

On the computational complexity of spatial logics with connectedness constraints

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joint work with

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Motivation

Connectedness

- is one of the most **fundamental** concepts of topology
(any textbook in the field contains a substantial chapter on connectedness)
- in spatial KR&R, the distinction between
connected and *disconnected* regions
is recognized as **indispensable** for various modelling and representation tasks

So far only sporadic attempts have been made to investigate the computational complexity of spatial logics with connectedness constraints

$\mathcal{S}4_u$: syntax and semantics

terms: **subsets of T**

$$\tau ::= v_i \mid \bar{\tau} \mid \tau_1 \cap \tau_2 \mid \tau^\circ \mid \tau^-$$

complement interior closure

formulas: **true** or **false**

$$\varphi ::= \tau_1 = \tau_2 \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2$$

e.g., $\mathfrak{M} \models \tau_1 = \tau_2$ iff $\tau_1^{\mathfrak{M}} = \tau_2^{\mathfrak{M}}$

topological model $\mathfrak{M} = (T, \cdot^{\mathfrak{M}})$
 T a topological space
 $\cdot^{\mathfrak{M}}$ a valuation

NB. This definition is as expressive as the 'standard' one

A space is called **Aleksandrov** if *arbitrary* intersections of open sets are open

Aleksandrov spaces = Kripke frames $F = (W, R)$, R is a quasi-order on W

(Shehtman 99, Areces et. al 00): $\text{Sat}(\mathcal{S}4_u, \text{ALL}) = \text{Sat}(\mathcal{S}4_u, \text{ALEK})$,

and this set is **PSPACE**-complete

NB. $\text{Sat}(\mathcal{S}4_u, \text{ALL}) \neq \text{Sat}(\mathcal{S}4_u, \text{CON})$ (in contrast with $\mathcal{S}4$)

Connectedness

A topological space is **connected** iff

it is not the union of two non-empty, disjoint, open sets

Example:

$$(v_1 \neq \mathbf{0}) \wedge (v_2 \neq \mathbf{0}) \wedge (v_1 \cup v_2 = \mathbf{1}) \wedge (v_1^- \cap v_2 = \mathbf{0}) \wedge (v_1 \cap v_2^- = \mathbf{0})$$

is satisfiable in a topological space T iff T is not connected

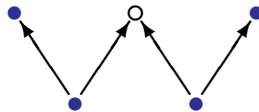
$X \subseteq T$ is **connected in T** just in case either it is empty,

or the topological space X (with the subspace topology) is connected

A maximal connected subset of X is called a **component** of X

An **Aleksandrov space** induced by $F = (W, R)$ is **connected** iff F is connected

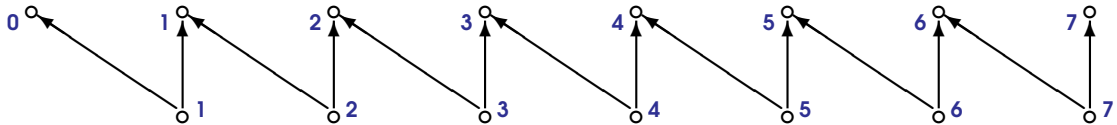
(i.e., between any two points $x, y \in W$ there is a path along the relation $R \cup R^{-1}$)



$\mathcal{S}4_u$ over connected topological spaces

(Shehtman 99): $\mathbf{Sat}(\mathcal{S}4_u, \mathbf{CON}) = \mathbf{Sat}(\mathcal{S}4_u, \mathbf{CONALEK}) = \mathbf{Sat}(\mathcal{S}4_u, \mathbb{R}^n)$, $n \geq 1$,
and this set is **PSPACE**-complete

Example: generating all numbers from $\mathbf{0}$ to $\mathbf{2}^n - \mathbf{1}$:



$\mathbf{0}$ and $\mathbf{2}^n - \mathbf{1}$ are non-empty:

$$\overline{v_n} \cap \dots \cap \overline{v_1} \neq \mathbf{0}, \quad v_n \cap \dots \cap v_1 \neq \mathbf{0}$$

the closure of \mathbf{m} can share points only with $\mathbf{m} + \mathbf{1}$, for $\mathbf{0} \leq \mathbf{m} < \mathbf{2}^n - \mathbf{1}$:

$$\begin{aligned} (v_j \cap \overline{v_k})^- &\subseteq v_j, & (\overline{v_j} \cap \overline{v_k})^- &\subseteq \overline{v_j}, & \text{for } n \geq j > k \geq 1 \\ (\overline{v_k} \cap v_{k-1} \cap \dots \cap v_1)^- &\subseteq (v_k \cap \overline{v_i}) \cup (\overline{v_k} \cap v_i), & \text{for } n \geq k > i \geq 1 \end{aligned}$$

