On dynamic topological logics

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joint work with

Boris Konev, Frank Wolter and Michael Zakharyaschev
The Story

S. Artemov, J. Davoren and A. Nerode.  
*Topological semantics for hybrid systems.*  

J. Davoren. *Modal logics for continuous dynamics.*  
Ph.D. Thesis, Department of Mathematics, Cornell University, 1998


*Bimodal logics for reasoning about continuous dynamics.*  
Advances in Modal Logic, Volume 3, pp. 91–110. World Scientific, 2002

*On dynamic topological and metric logics.*  
Dynamical systems

‘space’ + $f$

Orb$_f$(x) = { $f(x)$, $f^2(x)$, ... } — the orbit of x

Temporal logic $\times$ logic of topology

to describe and reason about the (asymptotic) behaviour of orbits
Dynamic topological structures

Dynamic topological structure \( \mathcal{F} = \langle \mathcal{T}, f \rangle \)

\[ \mathcal{T} = \langle T, I \rangle \quad \text{a topological space} \]

- arbitrary topologies
- Aleksandrov: arbitrary (not only finite) intersections of open sets are open — every Kripke frame \( \mathcal{G} = \langle U, R \rangle \), where \( R \) is a quasi-order, induces the Aleksandrov topological space \( \langle U, I_\mathcal{G} \rangle \):
  \[ I_\mathcal{G} X = \{ x \in U \mid \forall y (xRy \implies y \in X) \} \]
  — conversely, every Aleksandrov space is induced by a quasi-order

- Euclidean spaces \( \mathbb{R}^n, n \geq 1 \)
- \( \ldots \)

\[ f : T \rightarrow T \quad \text{a continuous function} \quad (X \text{ open } \implies f^{-1}(X) \text{ open}) \]

- continuous
- homeomorphisms (continuous bijections with continuous inverses)
Dynamic topological logic $\text{DTL}$

Formulas:

- propositional variables $p, q, \ldots$
- the Booleans $\neg, \land$ and $\lor$
- topological (‘modal’) operators $\mathbb{I}$ and $\mathbb{C}$
- temporal operators $\mathbb{O}$, $\Box_F$ and $\Diamond_F$

$\mathcal{V}$ a valuation in $\langle \langle T, \mathbb{I} \rangle, f \rangle$

subsets of $T$
$\neg, \land$ and $\lor$
$\mathbb{I}$ and $\mathbb{C}$

$\mathcal{V}(\mathbb{O}\varphi) = f^{-1}(\mathcal{V}(\varphi))$

$\mathcal{V}(\Box_F \varphi) = \bigcap_{n=1}^{\infty} f^{-n}(\mathcal{V}(\varphi)) = \{ x \in T \mid \text{Orb}_f(x) \subseteq \mathcal{V}(\varphi) \}$

$\mathcal{V}(\Diamond_F \varphi) = \bigcup_{n=1}^{\infty} f^{-n}(\mathcal{V}(\varphi)) = \{ x \in T \mid \text{Orb}_f(x) \cap \mathcal{V}(\varphi) \neq \emptyset \}$

Example: every $\psi$ satisfies $\varphi$ infinitely often

$\psi \rightarrow \Box_F \Diamond_F \varphi$

Oxford 05/08/07
Known results: no ‘infinite’ operations

\( DT\mathcal{L}_0 \) — subset of \( DT\mathcal{L} \) containing no ‘infinite’ operators (\( \Box_F \) and \( \diamond_F \))

**Artemov, Davoren & Nerode (1997):** The two dynamic topo-logics

\[ \text{Log}_o \{ \langle F, f \rangle \} \quad \text{and} \quad \text{Log}_o \{ \langle F, f \rangle \mid F \text{ an Aleksandrov space} \} \]

coincide, have the \textbf{fmp}, are finitely \textbf{axiomatisable}, and so decidable

**NB.** \( \text{Log}_o \{ \langle F, f \rangle \} \subsetneq \text{Log}_o \{ \langle \mathbb{R}, f \rangle \} \) (Slavnov 2003, Kremer & Mints 2003)

