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Complete Calculus for Conjugated Arrow Logic

SZABOLCS MIKULÁS

ABSTRACT. We will give a strongly complete Hilbert-style inference system for a variant of arrow logic. Namely, we will consider a logic whose connectives, beside the Booleans, are identity, the binary modality •, and its two conjugate modalities ▶ and ◀. The models for this logic are Kripke models with one ternary accessibility relation corresponding to the modality •. A Hilbert-style inference system will be defined; and we will prove the completeness of this calculus with respect to the above semantics. We will also prove that this logic is decidable for it has the finite model property. The completeness proof uses an algebraic representation theorem, which will be proved in the paper as well.

1 Introduction

Arrow logic is treated for instance in van Benthem 1991 and chapter 1. That version of arrow logic contains all the Boolean connectives, and three modal operators: identity, converse and composition. The proposed models for that logic are Kripke models with three accessibility relations corresponding to the modalities above.

We will investigate a variant of arrow logic, called *conjugated arrow logic*, CARL for short, which is given by forgetting the modality converse and adding two new binary (or dyadic) modalities: ▶ and ◀. The models for CARL are Kripke models with one ternary accessibility relation corresponding to composition •. The other two modalities are conjugate modalities of • as, in temporal logic, *sometimes-in-the-past* is a conjugate of *sometimes-in-the-future*. Since • is a binary modality, the accessibility relation inter-

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preting it is a ternary relation. Thus we can permute the arguments of the relation in many ways; the interpretations of ▶ and ◀ are given by two such permutations. The idea that the binary connectives •, (and the duals of) ▶ and ◀ can be considered as (dyadic) modalities is, e.g., in Dirk Roorda's dissertation on resource logics Roorda 1991. However, his semantics uses three accessibility relations, one for each modality. See also Kosta Došen's survey paper Došen 1992. The connectives ▶ and ◀ are interesting for several reasons. Their presence is equivalent with that of the *residuals* \ and / of • as we will see later. (The syntactic nature of ▶ and ◀ and their relationship to • are investigated, e.g., in Jónsson and Tsinakis 1992.) The residuals \ and / are the well-known connectives of *categorial grammars*, e.g., of the *Lambek calculus*. These are also connectives of many *substructural logics* and Girard's *linear logic*. Further motivation is that \ and / can be considered as some kinds of implication as well: $\varphi\backslash\psi$ is *preimplication* meaning *had* φ *then* ψ and φ/ψ is *postimplication* φ *if-ever* ψ, see Vaughan Pratt's paper 1990. Further, identity is interpreted using the above accessibility relation too: identity holds at a world iff it is the composition of itself.

We will define a Hilbert-style inference system and show its strong completeness for CARL. We will also prove by filtration that CARL has the finite model property, hence it is decidable.

One direction of further investigations would be adding more and more connectives to our logic; another interesting direction is to consider more "concrete" semantics for our logics, e.g., where the set of possible worlds is a binary relation and the accessibility relation is relational composition. Actually, Maarten Marx proved in Marx 1995 completeness for the pair version of CARL, i.e., where the models are arbitrary binary relations and the accessibility relation is relational composition restricted to the universe of the model. Similar investigations are, e.g., in chapter 2 and Andréka and Mikulás 1994. We could also restrict the class of models by requiring that some further axioms should hold, as suggested, e.g., in van Benthem 1994.

2 Who is CARL?

In this section we give the precise definition of CARL.

Definition 2.1 (Conjugated Arrow Logic) Let P be a set, called the set of *parameters*, and $Cn = \{\wedge, \neg, \bullet, \blacktriangleright, \blacktriangleleft, \iota\delta\}$. The set of *formulas* of CARL is built up from P using the elements of Cn as connectives, where $\iota\delta$ is a 0-ary and $\neg$ is a unary connective and the others are binary connectives, in the usual way. The notion of *subformula* (of a formula) is assumed to be known. We will also use the well-known derived connectives: $\rightarrow$, $\leftrightarrow$, $\vee$, and the formula schemes: *False*, $\bot$, and *True*, $\top$.

A *Kripke frame* for CARL is an ordered pair $\langle W, C\rangle$, where W, called the set of *possible worlds*, is a non-empty set and C, called *accessibility relation*, is a ternary relation on W, i.e., $C \subseteq W \times W \times W$.

A *Kripke model* for CARL is a frame enriched with a valuation v. More precisely, it is an ordered triple $\langle W, C, v\rangle$, where $v : P \longrightarrow \mathcal{P}(W)$, i.e., v associates to every parameter a subset of W.

(Local) Truth of a formula φ at a world $w \in W$ in a model $\langle W, C, v\rangle$, denoted by $w \Vdash_v \varphi$, is defined by recursion as follows.

- If $p \in P$, then $w \Vdash_v p \stackrel{\text{def}}{\iff} w \in v(p)$.
- $w \Vdash_v \varphi \wedge \psi \stackrel{\text{def}}{\iff} w \Vdash_v \varphi \;\&\; w \Vdash_v \psi$.
- $w \Vdash_v \neg\varphi \stackrel{\text{def}}{\iff}$ not $w \Vdash_v \varphi$ (also denoted as $w \nVdash_v \varphi$).
- $w \Vdash_v \varphi \bullet \psi \stackrel{\text{def}}{\iff} (\exists w_1, w_2 \in W) C w_1 w_2 w \;\&\; w_1 \Vdash_v \varphi \;\&\; w_2 \Vdash_v \psi$.
- $w \Vdash_v \varphi \blacktriangleright \psi \stackrel{\text{def}}{\iff} (\exists w_1, w_2 \in W) C w_1 w w_2 \;\&\; w_1 \Vdash_v \varphi \;\&\; w_2 \Vdash_v \psi$.
- $w \Vdash_v \varphi \blacktriangleleft \psi \stackrel{\text{def}}{\iff} (\exists w_1, w_2 \in W) C w w_2 w_1 \;\&\; w_1 \Vdash_v \varphi \;\&\; w_2 \Vdash_v \psi$.
- $w \Vdash_v \iota\delta \stackrel{\text{def}}{\iff} Cwww$.

(Global) Truth in a model and *validity* in a frame are defined in the usual way. That is,

- $\langle W, C, v\rangle \models \varphi \stackrel{\text{def}}{\iff}$ for every world $w \in W$, $w \Vdash_v \varphi$
- $\langle W, C\rangle \models \varphi \stackrel{\text{def}}{\iff}$ for every valuation v, $\langle W, C, v\rangle \models \varphi$.

