-in predicates can be evaluated by standard relational or stream database management systems, and so are an appropriate target for query rewriting. Recall also that if $\boldsymbol{q}$ is FO($<, +$)-rewritable then the evaluation problem for $\boldsymbol{q}$ is in LOGTIME-uniform AC^0 for data complexity [Barrington and Straubing, 1995].

In the definitions above we do not allow any constants from $\mathbb{Z}$ to be used in queries or rewritings. Note, however, that the formulas $\min(t) = \neg\exists t'\,(t' < t)$ and $\max(t) = \neg\exists t'\,(t' > t)$ define the minimal and maximal numbers that occur in any given data instance. So, we can allow the constants 0 (for min) and max in FO($<$)-rewritings as syntactic sugar. We also freely use terms of the form $t + 1$ because they are definable by the FO($<$)-formula $(t' > t) \land \neg\exists t''\,((t < t'') \land (t'' < t'))$ (in particular, 1 is definable and exists in any data instance). In FO($<, +$)-rewritings, in addition, we can use terms of the form $t + t'$ because they are definable by means of PLUS.

Example 3. Consider $\boldsymbol{q} = (\mathcal{O}, A)$, where $\mathcal{O}$ is the same as in Example 2. It is not hard to see that

$$\exists s, n\,[(A(x, s) \land (t - s = 2n \geq 0)) \lor (B(x, s) \land (t - s = 2n + 1 \geq 0))]$$

is an FO($<, +$)-rewriting of $\boldsymbol{q}$, where $t - s = 2n \geq 0$ stands for $\exists k\,\big((k = n + n) \land (t = s + k) \land (k \geq 0)\big)$ and $t - s = 2n + 1 \geq 0$ is defined similarly. (The constant 0 is obviously definable.) Note that $\boldsymbol{q}$ is not FO($<$)-rewritable since properties such as 't is even' are not definable by FO($<$)-formulas [Libkin, 2004].

Example 4. For a word $\boldsymbol{e} = (e_0, \dots, e_{n-1}) \in \{0, 1\}^n$, take $\mathcal{A}_{\boldsymbol{e}} = \{B_0(a, n)\} \cup \{A_{e_i}(a, i) \mid i < n\}$ and let $\mathcal{O}'$ contain

$$\bigcirc_F B_k \sqcap A_0 \sqsubseteq B_k \quad \text{and} \quad \bigcirc_F B_k \sqcap A_1 \sqsubseteq B_{1-k}, \quad k = 0, 1.$$

One can check that $(a, 0)$ is a certain answer to $(\mathcal{O}', B_0)$ over $\mathcal{A}_{\boldsymbol{e}}$ iff the number of 1s in $\boldsymbol{e}$ is even (*parity*). It follows that $(\mathcal{O}', B_0)$ is not FO-rewritable even using arbitrary numeric predicates [Arora and Barak, 2009].

If we replace $\mathfrak{S}_{\mathcal{A}}^{\mathsf{tem}(\mathcal{A})}$ in the definition above with $\mathfrak{S}_{\mathcal{A}}^{\mathbb{Z}}$, then we call $\Phi(x, t)$ an FO$^{\mathbb{Z}}$($<$)- or FO$^{\mathbb{Z}}$($<, +$)-*rewriting of* $\boldsymbol{q}$. Finding FO$^{\mathbb{Z}}$-rewritings is often a first step in the construction of FO-rewritings, which are usually more involved.

Example 5. Suppose $\mathcal{O} = \{A \sqsubseteq \bigcirc_F^2 A,\ B \sqsubseteq \bigcirc_F^3 B\}$ and $\varkappa = \Diamond_F (A \sqcap B)$. Then

$$\exists s, u, v, n, m\,[(t < s) \land A(x, u) \land (s - u = 2n \geq 0) \land B(x, v) \land (s - v = 3m \geq 0)]$$

is an FO$^{\mathbb{Z}}$($<, +$)-rewriting of $(\mathcal{O}, \varkappa)$ but not an FO($<, +$)-rewriting because s can be outside the active domain $\mathsf{tem}(\mathcal{A})$.

A *temporal ontology-mediated query* (TOMQ) is a pair $(\mathcal{O}, \psi)$, where ψ is an FO-formula built from atoms $\varkappa(x, t)$, $\varrho(x, y, t)$ and $t < t'$, with $\varkappa$ and ϱ being any positive temporal concept and role, x and y *object variables*, and t and t' *temporal variables*. Given an ABox $\mathcal{A}$, $(\mathcal{O}, \psi)$ is evaluated over the two-sorted structure with the object domain $\mathsf{ind}(\mathcal{A})$ and the temporal domain $\mathsf{tem}(\mathcal{A})$. An assignment $\mathfrak{a}$ maps object and temporal variables to elements of the object and temporal domains, respectively. Then the relation $\mathfrak{S}_{\mathcal{O},\mathcal{A}}^{\mathsf{tem}(\mathcal{A})} \models^{\mathfrak{a}} \psi$ ('ψ is true in $\mathfrak{S}_{\mathcal{O},\mathcal{A}}^{\mathsf{tem}(\mathcal{A})}$ under $\mathfrak{a}$') is formally defined as follows:

- $\mathfrak{S}_{\mathcal{O},\mathcal{A}}^{\mathsf{tem}(\mathcal{A})} \models^{\mathfrak{a}} \varkappa(x, t)$ iff $(\mathfrak{a}(x), \mathfrak{a}(t)) \in \mathsf{ans}(\mathcal{O}, \varkappa, \mathcal{A})$,
- $\mathfrak{S}_{\mathcal{O},\mathcal{A}}^{\mathsf{tem}(\mathcal{A})} \models^{\mathfrak{a}} \varrho(x, y, t)$ iff $(\mathfrak{a}(x), \mathfrak{a}(y), \mathfrak{a}(t)) \in \mathsf{ans}(\mathcal{O}, \varrho, \mathcal{A})$,
- $\mathfrak{S}_{\mathcal{O},\mathcal{A}}^{\mathsf{tem}(\mathcal{A})} \models^{\mathfrak{a}} t < t'$ iff $\mathfrak{a}(t) < \mathfrak{a}(t')$,
- and the usual clauses for the Booleans and the quantifiers.

Let $(x_1, \dots, x_k, t_1, \dots, t_m)$ be the list of free variables of ψ (in some fixed order). We say that $(a_1, \dots, a_k, \ell_1, \dots, \ell_m)$ is an *answer* to OMQ $(\mathcal{O}, \psi)$ over $\mathcal{A}$ if $\mathfrak{S}_{\mathcal{O},\mathcal{A}}^{\mathsf{tem}(\mathcal{A})} \models^{\mathfrak{a}} \psi$, where $\mathfrak{a}(x_i) = a_i$, for all i, $1 \leq i \leq k$, and $\mathfrak{a}(t_j) = \ell_j$, for all

j, $1 \le j \le m$. Thus, similarly to the SPARQL 1.1 entailment regime, we interpret the DL and temporal constructs of TOMIQs in arbitrary temporal models (over $\mathbb{Z}$), while the object and temporal variables of TOMQs range over the active domains only.

Example 6. Suppose $\mathcal{O} = \{\mathit{Scheduled} \sqsubseteq \bigcirc_F^{24}\mathit{Scheduled}\}$, $\psi(x,t) = \mathit{Scheduled}(x,t) \land \neg \mathit{Serviced}(x,t)$ and $\mathcal{A}$ contains $\mathit{Scheduled}(a,0)$ as well as entries $\mathit{Serviced}(a,\ell)$ made by a serviceman. Then $(\mathcal{O},\psi)$ returns all pairs $(a, 24n)$ such that $24n \le \max \mathcal{A}$ and there is no entry $\mathit{Serviced}(a, 24n)$ in $\mathcal{A}$.

FO-rewritings of TOMQs are defined similarly to TOMIQs. We generalise [Calvanese *et al.*, 2007a]:

Theorem 7. *If all constituent TOMIQs of a TOMQ $\boldsymbol{q}$ are FO($<$)- or FO($<,+$)-rewritable, then $\boldsymbol{q}$ is also FO($<$)- or, respectively, FO($<,+$)-rewritable.*

From now on we only focus on rewritability of TOMIQs.

3 Rewriting *LTL* OMQs

We begin our study of FO-rewritability by considering ontologies without role names and assuming that all ABoxes contain a single object name, say a. To simplify notation, we will omit a from the ABox assertions, interpretations, certain answers, etc., and write $A(\ell)$ instead of $A(a,\ell)$ and assume that answers to TOMIQs are subsets of $\mathsf{tem}(A)$ rather than $\{a\} \times \mathsf{tem}(\mathcal{A})$, and that FO-rewritings $\Phi(x,t)$ are one-sorted FO-formulas $\varphi(t)$. Ontologies in this restricted language can be regarded as formulas of the *propositional temporal logic LTL* given in clausal normal form (1), and so we denote the corresponding restrictions of $\mathit{DL\text{-}Lite}_{c}^{\boldsymbol{o}}$ by $\mathit{LTL}_{c}^{\boldsymbol{o}}$. In this context, positive temporal concepts are simply negation-free *LTL*-formulas.

As shown by Example 4, some $\mathit{LTL}_{\mathit{horn}}^{\bigcirc}$ TOMAQs cannot be FO($<,+$)-rewritable. On the other hand, relying on the well-known fact that the semantics of temporal formulas can be encoded by *monadic second-order* (MSO) formulas (built from atoms of the form $A(t)$, $t = t'$ and $t < t'$ using the Booleans, first-order quantifiers $\forall t$, $\exists t$ and second-order quantifiers $\forall A$ and $\exists A$), one can prove the following:

Theorem 8. *All $\mathit{LTL}_{\mathit{bool}}^{\Box\bigcirc}$ TOMQs are MSO($<$)-rewritable.*

Proof. Denote by $\Sigma_{\mathcal{O}}^{\varkappa_0}$ the set of subconcepts of $\mathcal{O}$ and $\varkappa_0$. For every concept $\varkappa$ in $\Sigma_{\mathcal{O}}^{\varkappa_0}$, we take a fresh unary predicate denoted $\varkappa^*(t)$. Let Ψ be a conjunction of the following FO($<$)-sentences:

$$\bigwedge_{\varkappa_1 \sqcap \varkappa_2 \in \Sigma_{\mathcal{O}}^{\varkappa_0}} \forall t \, \big((\varkappa_1 \sqcap \varkappa_2)^*(t) \leftrightarrow \varkappa_1^*(t) \land \varkappa_2^*(t) \big),$$

$$\bigwedge_{\varkappa_1 \mathcal{U} \varkappa_2 \in \Sigma_{\mathcal{O}}^{\varkappa_0}} \forall t \, \big[(t < \max) \to \big((\varkappa_1 \,\mathcal{U}\, \varkappa_2)^*(t) \leftrightarrow \varkappa_2^*(t+1) \lor (\varkappa_1^*(t+1) \land (\varkappa_1 \,\mathcal{U}\, \varkappa_2)^*(t+1)) \big) \big],$$

and similar sentences for the other types of concepts in $\Sigma_{\mathcal{O}}^{\varkappa_0}$. Let $\boldsymbol{S}$ be the set of all maximal subsets S of $\Sigma_{\mathcal{O}}^{\varkappa_0}$ that are *consistent* with $\mathcal{O}$. Consider

$$\forall \boldsymbol{\varkappa} \, \Big[\bigwedge_{\text{concept name } A} \forall t \, \big(A(t) \to A^*(t) \big) \land \Psi \land \forall t \, \big(\bigvee_{S \in \boldsymbol{S}} \bigwedge_{\varkappa \in S} \varkappa^*(t) \big) \; \to \; \varkappa_0^*(t_0) \Big],$$

where $\boldsymbol{\varkappa}$ lists all predicate names $\varkappa^*(t)$, for $\varkappa \in \Sigma_{\mathcal{O}}^{\varkappa_0}$. It is easily seen that this second-order formula is an MSO($<$)-rewriting of $(\mathcal{O}, \varkappa_0)$. ❑

It follows that, for any $\mathit{LTL}_{\mathit{bool}}^{\Box\bigcirc}$ TOMQ $\boldsymbol{q}$, one can build an NFA accepting an ABox $\mathcal{A}$ and $\ell \in \mathsf{tem}(\mathcal{A})$ written on its tape iff $\ell \in \mathsf{ans}(\boldsymbol{q}, \mathcal{A})$ [Straubing and Weil, 2010, Theorem 1.1], and that the evaluation problem for $\boldsymbol{q}$ is in NC^1 for data complexity [Ladner and Fischer, 1980]. We next establish the matching lower bound for $\mathit{LTL}_{\mathit{horn}}^{\bigcirc}$ TOMAQs and $\mathit{LTL}_{\mathit{krom}}^{\Box}$ and $\mathit{LTL}_{\mathit{krom}}^{\bigcirc}$ TOMIQs by generalising Example 4.