$\mathbf{2}^n - \mathbf{1}$ is a closed set:

$$(v_n \cap \dots \cap v_1)^- \subseteq v_n \cap \dots \cap v_1$$

$S4_u c = S4_u + \text{connectedness predicate (1)}$

$S4_u c$ -formulas: $\varphi ::= \tau_1 = \tau_2 \quad | \quad c(\tau) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2$

$\mathfrak{M} \models c(\tau)$ iff $\tau^{\mathfrak{M}}$ is connected in T

↓ one occurrence of c

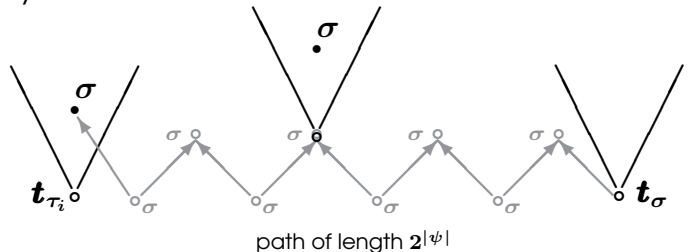
Theorem. $\text{Sat}(S4_u c^1, \text{ALL})$ is **PSPACE**-complete

Proof. Let $\psi = (\tau_0 = \mathbf{0}) \wedge \bigwedge_{i=1}^m (\tau_i \neq \mathbf{0}) \wedge (c(\sigma) \wedge (\sigma \neq \mathbf{0}))$ (conjunct of a full DNF)

1. guess a type (Hintikka set) t_σ containing σ and $\overline{\tau_0}^\circ$
and expand the tableau branch by branch (all points with σ are to be connected to t_σ)

2. for each i , guess a type t_{τ_i} containing τ_i and $\overline{\tau_0}^\circ$
and expand the tableau branch by branch

– if σ appears in the tableau
then we construct a path to t_σ
(by “divide and conquer”)



$\mathcal{S4}_u\mathcal{C} = \mathcal{S4}_u + \text{connectedness predicate (2)}$

Theorem. $\text{Sat}(\mathcal{S4}_u\mathcal{C}, \text{ALL})$ is in **EXPTIME**

Proof. Let $\psi = (\tau_0 = \mathbf{0}) \wedge \bigwedge_{i=1}^m (\tau_i \neq \mathbf{0}) \wedge \bigwedge_{i=1}^k (c(\sigma_i) \wedge (\sigma_i \neq \mathbf{0}))$ (conjunct of a full DNF)

The proof is by reduction to \mathcal{PDL} with converse and nominals [De Giacomo 95]

Let α and β be atomic programs and ℓ_i a nominal, for each σ_i

- the $\mathcal{S4}$ -box is simulated by $[\alpha^*]$:
 τ^\dagger is the result of replacing in τ each sub-term ϑ° with $[\alpha^*]\vartheta$
- the universal box is simulated by $[\gamma]$, where $\gamma = (\beta \cup \beta^- \cup \alpha \cup \alpha^-)^*$

$$\psi' = [\gamma]\neg\tau_0^\dagger \wedge \bigwedge_{i=1}^m \langle \gamma \rangle \tau_i^\dagger \wedge \bigwedge_{i=1}^k \left(\langle \gamma \rangle (\ell_i \wedge \sigma_i^\dagger) \wedge [\gamma](\sigma_i^\dagger \rightarrow \langle (\alpha \cup \alpha^-; \sigma_i^\dagger?)^* \rangle \ell_i) \right)$$

ψ' is satisfiable iff ψ is satisfiable

NB. Matching lower bound to follow. . .