**Kremer, Mints & Rybakov (1997):** The three dynamic topo-logics

\[ \text{Log}_o \{ \langle F, f \rangle \mid f \text{ a homeomorphism} \} , \]
\[ \text{Log}_o \{ \langle F, f \rangle \mid F \text{ an Aleksandrov space, } f \text{ a homeomorphism} \} , \]
\[ \text{Log}_o \{ \langle \mathbb{R}^n, f \rangle \mid f \text{ a homeomorphism} \} , \quad n \geq 1 , \]

coincide, have the \textbf{fmp}, are finitely \textbf{axiomatisable}, and so decidable
Homeomorphisms vs. continuous mappings

\( \mathcal{T} = \langle U, I \rangle \) is the Aleksandrov space induced by a quasi-order \( \mathcal{G} = \langle U, R \rangle \)

\[ f \text{ is a } \text{homeomorphism} \iff x \mathcal{R} y \iff f(x) \mathcal{R} f(y) \]

\[ f \text{ is continuous} \iff x \mathcal{R} y \Rightarrow f(x) \mathcal{R} f(y) \]

a DTM can be \text{unwound} into

\[ \text{a product model} \]

\[ \mathcal{G}_0 = \mathcal{G}_1 = \mathcal{G}_2 = \]

\[ \text{an } \text{e-product model} \]

\[ \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \]

\[ \text{(lcom)} \quad \text{(rcom)} \]

\[ \text{S4} \oplus \text{DAlt} \oplus (\mathcal{O} \mathcal{I} p \leftrightarrow \mathcal{I} \mathcal{O} p) \]

\[ \text{S4} \oplus \text{DAlt} \oplus (\mathcal{O} \mathcal{I} p \rightarrow \mathcal{I} \mathcal{O} p) \]

Oxford 05/08/07
DTLs with homeomorphisms

**Theorem 1 (AiML 2004).** No logic from the list below is recursively enumerable:

- \( \text{Log} \left\{ \langle S, f \rangle \mid f \text{ a homeomorphism} \right\} \),
- \( \text{Log} \left\{ \langle S, f \rangle \mid S \text{ an Aleksandrov space, } f \text{ a homeomorphism} \right\} \),
- \( \text{Log} \left\{ \langle \mathbb{R}^n, f \rangle \mid f \text{ a homeomorphism} \right\}, \ n \geq 1 \).

**Proof.** By reduction of the undecidable but r.e. Post’s Correspondence Problem to the satisfiability problem (more on the next slide)

**NB.** All these logics are different.
Encoding PCP

**PCP:** given a set of pairs \( \{(u_1, v_1), \ldots, (u_k, v_k)\} \) of nonempty finite words, decide whether there exists an \( N \geq 1 \) and a sequence \( i_1, \ldots, i_N \) such that

\[
u_{i_1} \cdot u_{i_2} \cdots \cdot u_{i_N} = v_{i_1} \cdot v_{i_2} \cdots \cdot v_{i_N}\]

**Post (1946):**
The PCP is undecidable and the set of PCP instances without solutions is not R.E.

- Aleksandrov space \( \langle U, \mathcal{I}\rangle \) (induced by \( \langle U, R\rangle \))

- ‘local’ formulas
  \[
  \square_{\mathcal{I}}^+ I(\psi_1 \rightarrow O \psi_2)\]

- plus
  \[
  \Diamond_{\mathcal{I}} \bigwedge_{a \in A} I(L_a \leftrightarrow R_a)\]

- arbitrary topological spaces and \( \mathbb{R}^n \):
  the formula requires only a finite number of iterations
  and thus the completeness results for \( \text{Log}_\circ \{ \cdots \} \) can be used
DTLs with continuous mappings

**Theorem 2.** No logic from the list below is **decidable**:

- \( \text{Log} \{ \langle \mathcal{F}, f \rangle \} \),
- \( \text{Log} \{ \langle \mathcal{F}, f \rangle \mid \mathcal{F} \text{ an Aleksandrov space} \} \),
- \( \text{Log} \{ \langle \mathbb{R}^n, f \rangle \}, \ n \geq 1 \).

