On Representable Ordered Residuated Semigroups

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Abstract

We show that the equational theory of representable lattice-ordered residuated semigroups is not finitely axiomatizable. We apply this result to the problem of completeness of substructural logics.

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1 Introduction

Residuated algebras have been extensively investigated in the literature, partly because of their connection to substructural logics. A residuated algebra is *representable* if it is isomorphic to a family of binary relations and the operations are interpreted as "natural" operations on binary relations — see the precise definition below. An important line of research is to determine precisely which classes of representable residuated algebras have finitely axiomatizable equational or quasiequational theories, since such axiomatizability results yield weak or strong completeness results for substructural logics. Indeed, families of binary relations as semantics for substructural logics have been proposed by various researchers, e.g., for the Lambek calculus (LC) by van Benthem and for relevance logics by Dunn and Maddux. Completeness results of this kind include the completeness of the LC [AM94] and the completeness of the relevance logic with mingle **RM** [Ma10]. We address a similar problem here by looking at the problem of expanding the similarity type of LC with (static) conjunction and disjunction.

In the remainder of this section we define representable algebras and recall a finite axiomatization of representable lower semilattice-ordered residuated semigroups. In the next section, we look at the possibility of extending the similarity type by including join as well. Finally, we will look at the implications of the main result to substructural logics.

Definition 1.1 1. *A* proper relation algebra *(a* PRA*) is* $\mathfrak{A} = (A, +, \cdot, -, ;, ^\smile, 1', 0, 1)$ *such that A consists of subsets of an equivalence relation* W*, the* unit *of* $\mathfrak{A}$*, and* $+$ *is union,* $\cdot$ *is intersection,* $-$ *is set difference w.r.t.* W*,* $0 = \emptyset$*,* $1 = W$ *and*

$$x \,;\, y = \{(u, v) \in W : \exists w((u, w) \in x \ \&\ (w, v) \in y)\} \tag{1}$$

$$x^\smile = \{(u, v) \in W : (v, u) \in x\} \tag{2}$$

$$1' = \{(u, v) \in W : u = v\} \tag{3}$$

The class Rs *of* relation set algebras *is the subclass of* PRA *with square units:* $W = U \times U$ *for some set* U*.*

The class RRA *of* representable relation algebras *is the closure of* PRA *under isomorphic copies:* RRA = **I**PRA*.*

2. *Given a class K of algebras of similarity type σ and a similarity type τ such that the elements of τ are definable (by fixed terms of σ) in K, the* τ-subreduct *of K is the class of subalgebras of the τ-reducts of the elements of K. We denote the τ-subreduct of K as $\mathsf{K}(\tau)$.*

We refer the reader to [HH02] for more details on RRA. We recall that

$$\mathsf{RRA} = \mathbf{SP}\mathsf{Rs}$$

i.e., we close Rs under (isomorphic copies of) subalgebras of products. Hence the (quasi)equational theories of $\mathsf{RRA}(\tau)$ and $\mathsf{Rs}(\tau)$ coincide.

Note that RRAs are residuated: the following definition defines the right and left residuals of composition

$$x \setminus y = -(x^{\smile} \,;\, -y) \text{ and } x \,/\, y = -(-x \,;\, y^{\smile}) \tag{4}$$

and we can define converse-complement as well: $\sim x = -(x^{\smile})$. The extension of these operations in a PRA with unit W is as follows:

$$x \setminus y = \{(u,v) \in W : \forall w((w,u) \in x \Rightarrow (w,v) \in y)\} \tag{5}$$

$$x \,/\, y = \{(u,v) \in W : \forall w((v,w) \in y \Rightarrow (u,w) \in x)\} \tag{6}$$

$$\sim x = \{(u,v) \in W : (v,u) \notin x\} \tag{7}$$

We recall from [AM94] that representable lower semilattice-ordered residuated semigroups are finitely axiomatizable.

Theorem 1.2 *The variety $\mathsf{RRA}(\cdot\,, ;, \setminus, /)$ generated by $\mathsf{Rs}(\cdot\,, ;, \setminus, /)$ is finitely axiomatizable.*

Indeed, we showed in [AM94] that the following set of quasiequations axiomatizes the class $\mathsf{RRA}(\cdot\,, ;, \setminus, /)$. Below $x \leq y$ abbreviates $x \cdot y = x$.