We say that a formula φ is a *semantical consequence* of the set Γ of formulas, in symbols $\Gamma \models \varphi$, iff for every model $\langle W, C, v\rangle$,

$$\langle W, C, v\rangle \models \Gamma \Rightarrow \langle W, C, v\rangle \models \varphi$$

where $\langle W, C, v\rangle \models \Gamma$ abbreviates that, for every $\psi \in \Gamma$, $\langle W, C, v\rangle \models \psi$.

We recall the definitions of the *residuals*, denoted by \ and /, of $\bullet$:

- $w \Vdash_v \varphi\backslash\psi \stackrel{\text{def}}{\iff} (\forall w_1, w_2 \in W)(C w_1 w w_2 \;\&\; w_1 \Vdash_v \varphi \Rightarrow w_2 \Vdash_v \psi)$
- $w \Vdash_v \varphi/\psi \stackrel{\text{def}}{\iff} (\forall w_1, w_2 \in W)(C w w_2 w_1 \;\&\; w_2 \Vdash_v \psi \Rightarrow w_1 \Vdash_v \varphi)$.

Then the following four formulas are semantically valid:

$$\begin{array}{lclclcl} \varphi\backslash\psi & \leftrightarrow & \neg(\varphi \blacktriangleright \neg\psi) & \quad & \varphi/\psi & \leftrightarrow & \neg(\neg\varphi \blacktriangleleft \psi) \\ \varphi \blacktriangleright \psi & \leftrightarrow & \neg(\varphi\backslash\neg\psi) & & \varphi \blacktriangleleft \psi & \leftrightarrow & \neg(\neg\varphi/\psi). \end{array}$$

3 Some Logic

The main result of this paper is the following strong completeness theorem, which we will prove later using an algebraic representation theorem (Theorem 4.3).

Theorem 3.1 (Strong Completeness) *There is a Hilbert-style inference system $\vdash$ which is strongly complete and strongly sound with respect to*

CARL. *That is, for every set* Γ *of formulas and formula* φ,

$$\Gamma \models \varphi \iff \Gamma \vdash \varphi.$$

In fact, the inference system $\vdash$ *defined in Definition 3.4 below has the above property.*

We will prove the following two theorems as well.

Theorem 3.2 (Decidability) CARL *is decidable, i.e., the set of valid formulas is a decidable set.*

Theorem 3.3 (Finite Model Property) CARL *has the finite model property, i.e., for every formula* φ, *if* φ *is not valid, then there is a finite Kripke model which refutes it.*

Definition 3.4 (Inference System $\vdash$) The Hilbert-style inference system $\vdash$ is given by the following axiom schemes and inference rules, where capital Greek letters denote formula schemes (metavariables which can be substituted by formulas) and $\bot$ stands for the formula scheme *False*.

Axiom Schemes.

(i) axiom schemes for classical propositional logic

(ii) $\bot \bullet \Phi \leftrightarrow \Phi \bullet \bot \leftrightarrow \bot$

(iii) $\Phi \bullet (\Psi \vee \Theta) \leftrightarrow (\Phi \bullet \Psi) \vee (\Phi \bullet \Theta)$

(iv) $(\Phi \vee \Psi) \bullet \Theta \leftrightarrow (\Phi \bullet \Theta) \vee (\Psi \bullet \Theta)$

(v) $\Phi \wedge \iota\delta \to \Phi \bullet \Phi.$

Inference Rules.

(vi) $$\frac{(\Phi \bullet \Psi) \wedge \Theta \leftrightarrow \bot}{(\Phi \blacktriangleright \Theta) \wedge \Psi \leftrightarrow \bot}$$ (double bar)

(vii) $$\frac{(\Phi \bullet \Psi) \wedge \Theta \leftrightarrow \bot}{(\Theta \blacktriangleleft \Psi) \wedge \Phi \leftrightarrow \bot}$$ (double bar)

(viii) $$\frac{\Phi \to \Psi \quad \Phi}{\Psi}$$

(ix) $$\frac{\Phi \leftrightarrow \Psi \quad \Theta \leftrightarrow \Lambda}{(\Phi \bullet \Theta) \leftrightarrow (\Psi \bullet \Lambda)}$$

where double bar indicates that we have both the downward and the upward rules. The definition of derivability of a formula φ from a set Γ of formulas, $\Gamma \vdash \varphi$, is the usual.

The following formula is an equivalent version of *(v)* (in the presence of *(i)*, *(iii)* and *(iv)*), and we will use it sometimes:

$$\Phi \wedge \Psi \wedge \iota\delta \to \Phi \bullet \Psi.$$

Remark 3.5 Instead of the conjugates $\blacktriangleright$ and $\blacktriangleleft$ we could choose the residuals $\backslash$ and $/$. Then the above three theorems still hold. This is true because the conjugates and the residuals are definable by each other as we mentioned after Definition 2.1. Thus we can replace each occurrence of $\varphi \blacktriangleright \psi$ by its definition $\neg(\varphi\backslash\neg\psi)$ and similarly for $\blacktriangleleft$.

Clearly, the conjugates and the residuals can be present at the same time too. Then we have to add the definitions of the residuals as new axioms to the inference system in Definition 3.4.

Further, we could define a very weak *converse* too:

$$\varphi^{\smile} \overset{\text{def}}{\iff} (\varphi \blacktriangleright \iota\delta) \wedge (\iota\delta \blacktriangleleft \varphi).$$

4 Some Algebra

In this section, we define the algebraic counterparts both of syntax (the class KA of algebras) and of semantics (the class RKA of algebras), and prove that they are identical (Theorem 4.3).

Note that by RKA we will denote a class of algebras, and we will use it as the abbreviation of 'representable Kripke algebra' too. Given a class K of algebras, we denote by **I**K and by **S**K the class of isomorphic copies and subalgebras of members of K, respectively.

Definition 4.1 (Representable Kripke Algebra) The class of *representable Kripke algebras* is defined as

$$\mathsf{RKA} \overset{\text{def}}{=} \mathbf{S}\{\langle \mathcal{P}(W), \cap, \sim, \circ^C, \rhd^C, \lhd^C, Id^C \rangle : W \text{ a set } \& \ C \subseteq W \times W \times W\}$$

where $\cap$ is intersection, $\sim$ is set-theoretic complementation with respect to W, i.e., $\sim a = W \smallsetminus a$, and

$$\begin{aligned}
a \circ^C b &\overset{\text{def}}{=} \{z \in W : (\exists x, y \in W) Cxyz \ \& \ x \in a \ \& \ y \in b\} \\
a \rhd^C b &\overset{\text{def}}{=} \{z \in W : (\exists x, y \in W) Cxzy \ \& \ x \in a \ \& \ y \in b\} \\
a \lhd^C b &\overset{\text{def}}{=} \{z \in W : (\exists x, y \in W) Czyx \ \& \ x \in a \ \& \ y \in b\} \\
Id^C &\overset{\text{def}}{=} \{z \in W : Czzz\}
\end{aligned}$$

for all elements a, b.