Theorem 9. *There exist an $\mathit{LTL}_{\mathit{horn}}^{\bigcirc}$ TOMAQ and $\mathit{LTL}_{\mathit{krom}}^{\Box}$ and $\mathit{LTL}_{\mathit{krom}}^{\bigcirc}$ TOMIQs that are NC^1-hard for data complexity; in particular, they are not FO-rewritable even with arbitrary numeric predicates.*

Proof sketch. It is known that there exist NC^1-complete regular languages [Barrington *et al.*, 1992]. Given an NFA $\mathfrak{A}$ and input $\boldsymbol{a} = a_0 \dots a_{n-1}$, we take concept names A_a and B_q for tape symbols a and states q of $\mathfrak{A}$, and then set $\mathcal{A}_{\boldsymbol{a}} = \{B_{q_1}(n)\} \cup \{A_{a_i}(i) \mid i < n\}$, where q_1 is the accepting state, and $\mathcal{O} = \{ \bigcirc_F B_{q'} \sqcap A_a \sqsubseteq B_q \mid q \to_a q' \}$. Then $\mathfrak{A}$ accepts $\boldsymbol{a}$ iff $0 \in \mathsf{ans}(\mathcal{O}, B_{q_0}, \mathcal{A}_{\boldsymbol{a}})$, for the initial state q_0. Next, let $\mathcal{O}'$ contain $N_q \sqcap B_q \sqsubseteq \bot$ and $\top \sqsubseteq B_q \sqcup N_q$, for each state q, and let $\varkappa = B_{q_0} \sqcup \bigsqcup_{q \to_a q'} \Diamond_P \Diamond_F (\bigcirc_F B_{q'} \sqcap A_a \sqcap N_q)$. Then $\mathfrak{A}$ accepts $\boldsymbol{a}$ iff $0 \in \mathsf{ans}(\mathcal{O}', \varkappa, \mathcal{A}_{\boldsymbol{a}})$. ❑

Next, we show the FO($<$)- and FO($<,+$)-rewritability results from the table in the introduction. To begin with, we focus on temporal ontology-mediated *atomic* queries (TOMAQs) of the form $(\mathcal{O}, A)$, where A is a concept name.

Theorem 10. *Any $\mathit{LTL}_{\mathit{krom}}^{\bigcirc}$ TOMAQ is FO($<,+$)-rewritable.*

Proof. Suppose $\boldsymbol{q} = (\mathcal{O}, A)$ is an $\mathit{LTL}_{\mathit{krom}}^{\bigcirc}$ OMAQ. By a *literal*, L, we mean a concept name from $\boldsymbol{q}$ or its negation (interpreted as the set-theoretic complement). We use $\bigcirc^n L$ in place of $\bigcirc_F^n L$ if $n > 0$, L if $n = 0$, and $\bigcirc_P^{-n} L$ if $n < 0$. We write $\mathcal{O} \models L \sqsubseteq \bigcirc^k L'$ if $L \sqsubseteq \bigcirc^k L'$ holds globally in any model of $\mathcal{O}$. For any ABox $\mathcal{A}$, we then have:

(*i*) if $(\mathcal{O}, \mathcal{A})$ is consistent then $\ell \in \mathsf{ans}^{\mathbb{Z}}(\boldsymbol{q}, \mathcal{A})$ iff either $\mathcal{O} \models \top \sqsubseteq A$ or $\mathcal{O} \models B \sqsubseteq \bigcirc^{\ell - n} A$, for some $B(n) \in \mathcal{A}$;

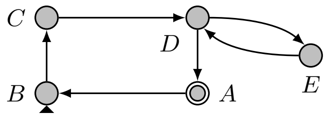
(*ii*) $(\mathcal{O}, \mathcal{A})$ is inconsistent iff either $\mathcal{O}$ is inconsistent or $\mathcal{O} \models B \sqsubseteq \bigcirc^{n'-n} \neg B'$, for some $B(n), B'(n') \in \mathcal{A}$.

Given literals L and L', let $\mathfrak{A}_{L,L'}$ be an NFA whose tape alphabet is $\{0\}$, the states are the literals, with L initial and L' accepting, and whose transitions are of the form $L_1 \to_0 L_2$, for $\mathcal{O} \models L_1 \sqsubseteq \bigcirc L_2$ (without loss of generality we assume that $\mathcal{O}$ does not contain nested $\bigcirc$). It is easy to see that $\mathfrak{A}_{L,L'}$ accepts 0^k ($k > 0$) iff $\mathcal{O} \models L \sqsubseteq \bigcirc^k L'$. By [Chrobak, 1986; To, 2009], there are $N = O(|\mathfrak{A}_{L,L'}|^2)$ arithmetic progressions $a_i + b_i\mathbb{N} = \{a_i + b_i \cdot m \mid m \ge 0\}$, $1 \le i \le N$, such that $0 \le a_i, b_i \le |\mathfrak{A}_{L,L'}|$ and $\mathfrak{A}_{L,L'}$ accepts 0^k iff $k \in a_i + b_i\mathbb{N}$ for some i, $1 \le i \le N$. These progressions give rise to the FO($<,+$)-rewriting we need.

Example 11. To illustrate, suppose $\boldsymbol{q} = (\mathcal{O}, A)$ and

$$\mathcal{O} = \{A \sqsubseteq \bigcirc B,\ B \sqsubseteq \bigcirc C,\ C \sqsubseteq \bigcirc D,\ D \sqsubseteq \bigcirc A,\ D \sqsubseteq \bigcirc E,\ E \sqsubseteq \bigcirc D\}.$$

The NFA $\mathfrak{A}_{B,A}$ (more precisely, the states reachable from B) is shown below, and for $L \in \{A, C, D, E\}$, $\mathfrak{A}_{L,A}$ is the same



NFA but with the initial state L. It is readily seen that $\mathfrak{A}_{B,A}$ accepts 0^k iff $k \in 3 + 2\mathbb{N}$, which can be described using

$$\varphi_{B,A}(t) = \exists s, n\, [B(s) \land (t - s = a + bn \geq 0)],$$

where $a = 3$, $b = 2$ and $t - s = a + bn \geq 0$ is defined as in Example 3. Similarly, for $\mathfrak{A}_{E,A}$, we have $a = b = 2$. (Note that in general more than one progression is needed to characterise automata $\mathfrak{A}_{L,A}$.)

In general, consider the progression $a + b\mathbb{N}$. If $b > 0$ then $v \in a + b\mathbb{N}$ abbreviates FO($<, +$)-formula

$$\exists y, u_1, \dots, u_b \big[\mathrm{PLUS}(v, a, u_b) \land \\ \big((u_1 = y) \land \bigwedge_{1 \leq j < b} \mathrm{PLUS}(u_{j+1}, u_j, y)\big)\big]$$

(the second line encodes $u_b = b \cdot y$); otherwise, if $b = 0$, then $v \in a + b\mathbb{N}$ stands for $v = a$. Next, let $\mathsf{entails}^0_{L,L'}(s, s')$ be $(s = s')$ if $\mathcal{O} \models L \sqsubseteq L'$ and $\bot$ otherwise. Denote by $\mathsf{entails}_{L,L'}(s, s')$ a disjunction of $\mathsf{entails}^0_{L,L'}(s, s')$ and

$$\exists v\, \big(\mathrm{PLUS}(s, s', v) \land (v \in a_i + b_i\mathbb{N})\big),$$

for each arithmetic progression $a_i + b_i\mathbb{N}$ associated with $\mathfrak{A}_{L,L'}$. Now, conditions (*i*) and (*ii*) can be encoded as an FO($<, +$)-formula $\varphi(t)$, which is $\top$ if $\mathcal{O} \models \top \sqsubseteq A$, and otherwise a disjunction of the following formulas:

$$\bigvee_{B \text{ occurs in } \mathcal{O}} \exists s\big(B(s) \land (\mathsf{entails}_{B,A}(s, t) \lor \mathsf{entails}_{\neg A, \neg B}(t, s))\big)$$

$$\bigvee_{B, B' \text{ occur in } \mathcal{O}} \exists s, s'\big(B(s) \land B'(s') \land \mathsf{entails}_{B, \neg B'}(s, s')\big).$$

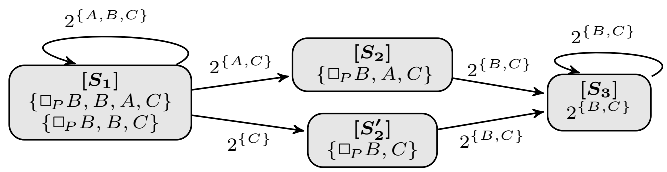
Thus, $\varphi(t)$ is an FO($<, +$)- and FO$^{\mathbb{Z}}$($<, +$)-rewriting of $\boldsymbol{q}$. ❑

Theorem 12. *Any $LTL^{\Box}_{bool}$ OMAQ is FO($<$)-rewritable.*

Proof. Given an $LTL^{\Box}_{bool}$-ontology $\mathcal{O}$, we construct an NFA $\mathfrak{A}$ that takes as input an ABox $\mathcal{A}$ written as the word $\mathcal{A}_0, \dots, \mathcal{A}_k$, where $k = \max \mathcal{A}$ and $\mathcal{A}_i = \{A \mid A(i) \in \mathcal{A}\}$. Let Σ be the set of temporal concepts in $\mathcal{O}$ and their negations. Each state of $\mathfrak{A}$ is a maximal set $S \subseteq \Sigma$ that is consistent with $\mathcal{O}$ (in particular, it contains either C or $\neg C$, for each concept name C in $\mathcal{O}$); let $\boldsymbol{S}$ be the set of all such states. For $S, S' \in \boldsymbol{S}$ and a tape symbol (a set of concept names) X, we set $S \to_X S'$ just in case (a) $X \subseteq S'$, (b) $\Box_F \beta \in S$ iff $\beta, \Box_F \beta \in S'$, and (c) $\Box_P \beta \in S'$ iff $\beta, \Box_P \beta \in S$. A state $S \in \boldsymbol{S}$ is *accepting* if $\mathfrak{A}$ has an infinite 'ascending' chain $S \to_\emptyset S_1 \to_\emptyset \dots$; S is *initial* if $\mathfrak{A}$ has an infinite 'descending' chain $\dots \to_\emptyset S_1 \to_\emptyset S$. Then $\mathfrak{A}$ *simulates* $\mathcal{O}$ in the following sense: for any ABox $\mathcal{A}$, concept name A and $\ell \in \mathbb{Z}$, we have $\ell \in \mathsf{ans}^{\mathbb{Z}}(\mathcal{O}, A, \mathcal{A})$ iff $\mathfrak{A}$ does not contain an *accepting path* $S_0 \to_{X_1} \cdots \to_{X_m} S_m$ (with initial S_0 and accepting S_m) where $A \notin S_\ell$ and there is i_0, $0 \leq i_0 \leq m - k$, such that $X_j = \mathcal{A}_{j - i_0}$, for j with $i_0 \leq j \leq i_0 + k$, and $X_j = \emptyset$, for all other j from 0 to m.

Define an equivalence relation, $\sim$, on $\boldsymbol{S}$ by taking $S \sim S'$ iff $S = S'$ or $\mathfrak{A}$ has a cycle with both S and S'. Let $[S]$ be the $\sim$-equivalence class of S. One can check that $S \to_X S'$ implies $S_1 \to_X S'$, for any $S_1 \in [S]$. Let $\mathfrak{A}'$ be an NFA with states $[S]$, for $S \in \boldsymbol{S}$, and transitions $[S] \to_X [S']$ iff $S_1 \to_X S'_1$, for some $S_1 \in [S]$ and $S'_1 \in [S']$. The initial (accepting) states of $\mathfrak{A}'$ are all $[S]$ with initial (accepting) S. The NFA $\mathfrak{A}'$ also simulates $\mathcal{O}$ and, moreover, contains no cycles other than trivial loops, which makes it possible to express the simulation condition by an FO($<$)-formula.