- $\cdot$ is a (lower) semilattice
- ; is associative and monotone w.r.t. $\leq$:

$$(x \cdot x') \,;\, (y \cdot y') \leq x \,;\, y \tag{8}$$

- if $x \leq y$, then $x \setminus y$ behaves similarly to a unit element for ;:

$$z \leq z \,;\, ((x \cdot y) \setminus y) \text{ and } z \leq ((x \cdot y) \setminus y) \,;\, z \tag{9}$$

  and similarly for /:

$$z \leq z \,;\, (y \,/\, (x \cdot y)) \text{ and } z \leq (y \,/\, (x \cdot y)) \,;\, z \tag{10}$$

- $\setminus$ and / are the right and left residuals of ;:

$$x \,;\, y \leq z \iff y \leq x \setminus z \iff x \leq z \,/\, y \tag{11}$$

Pratt [Pr90] observed that the quasiequations (11) can be replaced by the following equations:

$$x \setminus (y \cdot y') \leq x \setminus y \tag{12}$$

$$x \,;\, (x \setminus y) \leq y \leq x \setminus (x \,;\, y) \tag{13}$$

and the corresponding equations for /. Indeed, assuming $x;y \leq z$, we have $x \setminus (x;y) = x \setminus (x;y \cdot z) \leq x \setminus z$ by (12), and hence $y \leq x \setminus z$ by the second part of (13). If we assume $y \leq x \setminus z$, then $x \,;\, y \leq x \,;\, (x \setminus z)$ by monotonicity, and hence $x \,;\, y \leq z$ by the first part of (13). Since (12) and (13) are easily seen to be valid in $\mathsf{RRA}(\cdot\,, ;, \setminus, /)$, we are done.

In the next section, we look at the possibility of extending the similarity type by including join as well.

2 Main result

We observed in [AM94] that the quasiequational theory of representable lattice-ordered residuated semigroups is not finitely axiomatizable. We strengthen this result below.

Theorem 2.1 *The equational theory of* $\mathsf{RRA}(+,\cdot,;,\backslash,/)$ *is not finitely axiomatizable. The same holds if we expand the similarity type by any set of operations definable in* RRA.

Proof: Maddux [Ma89] defines non-representable, finite, integral ($1'$ is an atom), symmetric (every element is self converse) relation algebras $\mathfrak{A}_n$ (for $n \in \omega$) whose ultraproduct is representable. Andréka [An91] shows that already the $\{+,\cdot,;\}$-reduct of $\mathfrak{A}_n$ is not representable, whence τ-subreducts of RRA such that $\{+,\cdot,;\} \subseteq \tau$ are not finitely axiomatizable. Hence the quasivariety $\mathsf{RRA}(+,\cdot,;,\backslash,/)$ is not finitely axiomatizable. Here we show that the non-representability of the $\{+,\cdot,;,\backslash,/\}$-reduct $\mathfrak{B}_n$ of $\mathfrak{A}_n$ is witnessed by an equation.

We recall that $\mathfrak{A}_n$ has the following atoms (minimal, non-zero elements): identity $1'$, q_i for $1 \leq i \leq m$, and p_j for $1 \leq j \leq n$ with $m = 3 \cdot n!$. Composition is defined so that

$$q_{i+1} \leq p_1 \,;\, q_i \qquad \text{for every } 1 \leq i < m \tag{14}$$

$$0 = q_r \cdot q_s \,;\, q_t \qquad \text{for every } 1 \leq r,s,t \leq m \tag{15}$$

$$0 = p_l \cdot p_l \,;\, p_l \qquad \text{for every } 1 \leq l \leq n \tag{16}$$

and $q_m \backslash q_m = 1'$. Now assume for contradiction that the $\{+,\cdot,;\}$-reduct of $\mathfrak{A}_n$ would be representable. Since $q_m \neq 0$, we have $(u_m, v) \in q_m$ for some u_m and v. By (14), we get u_i such that $(u_{i+1}, u_i) \in p_1$ and $(u_i, v) \in q_i$ for every $1 \leq i < m$. Since $q_i \neq q_j$ for $i \neq j$, $U = \{u_1, \ldots, u_m\}$ consists of distinct points. Hence $(u_i, u_j) \in x$ for some non-identity atom x. By (15), x cannot be any q_k, whence x must be p_l for some $1 \leq l \leq n$. It follows that a representation of the $\{+,\cdot,;\}$-reduct of $\mathfrak{A}_n$ would require a coloring of the edges of the full graph on U with $|U| \geq 3 \cdot n!$ using the colors p_i (for $1 \leq i \leq n$) but without monochromatic triangles by (16) — an impossible task, see [Di05]. See Figure 1, where every dotted arrow should have a color p_i for some $1 \leq i \leq n$.

