On dynamic topological logics

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joint work with

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The Story

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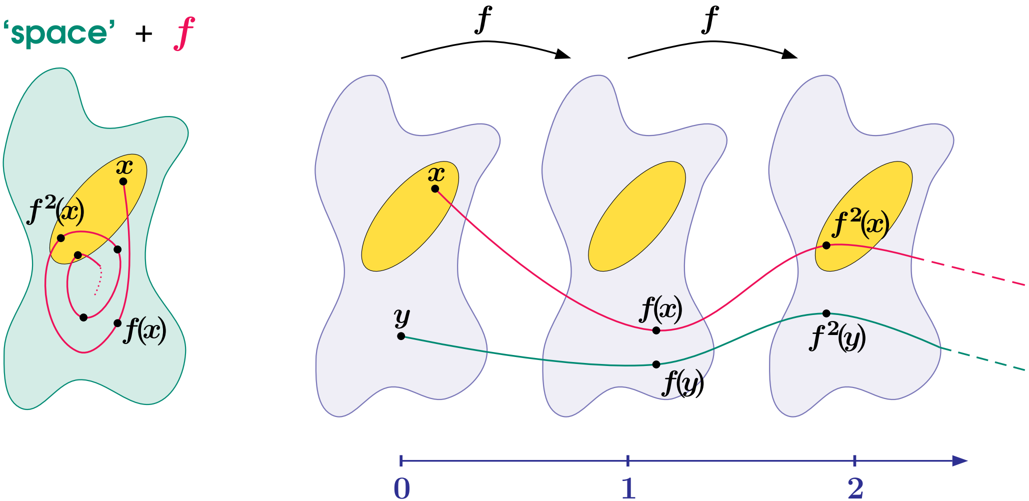
Proceedings of AiML 2004, pp. 182–196, Manchester, U.K., September 2004

Dynamical systems



$$\textit{Orb}_f(x) = \{ f(x), f^2(x), \ldots \}$$ — the **orbit** of x

Temporal logic × logic of topology
to describe and reason about the (asymptotic) behaviour of orbits

Dynamic topological structures

Dynamic topological structure $\mathfrak{F} = \langle \mathfrak{T}, f \rangle$

$\mathfrak{T} = \langle T, \mathbb{I} \rangle$ a **topological space**

T is the universe of $\mathfrak{T}$
$\mathbb{I}$ is the interior operator on $\mathfrak{T}$
$\mathbb{C}$ is the closure operator on $\mathfrak{T}$
($\mathbb{C}X = -\mathbb{I} - X$)

- arbitrary topologies
- Aleksandrov: **arbitrary** (not only finite) intersections of open sets are open
  — every Kripke frame $\mathfrak{G} = \langle U, R \rangle$, where R is a **quasi-order**, induces the Aleksandrov topological space $\langle U, \mathbb{I}_{\mathfrak{G}} \rangle$:
  $$\mathbb{I}_{\mathfrak{G}} X = \{x \in U \mid \forall y \, (xRy \rightarrow y \in X)\}$$
  — conversely, every Aleksandrov space is induced by a quasi-order
- Euclidean spaces $\mathbb{R}^n$, $n \geq 1$
- ...

$f : T \longrightarrow T$ a **continuous function** (X open $\Rightarrow$ $f^{-1}(X)$ open)

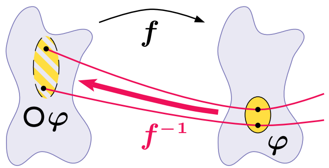
- continuous
- homeomorphisms (continuous bijections with continuous inverses)

Dynamic topological logic $\mathcal{DTL}$

Formulas: $\mathfrak{V}$ a **valuation** in $\langle\langle T, \mathbb{I}\rangle, f\rangle$