Definition 4.2 (Kripke Algebra) The class of (abstract) *Kripke algebras*, denoted by KA, is defined as the class of algebras similar to RKA's satisfying the axioms below. More precisely, every element of KA has the form

$$\langle A, \wedge, \neg, \bullet, \blacktriangleright, \blacktriangleleft, \iota\delta \rangle$$

where A is a non-empty set, $\iota\delta$ is a constant, $\neg$ is a unary operation on A, and $\wedge, \bullet, \blacktriangleright, \blacktriangleleft$ are binary operations on A; and the following set Ax of quasi-equations is valid in KA:

(1) Boolean axioms
(2) $0 \bullet x = x \bullet 0 = 0$
(3) $x \bullet (y \vee z) = (x \bullet y) \vee (x \bullet z)$
(4) $(x \vee y) \bullet z = (x \bullet z) \vee (y \bullet z)$
(5) $x \wedge \iota\delta \leq x \bullet x$
(6) $(x \bullet y) \wedge z = 0 \iff (x \blacktriangleright z) \wedge y = 0$
(7) $(x \bullet y) \wedge z = 0 \iff (z \blacktriangleleft y) \wedge x = 0,$

where 0 is Boolean zero, $x \vee y$ abbreviates $\neg(\neg x \wedge \neg y)$ and $x \leq y$ abbreviates $x \wedge y = x$.

Again, an axiom equivalent to (5) is

$$x \wedge y \wedge \iota\delta \leq x \bullet y.$$

The main algebraic result of this paper is the following theorem that we will prove in the following section as well.

Theorem 4.3 (Representation Theorem) *For every algebra $\mathcal{A}$,*

$$\mathcal{A} \in \mathsf{KA} \iff \mathcal{A} \in \mathbf{I}\mathsf{RKA}.$$

Moreover, if $\mathcal{A}$ is a finite algebra, then there are a finite set W and a relation $C \subseteq W \times W \times W$ such that $\mathcal{A}$ is isomorphic to a subalgebra of the RKA

$$\langle \mathcal{P}(W), \cap, \sim, \circ^C, \rhd^C, \lhd^C, Id^C \rangle.$$

5 Proofs

To prove Theorems 3.1 and 4.3 we need some lemmas (the proof of which we postpone till the end of this section) and some definitions.

Let Γ be an arbitrary set of formulas of CARL; then for any formulas φ and ψ, we set

$$\varphi \equiv_\Gamma \psi \stackrel{\text{def}}{\iff} \Gamma \vdash \varphi \leftrightarrow \psi.$$

The *formula algebra* $\mathcal{F}$ is defined as

$$\mathcal{F} \stackrel{\text{def}}{=} \langle F, \wedge, \neg, \bullet, \blacktriangleright, \blacktriangleleft, \iota\delta \rangle,$$

where F denotes the set of formulas of CARL.

Lemma 5.1 *$\equiv_\Gamma$ is a congruence relation on $\mathcal{F}$.*

Let $\mathcal{F}_\Gamma$ be the factor algebra of $\mathcal{F}$ by $\equiv_\Gamma$, i.e., $\mathcal{F}_\Gamma \stackrel{\text{def}}{=} \mathcal{F}/\equiv_\Gamma$. Further, for any formula φ, we let $\overline{\varphi} \stackrel{\text{def}}{=} \{\psi : \varphi \equiv_\Gamma \psi\}$. Let $\top$ denote the formula *True*.

Lemma 5.2 1. *For every formula φ, $\Gamma \vdash \varphi \iff \overline{\varphi} = \overline{\top}$.*

2. *$\mathcal{F}_\Gamma \models Ax$.*

Lemma 5.3 *Let W be a non-empty set, $\mathcal{A}$ be a subalgebra of the full* RKA *$\langle \mathcal{P}(W), \cap, \sim, \circ^C, \rhd^C, \lhd^C, Id^C \rangle$, and v be a valuation. Then*

$$\langle \mathcal{A}, v \rangle \models \varphi = 1 \iff \langle W, C, v \rangle \models \varphi,$$

where 1 denotes Boolean unit.

Proof. (Theorem 3.1) Soundness is easy to check.

For the completeness direction, we assume that $\Gamma \not\vdash \varphi$. Then, by Lemma 5.2, $\overline{\varphi} \neq \overline{\top}$. Now, let n be that valuation which associates its equivalence class to each formula, i.e., $n(\varphi) = \overline{\varphi}$ for every φ. Then $\langle \mathcal{F}_\Gamma, n \rangle \not\models \varphi = \top$.

By Lemma 5.2, for every $\psi \in \Gamma$, $\langle \mathcal{F}_\Gamma, n\rangle \models \psi = \top$. Since, by Lemma 5.2, $\mathcal{F}_\Gamma \models Ax$, Theorem 4.3 says that $\mathcal{F}_\Gamma \in$ IRKA. Thus we have an $\mathcal{A} \in$ RKA and a valuation v such that $\langle \mathcal{A}, v\rangle \not\models \varphi = 1$, while for every $\psi \in \Gamma$, $\langle \mathcal{A}, v\rangle \models \psi = 1$. Whence, by Lemma 5.3, there is a Kripke model $\mathcal{M}$ such that $\mathcal{M} \not\models \varphi$, while for every $\psi \in \Gamma$, $\mathcal{M} \models \psi$, i.e., $\Gamma \not\models \varphi$, which was to be proved. □

Next we prove that CARL is decidable.

Proof. (Theorem 3.2) By the Completeness Theorem 3.1, $\{\varphi : \models \varphi\} = \{\varphi : \vdash \varphi\}$; thus the set of valid formulas form a recursively enumerable set.

By Theorem 3.3, for every formula, we have a finite model that refutes it. Since we can enumerate the set of finite models and the semantic value of a formula is computable, the set $\{\varphi : \not\models \varphi\}$ is recursively enumerable.

Since both $\{\varphi : \models \varphi\}$ and its complement are recursively enumerable, $\{\varphi : \models \varphi\}$ is a decidable set. □

Now we prove the Representation Theorem. We will use an idea of Németi 1992.

Proof. (Theorem 4.3) It is easy to verify that RKA $\models Ax$, so we will omit it.