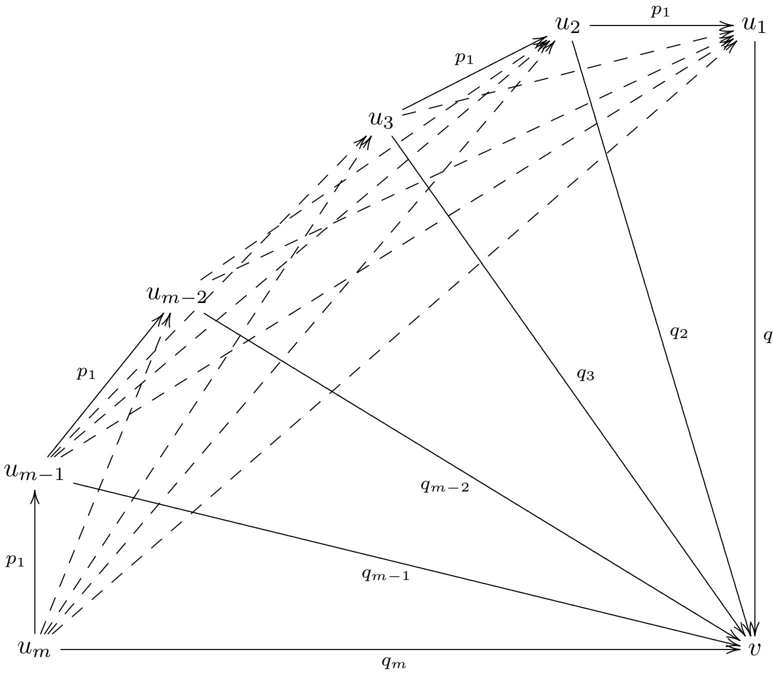

Figure 1: The reason for non-representability

Next we define the equation e_n witnessing the non-representability of $\mathfrak{B}_n$. Below we use the abbreviation x^{i+1} that is recursively defined by $x^{i+1} = x\,;x^i$ with the convention that $x^0\,;y = y$ for every term y. We also assume that the operations bind in this order: residuals, composition, meet and join. For instance, $a \cdot [(b \setminus c)\,;d]$ can be written as $a \cdot b \setminus c\,;d$.

Let x_i, y_j and z_0 be distinct variables for $1 \leq i \leq n$ and $1 \leq j \leq m$ and let Z stand for $\sum\{x_i : 1 \leq i \leq n\} + \sum\{y_j : 1 \leq i \leq m\} + (y_m \setminus y_m)$. We define the equation e_n for $n \in \omega$ as

$$\sigma_n\,;(\alpha_n \cdot \beta_n \cdot \gamma_n \cdot \delta_n) \leq z_0$$

where

$$\begin{aligned}
\sigma_n &= y_m \cdot [x_1\,;[y_{m-1} \cdot [(x_1 \cdot x_1 \setminus Z)\,;[y_{m-2} \cdot (x_1 \cdot x_1 \setminus Z \cdot (x_1\,;x_1) \setminus Z)\,;[\ldots]]]]] \\
\alpha_n &= \prod\{[x_1^{m-i}\,;(y_i \cdot y_j)] \setminus z_0 : 1 \leq i \neq j \leq m\} \\
\beta_n &= \prod\{[x_1^{m-i}\,;(x_l \cdot x_l\,;x_l)\,;y_j] \setminus z_0 : 1 \leq i,j \leq m, 1 \leq l \leq n\} \\
\gamma_n &= \prod\{[x_1^{m-i}\,;(y_i \cdot y_m \setminus y_m\,;y_{m-j})] \setminus z_0 : 0 < i \neq j < m\} \\
\delta_n &= \prod\{[x_1^{m-i}\,;(y_i \cdot y_k\,;y_j)] \setminus z_0 : 1 \leq i < m, 1 \leq j,k \leq m\}
\end{aligned}$$

We claim that e_n

1. fails in $\mathfrak{B}_n$
2. is valid in representable algebras.