- propositional variables $p, q, \ldots$ — subsets of T
- the Booleans $\neg$, $\wedge$ and $\vee$ — $-$, $\cap$ and $\cup$
- topological ('modal') operators **I** and **C** — $\mathbb{I}$ and $\mathbb{C}$
- temporal operators $\bigcirc$, $\Box_F$ and $\Diamond_F$ — $\mathfrak{V}(\bigcirc\varphi) = f^{-1}(\mathfrak{V}(\varphi))$

$$\mathfrak{V}(\Box_F\varphi) = \bigcap_{n=1}^{\infty} f^{-n}(\mathfrak{V}(\varphi)) = \{x \in T \mid \mathit{Orb}_f(x) \subseteq \mathfrak{V}(\varphi)\}$$



$$\mathfrak{V}(\Diamond_F\varphi) = \bigcup_{n=1}^{\infty} f^{-n}(\mathfrak{V}(\varphi)) = \{x \in T \mid \mathit{Orb}_f(x) \cap \mathfrak{V}(\varphi) \neq \emptyset\}$$

Example: every ψ satisfies φ infinitely often $\qquad \psi \longrightarrow \Box_F\Diamond_F\varphi$

Known results: no 'infinite' operations

$\mathcal{DTL}_{\bigcirc}$ — subset of $\mathcal{DTL}$ containing **no 'infinite'** operators ($\Box_F$ and $\Diamond_F$)

Artemov, Davoren & Nerode (1997): The two dynamic topo-logics

$$\mathrm{Log}_{\bigcirc}\{\langle \mathfrak{F}, f\rangle\} \qquad \text{and} \qquad \mathrm{Log}_{\bigcirc}\{\langle \mathfrak{F}, f\rangle \mid \mathfrak{F} \text{ an Aleksandrov space}\}$$

coincide, have the **fmp**, are finitely **axiomatisable**, and so decidable

NB. $\mathrm{Log}_{\bigcirc}\{\langle \mathfrak{F}, f\rangle\} \subsetneqq \mathrm{Log}_{\bigcirc}\{\langle \mathbb{R}, f\rangle\}$ (Slavnov 2003, Kremer & Mints 2003)

Kremer, Mints & Rybakov (1997): The three dynamic topo-logics

$\mathrm{Log}_{\bigcirc}\{\langle \mathfrak{F}, f\rangle \mid f \text{ a homeomorphism}\}$,

$\mathrm{Log}_{\bigcirc}\{\langle \mathfrak{F}, f\rangle \mid \mathfrak{F} \text{ an Aleksandrov space}, \ f \text{ a homeomorphism}\}$,

$\mathrm{Log}_{\bigcirc}\{\langle \mathbb{R}^n, f\rangle \mid f \text{ a homeomorphism}\}$, $n \geq 1$,

coincide, have the **fmp**, are finitely **axiomatisable**, and so decidable

Homeomorphisms vs. continuous mappings

$\mathfrak{T} = \langle U, \mathbb{I} \rangle$ is the Aleksandrov space induced by a quasi-order $\mathfrak{G} = \langle U, R \rangle$

f is a **homeomorphism** iff

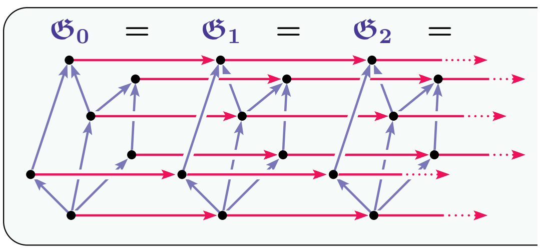
$$xRy \quad \Leftrightarrow \quad f(x)Rf(y)$$

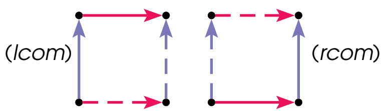
f is **continuous** iff

$$xRy \quad \Rightarrow \quad f(x)Rf(y)$$

a DTM can be **unwound** into

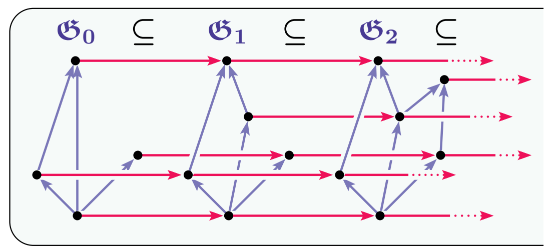
a product model

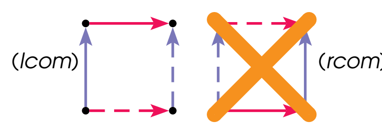




$\mathbf{S4} \oplus \mathbf{DAlt} \oplus (\bigcirc \mathsf{I} p \leftrightarrow \mathsf{I} \bigcirc p)$