$\mathcal{S4}_{ucc} = \mathcal{S4}_u + \text{component counting predicates}$

$\mathcal{S4}_{ucc}$ -formulas: $\varphi ::= \tau_1 = \tau_2 \mid c^{\leq k}(\tau) \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2$
 $\mathfrak{M} \models c^{\leq k}(\tau)$ iff $\tau^{\mathfrak{M}}$ has at most k components in T

reduction to $\mathcal{S4}_u$: (the v_i are fresh variables) **exponential** if k coded in binary!

- $c^{\leq k}(\tau) \rightarrow (\tau = \bigcup_{1 \leq i \leq k} v_i) \wedge \bigwedge_{1 \leq i \leq k} c(v_i)$
- $\neg c^{\leq k}(\tau) \rightarrow (\tau = \bigcup_{1 \leq i \leq k+1} v_i) \wedge \bigwedge_{1 \leq i \leq k+1} (v_i \neq \mathbf{0}) \wedge \bigwedge_{1 \leq i < j \leq k+1} (\tau \cap v_i^- \cap v_j^- = \mathbf{0})$

(Pratt-Hartmann 02): $\text{Sat}(\mathcal{S4}_{ucc}, \text{ALL}) = \text{Sat}(\mathcal{S4}_{ucc}, \text{ALEK})$; this set is in **NEXPTIME**

Proof. 1. Full $\mathcal{S4}_{ucc}$ is logspace-reducible to

its fragment with no negative occurrences of $c^{\leq k}(\tau)$

2. This fragment of $\mathcal{S4}_{ucc}$ has exponential fmp (by continuous topological filtration)

$\mathcal{S4}_u\mathcal{C}$ in Euclidean spaces

- satisfiable in \mathbb{R}^2 but not in \mathbb{R} :

$$\bigwedge_{1 \leq i \leq 3} c(v_i) \wedge \bigwedge_{1 \leq i < j \leq 3} (v_i \cap v_j \neq \mathbf{0}) \wedge (v_1 \cap v_2 \cap v_3 = \mathbf{0})$$

- satisfiable in \mathbb{R}^3 but not in \mathbb{R}^2 :

$$\bigwedge_{i \in \{j,k\}} (v_i \subseteq e_{j,k}^\circ) \wedge \bigwedge_{1 \leq i \leq 5} (v_i \neq \mathbf{0}) \wedge \bigwedge_{\{i,j\} \cap \{k,l\} = \emptyset} (e_{i,j} \cap e_{k,l} = \mathbf{0}) \wedge \bigwedge_{1 \leq i < j \leq 5} c(e_{i,j}^\circ)$$

- satisfiable in connected spaces but not in \mathbb{R}^n , for any $n \geq 1$:

$$(v_1 \cap v_2 = \mathbf{0}) \wedge \bigwedge_{i=1,2} ((v_i^- \subseteq v_i) \wedge c(\bar{v}_i)) \wedge \neg c(\bar{v}_1 \cap \bar{v}_2)$$

Theorem. $\text{Sat}(\mathcal{S4}_u\mathcal{CC}, \mathbb{R})$ is **PSPACE**-complete

Proof. Encoding in temporal logic with \mathcal{S} and \mathcal{U} over $(\mathbb{R}, <)$

Regular closed sets and \mathcal{B}

$X \subseteq T$ is **regular closed** if $X = X^{\circ-}$

$\mathbf{RC}(T)$ regular closed subsets of T

$\mathbf{RC}(T) =$ sets of the form $X^{\circ-}$, for $X \subseteq T$

$\mathbf{RC}(T)$ is a Boolean algebra $(\mathbf{RC}(T), +, -, \emptyset, T)$

where $X + Y = X \cup Y$ and $-X = (\overline{X})^-$

\mathcal{B} -terms: $\tau ::= r_i \mid -\tau \mid \tau_1 \cdot \tau_2$ regular closed sets!

\mathcal{B} -formulas: $\varphi ::= \tau_1 = \tau_2 \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2$

\mathcal{B} is a **fragment** of $\mathcal{S}4_u$: \mathcal{B} -terms \xrightarrow{h} $\mathcal{S}4$ -terms

$$h(r_i) = v_i^{\circ-}, \quad h(\tau_1 \cdot \tau_2) = (h(\tau_1) \cap h(\tau_2))^{\circ-}, \quad h(-\tau_1) = (\overline{h(\tau_1)})^-$$