For item 1, let ι be the evaluation of the variables

$$\iota(x_i) = p_i \qquad \iota(y_j) = q_j \qquad \iota(z_0) = 0$$

Then $\iota(RHS(e_n)) = \iota(z_0) = 0$. On the other hand, $\iota(LHS(e_n)) \neq 0$ because of the following. First note that $\iota(Z) = 1$, since $\iota(y_m \setminus y_m) = q_m \setminus q_m = 1'$, i.e., Z is the sum of all atoms. Then we have $\iota(\tau \setminus Z) = 1$ for any term τ. In $\mathfrak{B}_n$, $q_{i+1} \leq p_1\,;q_i$ for every $1 \leq i < m$, whence $\iota(\sigma_n) \neq 0$. Furthermore, the terms on the left of $\setminus z_0$ in α_n, β_n, γ_n and δ_n evaluate to 0, since the meet of distinct atoms is 0 (for α_n), there are no monochromatic triangles for p_i (for β_n), $q_m \setminus q_m \leq 1'$ (for γ_n) and there are no q-triangles (for δ_n). Hence $\iota(\alpha_n) = \iota(\beta_n) = \iota(\gamma_n) = \iota(\delta_n) = 1$. Thus $\iota(LHS(e_n)) \neq 0$.

For item 2, let $\mathfrak{C} = (C, +, \cdot, ;, \setminus, /)$ be a representable algebra and ι be an arbitrary valuation of the variables. Assume that $(u_m, u) \in \iota(LHS(e_n))$. Then there is v such that $(u_m, v) \in \iota(\sigma_n)$ and $(v, u) \in \iota(\alpha_n \cdot \beta_n \cdot \gamma_n \cdot \delta_n)$. By $(u_m, v) \in \iota(\sigma_n)$, we have $(u_{i+1}, u_i) \in \iota(x_1)$, whence $(u_m, u_i) \in \iota(x_1^{m-i})$, and $(u_i, v) \in \iota(y_i)$ for every $1 \leq i < m$. See Figure 2.

First we consider the case where there are $i \neq j$ such that $u_i = u_j$. In this case, we have $(u_i, v) \in \iota(y_i) \cdot \iota(y_j)$ and thus $(u_m, v) \in \iota(x_1^{m-i}\,;(y_i \cdot y_j))$, by $(u_m, u_i) \in \iota(x_1^{m-i})$. Since $(v, u) \in \iota(\alpha_n)$, we get $(u_m, u) \in \iota(z_0) = \iota(RHS(e_n))$.

Now let us assume that all the u_is are different. Note that

$$(u_i, u_j) \in \iota(Z) = \iota(\sum\{x_k : 1 \leq k \leq n\} + \sum\{y_l : 1 \leq l \leq m\} + (y_m \setminus y_m))$$

for every $1 \leq j < i \leq m$, by $(u_m, v) \in \iota(\sigma_n)$ and $(u_k, u_i) \in \iota(x_1^{k-i})$ for every $i < k \leq m$. If there are $j < i$ and k such that $(u_i, u_j) \in \iota(y_k)$, then $(u_m, v) \in \iota(x_1^{m-i}\,;(y_i \cdot y_k\,;y_j))$, since $(u_m, u_i) \in \iota(x_1^{m-i})$, $(u_i, v) \in \iota(y_i)$ and $(u_j, v) \in \iota(y_j)$. Hence $(u_m, u) \in \iota(z_0)$ by $(v, u) \in \iota(\delta_n)$, i.e., $(u_m, u) \in \iota(RHS(e_n))$. If there are $j < i$ such that $(u_i, u_j) \in \iota(y_m \setminus y_m)$, then $(u_m, v) \in \iota(x_1^{m-i}\,;(y_i \cdot y_m \setminus y_m\,;y_j))$ by a similar argument. Hence $(u_m, u) \in \iota(z_0)$ by $(v, u) \in \iota(\gamma_n)$, i.e., $(u_m, u) \in \iota(RHS(e_n))$. It remains to consider the case where, for all $j < i$, there is l such that $(u_i, u_j) \in \iota(x_l)$. By $m = 3 \cdot n!$, we have that for some $1 \leq k < j < i \leq m$, (u_i, u_j, u_k) is a monochromatic triangle the edges of which are colored with some $\iota(x_l)$. Hence $(u_m, v) \in \iota(x_1^{m-i}\,;(x_l \cdot x_l\,;x_l)\,;y_k)$ for some l. Then $(u_m, u) \in \iota(z_0)$, by $(v, u) \in \iota(\beta_n)$. Hence $(u_m, u) \in \iota(RHS(e_n))$ in this case as well. Thus e_n is indeed valid in representable algebras. ∎