For the other direction, let us assume that $\mathcal{A} \in$ KA. Then we will represent this $\mathcal{A}$ as an algebra $\mathcal{B}$ whose operations are almost good. Indeed, there will be a ternary relation C on the set of ultrafilters $Uf(\mathcal{A})$ such that the unary and binary operations can be defined via this C. Namely, if we let

$$CFGH \overset{\text{def}}{\iff} F \bullet G \subseteq H,$$

for every $F, G, H \in Uf(\mathcal{A})$, and for every $a \in A$,

$$rep(a) \overset{\text{def}}{=} \{F \in Uf(\mathcal{A}) : a \in F\},$$

then *rep* is almost an isomorphism. The only problem is that there may be ultrafilters F such that $F \bullet F \subseteq F$ but $\iota\delta \notin F$, i.e., $rep(\iota\delta) \neq \{F : CFFF\}$. If we split these "bad" ultrafilters into two parts carefully, then we will get a $\mathcal{B}' \in$ RKA isomorphic to the original algebra $\mathcal{A}$.

First we state and prove a lemma, where $Uf(\mathcal{A})$ denotes the set of Boolean ultrafilters of the algebra $\mathcal{A}$, and for any subsets F and G of A (the universe of $\mathcal{A}$),

$$F \bullet G \overset{\text{def}}{=} \{a \in A : (\exists f \in F)(\exists g \in G) a = f \bullet g\}.$$

$F \blacktriangleright G$ and $F \blacktriangleleft G$ are defined similarly.

Lemma 5.4 *Let $\mathcal{A} \in$ KA and F_0, G_0 be subsets of A with the finite intersection property (i.e., $x, y \in F_0 \Rightarrow x \wedge y \neq 0$). Then*

$$(\exists H \in Uf(\mathcal{A})) F_0 \bullet G_0 \subseteq H \Rightarrow$$
$$(\exists F \supseteq F_0)(\exists G \supseteq G_0) F \in Uf(\mathcal{A}) \;\&\; G \in Uf(\mathcal{A}) \;\&\; F \bullet G \subseteq H.$$

Proof. Let F_0, G_0 and H be as in the formulation of the lemma. Assume that F_0 is not an ultrafilter, i.e., $\exists x(x \notin F_0 \;\&\; \neg x \notin F_0)$. Let F' be the filter generated by $F_0 \cup \{x\}$ and F'' be the filter generated by $F_0 \cup \{\neg x\}$. Assume that $F' \bullet G_0 \not\subseteq H$ and $F'' \bullet G_0 \not\subseteq H$, i.e., $\exists f_1 \exists f_2 (\exists f, f' \in F_0)(\exists g, g' \in G_0)$

$$x \wedge f \leq f_1 \;\&\; f_1 \bullet g \notin H \;\&\; \neg x \wedge f' \leq f_2 \;\&\; f_2 \bullet g' \notin H.$$

Then, since H is upward closed, we have, by (4), $(x \wedge f) \bullet g \notin H$. Similarly, by (3) and (4), we get $(x \wedge f \wedge f') \bullet (g \wedge g') \notin H$. By the same argument, $(\neg x \wedge f \wedge f') \bullet (g \wedge g') \notin H$. Putting together,

$$\begin{aligned}(f \wedge f') \bullet (g \wedge g') &\overset{(1)}{=} ((x \wedge f \wedge f') \vee (\neg x \wedge f \wedge f')) \bullet (g \wedge g') \overset{(4)}{=} \\ &= ((x \wedge f \wedge f') \bullet (g \wedge g')) \vee ((\neg x \wedge f \wedge f') \bullet (g \wedge g')) \notin H,\end{aligned}$$

a contradiction, since $f \wedge f' \in F_0$ and $g \wedge g' \in G_0$ and $F_0 \bullet G_0 \subseteq H$. So either $F' \bullet G_0 \subseteq H$ or $F'' \bullet G_0 \subseteq H$. Note that if $F' \bullet G_0 \subseteq H$, then $0 \notin F'$, by (2), so F' is a proper filter. Using recursion, one can extend F_0 to an ultrafilter F with the property $F \bullet G_0 \subseteq H$. Then, in the same way, G_0 can be extended to an ultrafilter G such that $F \bullet G \subseteq H$. Thus the lemma has been proved. □

As we mentioned before, let for every $a \in A$,

$$rep(a) \overset{\text{def}}{=} \{F \in Uf(\mathcal{A}) : a \in F\},$$

and for every $F, G, H \in Uf(\mathcal{A})$,

$$CFGH \overset{\text{def}}{\Longleftrightarrow} F \bullet G \subseteq H.$$

Let $rep''A \overset{\text{def}}{=} \{rep(a) : a \in A\}$ and

$$\mathcal{B} \overset{\text{def}}{=} \langle rep''A, \cap, \sim, \circ^C, \rhd^C, \lhd^C, rep(\iota\delta)\rangle$$

where $\sim$ is set-theoretic complementation with respect to $Uf(\mathcal{A})$, and $\circ^C$, $\rhd^C$, $\lhd^C$ are defined by the frame $\langle Uf(\mathcal{A}), C\rangle$, i.e.,

$$\begin{aligned}rep(a) \circ^C rep(b) &\overset{\text{def}}{=} \{F \in Uf(\mathcal{A}) : (\exists G \in rep(a))(\exists H \in rep(b))CGHF\} \\ rep(a) \rhd^C rep(b) &\overset{\text{def}}{=} \{F \in Uf(\mathcal{A}) : (\exists G \in rep(a))(\exists H \in rep(b))CGFH\} \\ rep(a) \lhd^C rep(b) &\overset{\text{def}}{=} \{F \in Uf(\mathcal{A}) : (\exists G \in rep(a))(\exists H \in rep(b))CFHG\}.\end{aligned}$$

Now we will show that $rep : \mathcal{A} \longrightarrow \mathcal{B}$ is an isomorphism. rep is a Boolean isomorphism by Stone's representation theorem.

$$\begin{aligned}&rep(a) \circ^C rep(b) = \\ &= \{F \in Uf(\mathcal{A}) : (\exists G \in rep(a))(\exists H \in rep(b))CGHF\} = \\ &= \{F \in Uf(\mathcal{A}) : \exists G \exists H (a \in G \;\&\; b \in H \;\&\; G \bullet H \subseteq F)\} \overset{(a)}{=} \\ &= \{F \in Uf(\mathcal{A}) : a \bullet b \in F\} \\ &= rep(a \bullet b).\end{aligned}$$

$(a) : (\subseteq) : a \in G \;\&\; b \in H \Rightarrow a \bullet b \in F.$

$(\supseteq)$: If $a \bullet b \in F$, then, by Lemma 5.4, we can construct two ultrafilters G and H such that $a \in G$, $b \in H$ and $G \bullet H \subseteq F$.

$$\begin{aligned} & rep(a) \rhd^C rep(b) = \\ & = \{F \in Uf(\mathcal{A}) : (\exists G \in rep(a))(\exists H \in rep(b))CGFH\} = \\ & = \{F \in Uf(\mathcal{A}) : \exists G \exists H(a \in G \,\&\, b \in H \,\&\, G \bullet F \subseteq H)\} \stackrel{(b)}{=} \\ & = \{F \in Uf(\mathcal{A}) : a \blacktriangleright b \in F\} = \\ & = rep(a \blacktriangleright b). \end{aligned}$$

(b) : $(\subseteq)$: Assume $\neg(a \blacktriangleright b) \in F$. Then $a \bullet (\neg(a \blacktriangleright b)) \in H$, so $(a \bullet (\neg(a \blacktriangleright b))) \wedge b \neq 0$. Thus, by (6), $(a \blacktriangleright b) \wedge \neg(a \blacktriangleright b) \neq 0$, a contradiction. So $a \blacktriangleright b \in F$.