Theorem. $\mathbf{Sat}(\mathcal{B}, \text{REG}) = \mathbf{Sat}(\mathcal{B}, \text{CONREG}) = \mathbf{Sat}(\mathcal{B}, \mathbf{RC}(\mathbb{R}^n)), n \geq 1,$

and this set is **NP**-complete

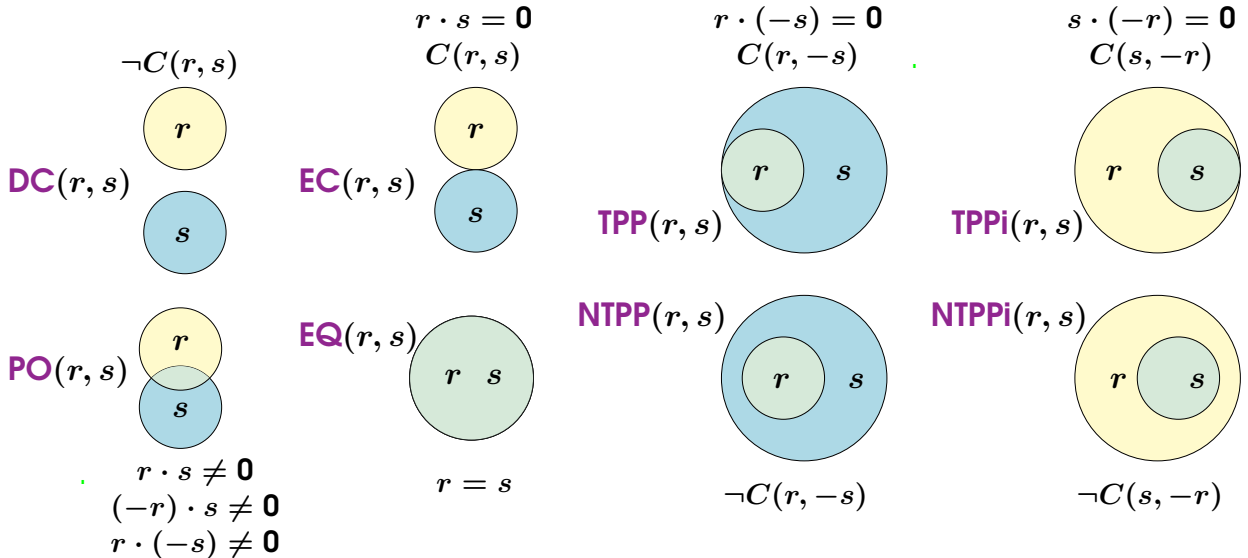
Proof. Every satisfiable \mathcal{B} -formula φ is satisfied

in a discrete topological space with $\leq |\varphi|$ points

$\mathcal{C} = \mathcal{B} + \text{contact predicate}$

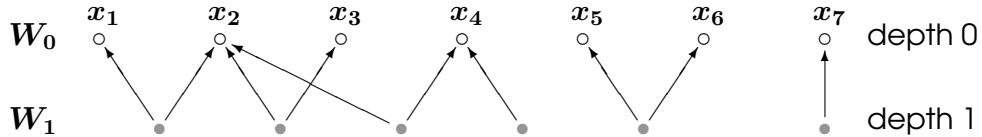
↓ Whitehead's 'connection' relation

\mathcal{C} -formulas: $\varphi ::= \tau_1 = \tau_2 \mid C(\tau_1, \tau_2) \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2$
 $\mathfrak{M} \models C(\tau_1, \tau_2) \text{ iff } \tau_1^{\mathfrak{M}} \cap \tau_2^{\mathfrak{M}} \neq \emptyset$ a.k.a. **BRCC-8**



Quasi-saw models for \mathcal{C}_{cc}

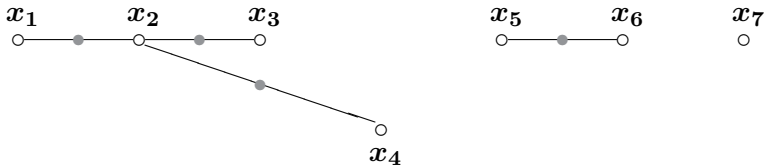
Lemma. Every satisfiable \mathcal{C}_{cc} -formula is satisfied in a quasi-saw model



A valuation may be defined only on points of depth 0

and 'computed' on points of depth 1

$$z \in \tau^{\mathfrak{M}} \cap W_1 \text{ iff there is } x \in \tau^{\mathfrak{M}} \cap W_0 \text{ with } zRx$$