Example 13. Consider $\mathcal{O} = \{A \sqsubseteq \Box_P B,\ \Box_P B \sqsubseteq C\}$. Then $\mathfrak{A}'$ is shown below, where all states are initial and accepting and negated concepts are omitted:



Let $\boldsymbol{q} = (\mathcal{O}, C)$. Take all accepting paths π in $\mathfrak{A}'$ with pairwise distinct states at least one of which contains a set without C. In $\pi = [S_1] \to_{\{A\}} [S_2] \to_\emptyset [S_3]$, a set in $[S_3]$ has no C, and the simulation condition for π, which makes sure that $\neg C$ holds at t, can be written as

$$\exists t_1, t_2 \big[\forall t' \big((t' < t_1) \to \mathsf{loop}_{[S_1]}(t')\big) \land \mathsf{sym}_{\{A\}}(t_1) \land \\ \forall t' \,\big((t_1 < t' < t_2) \to \mathsf{loop}_{[S_2]}(t')\big)\big) \land \mathsf{sym}_\emptyset(t_2) \land \\ \forall t' \,\big((t' > t_2) \to \mathsf{loop}_{[S_3]}(t')\big) \land (t \geq t_2) \land \neg C(t)\big],$$

where $\mathsf{sym}_{\{A\}}(t) = A(t) \land \neg B(t) \land \neg C(t)$ and $\mathsf{sym}_\emptyset(t) = \neg A(t) \land \neg B(t) \land \neg C(t)$ define transitions $\to_{\{A\}}$ and $\to_\emptyset$ in π, and $\mathsf{loop}_{[S_1]} = \top$, $\mathsf{loop}_{[S_2]} = \bot$ and $\mathsf{loop}_{[S_3]}(t) = \neg A(t)$ say that $[S_1]$ and $[S_3]$ have loop transitions with any input symbol and any input symbol but A, respectively, but $[S_2]$ does not have a loop. To obtain an FO($<$) and FO$^{\mathbb{Z}}$($<$)-rewriting of $\boldsymbol{q}$, we take a disjunction of such formulas for paths π in $\mathfrak{A}'$ and then negate it.

We now describe a general procedure that, given an OMAQ $(\mathcal{O}, A)$, uses $\mathfrak{A}'$ to construct an FO($<$)-formula $\varphi_{\mathcal{O},A}(t)$ such that $\mathfrak{A}'$ does not have accepting computations coming through a state without A at moment t.

For each tape symbol (a set of concept names) X, let $\mathsf{symb}_X(t)$ be a conjunction of all $C(t)$ such that $C \in X$ and $\neg C(t)$ such that $C \notin X$, for C occurring in $\mathcal{O}$. Let π be a path

$$[S_0] \to_{X_1} \cdots \to_{X_n} [S_n]$$

in $\mathfrak{A}'$ such that π contains no repeating $[S_i]$ and S_0 is an initial state and S_n an accepting one; we say that π is *a stutter-free accepting path*. For $0 \leq i \leq n$,

$$\mathsf{loop}_{[S_i]}(t) = \bigvee_{X \text{ with } [S_i] \to_X [S_i]} \mathsf{sym}_X(t)$$

If $n = 0$ then we take $\mathsf{path}_\pi = \forall t\, \mathsf{loop}_0(t)$ and if $n > 0$ then we take $\mathsf{path}_\pi(t_1, \dots, t_n)$ to be a conjunction of the following formulas:

$$\mathsf{sym}_{X_i}(t_i), \qquad \text{for } 1 \le i \le n,$$
$$\forall t\,\big((t < t_1) \to \mathsf{loop}_{[S_0]}(t)\big),$$
$$\forall t\,\big((t_i < t < t_{i+1}) \to \mathsf{loop}_{[S_i]}(t)\big), \qquad \text{for } 1 \le i < n,$$
$$\forall t\,\big((t > t_n) \to \mathsf{loop}_{[S_n]}(t)\big).$$

(Note that, if $[S_i]$ has no self-loop then $\mathsf{loop}_{[S_i]}(t)$ is the empty disjunction, $\bot$, and therefore, $t_{i+1} - t_i = 1$; in particular, both the initial and the accepting states must have a self-loop.) Now, we consider the concept A and let

$$\mathsf{loop}^{\neg A}_{[S_i]}(t) = \bigvee_{\substack{X \text{ with } S' \to_X S\\ S', S \in [S_i] \text{ and } A \notin S}} \mathsf{symb}_X(t),$$

$$\mathsf{tr}^{\neg A}_{[S_{i-1}] \to_{X_i} [S_i]} = \begin{cases} \top, & \text{if } S' \to_{X_i} S \text{ for some } S' \in [S_{i-1}] \\ & \quad \text{and } S \in [S_i] \text{ with } A \notin S, \\ \bot, & \text{otherwise.} \end{cases}$$

If $n = 0$ then we set $\mathsf{ans}^A_\pi(t) = \neg\mathsf{loop}^{\neg A}_{[S_0]}(t)$ and if $n > 0$ then we take $\mathsf{ans}^A_\pi(t, t_1, \dots, t_n)$ is a conjunction of the following formulas:

$$(t = t_i) \to \neg\mathsf{tr}^{\neg A}_{[S_{i-1}] \to_{X_i} [S_i]}, \qquad \text{for } 1 \le i \le n,$$
$$(t < t_1) \to \neg\mathsf{loop}^{\neg A}_{[S_0]}(t),$$
$$(t_i < t < t_{i+1}) \to \neg\mathsf{loop}^{\neg A}_{[S_i]}(t), \qquad \text{for } 1 \le i < n,$$
$$(t > t_n) \to \neg\mathsf{loop}^{\neg A}_{[S_n]}(t).$$

Clearly, the following formula

$$\varphi_{\mathcal{O},A}(t) = \bigwedge_{\substack{\pi \text{ a stutter-free}\\ \text{accepting path in } \mathfrak{A}'}} \forall t_1, \dots, t_n\,\big(\mathsf{path}_\pi(t_1, \dots, t_n) \to \mathsf{ans}^A_\pi(t, t_1, \dots, t_n)\big)$$

is an FO($<$)-rewriting of $(\mathcal{O}, A)$ we require. ❑

3.1 From OMAQs to TOMIQs

We turn now to more general ontology-mediated instance queries (TOMIQs), which can contain temporal operators and monotone Boolean connectives. We show how to construct FO-rewritings of $\mathit{LTL}^{\bigcirc}_{core}$ and $\mathit{LTL}^{\Box}_{horn}$ TOMIQs by induction using the fact that every consistent $(\mathcal{O}, \mathcal{A})$ in these languages has a single canonical model containing answers to all TOMIQs with the given ontology (it will formally be defined below). For example, an FO$^{\mathbb{Z}}(<)$-rewriting of $(\mathcal{O}, A_1 \,\mathcal{U}\, A_2)$ is defined as

$$\exists s\,\big[\big((s > t) \land \varphi_{A_2}(s)\big) \land \forall u\,\big((t < u < s) \to \varphi_{A_1}(u)\big)\big], \quad (2)$$

where $\varphi_{A_i}(t)$ is an FO$^{\mathbb{Z}}(<)$-rewriting of $(\mathcal{O}, A_i)$. Note, however, that FO$^{\mathbb{Z}}(<)$-rewritings obtained in this way are not in general FO($<$)-rewritings because they may refer to time instants outside the active domain $\mathsf{tem}(\mathcal{A})$ (cf. Example 5). To cope with this problem, we require (constant-free) sentences φ^k_A, for any concept name A and $k \neq 0$, such that, for every ABox $\mathcal{A}$ consistent with $\mathcal{O}$,

– $\mathfrak{S}^{\mathsf{tem}(\mathcal{A})}_{\mathcal{A}} \models \varphi^k_A$ iff $k + \max\mathcal{A} \in \mathsf{ans}^{\mathbb{Z}}(\mathcal{O}, A, \mathcal{A})$, for $k \ge 1$,

– $\mathfrak{S}^{\mathsf{tem}(\mathcal{A})}_{\mathcal{A}} \models \varphi^k_A$ iff $k \in \mathsf{ans}^{\mathbb{Z}}(\mathcal{O}, A, \mathcal{A})$, for $k \le -1$.

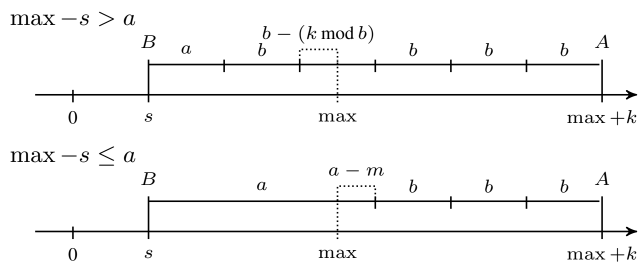
The existence of such sentences, called *witnesses* for $(\mathcal{O}, A)$ and k, follows from the proofs of Theorems 10 and 12.

Corollary 14. *For any $\mathit{LTL}^{\bigcirc}_{krom}$ OMAQ $(\mathcal{O}, A)$ and any $k \neq 0$, there exists an FO$(<, +)$-witness.*

Proof. Suppose first that $k \ge 1$. For any concept name B in $\mathcal{O}$ and progression $a_i + b_i\mathbb{N}$, $1 \le i \le N$, associated in the proof of Theorem 10 with NFA $\mathfrak{A}_{B,A}$, we denote by $\varphi^k_{B,i}$ the following sentence

$$\exists s\,\Big[B(s) \land \Big(\big((\max - s \in a_i - (k \bmod b_i) + b_i\mathbb{N}) \land (b_i \neq 0)\big) \lor \bigvee_{m \le a_i} \big((\max - s = m) \land (k \in (a_i - m) + b_i\mathbb{N})\big)\Big)\Big]$$

(note that the last conjunct does not depend on s and is, in fact, a constant: given particular k, m, a_i and b_i, it is either true or false). Finally, we define φ^k_A to be a disjunction of $\varphi^k_{B,i}$ for all B in $\mathcal{O}$ and all corresponding i. It follows from the proof of Theorem 10 that φ^k_A is a witness for $(\mathcal{O}, A)$ and $k \ge 1$. The other case, $k \le -1$, is similar. ❑



Corollary 15. *For any $\mathit{LTL}^{\Box}_{bool}$ OMAQ $(\mathcal{O}, A)$ and any $k \neq 0$, there exists an FO$(<)$-witness.*

Proof. Suppose first that $k \ge 1$. For a stutter-free accepting path π (see the proof of Theorem 12), we write $\pi \to^k_\emptyset [S]$ to say that $[S]$ is accepting in $\mathfrak{A}'$ and can be reached from the last state of π by k transitions $\to_\emptyset$. It is readily seen then that

$$\varphi^k_{\mathcal{O},A} = \bigwedge_{\pi \to^k_\emptyset [S],\ A \notin S} \forall t_1, \dots, t_n\,\big(\mathsf{path}_\pi(t_1, \dots, t_n) \to \bot\big)$$

is as required. The other case, $k \le -1$, is similar. ❑

Now, in order to construct an FO($<$)-rewriting, we could, instead of (2), consider the *infinite* 'formula':

$$(2) \lor \Big[\forall u\,((t < u) \to \varphi_{A_1}(u)) \land \bigvee_{k > 0} \big(\varphi^k_{A_2} \land \bigwedge_{0 < i < k} \varphi^i_{A_1}\big)\Big].$$

It turns out that we can make it finite by observing that the canonical models are ultimately periodical with at most exponential period. Thus, we obtain

Theorem 16. (*i*) *Any $\mathit{LTL}^{\bigcirc}_{core}$ OMQ is FO$(<, +)$-rewritable.*
(*ii*) *Any $\mathit{LTL}^{\Box}_{horn}$ OMQ is FO$(<)$-rewritable.*

Proof. Given an $LTL^{\Box\bigcirc}_{horn}$-ontology $\mathcal{O}$ and a data instance $\mathcal{A}$ consistent with $\mathcal{O}$, we define a temporal interpretation $\mathcal{C}_{\mathcal{O},\mathcal{A}}$ by taking

$$A^{\mathcal{C}_{\mathcal{O},\mathcal{A}}(n)} \quad \text{iff} \quad A^{\mathcal{I}(n)} \quad \text{for all models } \mathcal{I} \text{ of } (\mathcal{O},\mathcal{A}).$$

It is not hard to check that $\mathcal{C}_{\mathcal{O},\mathcal{A}}$ is a model of $(\mathcal{O},\mathcal{A})$ with the following properties:

1) for any positive temporal concept $\varkappa$, we have $\varkappa^{\mathcal{C}_{\mathcal{O},\mathcal{A}}(n)}$ iff $\varkappa^{\mathcal{I}(n)}$ for all models $\mathcal{I}$ of $(\mathcal{O},\mathcal{A})$;
2) for any positive temporal concept $\varkappa$ and any m with $\max\mathcal{A} < \ell < m$, if $\varkappa^{\mathcal{C}_{\mathcal{O},\mathcal{A}}(m)}$ then $\varkappa^{\mathcal{C}_{\mathcal{O},\mathcal{A}}(k)}$ for some $k > \ell$ with $k - \ell = O(2^{|\Sigma^{\varkappa}_{\mathcal{O}}|})$;
3) and symmetrically for $m < \ell < \min\mathcal{A}$.