$(\supseteq)$: Let $a \blacktriangleright b \in F$. Let $G_0 = \{x \in A : x \geq a\}$ and $H_0 = \{x \in A : x \geq b\}$. First we show that $(G_0 \bullet F) \cup H_0$ can be extended to an ultrafilter H. Let

$$\uparrow (G_0 \bullet F) \stackrel{\text{def}}{=} \{x \in A : (\exists g \in G_0)(\exists f \in F) x \geq g \bullet f\}$$

and $x_1, x_2 \in\ \uparrow (G_0 \bullet F)$. Then, using monotonicity of $\bullet$, $x_1 \wedge x_2 \geq (g_1 \bullet f_1) \wedge (g_2 \bullet f_2) \geq (a \bullet f_1) \wedge (a \bullet f_2) \geq a \bullet (f_1 \wedge f_2) \in G \bullet F$ whence $x_1 \wedge x_2 \in\ \uparrow (G_0 \bullet F)$. So $\uparrow (G_0 \bullet F)$ is $\wedge$-closed, and so is H_0. Now we show that $(\uparrow (G_0 \bullet F)) \cup H_0$ has the finite intersection property, so it can be extended to an ultrafilter H. Assume to the contrary that $x \in\ \uparrow (G_0 \bullet F)$, $h \in H_0$ and $x \wedge h = 0$, then $0 = x \wedge h \geq (g \bullet f) \wedge b \geq (a \bullet f) \wedge b$, i.e., $0 = (a \bullet f) \wedge b$. Then, by (6), $0 = (a \blacktriangleright b) \wedge f \in F$, a contradiction. Clearly, $b \in H$ and $G_0 \bullet F \subseteq H$. Then, by Lemma 5.4, we can extend G_0 to an ultrafilter G such that $G \bullet F \subseteq H$, as desired.

Similar argument shows that $rep(a) \lhd^C rep(b) = rep(a \blacktriangleleft b)$. Since $rep(\iota\delta)$ is the identity constant in $\mathcal{B}$, $\mathcal{A} \cong \mathcal{B}$.

Later we will need the following fact.

$$(*) \qquad (\forall F \in Uf(\mathcal{A}))\iota\delta \in F \Rightarrow CFFF.$$

This holds, since $x, y, \iota\delta \in F$ implies, by (5), $x \bullet y \geq x \wedge y \wedge \iota\delta \in F$, i.e., $F \bullet F \subseteq F$.

Now we define a Kripke frame $\langle W', C' \rangle$ and the corresponding RKA $\mathcal{B}'$, which will turn out to be isomorphic to our original algebra $\mathcal{A}$. First we split the "bad" ultrafilters on $\mathcal{A}$ into two parts. Let

$$D \stackrel{\text{def}}{=} \{F \in Uf(\mathcal{A}) : \iota\delta \notin F \,\&\, CFFF\}.$$

Let for every $F \in D$, F_1 and F_2 be two distinct elements (of our set-theoretic universe) not occurring in $Uf(\mathcal{A})$, and

$$s(F) \stackrel{\text{def}}{=} \begin{cases} \{F_1, F_2\} & \text{if } F \in D \\ \{F\} & \text{if } F \notin D. \end{cases}$$

We assume also that for different F's from D the F_i's are completely different. If $F \notin D$, then by both F_1 and F_2 we mean F. Let

$$W' \stackrel{\text{def}}{=} \{F_1, F_2 : F \in \mathit{Uf}(\mathcal{A})\}$$

and

$$C' = (\bigcup\{s(F)\times s(G)\times s(H) : CFGH\})\smallsetminus\{\langle F_i, F_i, F_i\rangle : F \in D,\ i \in \{1,2\}\}.$$

Now we define a representation function *Rep* as

$$\mathit{Rep}(b) \stackrel{\text{def}}{=} \{F_1, F_2 : F \in b\}$$

for every $b \in B$. Then let B' be the *Rep*-image of B, i.e., $B' \stackrel{\text{def}}{=} \mathit{Rep}''B$ and

$$\mathcal{B}' \stackrel{\text{def}}{=} \langle B', \cap, \sim, \circ^{C'}, \rhd^{C'}, \lhd^{C'}, \mathit{Id}^{C'}\rangle,$$

where $\sim$ is set-theoretic complementation with respect to W' and the other operations are defined as in Definition 4.1 (using W' and C' instead of W and C, respectively). Clearly, $\mathcal{B}'$ is an RKA; so it remains to show that $\mathit{Rep} : \mathcal{B} \longrightarrow \mathcal{B}'$ is an isomorphism, for then $\mathcal{A} \cong \mathcal{B}'$, by $\mathcal{A} \cong \mathcal{B}$.

Clearly, *Rep* is a Boolean isomorphism. For the identity we have:

$$\begin{aligned}
\mathit{Rep}(\mathit{rep}(\iota\delta)) &= \bigcup\{s(F) : F \in \mathit{rep}(\iota\delta)\} \stackrel{(c)}{=} \\
&= \bigcup\{s(F) : \iota\delta \in F \;\&\; CFFF\} = \\
&= \{F : F \in \mathit{Uf}(\mathcal{A}) \smallsetminus D \;\&\; \iota\delta \in F \;\&\; CFFF\} \cup \\
&\quad\ \{F_1, F_2 : F \in D \;\&\; \iota\delta \in F \;\&\; CFFF\} \stackrel{(d)}{=} \\
&= \{F : F \in W' \;\&\; \iota\delta \in F \;\&\; C'FFF\} \stackrel{(e)}{=} \\
&= \{F : F \in W' \;\&\; C'FFF\} = \\
&= \mathit{Id}^{C'}.
\end{aligned}$$

(c): $F \in \mathit{rep}(\iota\delta)$ implies $\iota\delta \in F$, so, by $(*)$, $CFFF$.

(d): By the definition of D, $F \in D$ implies not $CFFF$; then apply the definition of C'.

(e): $C'FFF$ implies $F \in \mathit{Uf}(\mathcal{A}) \smallsetminus D$ and $CFFF$, whence $\iota\delta \in F$, by the definition of D.