**Proof.** By reduction of the undecidable \( \omega \)-reachability problem for lossy channels to the satisfiability problem (more on the next slide)

**NB.** All these logics are **different**.
Encoding lossy channels backwards

Single channel system

\[ S = \langle Q, \Sigma, \Delta \rangle \]

send

\[ \langle q, w \rangle \xrightarrow{\langle q', w' \rangle} \ell \langle q', w' \rangle \]

iff \( w' \subseteq a \cdot w \)

receive

\[ \langle q, w \cdot a \rangle \xrightarrow{\langle q', w' \rangle} \ell \langle q', w' \rangle \]

iff \( w' \subseteq w \)

backward encoding: loss of messages = introduction of new points
Encoding lossy channels: $\omega$-reachability (1)

$\omega$-reachability:

given a single channel lossy system $S$ and two states $q_0$ and $q_{rec}$,

decide whether, for every $n > 0$, there is a computation

$$
\langle q_0, \epsilon \rangle \xrightarrow{\delta_1} \ell \langle q_{i_1}, w_1 \rangle \xrightarrow{\delta_2} \ell \langle q_{i_2}, w_2 \rangle \xrightarrow{\delta_3} \ell \ldots
$$

to reach $q_{rec}$ at least $n$ times

Schnoebelen (2004): $\omega$-reachability is undecidable

The $\omega$-reachability problem can be encoded

using only ‘local’ formulas $\square_F^+ I(\psi_1 \rightarrow O \psi_2)$ plus $\square_F \diamond_F m$ plus...
Encoding lossy channels: $\omega$-reachability  (2)

\[ light \land \square^+_{\mathcal{F}}(\text{light} \rightarrow \bigcirc \text{light}) \]
\[ \square^+_{\mathcal{F}}(m \rightarrow \bigcirc \mathcal{I}(\text{light} \rightarrow \text{on})) \]
\[ \square^+_{\mathcal{F}}(\mathcal{C}(\text{light} \land \text{on} \land \bigcirc \neg \text{on}) \rightarrow q_{\text{rec}}) \]
\[ \square^+_{\mathcal{F}}(m \rightarrow \mathcal{I}(\text{light} \rightarrow \neg \text{on})) \]
\[ \square_{\mathcal{F}}(m \rightarrow \mathcal{I}(\text{light} \rightarrow \bigcirc \mathcal{S} \text{light})) \]
\[ \square^+_{\mathcal{F}}\mathcal{I}(\text{(light} \land \text{on} \land \bigcirc \neg \text{on}) \rightarrow \neg \mathcal{S}(\text{light} \land \text{on} \land \bigcirc \neg \text{on})) \]
Finite iterations

- arbitrary finite flows of time
- finite change assumption (the system eventually stabilises)

**Theorem 3 (APAL 2006).** The two topo-logics

\[
\text{Log}_{\text{fin}} \{\langle \mathcal{F}, f \rangle \} \quad \text{and} \quad \text{Log}_{\text{fin}} \{\langle \mathcal{F}, f \rangle \mid \mathcal{F} \text{ an Aleksandrov space} \}
\]

coincide and are **decidable**, but **not in primitive recursive** time

**Proof.** By Kruskal’s tree theorem and

reduction of the reachability problem for lossy channels

*(decidable but not in primitive recursive time)*

**However:**

**Theorem 4 (AiML 2004).** The two topo-logics

\[
\text{Log}_{\text{fin}} \{\langle \mathcal{F}, f \rangle \mid f \text{ a homeomorphism} \} \quad \text{and} \quad \text{Log}_{\text{fin}} \{\langle \mathcal{F}, f \rangle \mid \mathcal{F} \text{ an Aleksandrov space, } f \text{ a homeomorphism} \}
\]

coincide but are **not recursively enumerable**
Open problems

- Axiomatisation of DTL over Euclidean spaces (without $\square_F, \Diamond_F$)
- Are full DTLs with continuous mappings r.e.?
- If so, are they finitely axiomatisable? Axiomatisations?
- ...
Publications (all available on the web)

Dynamic topological logics over spaces with continuous functions

Non-primitive recursive decidability of products of modal logics with expanding domains

On dynamic topological and metric logics
Studia Logica, 84:127–158, 2006

4) B. Konev, F. Wolter and M. Zakharyaschev.
Temporal logics over transitive states

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