Undecidability of first-order intuitionistic (and modal) logic with two variables

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Classical vs. Intuitionistic logic

Classical (Frege, Hilbert, . . .)

all 'convincing' proofs are permitted

- *tertium non datur*:

  $\vdash \varphi \lor \neg\varphi$

- *reducto ad absurdum*:

  if $\Gamma \cup \{\neg\varphi\} \vdash \bot$ then $\Gamma \vdash \varphi$

Intuitionistic (Brouwer, Heyting, . . .)

allows only 'constructive' proofs

- *disjunction property*:

  $\vdash \varphi \lor \psi$ iff $\vdash \varphi$ or $\vdash \psi$

- *existence property*:

  if $\vdash \exists x \varphi(x)$ then $\vdash \varphi(t)$ for some term t

Intuitionism: "A statement is true if we have a constructive proof of it, and false if we can show that the assumption that there is a proof leads to a contradiction"

$$\textbf{Int} = \textbf{Cl} - \{\neg\neg\varphi \to \varphi, \quad \varphi \lor \neg\varphi, \quad \neg\forall x\, \neg\varphi(x) \to \exists x\, \varphi(x), \quad \ldots\}$$

Intuitionistic logic: Kripke semantics

Intuitionistic Kripke model $\mathfrak{M} = (\mathfrak{F}, \Delta, \delta, I)$:

- $\mathfrak{F} = (W, \leq)$
- $\delta(w) \subseteq \Delta$ (the domain at w)
- $I(w) = (\Delta, P^w, Q^w, \dots)$

if $u \leq v$ then $\delta(u) \subseteq \delta(v)$ and $P^u \subseteq P^v$

- $w \models^{\mathfrak{a}} P(x_1, \dots, x_n)$ iff $P^w(\mathfrak{a}(x_1), \dots, \mathfrak{a}(x_n))$
- $w \not\models^{\mathfrak{a}} \bot$
- $w \models^{\mathfrak{a}} \varphi \vee \psi$ iff $w \models^{\mathfrak{a}} \varphi$ or $w \models^{\mathfrak{a}} \psi$
- $w \models^{\mathfrak{a}} \varphi \wedge \psi$ iff $w \models^{\mathfrak{a}} \varphi$ and $w \models^{\mathfrak{a}} \psi$
- $w \models^{\mathfrak{a}} \exists x\, \varphi$ iff $w \models^{\mathfrak{b}} \varphi$ for some $\mathfrak{b}$ such that $\mathfrak{b} \sim_x \mathfrak{a}$ and $\mathfrak{b}(x) \in \delta(w)$
- $w \models^{\mathfrak{a}} \varphi \to \psi$ iff $v \models^{\mathfrak{a}} \varphi$ implies $v \models^{\mathfrak{a}} \psi$ **for all** $v \geq w$
- $w \models^{\mathfrak{a}} \forall x\, \varphi$ iff $v \models^{\mathfrak{b}} \varphi$ **for all** $v \geq w$ and all $\mathfrak{b}$ such that $\mathfrak{b} \sim_x \mathfrak{a}$ and $\mathfrak{b}(x) \in \delta(v)$

NB. All connectives and quantifiers are **independent** (note that $\neg\varphi = \varphi \to \bot$)

Das Entscheidungsproblem

Classical

Cl(n) — logic with n variables

Cl(1) is decidable (*Wajsberg 1933*)

Cl(1) = **S5**

Cl(2) is decidable (*Scott 1962, Mortimer 1975*)

Cl(3) is **undecidable** (*Syrány 1943*)

Intuitionistic

Int(n) — logic with n variables

Int(1) is decidable (*Bull 1966, Mints 1968, Ono 1977*)

Int(1) = **MIPC**

Int(2) $+ \ \forall x(P(x) \vee q) \to \forall x\, P(x) \vee q$ **(constant domains)**

is **undecidable**

(*Gabbay & Shehtman 1993*)

Int(3) is undecidable

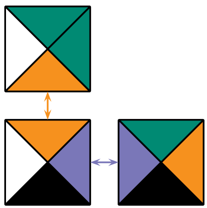
Cl is reducible to **Int** 'by prefixing ¬¬' (*Gödel 1933, etc.*)

Open problem: **Is Int(2) decidable?**

$\mathbb{N} \times \mathbb{N}$ tiling problem

Given a finite set T of tile types $t = (\mathit{left}(t), \mathit{right}(t), \mathit{up}(t), \mathit{down}(t))$

decide whether there exists $\tau : \mathbb{N} \times \mathbb{N} \longrightarrow T$ such that, for all $i, j \in \mathbb{N}$,

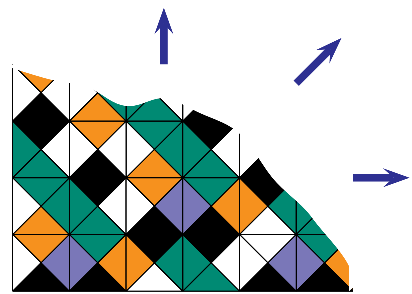


$$\mathit{up}(\tau(i,j)) = \mathit{down}(\tau(i, j+1))$$

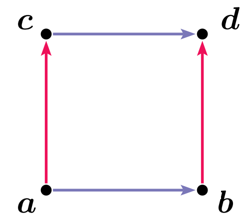
and

$$\mathit{left}(\tau(i,j)) = \mathit{right}(\tau(i+1, j)).$$

(Berger 1966): The $\mathbb{N} \times \mathbb{N}$ tiling problem is **undecidable**.