In checking that *Rep* preserves $\circ$, $\rhd$ and $\lhd$, we will use the following lemma.

Lemma 5.5 $(\forall a, b \in B)(\forall z \in W')$

$$(\exists x \in \mathit{Rep}(a))(\exists y \in \mathit{Rep}(b))C'xyz \iff$$
$$(\exists F \in a)(\exists G \in b)(\exists H \in \mathit{Uf}(\mathcal{A}))z \in s(H) \;\&\; CFGH.$$

The same holds for $C'xzy$ and $C'zyx$.

Proof. $(\Rightarrow)$: Assume $C'xyz$. Let $F \in a$, $G \in b$ with $x \in s(F)$ and $y \in s(G)$, and let $H \in \mathit{Uf}(\mathcal{A})$ such that $z \in s(H)$. Then $\langle x, y, z\rangle \in s(F)\times s(G)\times s(H)$, thus, by the definition of C', $CFGH$ holds.

$(\Leftarrow)$: We have two cases.

CASE 1: $F = G = H \in D$. Then $H_1 \neq H_2$, so we can choose $x, y \in s(H)$

with $x = y \neq z$. Now, $\langle x, y, z\rangle \in s(H) \times s(H) \times s(H)$ whence $C'xyz$.
CASE 2: not Case 1. Then let $x \in s(F)$, $y \in s(G)$. Again, $\langle x, y, z\rangle \in s(F) \times s(G) \times s(H)$. Thus $C'xyz$ holds, which finishes the proof of the lemma. □

$$\begin{aligned}
&\mathit{Rep}(a) \circ^{C'} \mathit{Rep}(b) = \\
&= \{z \in W' : (\exists x \in \mathit{Rep}(a))(\exists y \in \mathit{Rep}(b))C'xyz\} \stackrel{\text{L.5.5}}{=} \\
&= \{z \in W' : (\exists F \in a)(\exists G \in b)(\exists H \in \mathit{Uf}(\mathcal{A}))z \in s(H)\ \&\ CFGH\} = \\
&= \bigcup\{s(H) : H \in \mathit{Uf}(\mathcal{A})\ \&\ (\exists F \in a)(\exists G \in b)CFGH\} = \\
&= \bigcup\{s(H) : H \in a \circ^{C} b\} = \\
&= \mathit{Rep}(a \circ^{C} b).
\end{aligned}$$

$$\begin{aligned}
&\mathit{Rep}(a) \rhd^{C'} \mathit{Rep}(b) = \\
&= \{z \in W' : (\exists x \in \mathit{Rep}(a))(\exists y \in \mathit{Rep}(b))C'xzy\} \stackrel{\text{L.5.5}}{=} \\
&= \{z \in W' : (\exists F \in a)(\exists G \in b)(\exists H \in \mathit{Uf}(\mathcal{A}))z \in s(H)\ \&\ CFHG\} = \\
&= \bigcup\{s(H) : H \in \mathit{Uf}(\mathcal{A})\ \&\ (\exists F \in a)(\exists G \in b)CFHG\} = \\
&= \bigcup\{s(H) : H \in a \rhd^{C} b\} = \\
&= \mathit{Rep}(a \rhd^{C} b).
\end{aligned}$$

The case of $\lhd$ is similar.

Thus $\mathit{Rep} : \mathcal{B} \longrightarrow \mathcal{B}'$ is an isomorphism, whence $\mathcal{A} \cong \mathcal{B} \cong \mathcal{B}' \in \mathsf{RKA}$, i.e., $\mathcal{A} \in \mathbf{I}\mathsf{RKA}$.

If $|A| < \omega$, then $|\mathit{Uf}(\mathcal{A})| < \omega$, and so $|B'| < \omega$. Thus finite algebras are represented on finite bases. So we have proved Theorem 4.3. □

In the following proof of the finite model property, we will follow the strategy in the proofs of Theorems 3.6.2 and 3.6.3 in Roorda 1991. There, Dirk Roorda proves finite model property for logics with dyadic modalities using filtration.

Proof. (Theorem 3.3) Assume that $\nvdash \varphi$ and let $\langle W, C, v\rangle$ be any model refuting φ. Let Σ be the set consisting of $\iota\delta$ and the subformulas of φ. We define the equivalence relation $\approx$ on W as

$$x \approx y \stackrel{\text{def}}{\Longleftrightarrow} (\forall \psi \in \Sigma)(x \Vdash_v \psi \Longleftrightarrow y \Vdash_v \psi).$$

We choose an arbitrary but fixed element w' from every equivalence class $\{x \in W : x \approx w\}$. Let

$$W' \stackrel{\text{def}}{=} \{w' : w \in W\}$$

and

$$v'(p_i) \stackrel{\text{def}}{=} \{w' : w \in v(p_i)\}$$

for every $p_i \in P$. Clearly, W' is finite and v' is a valuation. Let $C' \subseteq$

$W' \times W' \times W'$ be defined as

$$\begin{gathered} C'x'y'w' \stackrel{\text{def}}{\iff} \\ [((\forall \psi_1 \bullet \psi_2 \in \Sigma)(x' \Vdash_v \psi_1 \mathbin{\&} y' \Vdash_v \psi_2 \Rightarrow w' \Vdash_v \psi_1 \bullet \psi_2) \mathbin{\&} \\ (\forall \psi_1 \blacktriangleright \psi_2 \in \Sigma)(x' \Vdash_v \psi_1 \mathbin{\&} w' \Vdash_v \psi_2 \Rightarrow y' \Vdash_v \psi_1 \blacktriangleright \psi_2) \mathbin{\&} \\ (\forall \psi_1 \blacktriangleleft \psi_2 \in \Sigma)(w' \Vdash_v \psi_1 \mathbin{\&} y' \Vdash_v \psi_2 \Rightarrow x' \Vdash_v \psi_1 \blacktriangleleft \psi_2)) \text{ or} \\ (x' = y' = w' \mathbin{\&} Cw'w'w')] . \end{gathered}$$

Let (min) be the following formula:

$$\begin{aligned} &(w' \in W' \mathbin{\&} x, y \in W \mathbin{\&} Cxyw' \Rightarrow C'x'y'w') \mathbin{\&} \\ &(w' \in W' \mathbin{\&} x, y \in W \mathbin{\&} Cxw'y \Rightarrow C'x'w'y') \mathbin{\&} \\ &(w' \in W' \mathbin{\&} x, y \in W \mathbin{\&} Cw'yx \Rightarrow C'w'y'x') \end{aligned}$$

where x' denotes the distinguished element of the equivalence class of x, and similarly for y.