We now use these properties to define the required rewriting for any given $LTL^{\bigcirc}_{core}$ or $LTL^{\Box}_{horn}$ TOMIQ $(\mathcal{O},\varkappa)$. We proceed by induction on the structure of $\varkappa$ and construct rewritings and witnesses for all its subconcepts (witnesses for complex positive temporal concepts are defined in the same way as for concepts names, that is, OMAQs). For $\varkappa = A$, we take the rewriting $\varphi_A(t)$ provided by Theorems 10 and 12 as well the witnesses φ^k_A given by Corollaries 14 and 15. Consider next $\varkappa = \varkappa_1 \,\mathcal{U}\, \varkappa_2$ and suppose that we have rewritings $\varphi_{\varkappa_i}(t)$ of $(\mathcal{O},\varkappa_i)$ and witnesses $\varphi^k_{\varkappa_i}(t)$ for $(\mathcal{O},\varkappa_i)$ and k, with $i = 1, 2$. The intended meaning of the required rewriting $\varphi_{\varkappa_1\mathcal{U}\varkappa_2}(t)$ of $(\mathcal{O},\varkappa_1 \,\mathcal{U}\, \varkappa_2)$ can be expressed by the following infinite 'formula':

$$\exists s\,\big[\big((s > t) \land \varphi_{\varkappa_2}(s)\big) \land \forall u\,\big((t < u < s) \to \varphi_{\varkappa_1}(u)\big)\big] \\ \lor \big[\forall u\,((u < t) \to \varphi_{\varkappa_1}(u)) \land \bigvee_{k>0} \big(\varphi^k_{\varkappa_2} \land \bigwedge_{0<i<k} \varphi^i_{\varkappa_1}\big)\big].$$

Witnesses $\varphi^k_{\varkappa_1\mathcal{U}\varkappa_2}(t)$, for $(\mathcal{O},\varkappa_1 \,\mathcal{U}\, \varkappa_2)$ and $k > 0$, can be expressed by the following infinite 'formula':

$$\bigvee_{i>k} \big(\varphi^i_{\varkappa_2} \land \bigwedge_{k<j<i} \varphi^j_{\varkappa_1}\big).$$

(the case $k < 0$ is similar and left to the reader). It remains to observe that we can make these infinite formulas finite by using property 2) above: for example, the disjunction $\bigvee_{k>0}$ can be replaced by $\bigvee_{0<k<O(2^{|\Sigma^{\varkappa}_{\mathcal{O}}|})}$. The remaining temporal operators and Boolean connectives are treated analogously. ❑

4 OBDA with Temporal *DL-Lite*

Now we transfer the rewritability results obtained above to certain $DL\text{-}Lite^{\Box\bigcirc}_{horn}$ TOMIQs.

Observe first that rewritability of TOMIQs of the form $(\mathcal{O},\varrho)$ can be easily reduced to rewritability of $LTL^{\Box\bigcirc}_{horn}$ TOMIQs because only the RBox $\mathcal{R}$ in $\mathcal{O}$ has to be taken into account. We assume that with every role inclusion (e.g., $\Box_F R \sqcap \bigcirc_F Q^- \sqsubseteq T$) the RBox also contains the corresponding inclusion for the inverse roles (that is, $\Box_F R^- \sqcap \bigcirc_F Q \sqsubseteq T^-$). We treat the roles in $\mathcal{R}$ and ϱ as *concept names* and denote the resulting $LTL^{\Box\bigcirc}_{horn}$ TOMIQ by $(\mathcal{R}^*,\varrho^*)$. If it has an FO-rewriting $\varphi_{\mathcal{R}^*,\varrho^*}(t)$, then we replace every $P(s)$ in it with $P(x,y,s)$, every $P^-(s)$ with $P(y,x,s)$, and denote the resulting FO-formula by $\Phi_{\mathcal{O},\varrho}(x,y,t)$.

Theorem 17. *For any $DL\text{-}Lite^{\Box\bigcirc}_{horn}$ TOMIQ $(\mathcal{O},\varrho)$, $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$, if $\varphi_{\mathcal{R}^*,\varrho^*}(t)$ is an F-rewriting of $(\mathcal{R}^*,\varrho^*)$ then $\Phi_{\mathcal{O},\varrho}(x,y,t)$ is an F-rewriting of $(\mathcal{O},\varrho)$, where F is FO$(<)$ or FO$(<,+)$.*

However, this simple reduction to *LTL* does not work for TOMIQs of the form $(\mathcal{O},\varkappa)$. The main reason is that the RBox in $\mathcal{O}$ can have a strong impact on its TBox.

Example 18. Let $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$, $\mathcal{T} = \{B \sqsubseteq \exists R, \exists Q \sqsubseteq A\}$ and $\mathcal{R} = \{R \sqsubseteq \bigcirc_F Q\}$. It is not hard to see that the following is an FO$(<)$-rewriting of $(\mathcal{O}, A)$:

$$A(x,t) \lor \exists y\, Q(x,y,t) \lor \exists y\, R(x,y,t-1) \lor B(x,t-1).$$

It can also be constructed by treating $\mathcal{T}$ as an $LTL^{\bigcirc}_{core}$-ontology, but only if the *connection axiom* $\exists R \sqsubseteq \bigcirc_F \exists Q$ is added to $\mathcal{T}$.

To understand what connection axioms are required for any given $DL\text{-}Lite^{\Box\bigcirc}_{horn}$ TOMIQ $(\mathcal{O},\varkappa)$, we first define a canonical model (or chase) for $(\mathcal{O},\mathcal{A})$, which will also be used to prove our main Theorems 21 and 26. Let $\mathcal{C}$ be a set of atoms of the form $A(a,n)$, $\exists R(a,n)$ and $R(a_1,a_2,n)$, possibly prefixed by temporal operators and such that $P(a_1,a_2,n) \in \mathcal{C}$ iff $P^-(a_2,a_1,n) \in \mathcal{C}$. Denote by $\mathsf{cl}(\mathcal{C})$ the result of applying non-recursively the following rules to (the same) $\mathcal{C}$:

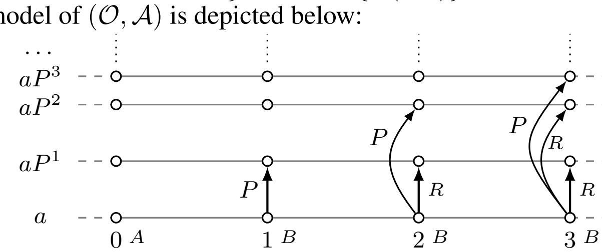
– if $R(a,b,n) \in \mathcal{C}$ then we add $\exists R(a,n)$ to $\mathcal{C}$;
– if $(\lambda_1 \sqcap \dots \sqcap \lambda_k \sqsubseteq \lambda) \in \mathcal{O}$ and $\lambda_i(\boldsymbol{a},n) \in \mathcal{C}$, for all $i = 1,\dots,k$, then we add $\lambda(\boldsymbol{a},n)$ to $\mathcal{C}$;
– if $\exists R(a,n) \in \mathcal{C}$ then we add $R(a, aR^n, n)$ to $\mathcal{C}$, where aR^n is a new object name (called a *witness* for $\exists R(a,n)$);
– if $\Box_F \lambda(\boldsymbol{a},n) \in \mathcal{C}$ then add $\lambda(\boldsymbol{a},m)$ to $\mathcal{C}$ for all $m > n$;
– if $\lambda(\boldsymbol{a},m) \in \mathcal{C}$ for all $m > n$, then add $\Box_F \lambda(\boldsymbol{a},n)$ to $\mathcal{C}$;
– if $\bigcirc_F \lambda(\boldsymbol{a},n) \in \mathcal{C}$ then add $\lambda(\boldsymbol{a},n+1)$ to $\mathcal{C}$;
– if $\lambda(\boldsymbol{a},n) \in \mathcal{C}$ then add $\bigcirc_F \lambda(\boldsymbol{a},n-1)$ to $\mathcal{C}$;

and symmetrical rules for $\Box_P$ and $\bigcirc_P$. Then we set $\mathsf{cl}^0(\mathcal{C}) = \mathcal{C}$ and, for any successor ordinal $\xi + 1$ and limit ordinal ζ,

$$\mathsf{cl}^{\xi+1}(\mathcal{C}) = \mathsf{cl}(\mathsf{cl}^{\xi}(\mathcal{C})) \quad \text{and} \quad \mathsf{cl}^{\zeta}(\mathcal{C}) = \bigcup_{\xi<\zeta} \mathsf{cl}^{\xi}(\mathcal{C}).$$

Let N be the number of temporal operators in $\mathcal{O}$. We regard $\mathcal{C}_{\mathcal{O},\mathcal{A}} = \mathsf{cl}^{\omega\cdot N}(\mathcal{A})$ as an interpretation whose domain $\Delta^{\mathcal{C}_{\mathcal{O},\mathcal{A}}}$ comprises $\mathsf{ind}(\mathcal{A})$ and the witnesses aR^n and the interpretation function is defined by taking $a^{\mathcal{C}_{\mathcal{O},\mathcal{A}}} = a$, for $a \in \mathsf{ind}(\mathcal{A})$, and $\boldsymbol{a} \in S^{\mathcal{C}_{\mathcal{O},\mathcal{A}}(\ell)}$ iff $S(\boldsymbol{a},\ell) \in \mathcal{C}_{\mathcal{O},\mathcal{A}}$, for concept or role names S. We call $\mathcal{C}_{\mathcal{O},\mathcal{A}}$ the *canonical model* of $(\mathcal{O},\mathcal{A})$.

Example 19. Suppose that $\mathcal{O} = \{A \sqsubseteq \Box_F \exists P,\ \Box_F \exists R \sqsubseteq B,\ P \sqsubseteq \bigcirc_F R,\ R \sqsubseteq \bigcirc_F R\}$ and $\mathcal{A} = \{A(a,0)\}$. The canonical model of $(\mathcal{O},\mathcal{A})$ is depicted below:



Note that its construction requires $\omega + 1$ applications of cl.