Encoding the $\mathbb{N} \times \mathbb{N}$ tiling problem in Cl(3)



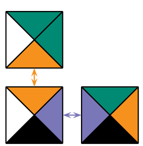
(γ): generating an $\mathbb{N} \times \mathbb{N}$-like grid

$$\forall x \exists y\ \mathit{succ}_H(x,y) \quad \wedge \quad \forall x \exists y\ \mathit{succ}_V(x,y)$$

$$\forall x \forall y \forall z\ \Big(\big(\mathit{succ}_H(x,y)\ \wedge\ \mathit{succ}_V(x,z)\big) \quad \rightarrow \quad \exists x \big(\mathit{succ}_V(y,x)\ \wedge\ \mathit{succ}_H(z,x)\big)\Big)$$

(Church-Rosser axiom)

(ϑ): encoding tiling rules



$$\forall x \bigvee_{t \in T} \Big(P_t(x) \wedge \bigwedge_{t' \neq t} \big(P_{t'}(x) \rightarrow \bot\big)\Big)$$

$$\bigwedge_{\mathit{right}(t) \neq \mathit{left}(t')} \forall x \forall y \big(\mathit{succ}_H(x,y) \wedge P_t(x) \wedge P_{t'}(y) \rightarrow \bot\big)$$

$$\bigwedge_{\mathit{up}(t) \neq \mathit{down}(t')} \forall x \forall y \big(\mathit{succ}_V(x,y) \wedge P_t(x) \wedge P_{t'}(y) \rightarrow \bot\big)$$

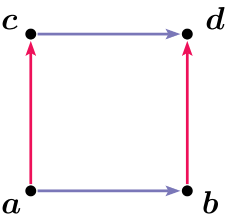
Theorem. $(\vartheta \wedge \gamma)$ is classically **satisfiable** iff T **tiles** $\mathbb{N} \times \mathbb{N}$

Generating an $\mathbb{N} \times \mathbb{N}$-like grid: revised

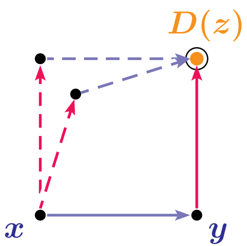
$$\forall x \exists y\ \mathit{succ}_H(x,y) \quad \wedge \quad \forall x \exists y\ \mathit{succ}_V(x,y)$$

$$\forall x \forall y \forall z\ \Big(\big(\mathit{succ}_H(x,y)\ \wedge\ \mathit{succ}_V(x,z)\big) \quad \rightarrow \quad \exists x \big(\mathit{succ}_V(y,x)\ \wedge\ \mathit{succ}_H(z,x)\big)\Big)$$

(Church-Rosser axiom)



Let a unary predicate D be true at point z:



$$\Big(\mathit{succ}_H(x,y)\ \wedge\ \exists z\ \big(D(z)\ \wedge\ \mathit{succ}_V(y,z)\big) \quad \rightarrow \quad \forall y\ \big(\mathit{succ}_V(x,y)\ \rightarrow\ \forall z\ \big(D(z)\ \rightarrow\ \mathit{succ}_H(y,z)\big)\big)\Big)$$

Encoding the $\mathbb{N} \times \mathbb{N}$ tiling problem in Int(2)

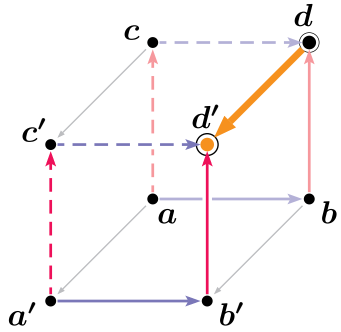
Theorem.

$$(\vartheta \wedge \gamma') \;\to\; \exists x\,(D(x) \to \bot)$$

does **not belong** to **Int** iff T **tiles** $\mathbb{N} \times \mathbb{N}$

(γ'): **generating an $\mathbb{N} \times \mathbb{N}$-like grid**

$$\forall x \exists y\; \mathit{succ}_H(x,y) \quad \wedge \quad \forall x \exists y\; \mathit{succ}_V(x,y)$$



$$\forall x \forall y\; \big(\mathit{succ}_H(x,y) \vee (\mathit{succ}_H(x,y) \to \bot)\big)$$

$$\forall x \forall y\; \Big(\mathit{succ}_H(x,y) \;\wedge\; \exists x\,\big(D(x) \;\wedge\; \mathit{succ}_V(y,x)\big) \;\to\; \forall y\,\big(\mathit{succ}_V(x,y) \;\to\; \forall x\,(D(x) \to \mathit{succ}_H(y,x))\big)\Big)$$

Quantified modal logics with expanding domains

Theorem. Let L be any propositional modal logic having a Kripke frame that contains a point with infinitely many successors.

Then the **two-variable** fragment of $\mathbf{Q}^e L$ is **undecidable**.

Using the Kripke trick:

$$R(x, y) \quad \rightsquigarrow \quad \Diamond\big(P(x) \wedge Q(y)\big)$$

Theorem. For almost all standard propositional modal logics L, the **monadic two-variable** fragment of $\mathbf{Q}^e L$ is **undecidable**.

Open problems

1. Is the **monadic two-variable** fragment of **Int** decidable?

   The monadic fragment with arbitrarily many variables is undecidable
   (Mints et al. 1965, Gabbay 1981, etc.)

2. What is the complexity of $\mathbf{Int}(1) = \mathbf{MIPC}$?

Conjectures

3. The two-variable fragment of the quantified extension of a propositional superintuitionistic or modal logic L is decidable iff L is tabular.

4. All logics of the form $(L_1 \times (L_2 \times L_3))^{\text{ex}}$, where L_1, L_2 and L_3 are any Kripke complete propositional modal logics between **K** and **S5**, are undecidable