C_c is ExpTime-hard

Theorem. $\text{Sat}(\mathcal{C}_c, \text{REG})$ is **EXPTIME**-hard

Proof. Let \mathcal{D}_2^f be the bimodal logic of the full infinite binary tree $\mathfrak{G} = (V, R_1, R_2)$
with **functional** R_1 and R_2

Reduction of the **global consequence relation** $\psi \models_2^f \chi$:

1. $(a \neq \mathbf{0}) \wedge c(f_0 + a) \wedge c(f_1 + a)$
2. every component of f_j contains a **sequence** of points in $s_j^0, s_j^1, \dots, s_j^5$
(provided it contains a point in s_j^0)
3. d marks points representing nodes of the binary tree, $d = s_0^0 + s_1^0$
for each φ , q_φ means ' **φ holds at the point**'
4. $q_{\neg\psi} \cdot s_0^0 \neq \mathbf{0}$ and $d \subseteq q_\chi$
5. $d \cdot q_{\neg\varphi} = d \cdot (-q_\varphi)$ and $d \cdot q_{\varphi_1 \wedge \varphi_2} = d \cdot (q_{\varphi_1} \cdot q_{\varphi_2})$
6. s_j^2 is the R_1 -successor, s_j^4 is the R_2 -successor: $s_j^2 \subseteq s_{j\oplus 1}^0$, $s_j^4 \subseteq s_{j\oplus 1}^0$, $j = 0, 1$
7. for each $\Box_i \varphi$, $m_\varphi^{i,j}$ means ' **φ holds at the R_i -successor**'
 $\neg C(f_j \cdot m_\varphi^{i,j}, f_j \cdot m_{\neg\varphi}^{i,j})$
 $(s_j^0 \cdot q_{\Box_i \varphi} \subseteq m_\varphi^{i,j})$ and $(m_\varphi^{i,j} \cdot s_j^{2i} \subseteq q_\varphi)$ (similarly for $m_{\neg\varphi}^{i,j}$)

Ccc is NExpTime-hard

Theorem. $\text{Sat}(\mathcal{C}_c, \text{REG})$ is **NEXPTIME**-hard

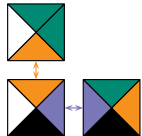
Proof. By reduction of the $2^n \times 2^n$ origin constrained tiling

Given $n \in \mathbb{N}$, a finite set \mathcal{T} of tile types $t = (\text{left}(t), \text{right}(t), \text{up}(t), \text{down}(t))$ and $t_0 \in \mathcal{T}$



decide whether there exists $\tau : [0, 2^n] \times [0, 2^n] \rightarrow \mathcal{T}$ such that

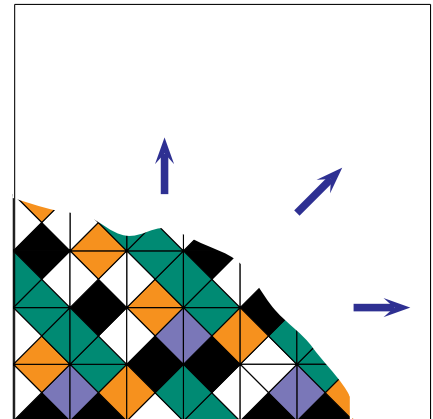
(i) for all i, j ,



$$\begin{aligned} \text{up}(\tau(i, j)) &= \text{down}(\tau(i, j + 1)) \\ &\text{and} \\ \text{left}(\tau(i, j)) &= \text{right}(\tau(i + 1, j)) \end{aligned}$$

(ii) $\tau(0, 0) = t_0$.