an e-product model





$\mathbf{S4} \oplus \mathbf{DAlt} \oplus (\bigcirc \mathsf{I} p \rightarrow \mathsf{I} \bigcirc p)$

DTLs with homeomorphisms

Theorem 1 (AiML 2004). **No logic** from the list below is **recursively enumerable**:

- Log $\{\langle \mathfrak{F}, f\rangle \mid f$ a homeomorphism$\}$,
- Log $\{\langle \mathfrak{F}, f\rangle \mid \mathfrak{F}$ an Aleksandrov space, f a homeomorphism$\}$,
- Log $\{\langle \mathbb{R}^n, f\rangle \mid f$ a homeomorphism$\}$, $n \geq 1$.

Proof. By reduction of the undecidable but r.e. Post's Correspondence Problem to the satisfiability problem

(more on the next slide)

NB. All these logics are **different**.

Encoding PCP

PCP: given a set of pairs $\{(u_1, v_1), \dots, (u_k, v_k)\}$ of nonempty finite words, decide whether there exists an $N \geq 1$ and a sequence $i_1, \dots, i_N$ such that

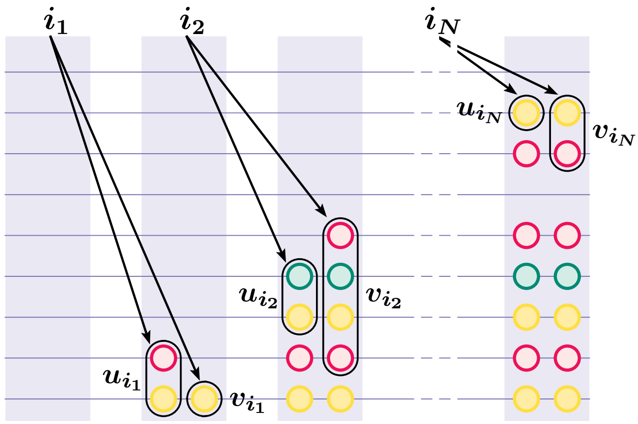
$$u_{i_1} \cdot u_{i_2} \cdot \dots \cdot u_{i_N} = v_{i_1} \cdot v_{i_2} \cdot \dots \cdot v_{i_N}$$

Post (1946):
The PCP is undecidable and the set of PCP instances without solutions is not R.E.

- Aleksandrov space $\langle U, \mathbb{I} \rangle$
  (induced by $\langle U, R \rangle$)

'local' formulas $\Box_F^+ \mathsf{I}(\psi_1 \to \bigcirc \psi_2)$

plus $\Diamond_F \bigwedge_{a \in A} \mathsf{I}(L_a \leftrightarrow R_a)$



- arbitrary topological spaces and $\mathbb{R}^n$:
  the formula requires only a **finite** number of iterations
  and thus the completeness results for $\mathrm{Log}_{\bigcirc}\{\cdots\}$ can be used

DTLs with continuous mappings

Theorem 2. **No logic** from the list below is **decidable**:

- Log $\{\langle \mathfrak{F}, f \rangle\}$,
- Log $\{\langle \mathfrak{F}, f \rangle \mid \mathfrak{F}$ an Aleksandrov space$\}$,
- Log $\{\langle \mathbb{R}^n, f \rangle\}$, $n \geq 1$.

Proof. By reduction of
the undecidable ω-reachability problem for lossy channels
to the satisfiability problem

(more on the next slide)

NB. All these logics are **different**.

