

Representable Domain Algebras are not Finitely Axiomatizable

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Abstract

The family of domain algebras provide an elegant formal system for automated reasoning about programme verification. Their primary models are algebras of relations, viz. representable domain algebras. We prove that, even for the minimal signature consisting of the domain and composition operations, the class of representable domain algebras is not finitely axiomatizable. Then we show similar results for extended similarity types of domain algebras.

Keywords: domain algebras, relation algebras, finite axiomatizability

1 Introduction

Domain algebras provide an elegant, one-sorted formalism for automated reasoning about program and system verification [DS08a, DS08b]. Traditionally, similar algebraic formalisms (like dynamic algebras [Pr90, Pr91] and Kleene algebras with test [Ko97]) used a two-sorted approach: there is one sort for states and another sort for actions, and some operations mapping between actions and states. Using the domain operation d , one sort (sort of actions) is enough, since states can be modelled as those actions a for which we have $a = d(a)$. Such a one-sorted approach is simpler and more suitable for automated reasoning, see [DS08a, DS08b] and the references therein for more details.

Using the domain operation we can express if an action is enabled at a certain state: $d(a)$ consists of those states at which action a can be taken. Besides the domain operation, domain algebras contain an operation for modelling sequential composition of actions (or processes), we will denote it by $;$. Other connectives that can be included are: join $+$ for modelling non-deterministic choice, identity $1'$ for modelling the ineffective action skip, zero 0 for modelling the abortive action, and the reflexive–transitive closure $*$ of $;$ for modelling iteration. If we want the set of states, or domain elements, to have a more expressive structure than a (semi)lattice, then we can include boolean negation on states. In the one-sorted approach, this can be done by including an antidomain operation a : $a(a)$ consists of those states where action a is not enabled. The dual of domain is given by the range operation: $r(a)$ consists of those states that can be reached via action a . Depending of the choice of operations, one can define (anti)domain(–range) semigroups/monoids/semirings, etc., see [DS08a, DS08b, DJS09] for axiomatizations of these classes of algebras.

Arguably the most intuitive and important semantics of domain algebras is provided by binary relations. An action is modelled as the binary relation of input–output pairs, states as subidentity relations, and the operations as “natural” operations on binary relations. For instance,

$$d(a) = \{(s, s) : (s, t) \in a \text{ for some } t\} \tag{1}$$

$$a ; b = \{(s, t) : (s, u) \in a \text{ and } (u, t) \in b \text{ for some } u\} \tag{2}$$

see Definition 2.2 below for exact details. One of the fundamental questions is whether domain algebras are complete with respect to this semantics, i.e., whether every semantically valid (quasi)equation can be derived from the axioms. The recent publication [DJS09] poses some open problems regarding the completeness of domain algebras. These questions boil down to the problem of whether the axiomatically defined domain algebras are isomorphic to algebras of binary relations, i.e., using the slogan of algebraic logic, whether they are representable. We can formulate the question in a more general setting: are the (quasi)equational theories of representable domain algebras finitely axiomatizable?

Representable domain algebras are fragments of Tarski's representable relation algebras RRAs, see, e.g., [HH02, Ma06]. Finite axiomatizability of these fragments have been extensively investigated, see [Sc91, Mi04] for surveys. We recall that a representable relation algebra is a boolean set algebra of binary relations expanded with the following extra boolean operators: the identity constant $1'$ (interpreted as the identity, or diagonal, relation), the unary converse operation \smile (interpreted as the inverse relation) and the binary composition operation $;$ (see (2) above). In RRAs, the domain, range and antidomain operations are definable as

$$\mathbf{d}(x) = 1' \cdot (x ; x^\smile) \quad \text{and} \quad \mathbf{r}(x) = 1' \cdot (x^\smile ; x) \quad \text{and} \quad \mathbf{a}(x) = -\mathbf{d}(x)$$

The problem is that this definition explicitly uses the converse and meet \cdot operations, and the $\{;, \cdot, \smile, 1'\}$ -reduct of RRA has non-finitely axiomatizable equational and quasiequational theories, cf. [HM00]. If we take \mathbf{d} and \mathbf{r} as basic operations, then we can avoid the use of converse and meet and hope for finite axiomatizability. As it turns out, the quasiequational theory is still not finitely based, cf. Theorem 2.3, but we conjecture that the equational theory is finitely based, see [Ho97] for axiomatizing the equational theory of antidomain monoids.

The rest of the paper is organized as follows. In the next section, we recall the precise definitions of domain algebras and representable algebras, and state our main result that representable domain algebras are not finitely axiomatizable even for the minimal signature $\{\mathbf{d}, ;\}$ consisting of domain and composition. In Section 3, we define domain algebras and show that they are not representable, and in Section 4, we characterize representability by a two-player game, and show that a non-principal ultraproduct of our domain algebras are representable, establishing the main result. In Section 5, we look at the problem of expanding the similarity type of domain algebras with extra operations, and in Section 6, we conclude by stating some open problems.

2 Basics and main result

First we recall the basic definitions of domain algebras from [DJS09].