Theorem 20. *For any DL-Lite$^{\Box\bigcirc}_{horn}$ TOMIQs* $(\mathcal{O}, \varkappa)$ *and* $(\mathcal{O}, \varrho)$*, any ABox* $\mathcal{A}$*,* $a, b \in \mathsf{ind}(\mathcal{A})$ *and* $\ell \in \mathbb{Z}$*, we have*

$$(a, \ell) \in \mathsf{ans}^{\mathbb{Z}}(\mathcal{O}, \varkappa, \mathcal{A}) \quad \textit{iff} \quad a \in \varkappa^{\mathcal{C}_{\mathcal{O},\mathcal{A}}(\ell)},$$
$$(a, b, \ell) \in \mathsf{ans}^{\mathbb{Z}}(\mathcal{O}, \varrho, \mathcal{A}) \quad \textit{iff} \quad (a, b) \in \varrho^{\mathcal{C}_{\mathcal{O},\mathcal{A}}(\ell)}.$$

Let $(\mathcal{O}, \varkappa)$ be a *DL-Lite$^{\Box\bigcirc}_{horn}$* TOMIQ with $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. For any role R in $\mathcal{O}$, we consider the canonical model $\mathcal{C}_{\mathcal{R},\{R(a,b,0)\}}$ and denote by $\boldsymbol{r}^R_n$, $n \in \mathbb{Z}$, the set of roles Q in $\mathcal{O}$ such that $Q(a, b, n) \in \mathcal{C}_{\mathcal{R},\{R(a,b,0)\}}$. It is known from temporal logic [Gabbay *et al.*, 1994] that there are positive numbers $n_r, n_l, k_r, k_l = O(2^{|\mathcal{R}|})$ such that

$$\boldsymbol{r}^R_i = \boldsymbol{r}^R_{i+k_r} \text{ for any } i \geq n_r, \quad \boldsymbol{r}^R_i = \boldsymbol{r}^R_{i-k_l} \text{ for any } i \leq -n_l.$$

We take, for each R and $0 \leq i \leq n_r + k_r$, fresh concept names D_i and *add to* $\mathcal{T}$ the concept inclusions (cf. Example 18):

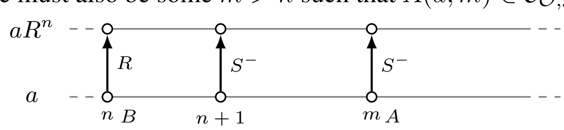
(con) $\exists R \sqsubseteq D_0$, $D_{i-1} \sqsubseteq \bigcirc_{\scriptscriptstyle F} D_i$ (if $i > 0$), $D_{n_r+k_r} \sqsubseteq D_{n_r}$ and $D_i \sqsubseteq \exists Q$, for $0 \leq i \leq n_r + k_r$ and each $Q \in \boldsymbol{r}^R_i$.

and symmetrical inclusions for $-n_l - k_l \leq i \leq 0$. Denote by $\mathcal{T}^*$ the *LTL$^{\Box\bigcirc}_{horn}$* TBox obtained from $\mathcal{T}$ by replacing the *basic* concepts in it with *concept names*.

Theorem 21. *Suppose* $(\mathcal{O}, \varkappa)$ *is a DL-Lite$^{\Box\bigcirc}_{horn}$ TOMIQ and* $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ *with* $\mathcal{T}$ *supplemented by* **(con)**. *If there exist F-rewritings of* $(\mathcal{T}^*, A)$ *and* $(\mathcal{R}^*, P)$*, for every concept name* A *and role name* P *in* $\mathcal{O}$*, then* $(\mathcal{O}, \varkappa)$ *is F-rewritable, where F is FO*$(<)$ *or FO*$(<, +)$*.*

We first illustrate the rather involved proof by an example.

Example 22. Let $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$, where $\mathcal{T} = \{B \sqsubseteq \exists R\}$ and $\mathcal{R} = \{R \sqsubseteq \Box_{\scriptscriptstyle F} S^-\}$, and $\boldsymbol{q} = (\mathcal{O}, \varkappa)$ with $\varkappa = \exists R.\Diamond_{\scriptscriptstyle F} \exists S.A$. By Theorem 20, certain answers to $\boldsymbol{q}$ over an ABox $\mathcal{A}$ are all $a \in \mathsf{ind}(\mathcal{A})$ and $n \in \mathsf{tem}(\mathcal{A})$ with $a \in \varkappa^{\mathcal{C}_{\mathcal{O},\mathcal{A}}(n)}$. Witnesses for $\exists R$ in $\varkappa$ are either located in $\mathsf{ind}(\mathcal{A})$ or of the form aR^n if $\mathcal{C}_{\mathcal{O},\mathcal{A}}$ contains $B(a, n)$, and so $R(a, aR^n, n)$, in which case there must also be some $m > n$ such that $A(a, m) \in \mathcal{C}_{\mathcal{O},\mathcal{A}}$:



This observation gives the following inductive rewriting:

$$\Phi_{\mathcal{O},\varkappa}(x, t) = \exists y\, [R(x, y, t) \land \Phi_{\mathcal{O},\Diamond_F \exists S.A}(y, t)] \quad \lor$$
$$[B(x, t) \land \exists t'\, ((t' > t) \land A(x, t'))],$$
$$\Phi_{\mathcal{O},\Diamond_F \exists S.A}(x, t) = \exists t'\, [(t' > t) \land \Phi_{\exists S.A}(x, t')],$$
$$\Phi_{\mathcal{O},\exists S.A}(x, t) = \exists y\, [A(y, t) \land (S(x, y, t) \lor$$
$$\exists t'\, ((t' < t) \land R(y, x, t')))].$$

Proof. Consider an TOMIQ $(\mathcal{O}, \varkappa)$. First, we define an FO-rewriting $\Phi_{\mathcal{O},\varkappa}(x, t)$ over $\mathbb{Z}$. We proceed by induction on the construction of $\varkappa$.

Case $\varkappa = \top$. We take $\top$.

Case $\varkappa = A$. We take a rewriting $\varphi_{\mathcal{T}^*,A}(t)$ and convert it to $\Phi_{\mathcal{O},A}(x, t)$ as described above Theorem 21.

Case $\varkappa = \exists Q.\varkappa'$. For each role R in $\mathcal{O}$ and each concept B of the form A or $\exists R'$, let $\Xi_{\exists R \leadsto B}(s)$ be an FO$(<, +)$-formula such that, for all $k \in \mathbb{Z}$,

$$\Xi_{\exists R \leadsto B}(k) \quad \text{iff} \quad B(a, k) \in \mathcal{C}_{\mathcal{T},\{R(a,a',0)\}}.$$

Informally, $\Xi_{\exists R \leadsto B}(k)$ stands for $\mathcal{O} \models \exists R \sqsubseteq \bigcirc^k B$. Due to **(con)**, these formulas can be obtained by constructing the canonical model around single element a (that is, by ignoring a' and any witnesses that in general need to be created). Also, for each pair of roles R and Q, let $\Lambda_{R \leadsto Q}(s)$ be an FO$(<, +)$-formula such that, for all $k \in \mathbb{Z}$,

$$\Lambda_{R \leadsto Q}(k) \quad \text{iff} \quad Q(a, b, k) \in \mathcal{C}_{\mathcal{R},\{R(a,b,0)\}}.$$

Informally, $\Lambda_{R \leadsto Q}(k)$ stands for $\mathcal{R} \models R \sqsubseteq \bigcirc^k Q$. These formulas can be obtained from the two-element canonical model $\mathcal{C}_{\mathcal{R},\{R(a,b,0)\}}$ (no witnesses are created in this case because there is no TBox; see also the definition of the sets $\boldsymbol{r}^R_i$). We define $\Phi_{\mathcal{O},\exists Q.\varkappa'}(x, t)$ as the following formula:

$$\exists y\, \big[\Phi_{\mathcal{O},Q}(x, y, t) \land \Phi_{\mathcal{O},\varkappa'}(y, t)\big] \quad \lor$$
$$\bigvee_{R_1} \exists t_1\, \big[\Phi_{\mathcal{O},\exists R_1}(x, t_1) \land$$
$$\Lambda_{R_1 \leadsto Q}(t - t_1) \land \Psi_{R_1,\varkappa'}(x, t, t_1)\big],$$

where $\Phi_{\mathcal{O},\exists R_1}(x, t_1)$ is obtained from the rewriting $\varphi_{\mathcal{T}^*,\exists R_1}(t_1)$ of $(\mathcal{T}^*, \exists R_1)$ in precisely the same way as described above Theorem 21, and $\Psi_{R_1,\varkappa'}(x, t, t_1)$ is defined below. More precisely, for a sequence of roles $R_1, \dots, R_n$ and a concept $\varkappa$, we define $\Psi_{R_1 \dots R_n,\varkappa}(x, t, t_1, \dots, t_n)$ by induction on the construction of $\varkappa$:

- *Case* $\varkappa = \top$. Then we take $\top$.
- *Case* $\varkappa = A$. Then we take $\Xi_{\exists R_n^- \leadsto A}(t_n - t)$.
- *Case* $\varkappa = \exists Q.\varkappa'$. Then we take a disjunction of

$$\bigvee_{R_{n+1}} \exists t_{n+1}\, \big[\Xi_{\exists R_n^- \leadsto \exists R_{n+1}}(t_{n+1} - t_n) \land$$
$$\Lambda_{R_{n+1} \leadsto Q}(t - t_{n+1}) \land$$
$$\Psi_{R_1 \dots R_n R_{n+1},\varkappa'}(x, t, t_1, \dots, t_n, t_{n+1})\big],$$

$$\Lambda_{R_n \leadsto Q^-}(t - t_n) \land \Psi_{R_1 \dots R_{n-1},\varkappa'}(x, t, t_1, \dots, t_{n-1}), \quad \text{if } n > 1,$$

$$\Lambda_{R_n \leadsto Q^-}(t - t_n) \land \Phi_{\mathcal{O},\varkappa'}(x, t), \qquad \text{if } n = 1.$$

- *Cases* $\varkappa = \varkappa_1 \sqcup \varkappa_2$ and $\varkappa = \varkappa_1 \sqcap \varkappa_2$. We take, respectively, the disjunction and the conjunction of the $\Psi_{R_1 \dots R_n,\varkappa_i}(x, t, t_1, \dots, t_n)$, for $i = 1, 2$.
- *Case* $\varkappa = \Box_{\scriptscriptstyle F} \varkappa'$. Then we take

$$\forall s\, [(s > t) \to \Psi_{R_1 \dots R_n,\varkappa'}(x, s, t_1, \dots, t_n)].$$

- *Case* $\varkappa = \Diamond_{\scriptscriptstyle F} \varkappa'$. Then we take

$$\exists s\, [(s > t) \land \Psi_{R_1 \dots R_n,\varkappa'}(x, s, t_1, \dots, t_n)].$$

- Other temporal operators are analogous.

To illustrate, consider $\varkappa = \exists Q_1.\exists Q_2.\exists Q_3.\exists Q_4.B$ with the following ontology $\mathcal{O}$:

$$A \sqsubseteq \bigcirc^{k_1} \exists R_1, \qquad R_1 \sqsubseteq \bigcirc^{-k_1} Q_1 \sqcap \bigcirc^{-k_1} Q_3^-,$$
$$\exists R_1^- \sqsubseteq \bigcirc^m \exists R_2, \qquad R_2 \sqsubseteq \bigcirc^{-k_2} Q_2 \sqcap \bigcirc^{-k_2} Q_4^-.$$

Then the rewriting $\Phi_{\mathcal{O},\varkappa}(x, t)$ of $\varkappa$ contains the following disjunct:

$$\exists t_1 \big[A(t_1 - k_1) \land$$
$$(t_1 - t = -k_1) \land \Psi_{R_1,\exists Q_2.\exists Q_3.\exists Q_4.B}(x, t, t_1)\big],$$

where $\Psi_{R_1,\exists Q_2.\exists Q_3.\exists Q_4.B}(x,t,t_1)$, in turn, contains the following disjunct:

$$\exists t_2\big[(t_2 - t_1 = m) \wedge \\ (t - t_2 = -k_2) \wedge \Psi_{R_1R_2,\exists Q_3.\exists Q_4.B}(x,t,t_1,t_2)\big].$$

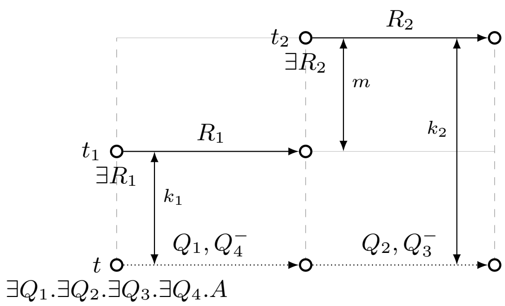
Then, $\Psi_{R_1R_2,\exists Q_3.\exists Q_4.B}(x,t,t_1,t_2)$ contains the following disjunct:

$$(t - t_2 = -k_2) \wedge \Psi_{R_1,\exists Q_4.B}(x,t,t_1),$$

where $\Psi_{R_1,\exists Q_4.B}(x,t,t_1)$ contains the following disjunct:

$$(t - t_1 = -k_1) \wedge \Phi_{\mathcal{O},B}(x,t).$$

The structure of the canonical model with a positive answer to $\varkappa$ is depicted below:



Cases $\varkappa = \varkappa_1 \sqcup \varkappa_2$ and $\varkappa = \varkappa_1 \sqcap \varkappa_2$. We take, respectively, the disjunction and the conjunction of $\Phi_{\mathcal{O},\varkappa_1}(x,t)$ and $\Phi_{\mathcal{O},\varkappa_2}(x,t)$.