By (max) we mean the following formula:

$$\begin{aligned} &((C'x'y'w' \mathbin{\&} \psi_1 \bullet \psi_2 \in \Sigma \mathbin{\&} x' \Vdash_v \psi_1 \mathbin{\&} y' \Vdash_v \psi_2) \Rightarrow w' \Vdash_v \psi_1 \bullet \psi_2) \mathbin{\&} \\ &((C'x'y'w' \mathbin{\&} \psi_1 \blacktriangleright \psi_2 \in \Sigma \mathbin{\&} x' \Vdash_v \psi_1 \mathbin{\&} w' \Vdash_v \psi_2) \Rightarrow y' \Vdash_v \psi_1 \blacktriangleright \psi_2) \mathbin{\&} \\ &((C'x'y'w' \mathbin{\&} \psi_1 \blacktriangleleft \psi_2 \in \Sigma \mathbin{\&} w' \Vdash_v \psi_1 \mathbin{\&} y' \Vdash_v \psi_2) \Rightarrow x' \Vdash_v \psi_1 \blacktriangleleft \psi_2). \end{aligned}$$

Lemma 5.6 *Let C' be defined as above. Then* (min) *and* (max) *hold.*

Proof. Let $w' \in W'$ & $x, y \in W$ & $Cxyw'$. If $\psi_1 \bullet \psi_2 \in \Sigma$ is arbitrary and $x' \Vdash_v \psi_1$ & $y' \Vdash_v \psi_2$, then $x \Vdash_v \psi_1$ & $y \Vdash_v \psi_2$, thus $w' \Vdash_v \psi_1 \bullet \psi_2$, by $Cxyw'$. If $\psi_1 \blacktriangleright \psi_2 \in \Sigma$ is arbitrary and $x' \Vdash_v \psi_1$ & $w' \Vdash_v \psi_2$, then $x \Vdash_v \psi_1$, thus $y \Vdash_v \psi_1 \blacktriangleright \psi_2$, by $Cxyw'$. So $y' \Vdash_v \psi_1 \blacktriangleright \psi_2$. Similarly, if $\psi_1 \blacktriangleleft \psi_2 \in \Sigma$ is arbitrary and $w' \Vdash_v \psi_1$ & $y' \Vdash_v \psi_2$, then $x' \Vdash_v \psi_1 \blacktriangleleft \psi_2$. Then, by the definition of C', we have $C'x'y'w'$.

Similar arguments prove the other two implications of (min). One can easily prove (max) using case distinction according to whether $x' = y' = w'$ holds. □

Now we define the relation $\Vdash'_{v'}$ between the set of possible worlds W' of the model $\langle W', C', v' \rangle$ and the set of formulas. The definition of $\Vdash'_{v'}$ is the usual for parameters, Boolean connectives, and for $\bullet$, $\blacktriangleright$ and $\blacktriangleleft$, cf. Definition 2.1; but for the identity we have:

$$w' \Vdash'_{v'} \iota\delta \stackrel{\text{def}}{\iff} w' \Vdash_v \iota\delta.$$

Then we can prove the following lemma.

Lemma 5.7 $(\forall \psi \in \Sigma)(\forall w \in W)$

$$w \Vdash_v \psi \iff w' \Vdash'_{v'} \psi.$$

Proof. Proving the lemma we will use without mentioning that, since $w' \approx w$, $w \Vdash_v \psi \iff w' \Vdash_v \psi$, and that $W' \subseteq W$. We will prove by induction and refer to the induction hypothesis by 'i.h.'.

$$(\forall p_i \in P)(w' \in v(p_i) \iff w' \in v'(p_i))$$

$$w \Vdash_v \neg\psi_1 \iff w' \nVdash_v \psi_1 \overset{\text{i.h.}}{\iff} w' \nVdash'_{v'} \psi_1 \iff w' \Vdash'_{v'} \neg\psi_1$$

$$\begin{aligned} w \Vdash_v \psi_1 \wedge \psi_2 &\iff w' \Vdash_v \psi_1 \;\&\; w' \Vdash_v \psi_2 \overset{\text{i.h.}}{\iff} \\ &\iff w' \Vdash'_{v'} \psi_1 \;\&\; w' \Vdash'_{v'} \psi_2 \iff \\ &\iff w' \Vdash'_{v'} \psi_1 \wedge \psi_2 \end{aligned}$$

$$\begin{aligned} w \Vdash_v \psi_1 \bullet \psi_2 &\Rightarrow (\exists x, y \in W) Cxyw' \;\&\; x \Vdash_v \psi_1 \;\&\; y \Vdash_v \psi_2 \overset{(\text{min})}{\Rightarrow} \\ &\Rightarrow (\exists x, y \in W) C'x'y'w' \;\&\; x \Vdash_v \psi_1 \;\&\; y \Vdash_v \psi_2 \overset{\text{i.h.}}{\Rightarrow} \\ &\Rightarrow (\exists x', y' \in W') C'x'y'w' \;\&\; x' \Vdash'_{v'} \psi_1 \;\&\; y' \Vdash'_{v'} \psi_2 \Rightarrow \\ &\Rightarrow w' \Vdash'_{v'} \psi_1 \bullet \psi_2 \end{aligned}$$

$$\begin{aligned} w' \Vdash'_{v'} \psi_1 \bullet \psi_2 &\Rightarrow (\exists x', y' \in W') C'x'y'w' \;\&\; x' \Vdash'_{v'} \psi_1 \;\&\; y' \Vdash'_{v'} \psi_2 \overset{\text{i.h.}}{\Rightarrow} \\ &\Rightarrow (\exists x', y' \in W') C'x'y'w' \;\&\; x' \Vdash_v \psi_1 \;\&\; y' \Vdash_v \psi_2 \overset{(\text{max})}{\Rightarrow} \\ &\Rightarrow w \Vdash_v \psi_1 \bullet \psi_2 \end{aligned}$$

$$\begin{aligned} w \Vdash_v \psi_1 \blacktriangleright \psi_2 &\Rightarrow (\exists x, y \in W) Cxw'y \;\&\; x \Vdash_v \psi_1 \;\&\; y \Vdash_v \psi_2 \overset{(\text{min})}{\Rightarrow} \\ &\Rightarrow (\exists x, y \in W) C'x'w'y' \;\&\; x \Vdash_v \psi_1 \;\&\; y \Vdash_v \psi_2 \overset{\text{i.h.}}{\Rightarrow} \\ &\Rightarrow (\exists x', y' \in W') C'x'w'y' \;\&\; x' \Vdash'_{v'} \psi_1 \;\&\; y' \Vdash'_{v'} \psi_2 \Rightarrow \\ &\Rightarrow w' \Vdash'_{v'} \psi_1 \blacktriangleright \psi_2 \end{aligned}$$

$$\begin{aligned} w' \Vdash'_{v'} \psi_1 \blacktriangleright \psi_2 &\Rightarrow (\exists x', y' \in W') C'x'w'y' \;\&\; x' \Vdash'_{v'} \psi_1 \;\&\; y' \Vdash'_{v'} \psi_2 \overset{\text{i.h.}}{\Rightarrow} \\ &\Rightarrow (\exists x', y' \in W') C'x'w'y' \;\&\; x' \Vdash_v \psi_1 \;\&\; y' \Vdash_v \psi_2 \overset{(\text{max})}{\Rightarrow} \\ &\Rightarrow w \Vdash_v \psi_1 \blacktriangleright \psi_2 \end{aligned}$$

and similar argument proves the case of $\blacktriangleleft$. Finally,

$$w' \Vdash'_{v'} \iota\delta \iff w' \Vdash_v \iota\delta$$

by definition. □

So far so good, but we would like to have

$$w' \Vdash'_{v'} \iota\delta \iff C'w'w'w'.$$

To achieve it we will apply the same trick as in the proof of the Representation Theorem 4.3.