Encoding lossy channels backwards

Single channel system

$$S = \langle Q, \Sigma, \Delta \rangle$$

Q — a set of *control states*
Σ — an alphabet of *messages*
$\Delta \subseteq Q \times \{?, !\} \times \Sigma \times Q$ — a set of *transitions*

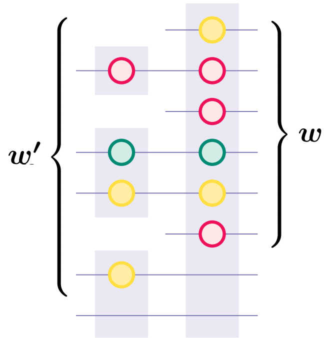
send

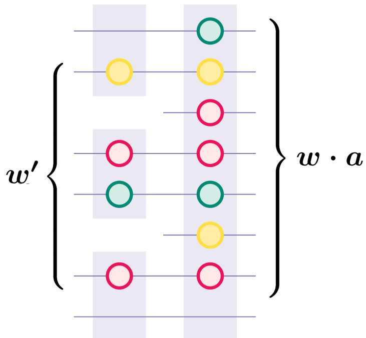
$$\langle q, w \rangle \xrightarrow{\langle q,!,a,q' \rangle}_\ell \langle q', w' \rangle \quad \text{iff} \quad w' \sqsubseteq a \cdot w$$

receive

$$\langle q, w \cdot a \rangle \xrightarrow{\langle q,?,a,q' \rangle}_\ell \langle q', w' \rangle \quad \text{iff} \quad w' \sqsubseteq w$$

backward encoding: loss of messages = introduction of new points





Encoding lossy channels: ω-reachability (1)

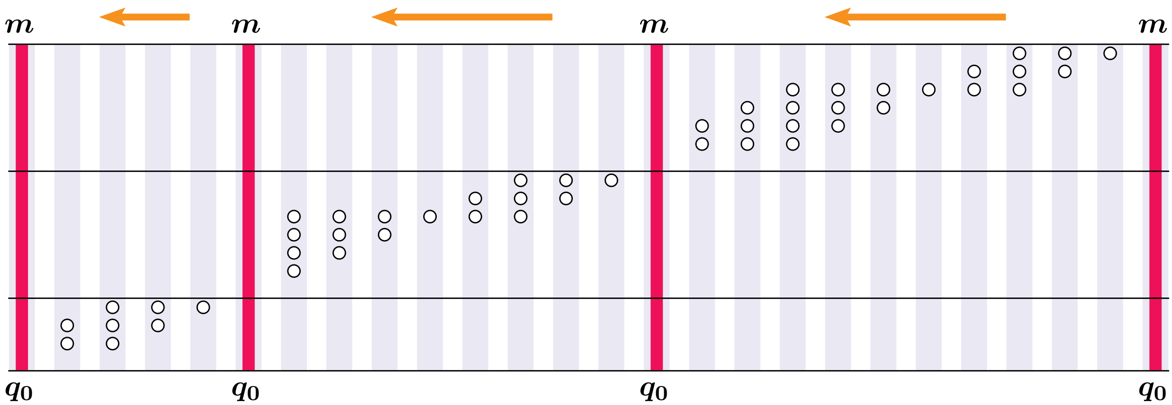
ω-reachability:

given a single channel lossy system S and two states q_0 and q_{rec}, decide whether, **for every $n > 0$**, there is a computation

$$\langle q_0, \epsilon \rangle \xrightarrow{\delta_1}_\ell \langle q_{i_1}, w_1 \rangle \xrightarrow{\delta_2}_\ell \langle q_{i_2}, w_2 \rangle \xrightarrow{\delta_3}_\ell \dots$$