Case $\varkappa = \Box_F \varkappa'$. We take

$$\forall s\,[(s > t) \to \Phi_{\mathcal{O},\varkappa'}(x,s)].$$

Cases of other temporal operators are analogous.

Now we turn to FO-rewritings over $[\mathcal{A}]$ and show how to obtain them from the FO-rewritings over $\mathbb{Z}$ constructed above.

An FO-formula $\Theta^k_{\mathcal{O},\varkappa}(x)$ is called a *witness* for an TOMIQ $(\mathcal{O},\varkappa)$ and $k \neq 0$ if, for any ABox $\mathcal{A}$ and any $a \in \mathsf{ind}(\mathcal{A})$, we have

– $\mathfrak{S}^{\mathsf{tem}(\mathcal{A})}_{\mathcal{A}} \models \Theta^k_{\mathcal{O},\varkappa}(a)$ iff $a \in \varkappa^{\mathcal{C}_{\mathcal{O},\mathcal{A}}(\max \mathcal{A}+k)}$, for $k \geq 1$,

– $\mathfrak{S}^{\mathsf{tem}(\mathcal{A})}_{\mathcal{A}} \models \Theta^k_{\mathcal{O},\varkappa}(a)$ iff $a \in \varkappa^{\mathcal{C}_{\mathcal{O},\mathcal{A}}(k)}$, for $k \leq -1$.

We will define $\Theta^i_{\mathcal{O},\varkappa}$ by induction on the structure of $\varkappa$ along the construction of the rewriting for $(\mathcal{O},\varkappa)$. It will follow that the $\Theta^k_{\mathcal{O},\varkappa}$ are ultimately periodic, in the sense that there are integers $\mathit{prfx}^{\mathcal{O}}_F(\varkappa) \geq 0$, $\mathit{prd}^{\mathcal{O}}_F(\varkappa) \geq 1$, $\mathit{prfx}^{\mathcal{O}}_P(\varkappa) \leq 0$ and $\mathit{prd}^{\mathcal{O}}_P(\varkappa) \leq -1$ such that, for any ABox $\mathcal{A}$ and $a \in \mathsf{ind}(\mathcal{A})$,

– $\mathfrak{S}^{\mathsf{tem}(\mathcal{A})}_{\mathcal{A}} \models \Theta^k_{\mathcal{O},\varkappa}(a)$ iff $\mathfrak{S}^{\mathsf{tem}(\mathcal{A})}_{\mathcal{A}} \models \Theta^{k+\mathit{prd}^{\mathcal{O}}_F(\varkappa)}_{\mathcal{O},\varkappa}(a)$ for all $k > \mathit{prfx}^{\mathcal{O}}_F(\varkappa)$;

– similarly for $k < \mathit{prfx}^{\mathcal{O}}_P(\varkappa)$ with $k + \mathit{prd}^{\mathcal{O}}_P(\varkappa)$.

Witnesses $\Theta^k_{\mathcal{R},\varrho}(x,y)$ for TOMIQs with a positive temporal role ϱ are defined analogously. We construct them in the same way as in the propositional case of the proofs of Corollaries 14 and 15.

Also, given a sequence $R_1 \dots R_n$ and a finite set of integers M different from 0, let μ be a *partial* mapping from the prefixes of $R_1 \dots R_n$ to M:

$$\mu\colon \{R_1 \dots R_i \mid 1 \leq i \leq n\} \to M.$$

The domain of μ is denoted by $\mathit{dom}\,\mu$. By by induction on the construction of our TOMIQ we will define formulas $\Psi^{\mu}_{R_1\dots R_n,\varkappa}(x,t,t_1,\dots,t_m)$ and $\Psi^{k,\mu}_{R_1\dots R_n,\varkappa}(x,t_1,\dots,t_m)$, where $m = n - |\,\mathit{dom}\,\mu|$. The construction will imply that the formulas are ultimately periodic in the sense that, given a set D of prefixes of $R_1 \dots R_n$, there exist numbers

$$\mathit{prfx}^{\mathcal{O}}_F(R_1 \dots R_n,\varkappa,D) \geq 0,\\ \mathit{prd}^{\mathcal{O}}_F(R_1 \dots R_n,\varkappa,D) \geq 1\\ \mathit{prfx}^{\mathcal{O}}_P(R_1 \dots R_n,\varkappa,D) \leq 0,\\ \mathit{prd}^{\mathcal{O}}_P(R_1 \dots R_n,\varkappa,D) \leq -1,$$

such that

$$\mathfrak{S}^{\mathsf{tem}(\mathcal{A})}_{\mathcal{A}} \models \Psi^{\mu}_{R_1\dots R_n,\varkappa}(a,\boldsymbol{\ell}) \text{ iff } \mathfrak{S}^{\mathsf{tem}(\mathcal{A})}_{\mathcal{A}} \models \Psi^{\mu'}_{R_1\dots R_n,\varkappa}(a,\boldsymbol{\ell}),$$

for all $a \in \mathsf{ind}(\mathcal{A})$, all $(n - |D|)$-vectors $\boldsymbol{\ell}$ in $\mathsf{tem}(\mathcal{A})$ and all μ and μ' defined on D that are identical except for

$$\mu(R_1 \dots R_i) = s > \mathit{prfx}^{\mathcal{O}}_F(R_1 \dots R_n,\varkappa,D),\\ \mu'(R_1 \dots R_i) = s + \mathit{prd}^{\mathcal{O}}_F(R_1 \dots R_n,\varkappa,D);$$

and similarly for the past-time counterparts. Formulas $\Psi^{k,\mu}_{R_1\dots R_n,\varkappa}(x,t_1,\dots,t_m)$ play the role of witnesses for $\Psi^{\mu}_{R_1\dots R_n,\varkappa}(x,t_1,\dots,t_m)$ and the periodicity condition is defined analogously.

Case $\varkappa = A$. Observe first that we cannot construct the FO-rewriting $\Phi_{\mathcal{O},A}(x,t)$ in the same way we constructed the FO$^{\mathbb{Z}}$-rewriting for A in the previous section.

Example 23. Consider $\mathcal{O}$ with the following inclusions: $\Diamond_P P \sqcap \Diamond_P \Diamond_P R \sqsubseteq Q$ and $\Diamond_F \exists Q \sqsubseteq A$. Then the FO$^{\mathbb{Z}}$- and the FO-rewriting of $(\mathcal{O},Q)$ is

$$\Phi_{\mathcal{O},Q}(x,y,t) = Q(x,y,t) \vee \\ \exists t' \exists t'' \big((t' < t) \wedge P(x,y,t') \wedge \\ (t'' < t-1) \wedge R(x,y,t'')\big)$$

and the FO$^{\mathbb{Z}}$- and FO-rewriting for $(\mathcal{T}^*,A)$ is

$$\varphi_{\mathcal{T}^*,A}(t) = A(t) \vee \exists t' \,\big((t' > t) \wedge (\exists R)(t')\big).$$

Now, even though

$$\Phi(x,t) = A(x,t) \vee \exists t' \,\big((t' > t) \wedge \exists y\, \Phi_{\mathcal{O},Q}(x,y,t')\big)$$

is an FO$^{\mathbb{Z}}$-rewriting of $(\mathcal{O},A)$, it is not an FO-rewriting, which is demonstrated by $\mathcal{A} = \{R(a,b,0), P(a,b,1)\}$ and $(\mathcal{O},\mathcal{A}) \models A(1)$.

In order to define the rewriting $\Phi_{\mathcal{O},A}(x,t)$, we require the following notion. A *role shrub* $\boldsymbol{p}$ *for* $\mathcal{O}$ is a tuple $(\boldsymbol{r}^R_1,\dots,\boldsymbol{r}^R_n)$ of pairwise distinct sets $\boldsymbol{r}^R_i$ of roles from $\mathcal{O}$ (without loss of generality, we assume that, for each 'complex' role prefixed by temporal operators, $\mathcal{O}$ contains a role that is equivalent to the complex role). Clearly, there are at most $O(2^{2^{|\mathcal{O}|}})$ role shrubs for $\mathcal{O}$. Consider the canonical model $\mathcal{C}_{\mathcal{O},\mathcal{A}_{\boldsymbol{p}}}$ of the ABox of $\boldsymbol{p}$, which is defined by taking

$$\mathcal{A}_{\boldsymbol{p}} = \{R(a, b_i, 0) \mid R \in \boldsymbol{r}_i^R,\ 1 \leq i \leq n\}.$$

It is easy to see that due to ultimate periodicity we can obtain $\mathcal{T}^*$ by extending $\mathcal{T}$ with fresh concept symbols $E_{\boldsymbol{p}}$ and axioms similar to **(con)** such that the following property holds:

$$\exists Q(a,n) \in \mathcal{C}_{\mathcal{O},\{E_{\boldsymbol{p}}(a,0)\}} \quad \text{iff} \quad \exists Q(a,n) \in \mathcal{C}_{\mathcal{O},\mathcal{A}_{\boldsymbol{p}}}.$$

Consider now the FO-rewriting $\varphi_{\mathcal{T}^*,A}(t)$ for the *LTL* ontology $\mathcal{T}^*$. We define $\Phi_{\mathcal{O},A}(x,t)$ as the result of replacing in $\varphi_{\mathcal{T}^*,A}(t)$ each $B(t)$ by $B(x,t)$ and each $E_{\boldsymbol{p}}(t)$ by

$$\Sigma_{\boldsymbol{p}}(x,t) \;=\; \bigwedge_{i=1}^{n} \exists y_i \bigwedge_{\varrho \in \boldsymbol{r}_i^R} \Phi_{\mathcal{O},\varrho}(x, y_i, t),$$

which, intuitively, means that $\boldsymbol{p}$ is realised on x at moment t.

For the witnesses $\Theta^i_{\mathcal{O},A}(x)$, we need to extend the notion of role shrubs. A *shrub $\boldsymbol{s}$ for $\mathcal{O}$* is a tuple $(\boldsymbol{t},\boldsymbol{p})$ comprising a set $\boldsymbol{t}$ of concept names from $\mathcal{O}$ and a role shrub $\boldsymbol{p}$ (we again assume that, for each 'complex' concept prefixed by temporal operators, $\mathcal{O}$ contains a concept name that is equivalent to the complex concept). We consider the canonical model for $(\mathcal{O},\mathcal{A}_{\boldsymbol{s}})$, where the ABox of $\boldsymbol{s}$ is defined by taking

$$\mathcal{A}_{\boldsymbol{s}} = \{B(a,0) \mid B \in \boldsymbol{t}\} \cup \mathcal{A}_{\boldsymbol{p}}.$$

Clearly, there are at most $O(2^{2^{|\mathcal{O}|}})$ shrubs for $\mathcal{O}$. For any shrub $\boldsymbol{s} = (\boldsymbol{t},\boldsymbol{p})$ for $\mathcal{O}$, let

$$\Sigma_{\boldsymbol{s}}(x,t) \;=\; \bigwedge_{B \in \boldsymbol{t}} \Phi_{\mathcal{O},B}(x,t) \;\wedge\; \Sigma_{\boldsymbol{p}}(x,t),$$

which, intuitively, means that $\boldsymbol{s}$ is realised on x at moment t. Next, observe that, for any shrub $\boldsymbol{s}$ for $\mathcal{O}$ and any $n \in \mathbb{Z}$, there is an FO-sentence $\Gamma^n_{\boldsymbol{s},A}$ (for a fixed A) such that

$$\Gamma^n_{\boldsymbol{s},A} \quad \text{iff} \quad A(a,n) \in \mathcal{C}_{\mathcal{O},\mathcal{A}_{\boldsymbol{s}}}.$$

So, for $k \geq 1$, we define

$$\Theta^k_{\mathcal{O},A}(x) = \bigvee_{\boldsymbol{s} \text{ a shrub for } \mathcal{O}} \big(\Sigma_{\boldsymbol{s}}(x, \max \mathcal{A}) \wedge \Gamma^k_{\boldsymbol{s},A}\big).$$

From the canonical model of $(\mathcal{O},\mathcal{A}_{\boldsymbol{s}})$ we obtain $\mathit{prfx}^{\mathcal{O}}_{F}(\varkappa)$ and $\mathit{prd}^{\mathcal{O}}_{F}(\varkappa)$. Witnesses for $k \leq -1$ and numbers $\mathit{prfx}^{\mathcal{O}}_{P}(\varkappa)$ and $\mathit{prd}^{\mathcal{O}}_{P}(\varkappa)$ are defined symmetrically.