The $2^n \times 2^n$ origin constrained tiling
is **NEXPTIME**-complete



\mathcal{C}_{cc} is NExpTime-hard

Theorem. $\text{Sat}(\mathcal{C}_{cc}, \text{REG})$ is **NEXPTIME**-hard

Proof. By reduction of the $2^n \times 2^n$ origin constrained tiling

1. 2^n -counter formulas X_n, \dots, X_1 and 2^n -counter formulas for Y_n, \dots, Y_1
2. 4-neighbours: $\neg C(X_j \cdot Y_k, (-X_j) \cdot (-Y_k))$ and $\neg C((-X_j) \cdot Y_k, X_j \cdot (-Y_k))$
3. perimeter: $0_X \cdot 0_Y \neq \mathbf{0}, \quad (2^d - 1)_X \cdot (2^d - 1)_Y \neq \mathbf{0},$
 $c(0_X + (2^d - 1)_Y), \quad c((2^d - 1)_X + 0_Y)$
4. interior: $c((-X_1) + 0_Y), \quad c(X_1 + 0_Y), \quad c(0_X + (-Y_1)), \quad c(0_X + Y_1)$
5. chessboard: $\mathbf{b} = (X_1 \cdot (-Y_1)) + ((-X_1) \cdot Y_1) \quad c \leq 2^{n-1}(\mathbf{b})$
 $\mathbf{w} = ((-X_1) \cdot (-Y_1)) + (X_1 \cdot Y_1) \quad c \leq 2^{n-1}(\mathbf{w})$

Note that (1)–(4) imply that each \mathbf{b} and \mathbf{w} contains **at least** 2^{n-1} components

6. $\neg C(\mathbf{b} \cdot T, \mathbf{b} \cdot T')$ and $\neg C(\mathbf{w} \cdot T, \mathbf{w} \cdot T'), \quad \text{for } T \neq T'$

7. standard tiling formulas

Reduction from \mathcal{C}_c to \mathcal{B}_c

\mathcal{B}_c is a fragment of \mathcal{C}_c and the following formula is a \mathcal{C}_c -validity:

$$c(\tau_1) \wedge c(\tau_2) \rightarrow (c(\tau_1 + \tau_2) \leftrightarrow C(\tau_1, \tau_2))$$

Let φ be a \mathcal{C}_c -formula

- positive occurrence of $C(\tau_1, \tau_2)$:

$$\varphi^* = \varphi[t = \mathbf{0}]^+ \wedge ((t = \mathbf{0}) \rightarrow c(t_1 + t_2) \wedge \bigwedge_{i=1,2} (t_i \leq \tau_i) \wedge c(t_i))$$

- negative occurrence of $C(\tau_1, \tau_2)$:

$$\varphi^* = (\varphi[t = \mathbf{0}]^-)_{|s} \wedge (\neg(t = \mathbf{0}) \rightarrow \neg c(t_1 + t_2) \wedge \bigwedge_{i=1,2} c(t_i) \wedge (\tau_i \cdot s \leq t_i))$$

Then φ is satisfiable in an Aleksandrov space iff

φ^* is satisfiable in an Aleksandrov space

Summary of the results

	REG	CONREG	$RC(\mathbb{R}^n)$ $n > 2$	$RC(\mathbb{R}^2)$	$RC(\mathbb{R})$	
<i>RCC-8</i>	NP					
<i>RCC-8c</i>					?	$\leq PSPACE, \geq NP$
<i>RCC-8cc</i>					?	$\leq PSPACE, \geq NP$
B	NP					
<i>Bc</i>	EXPTIME	EXPTIME	?	?	$\leq PSPACE, \geq NP$	
<i>Bcc</i>	NEXPTIME	NEXPTIME	?	?	$\leq PSPACE, \geq NP$	
C	NP	PSPACE				
<i>Cc</i>	EXPTIME	EXPTIME	$\geq EXPTIME$	$\geq EXPTIME$	PSPACE	
<i>Ccc</i>	NEXPTIME	NEXPTIME	$\geq NEXPTIME$	$\geq NEXPTIME$	PSPACE	
C^m	NP	PSPACE		PSPACE	PSPACE	
<i>C^mc</i>	EXPTIME	EXPTIME	$\geq EXPTIME$	$\geq EXPTIME$	PSPACE	
<i>C^mcc</i>	NEXPTIME	NEXPTIME	$\geq NEXPTIME$	$\geq NEXPTIME$	PSPACE	
	ALL	CON	$\mathbb{R}^n, n > 2$	\mathbb{R}^2	\mathbb{R}	
S_{4u}	PSPACE	PSPACE				
<i>S_{4u}c</i>	EXPTIME	EXPTIME	$\geq EXPTIME$	$\geq EXPTIME$	PSPACE	
<i>S_{4u}cc</i>	NEXPTIME	NEXPTIME	$\geq NEXPTIME$	$\geq NEXPTIME$	PSPACE	