reaching q_{rec} **at least n times**

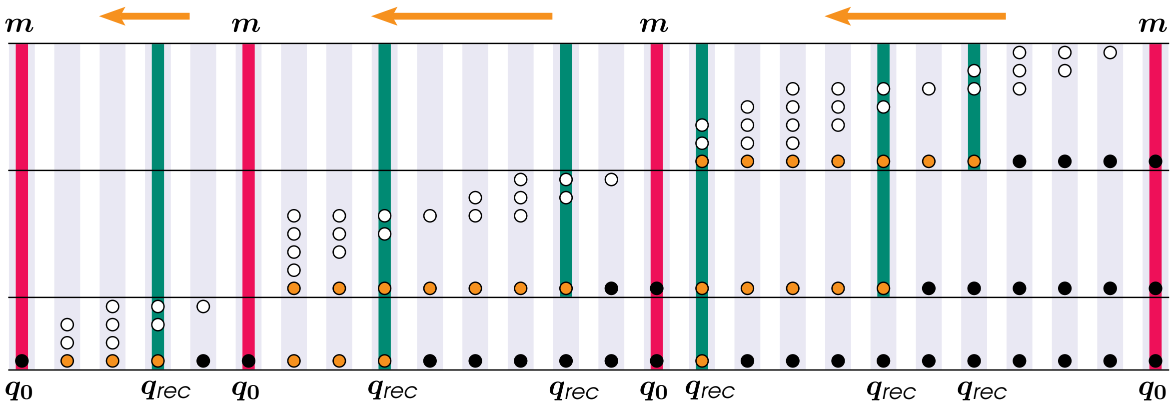
Schnoebelen (2004): ω-reachability is undecidable



The ω-reachability problem can be encoded

using only 'local' formulas $\Box_F^+ \mathsf{I}(\psi_1 \to \bigcirc \psi_2)$ plus $\Box_F \Diamond_F m$ plus...

Encoding lossy channels: ω-reachability (2)



$$light \wedge \Box^+_F(light \longrightarrow \mathsf{O}\, light)$$

$$\Box^+_F(m \longrightarrow \mathsf{O}\,\mathsf{I}(light \longrightarrow on))$$

$$\Box^+_F(\mathsf{C}(light \wedge on \wedge \mathsf{O}\neg on) \longrightarrow q_{rec})$$

$$\Box^+_F(m \longrightarrow \mathsf{I}(light \longrightarrow \neg on))$$

$$\Box_F(m \longrightarrow \mathsf{I}(light \longrightarrow \mathsf{O}\,\mathsf{S}\, light))$$

$$\Box^+_F\,\mathsf{I}((light \wedge on \wedge \mathsf{O}\neg on) \longrightarrow \neg\,\mathsf{S}(light \wedge on \wedge \mathsf{O}\neg on))$$

Finite iterations

- arbitrary finite flows of time
- finite change assumption (the system eventually stabilises)

Theorem 3 (APAL 2006). The two topo-logics

$$\mathrm{Log}_{\mathrm{fin}}\ \{\langle \mathfrak{F}, f\rangle\} \qquad \text{and} \qquad \mathrm{Log}_{\mathrm{fin}}\ \{\langle \mathfrak{F}, f\rangle \mid \mathfrak{F} \text{ an Aleksandrov space}\}$$

coincide and are **decidable**, but **not in primitive recursive** time

Proof. By Kruskal's tree theorem and
reduction of the reachability problem for lossy channels
(*decidable but not in primitive recursive time*)

<u>However:</u>

Theorem 4 (AiML 2004). The two topo-logics

$$\mathrm{Log}_{\mathrm{fin}}\ \{\langle \mathfrak{F}, f\rangle \mid f \text{ a homeomorphism}\} \qquad \text{and}$$

$$\mathrm{Log}_{\mathrm{fin}}\ \{\langle \mathfrak{F}, f\rangle \mid \mathfrak{F} \text{ an Aleksandrov space}, \ \ f \text{ a homeomorphism}\}$$

coincide but are **not recursively enumerable**

Open problems

- Axiomatisation of DTL over Euclidean spaces (without $\Box_{\boldsymbol{F}}$, $\Diamond_{\boldsymbol{F}}$)
- Are full DTLs with continuous mappings r.e.?
- If so, are they finitely axiomatisable? Axiomatisations?
- ...

Publications (all available on the web)

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