Cases $\varkappa = \varkappa_1 \sqcap \varkappa_2$ and $\varkappa = \varkappa_1 \sqcup \varkappa_2$ are trivial.

Case $\varkappa = \Box_F \varkappa'$. Let $N = \mathit{prfx}^{\mathcal{O}}_{F}(\mathcal{O},\varkappa') + \mathit{prd}^{\mathcal{O}}_{F}(\mathcal{O},\varkappa')$. Then we take

$$\Phi_{\mathcal{O},\Box_F\varkappa'}(x,t) = \forall s\, [(s > t) \to \Phi_{\mathcal{O},\varkappa'}(x,s)] \wedge \bigwedge_{0<k<N} \Theta^k_{\mathcal{O},\varkappa'}(x),$$

$$\Theta^k_{\mathcal{O},\Box_F\varkappa'}(x) = \bigwedge_{k<i<N} \Theta^i_{\mathcal{O},\varkappa'}(x), \qquad \text{for } k \geq 1,$$

$$\Theta^k_{\mathcal{O},\Box_F\varkappa'}(x) = \bigwedge_{k<i<0} \Theta^i_{\mathcal{O},\varkappa'}(x) \wedge \Phi_{\mathcal{O},\Box_F\varkappa'}(x,0), \text{ for } k \leq -1.$$

The numbers $\mathit{prfx}^{\mathcal{O}}_{F}(\Box_F\varkappa')$, $\mathit{prd}^{\mathcal{O}}_{F}(\Box_F\varkappa')$, $\mathit{prfx}^{\mathcal{O}}_{P}(\Box_F\varkappa')$ and $\mathit{prd}^{\mathcal{O}}_{P}(\Box_F\varkappa')$ follow from the construction.

Cases of other temporal operators are similar.

Case $\varkappa = \exists Q.\varkappa'$. We define $\Phi_{\mathcal{O},\exists Q.\varkappa'}(x,t)$ as a disjunction of the following formulas:

$$\exists y\, [\Phi_{\mathcal{R},Q}(x,y,t) \wedge \Phi_{\mathcal{O},\varkappa'}(y,t)],$$

$$\bigvee_{R_1} \exists t_1\, [\Phi_{\mathcal{O},\exists R_1}(x,t_1) \wedge \Lambda_{R_1 \rightsquigarrow Q}(t-t_1) \wedge \Psi^{\emptyset}_{R_1,\varkappa'}(x,t,t_1)],$$

$$\bigvee_{R_1} \bigvee_{\substack{N_P<i<N_F\\ i\neq 0}} [\Theta^i_{\mathcal{O},\exists R_1}(x) \wedge \Lambda_{R_1,i \rightsquigarrow Q}(t) \wedge \Psi^{\{R_1 \mapsto i\}}_{R_1,\varkappa'}(x,t)],$$

where N_P can be obtained from $\mathit{prfx}^{\mathcal{O}}_{P}(\exists R_1)$, $\mathit{prd}^{\mathcal{O}}_{P}(\exists R_1)$, $\mathit{prfx}^{\mathcal{O}}_{P}(R_1,\varkappa',\{R_1\})$, $\mathit{prd}^{\mathcal{O}}_{P}(R_1,\varkappa',\{R_1\})$ and the periodicity of the $\Lambda_{R_1,i\rightsquigarrow Q}$; N_F is defined similarly.

We define $\Theta^k_{\mathcal{O},\exists Q.\varkappa'}(x)$, for $k \neq 0$, as a disjunction of the following formulas:

$$\exists y\, [\Theta^k_{\mathcal{R},Q}(x,y) \wedge \Theta^k_{\mathcal{O},\varkappa'}(y)],$$

$$\bigvee_{R_1} \exists t_1\, [\Phi_{\mathcal{O},\exists R_1}(x,t_1) \wedge \Lambda_{R_1 \rightsquigarrow Q,k}(t_1) \wedge \Psi^{k,\emptyset}_{R_1,\varkappa'}(x,t_1)],$$

$$\bigvee_{R_1} \bigvee_{\substack{N_P<i<N_F\\ i\neq 0}} [\Theta^i_{\mathcal{O},\exists R_1}(x) \wedge \Lambda_{R_1,i \rightsquigarrow Q,k} \wedge \Psi^{k,\{R_1 \mapsto i\}}_{R_1,\varkappa'}(x)],$$

where N_P and N_F can be defined analogoulsy to the previous case. The meaning of $\Lambda_{R_1 \rightsquigarrow Q,k}(t)$ and $\Lambda_{R_1,i\rightsquigarrow Q,k}$ should be clear from the formulas above. These predicates are clearly definable.

We now define $\Psi^{\mu}_{R_1\dots R_n,\varkappa}(x,t,t_1,\dots,t_m)$ by induction on the structure of $\varkappa$ (as before, $m = n - |\mathit{dom}\,\mu|$).

Case $\varkappa = A$ is easy and left to the reader.

Case $\varkappa = \exists Q.\varkappa'$. If $R_1 \dots R_n \notin \mathit{dom}\,\mu$ then we take a disjunction of the following formulas:

$$\begin{aligned}\exists t_{m+1}[&\Xi_{\exists R_n^- \rightsquigarrow \exists R_{n+1}}(t_{m+1}-t_m) \wedge \\ &\Lambda_{R_{n+1}\rightsquigarrow Q}(t-t_{m+1}) \wedge \\ &\Psi^{\mu}_{R_1\dots R_n R_{n+1},\varkappa'}(x,t,t_1,\dots,t_m,t_{m+1})],\end{aligned}$$

$$\begin{aligned}\bigvee_{\substack{N_P<i<N_F\\ i\neq 0}} [&\Xi_{\exists R_n^- \rightsquigarrow \exists R_{n+1},i}(t_m) \wedge \Lambda_{R_{n+1},i\rightsquigarrow Q}(t_m) \wedge \\ &\Psi^{\mu\cup\{R_1\dots R_n R_{n+1}\mapsto i\}}_{R_1\dots R_n R_{n+1},\varkappa'}(x,t,t_1,\dots,t_m)],\end{aligned}$$

$$\Lambda_{R_n \rightsquigarrow Q^-}(t-t_m) \wedge \Psi_{R_1\dots R_{n-1},\varkappa'}(x,t,t_1,\dots,t_{m-1}), \quad \text{if } n > 1,$$

$$\Lambda_{R_n \rightsquigarrow Q^-}(t-t_m) \wedge \Phi_{\mathcal{O},\varkappa'}(x,t), \qquad \text{if } n = 1.$$

The meaning of $\Xi_{\exists R \rightsquigarrow \exists Q}(t)$, $\Xi_{\exists R \rightsquigarrow \exists Q,i}(t)$ and of $\Xi_{\exists R,k\rightsquigarrow \exists Q,i}$ below should be clear from the formulas, as well as their definability.

If $\mu(R_1 \dots R_n) = j$ then we take a disjunction of

$$\exists t_{m+1} [\Xi_{\exists R_n^-, j \rightsquigarrow \exists R_{n+1}}(t_{m+1}) \wedge \Lambda_{R_{n+1} \rightsquigarrow Q}(t - t_{m+1}) \wedge \Psi^{\mu}_{R_1 \dots R_n R_{n+1}, \varkappa'}(x, t, t_1, \dots, t_m, t_{m+1})],$$

$$\bigvee_{\substack{N_P < i < N_F \\ i \neq 0}} [\Xi_{\exists R_n^-, j \rightsquigarrow \exists R_{n+1}, i} \wedge \Lambda_{R_{n+1} \rightsquigarrow Q, i}(t) \wedge \Psi^{\mu \cup \{R_1 \dots R_n R_{n+1} \mapsto i\}}_{R_1 \dots R_n R_{n+1}, \varkappa'}(x, t, t_1, \dots, t_m)],$$

$$\Lambda_{R_n, j \rightsquigarrow Q^-}(t) \wedge \Psi^{\mu \setminus \{R_1 \dots R_n \mapsto j\}}_{R_1 \dots R_{n-1}, \varkappa'}(x, t, t_1, \dots, t_{m-1}), \qquad \text{if } n > 1,$$

$$\Lambda_{R_n, j \rightsquigarrow Q^-}(t) \wedge \Phi_{\mathcal{O}, \varkappa'}(x, t), \qquad \text{if } n = 1.$$

Case $\varkappa = \Box_F \varkappa'$. We take

$$\forall s \, [(s > t) \to \Psi^{\mu}_{R_1 \dots R_n, \varkappa'}(x, s, t_1, \dots, t_m)] \wedge \bigwedge_{0 < k < N} \Psi^{k, \mu}_{R_1 \dots R_n, \varkappa'}(x, t_1, \dots, t_m).$$

Cases of other temporal operators are analogous.

The definition of $\Psi^{k,\mu}_{R_1 \dots R_n, \varkappa}(x, t, t_1, \dots, t_m)$ is similar and left to the reader (note that the formulas $\Theta^i_{\mathcal{O}, \varkappa'}(x)$ will be used instead of the rewritings). ❑

Since the connection axioms **(con)** belong to *DL-Lite*$^{\bigcirc}_{core}$, as a consequence of Theorems 16 (*i*), 17 and 21 we obtain:

Corollary 24. *DL-Lite*$^{\bigcirc}_{core}$ *TOMIQs are FO*$(<,+)$*-rewritable.*

On the other hand, the following unexpected result shows, in particular, that **(con)** cannot be expressed in *DL-Lite*$^{\Box}_{horn}$:
On the other hand, the following unexpected result shows, in particular, that **(con)** cannot be expressed in *DL-Lite*$^{\Box}_{horn}$:

Theorem 25. *There is a DL-Lite*$^{\Box}_{horn}$ *OMAQ which is* NC^1*-hard for data complexity*; *in particular, it is not FO-rewritable even with arbitrary numeric predicates.*

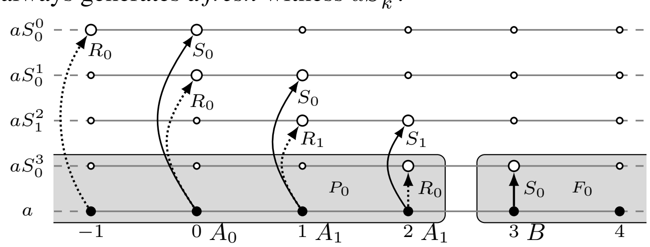
Proof. Here we only show non-rewritability by modifying the construction of Example 4. Consider the RBox $\mathcal{R}$ with

$$S_k \sqsubseteq F_k, \; S_k \sqsubseteq \Box_F F_k, \; S_k \sqsubseteq \Box_P P_k, \; \Box_F F_k \sqcap P_k \sqsubseteq R_k,$$

and the TBox $\mathcal{T}$ with

$$\exists R_k \sqcap A_0 \sqsubseteq \exists S_k, \quad \exists R_k \sqcap A_1 \sqsubseteq \exists S_{1-k}, \quad B \equiv \exists S_0,$$

where $k = 0, 1$. For $\boldsymbol{e} = (e_0, \dots, e_{n-1}) \in \{0,1\}^n$, we take $\mathcal{A}_{\boldsymbol{e}} = \{B(a,n)\} \cup \{A_{e_i}(a,i) \mid i < n\}$. Then $(a, 0)$ is a certain answer to $(\mathcal{T} \cup \mathcal{R}, B)$ over $\mathcal{A}_{\boldsymbol{e}}$ iff the number of 1s in $\boldsymbol{e}$ is even—see the picture below and note that $\exists S_k(a,n)$ always generates a *fresh* witness aS_k^n:



The same idea can be used to simulate arbitrary NFAs and show NC^1-hardness of *DL-Lite*$^{\Box}_{horn}$ TOMAQs; see Thm. 9. ❑

Thus, Theorem 16 (*ii*) cannot be lifted to *DL-Lite*$^{\Box}_{horn}$. In the proof above, using *Horn role inclusions* with $\Box_F$ and $\Box_P$, we have encoded $\bigcirc_F$ on concepts: $\mathcal{T} \cup \mathcal{R} \models \bigcirc_F \exists S_k \sqsubseteq \exists R_k$.