Let

$$\mathcal{T} \overset{\text{def}}{=} \langle \mathcal{P}(W'), \cap, \sim, \circ^{C'}, \rhd^{C'}, \lhd^{C'}, \{w' : w' \Vdash'_{v'} \iota\delta\} \rangle.$$

Note that we do not know whether $\mathcal{T} \in \mathsf{RKA}$. It is easy to check that for all formula ψ

$$\langle \mathcal{T}, v' \rangle \models \psi = 1 \iff \langle W', C', v' \rangle \models \psi,$$

and so $\langle \mathcal{T}, v' \rangle \not\models \varphi = 1$. First we note that

$$w' \Vdash'_{v'} \iota\delta \Rightarrow w' \in Id^{C'}.$$

Indeed,

$$w' \Vdash'_{v'} \iota\delta \iff w' \Vdash_v \iota\delta \iff Cw'w'w' \Rightarrow C'w'w'w' \iff w' \in Id^{C'}.$$

In the proof of the Representation Theorem 4.3 we met exactly the same problem. There we had an algebra $\mathcal{B}$ whose identity was not the "real" one. Then we could split the "bad" elements of the base of $\mathcal{B}$ and then we got a $\mathcal{B}' \in \mathsf{RKA}$ isomorphic to $\mathcal{B}$. Now, if we do the same trick with $\mathcal{T}$ instead of $\mathcal{B}$, then we get a $\mathcal{T}' \in \mathsf{RKA}$ isomorphic to $\mathcal{T}$, where W'' and C'' are defined using W' and C' precisely in the same way as in the representation proof. (There a certain W' and C' were defined using some W and C.) Note that since W' was finite, so is W''. Then $\mathcal{T}'$ is the subalgebra of the finitely based full RKA

$$\langle \mathcal{P}(W''), \cap, \sim, \circ^{C''}, \rhd^{C''}, \lhd^{C''}, Id^{C''} \rangle.$$

By $\mathcal{T} \cong \mathcal{T}'$, we have

$$\langle \mathcal{T}', v'' \rangle \not\models \varphi = 1,$$

where v'' is the valuation determined by v', i.e., $v''(\varphi)$ is the image of $v'(\varphi)$ along the isomorphism. Thus, by Lemma 5.3,

$$\langle W'', C'', v'' \rangle \not\models \varphi.$$

That is we constructed a finite model refuting the non-valid formula φ. □

Now we will prove the lemmas that were used in the proof of the Completeness Theorem 3.1.

Proof. (Lemma 5.1) By propositional axioms and Modus Ponens, $\equiv_\Gamma$ is an equivalence relation.

Now assume that $\Gamma \vdash A \leftrightarrow B$ & $\Gamma \vdash C \leftrightarrow D$. Then, by propositional calculus again, $\Gamma \vdash \neg A \leftrightarrow \neg B$ and $\Gamma \vdash A \wedge C \leftrightarrow B \wedge D$. By the substitution rule (ix) for $\bullet$, $\Gamma \vdash A \bullet C \leftrightarrow B \bullet D$. For $\blacktriangleright$ we have

$\Gamma \vdash \neg((B \blacktriangleright D) \wedge (\neg(B \blacktriangleright D)))$	$\Leftrightarrow$	(propositional calculus)
$\Gamma \vdash (B \blacktriangleright D) \wedge (\neg(B \blacktriangleright D)) \leftrightarrow \bot$	$\Leftrightarrow$	(vi)
$\Gamma \vdash (B \bullet (\neg(B \blacktriangleright D))) \wedge D \leftrightarrow \bot$	$\Leftrightarrow$	(propositional calculus and (ix))
$\Gamma \vdash (A \bullet (\neg(B \blacktriangleright D))) \wedge C \leftrightarrow \bot$	$\Leftrightarrow$	(vi)
$\Gamma \vdash (A \blacktriangleright C) \wedge (\neg(B \blacktriangleright D)) \leftrightarrow \bot$	$\Leftrightarrow$	(propositional calculus)
$\Gamma \vdash A \blacktriangleright C \rightarrow B \blacktriangleright D.$		

By symmetry, $\Gamma \vdash B \blacktriangleright D \rightarrow A \blacktriangleright C$.

The proof for $\blacktriangleleft$ is similar. Thus we have proved the lemma. □

Proof. (Lemma 5.2) 1 is true by propositional calculus.

The proof of 2 is as follows. Let $x_1 = x_2$ be an equation of Ax, v be an arbitrary valuation and $v(x_1) = \overline{p_1}$ and $v(x_2) = \overline{p_2}$. Then, since $p_1 \leftrightarrow p_2$ is an axiom of CARL, $\Gamma \vdash p_1 \leftrightarrow p_2$. Thus $\overline{p_1} = \overline{p_2}$, whence $\langle \mathcal{F}_\Gamma, v \rangle \models x_1 = x_2$.

Let $(x_1 = x_2) \Rightarrow (x_3 = x_4)$ be a quasi-equation of Ax. Let v be a valuation and $v(x_i) = \overline{p_i}$ for $1 \leq i \leq 4$. Then

$$\frac{p_1 \leftrightarrow p_2}{p_3 \leftrightarrow p_4}$$

is an instance of an inference rule of CARL. Now assume that $\overline{p_1} = \overline{p_2}$ is true in $\mathcal{F}_\Gamma$, whence $\Gamma \vdash p_1 \leftrightarrow p_2$. By the rule above, $\Gamma \vdash p_3 \leftrightarrow p_4$, hence $\overline{p_3} = \overline{p_4}$ in $\mathcal{F}_\Gamma$. Thus $\mathcal{F}_\Gamma \models x_1 = x_2 \Rightarrow x_3 = x_4$. □

Proof. (Lemma 5.3) Easy by definition. □

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