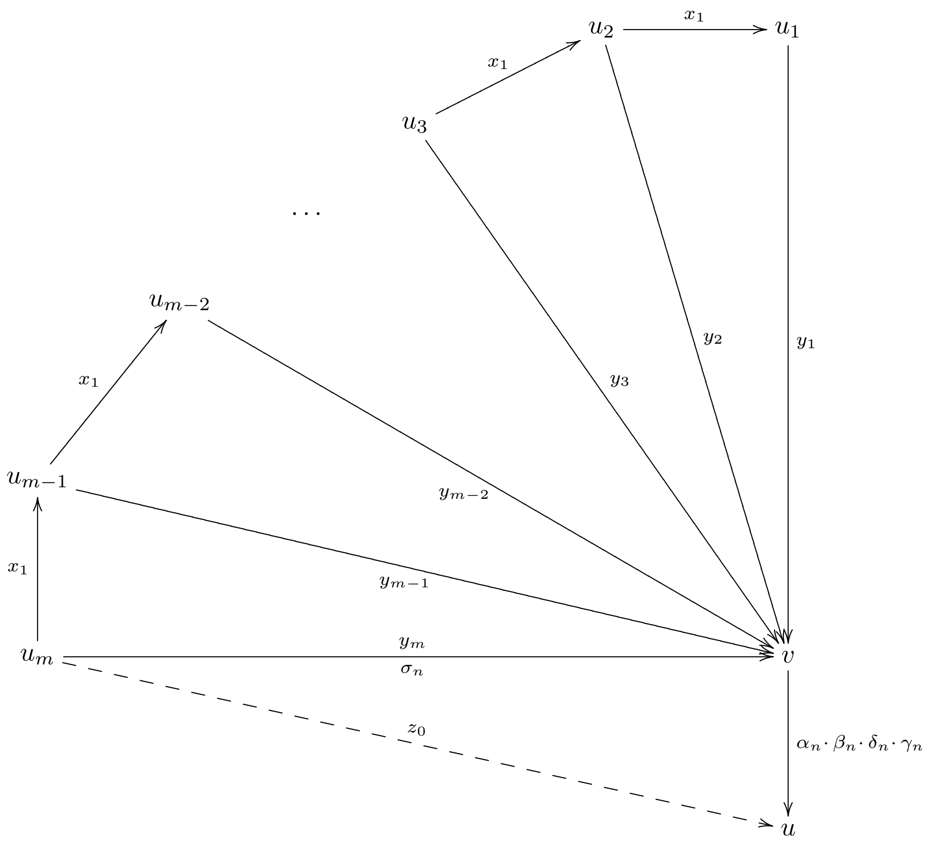

Figure 2: The validity of e_n

3 Substructural logics

In this section, we explain the connection of the main result to substructural logics.

Lambek calculus In [AM94], we showed that the Lambek calculus with static conjunction is complete w.r.t. relational semantics $\mathsf{Rel}(\mathrm{LC})$ consisting of binary relations, where fusion ($\bullet$ in the original formulation and $;$ in our notation) is interpreted as relation composition, conjunction ($\wedge$ or $\cdot$) as intersection and $\backslash$ and $/$ as the two residuals of composition. There is one subtle but significant difference between this semantics $\mathsf{Rel}(\mathrm{LC})$ and the subreduct $\mathsf{RRA}(\cdot\,, ;, \backslash, /)$ of RRA. Namely, in a PRA the residuals are defined as in (4), whence their extension is as in (5) and (6) where W is the unit of a PRA, an equivalence relation. On the other hand, in $\mathsf{Rel}(\mathrm{LC})$ we do not require that the residuals are computed w.r.t. an equivalence relation W. In fact, in the completeness proof for LC, W is transitive, irreflexive ($(x, x) \notin W$) and antisymmetric (for $x \neq y$, $(x, y) \in W$ implies $(y, x) \notin W$), and the residuals are defined as in (5) and (6) with this irreflexive W as a parameter. Since W is irreflexive, so are $x \,\backslash\, y$ and $x \,/\, y$, and (9) and (10) fail in accordance with the absence of the corresponding rules in the original LC [La58], while these equations are valid in $\mathsf{RRA}(\cdot\,, ;, \backslash, /)$. Thus Theorem 2.1 does not seem to answer the problem of the weak completeness of the LC extended with conjunction and disjunction. But we claim that essentially the same proof yields the following.