We now give a sufficient condition under which **(con)** can be expressed in *DL-Lite*$^{\Box}_{core}$. Call an RBox $\mathcal{R}$ *monotone* if, for any roles R, Q in it, $Q \in \boldsymbol{r}^R_n$ and $n \neq 0$ imply $Q \in \boldsymbol{r}^R_m$ for all $m \geq n$ or all $m \leq n$. Clearly, for a monotone $\mathcal{R}$ and any roles R and Q in it, one of the four options holds:

(*i*) $Q \in \boldsymbol{r}^R_n$ iff $n = 0$,

(*ii*) there is $m_r \in \mathbb{Z}$ such that $Q \in \boldsymbol{r}^R_n$ iff $m_r \leq n$ or $n = 0$,

(*iii*) there is $m_l \in \mathbb{Z}$ such that $Q \in \boldsymbol{r}^R_n$ iff $n \leq m_l$ or $n = 0$,

(*iv*) there are $m_l, m_r \in \mathbb{Z}$, $m_l \leq m_r$, such that $Q \in \boldsymbol{r}^R_n$ iff $n \leq m_l$ or $m_r \leq n$ or $n = 0$.

We encode (*i*) by $\exists R \sqsubseteq \exists Q$; (*ii*) by the following:

$$\exists R \sqsubseteq \exists Q, \qquad \text{if } \mathcal{R} \models R \sqsubseteq Q,$$
$$\exists R \sqsubseteq \Box_F^{m_r} \exists Q, \qquad \text{if } m_r > 0,$$
$$\exists R \sqsubseteq \Box_F D, \quad \Box_F^{-m_r+1} D \sqsubseteq \exists Q, \qquad \text{if } m_r \leq 0,$$

with fresh D; (*iii*) is symmetrical; (*iv*) combines (*ii*) and (*iii*). The set of all such connection axioms is denoted by **(con′)**.

Theorem 26. *Theorem 21 holds true for* $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ *with monotone* $\mathcal{R}$ *and* **(con′)** *in place of* **(con)**.

One can show that all *DL-Lite*$^{\Box}_{core}$ RBoxes as well as *DL-Lite*$^{\Box}_{horn}$ RBoxes without $\Box$-operators on the left-hand side of role inclusions are monotone. We denote by *DL-Lite*$^{mon\Box}_{horn}$ the fragment of *DL-Lite*$^{\Box}_{horn}$ whose role inclusions do not contain negative occurrences of $\Box_F$ and $\Box_P$; this language can be regarded as an extension of *TQL* [Artale *et al.*, 2013b].

Corollary 27. *All DL-Lite*$^{\Box}_{core}$ *as well as DL-Lite*$^{mon\Box}_{horn}$ *TOMIQs are FO*$(<)$*-rewritable.*

5 Conclusions

We have developed a two-step approach to analysing FO-rewritability of temporal ontology-mediated queries. First, we classified the FO-rewritability properties of TOMQs in fragments of *LTL*. Second, we proved two general transfer theorems identifying conditions under which FO-rewritability is preserved for combinations of *LTL* with *DL-Lite*. The transfer results show that, although temporal DLs are notorious for their high complexity, one can nevertheless find large 'islands' of tractable and expressive TOMQs. Many interesting and challenging research questions can be tackled based on the results of this paper:

– complexity and succinctness of rewritings (the size of our rewritings of TOMAQs varies from polynomial for $LTL^{\bigcirc}_{krom}$ to single, double and triple exponential in the size $|\mathcal{T}|$ of the TBox $\mathcal{T}$ for various temporalised versions of *DL-Lite*; rewritings of TOMQs are of size $S^{|\boldsymbol{q}|}$ and $S^{|\boldsymbol{q}||\mathcal{T}|}$ for, respectively, *LTL* and *DL-Lite* ontologies, where $|\boldsymbol{q}|$ is the size of the query and S the size of the rewritings for the underlying TOMAQs);

– generalising our FO rewritings of TOMIQs to (two-sorted) CQs using the methods of [Artale *et al.*, 2013b];

- data complexity of evaluating $\mathit{DL\text{-}Lite}_{horn}^{\Box\bigcirc}$ TOMQs; based on our proofs, we conjecture that it is NC^1-complete;
- the effect of the reflexive semantics of the temporal operators;
- extension of the transfer results for Horn logics to $\sqcup$ and $\neg$, possibly via a non-uniform analysis;
- our rewriting algorithms need to be optimised and evaluated in real-world applications.

References

[Arora and Barak, 2009] S. Arora and B. Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, New York, USA, 1st edition, 2009.

[Artale *et al.*, 2009] A. Artale, D. Calvanese, R. Kontchakov, and M. Zakharyaschev. The DL-Lite family and relations. *J. Artif. Intell. Res. (JAIR)*, 36:1–69, 2009.

[Artale *et al.*, 2013a] A. Artale, R. Kontchakov, V. Ryzhikov, and M. Zakharyaschev. The complexity of clausal fragments of LTL. In *Proc. of the 19th Int. Conf. on Logic for Programming, Artificial Intelligence and Reasoning (LPAR)*, volume 8312 of *LNCS*, pages 35–52. Springer, 2013.

[Artale *et al.*, 2013b] A. Artale, R. Kontchakov, F. Wolter, and M. Zakharyaschev. Temporal description logic for ontology-based data access. In *Proc. of the 23rd Int. Joint Conf. on Artificial Intelligence (IJCAI)*. IJCAI/AAAI, 2013.

[Baader *et al.*, 2013] F. Baader, S. Borgwardt, and M. Lippmann. Temporalizing ontology-based data access. In *Proc. of the 24th Int. Conf. on Automated Deduction (CADE-24)*, volume 7898 of *LNCS*, pages 330–344. Springer, 2013.

[Barrington and Straubing, 1995] D. Barrington and H. Straubing. Superlinear lower bounds for bounded-width branching programs. *J. Comput. Syst. Sci.*, 50(3):374–381, 1995.

[Barrington *et al.*, 1992] D. Barrington, K. Compton, H. Straubing, and D. Thérien. Regular languages in NC^1. *J. Comput. Syst. Sci.*, 44(3):478–499, 1992.

[Borgwardt *et al.*, 2013] S. Borgwardt, M. Lippmann, and V. Thost. Temporal query answering in the description logic DL-Lite. In *Proc. of the 9th Int. Symposium on Frontiers of Combining Systems (FroCoS'13)*, volume 8152 of *LNCS*, pages 165–180. Springer, 2013.

[Calbimonte *et al.*, 2012] J.-P. Calbimonte, H. Jeung, Ó. Corcho, and K. Aberer. Enabling query technologies for the semantic sensor web. *Int. J. Semantic Web Inf. Syst.*, 8(1):43–63, 2012.

[Calvanese *et al.*, 2007a] D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. EQL-Lite: Effective first-order query processing in description logics. In *Proc. of the 20th Int. Joint Conf. on Artificial Intelligence (IJCAI 2007)*, pages 274–279, 2007.

[Calvanese *et al.*, 2007b] D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. Tractable reasoning and efficient query answering in description logics: The *DL-Lite* family. *J. Autom. Reasoning*, 39(3):385–429, 2007.

[Chrobak, 1986] M. Chrobak. Finite automata and unary languages. *Theor. Comp. Sci.*, 47(2):149–158, 1986.

[Fisher *et al.*, 2001] M. Fisher, C. Dixon, and M. Peim. Clausal temporal resolution. *ACM Trans. Comput. Logic*, 2(1):12–56, 2001.

[Gabbay *et al.*, 1994] D. Gabbay, I. Hodkinson, and M. Reynolds. *Temporal Logic: Mathematical Foundations and Computational Aspects*, volume 1. Oxford University Press, 1994.

[Gutiérrez-Basulto and Klarman, 2012] Víctor Gutiérrez-Basulto and Szymon Klarman. Towards a unifying approach to representing and querying temporal data in description logics. In *Proc. of the 6th Int. Conf. on Web Reasoning and Rule Systems (RR 2012)*, volume 7497 of *LNCS*, pages 90–105. Springer, 2012.

[Gutierrez *et al.*, 2007] C. Gutierrez, C. A. Hurtado, and A. A. Vaisman. Introducing time into RDF. *IEEE Trans. Knowl. Data Eng.*, 19(2):207–218, 2007.

[Kharlamov *et al.*, 2014] E. Kharlamov, N. Solomakhina, Ö. Özçep, D. Zheleznyakov, T. Hubauer, S. Lamparter, M. Roshchin, A. Soylu, and S. Watson. How semantic technologies can enhance data access at Siemens Energy. In *Proc. of the 13th Int. Semantic Web Conf. (ISWC 2014), Part I*, volume 8796 of *LNCS*, pages 601–619. Springer, 2014.

[Klarman and Meyer, 2014] S. Klarman and T. Meyer. Querying temporal databases via OWL 2 QL. In *Proc. of the 8th Int. Conf. on Web Reasoning and Rule Systems (RR 2014)*, volume 8741 of *LNCS*, pages 92–107. Springer, 2014.

[Kontchakov *et al.*, 2014] R. Kontchakov, M. Rezk, M. Rodriguez-Muro, G. Xiao, and M. Zakharyaschev. Answering SPARQL queries over databases under OWL 2 QL entailment regime. In *Proc. of the 13th Int. Semantic Web Conf. (ISWC 2014), Part I*, volume 8796 of *LNCS*, pages 552–567. Springer, 2014.

[Ladner and Fischer, 1980] R. E. Ladner and M. J. Fischer. Parallel prefix computation. *J. ACM*, 27(4):831–838, 1980.

[Libkin, 2004] L. Libkin. *Elements Of Finite Model Theory*. Springer, 2004.

[Lutz *et al.*, 2008] C. Lutz, F. Wolter, and M. Zakharyaschev. Temporal description logics: A survey. In *Proc. of the 15th Int. Symposium on Temporal Representation and Reasoning (TIME 2008)*, pages 3–14, 2008.

[Motik, 2012] B. Motik. Representing and querying validity time in RDF and OWL: A logic-based approach. *J. Web Sem.*, 12:3–21, 2012.

[Özçep and Möller, 2014] Ö. Özçep and R. Möller. Ontology based data access on temporal and streaming data. In *Proc.*

of the 10th Int. Summer School on Reasoning Web (RW 2014), volume 8714 of *LNCS*, pages 279–312. Springer, 2014.

[Özcep *et al.*, 2013] Ö. Özcep, R. Möller, C. Neuenstadt, D. Zheleznyakov, and E. Kharlamov. A semantics for temporal and stream-based query answering in an OBDA context. Technical report, Deliverable D5.1, FP7-318338, EU, 2013.

[Straubing and Weil, 2010] H. Straubing and P. Weil. An introduction to finite automata and their connection to logic. *CoRR*, abs/1011.6491, 2010.

[To, 2009] A. W. To. Unary finite automata vs. arithmetic progressions. *Information Processing Letters*, 109(17):1010–1014, 2009.

[Zhang *et al.*, 2012] Y. Zhang, M.-D. Pham, Ó. Corcho, and J.-P. Calbimonte. SRBench: A streaming RDF/SPARQL benchmark. In *Proc. of the 11th Int. Semantic Web Conf. (ISWC 2012), Part I*, volume 7649 of *LNCS*, pages 641–657. Springer, 2012.