Corollary 3.1 *No extension with finitely many axioms of the Lambek calculus with disjunction and conjunction is weakly complete w.r.t. relational semantics.*

Proof: In this case, the representable algebras corresponding to $\mathsf{Rel}(\mathrm{LC})$ have the form $\mathfrak{D} = (D, +, \cdot, ;, \backslash, /)$ where D consists of subsets of a transitive relation W, $+$ is union, $\cdot$ is intersection, $;$ is relation composition as defined in (1) and the residuals are defined as in (5) and (6) using W as a parameter. An inspection of the proof of Theorem 2.1 reveals that we did not use anywhere that the unit of the representable algebra would be an equivalence relation, and that, in particular, $\iota(y_m \setminus y_m)$ would be reflexive.[1] Hence the same argument as above shows that $\mathfrak{B}_n$ is not representable over a transitive relation while a non-trivial ultraproduct of $\mathfrak{B}_n$ is representable, and that the equation e_n fails in $\mathfrak{B}_n$ while it is valid in algebras representable over transitive relations. ∎

The completeness of the Lambek calculus augmented with disjunction but without conjunction w.r.t. relational semantics seems to be an open problem.

Relevance logic Finally, let us mention a problem related to relevance logic [AB75, ABD92]. Recall that relevance logics can be soundly interpreted over families of binary relations. Let $\mathsf{Rs}(+, \cdot, ;, \backslash, \sim)^{cd}$ be that subclass of $\mathsf{Rs}(+, \cdot, ;, \backslash, \sim)$, the $\{+, \cdot, ;, \backslash, \sim\}$-subreduct of Rs, where each algebra is commutative $x;y = y;x$ and dense (or square-increasing) $x \le x;x$. Let $\mathfrak{A} \in \mathsf{Rs}(+, \cdot, ;, \backslash, \sim)^{cd}$ such that $A \subseteq \mathcal{P}(W)$ for some set W of the form $U \times U$ and v be a valuation of the propositional atoms into A. We extend v to compound formulas by interpreting conjunction as intersection, disjunction as union, fusion as relation composition (1), implication as the residual operation (5) (right and left residuals coincide by commutativity) and relevant negation as converse-complement (7).[2] We define

$$\mathfrak{A} \models \varphi \iff \mathsf{Id} \subseteq v(\varphi)$$

where $\mathsf{Id} = \{(u, v) \in W : u = v\}$. Then the relevance logic **R** [RM73] is sound w.r.t. $\mathsf{Rs}(+, \cdot, ;, \backslash, \sim)^{cd}$, and the relevance logic **RM** with the mingle axiom is sound and complete w.r.t. the semantics $\mathsf{Rs}(+, \cdot, ;, \backslash, \sim)^{cdt}$, that subclass of $\mathsf{Rs}(+, \cdot, ;, \backslash, \sim)^{cd}$ where each algebra is transitive (or square-decreasing) $x ; x \le x$, see [Ma10]. On the other hand, completeness fails without the mingle axiom: the logic of $\mathsf{Rs}(+, \cdot, ;, \backslash, \sim)^{cd}$ is not finitely axiomatizable [Mi09].

The question is whether we could achieve completeness in the absence of relevant negation. To formulate the problem algebraically:

Is the equational theory of $\mathsf{Rs}(+, \cdot, ;, \backslash)^{cd}$ finitely axiomatizable?

We note that the quasiequational theory of $\mathsf{Rs}(+, \cdot, ;, \backslash)^{cd}$ is not finitely axiomatizable, since the $\{+, \cdot, ;, \backslash\}$-reduct of the algebras $\mathfrak{A}_n$ from [Mi09] are not representable while their ultraproduct is. We answer the above problem negatively in the forthcoming paper [HM10]. In passing we note that the algebras $\mathfrak{A}_n$ in the proof of Theorem 2.1 are commutative, but they are not dense, since there are no monochromatic triangles. And this latter fact seems to be crucial in the above proof, thus the non-finite axiomatizability of $\mathsf{Rs}(+, \cdot, ;, \backslash)^{cd}$ requires a new construction. The problem of finite axiomatizability of the equational theory of $\mathsf{Rs}(+, ;, \backslash)^{cd}$ seems to be open.

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[1] We used transitivity of the unit, but that holds for W in this case as well.

[2] In passing we note that we could omit fusion from the language, since we can define $x ; y$ as $\sim (y \backslash \sim x)$.

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