Definition 2.1 A domain–range monoid $\mathfrak{A} = (A, \mathbf{d}, \mathbf{r}, ;, 1')$ consists of a monoid $(A, ;, 1')$ and unary operations $\mathbf{d}, \mathbf{r} : A \rightarrow A$ such that the axioms (D1) – (D5) and (R1) – (R5) are satisfied:

$$\begin{array}{ll} (D1) & \mathbf{d}(x) ; x = x \\ (D2) & \mathbf{d}(x ; y) = \mathbf{d}(x ; \mathbf{d}(y)) \\ (D3) & \mathbf{d}(\mathbf{d}(x) ; y) = \mathbf{d}(x) ; \mathbf{d}(y) \\ (D4) & \mathbf{d}(x) ; \mathbf{d}(y) = \mathbf{d}(y) ; \mathbf{d}(x) \\ (D5) & \mathbf{d}(\mathbf{r}(x)) = \mathbf{r}(x) \\ (R1) & x ; \mathbf{r}(x) = x \\ (R2) & \mathbf{r}(x ; y) = \mathbf{r}(\mathbf{r}(x) ; y) \\ (R3) & \mathbf{r}(x ; \mathbf{r}(y)) = \mathbf{r}(x) ; \mathbf{r}(y) \\ (R4) & \mathbf{r}(x) ; \mathbf{r}(y) = \mathbf{r}(y) ; \mathbf{r}(x) \\ (R5) & \mathbf{r}(\mathbf{d}(x)) = \mathbf{d}(x) \end{array}$$

A domain monoid $\mathfrak{A} = (A, \mathbf{d}, ;, 1')$ is defined similarly, by dropping all the conditions that involve \mathbf{r} . A domain- or domain–range semigroup is an $1'$ -free subreduct of the corresponding monoid.

A antidomain monoid is $\mathfrak{A} = (A, \mathbf{a}, ;, 1')$ with a monoid $(A, ;, 1')$ and unary operation $\mathbf{a} : A \rightarrow A$

satisfying the following axioms

- (A1) $\mathbf{a}(x); x = 0$
- (A2) $x; 0 = 0$
- (A3) $\mathbf{a}(x); \mathbf{a}(y) = \mathbf{a}(y); \mathbf{a}(x)$
- (A4) $\mathbf{a}(\mathbf{a}(x)); x = x$
- (A5) $\mathbf{a}(x) = \mathbf{a}(x; y); \mathbf{a}(x; \mathbf{a}(y))$
- (A6) $\mathbf{a}(x; y); x = \mathbf{a}(x; y); x; \mathbf{a}(y)$

where 0 is defined as $\mathbf{a}(1')$.

We will use the term domain algebra for any algebra \mathfrak{A} of similarity type τ such that $\{;, \mathbf{d}\} \subseteq \tau \subseteq \{;, \mathbf{d}, \mathbf{r}, \mathbf{a}\}$ and \mathfrak{A} satisfies the relevant axioms above.

The set of domain elements D of a domain algebra is the set of all elements e such that $\mathbf{d}(e) = e$.

If D is the set of domain elements of \mathfrak{A} then $(D, ;)$ is a lower semilattice. We write \leq for the ordering induced by this operation: $e \leq e'$ iff $e; e' = e$. In an antidomain monoid, we define $\mathbf{d}(x) = \mathbf{a}(\mathbf{a}(x))$. In an antidomain monoid, the domain elements D form a boolean algebra $\mathfrak{D} = (D, \cdot, -)$ with the operations defined as $x \cdot y = x; y$ and $-x = \mathbf{a}(x)$.

Definition 2.2 A representation \mathcal{M} of a domain algebra \mathfrak{A} consists of a set M (the base of the representation) and an interpretation $x^{\mathcal{M}} \subseteq M \times M$ of each element $x \in A$ such that $(x; y)^{\mathcal{M}}$ is the composition of the relations $x^{\mathcal{M}}$ and $y^{\mathcal{M}}$

$$(x; y)^{\mathcal{M}} = \{(u, v) \in M \times M : (u, w) \in x^{\mathcal{M}} \text{ and } (w, v) \in y^{\mathcal{M}} \text{ for some } w \in M\}$$

and for each operation in the signature of \mathfrak{A} ,

$$\begin{aligned} (\mathbf{d}(x))^{\mathcal{M}} &= \{(u, u) \in M \times M : (u, v) \in x^{\mathcal{M}} \text{ for some } v \in M\} \\ (\mathbf{r}(x))^{\mathcal{M}} &= \{(v, v) \in M \times M : (u, v) \in x^{\mathcal{M}} \text{ for some } u \in M\} \\ (\mathbf{a}(x))^{\mathcal{M}} &= \{(u, u) \in M \times M : (u, v) \notin x^{\mathcal{M}} \text{ for all } v \in M\} \\ 1'^{\mathcal{M}} &= \{(u, v) \in M \times M : u = v\} \end{aligned}$$

and $x \neq y$ implies $x^{\mathcal{M}} \neq y^{\mathcal{M}}$.

A domain algebra is representable if it has a representation. In general, if τ is a similarity type, we write $R(\tau)$ for the class of representable τ -algebras.

It is easy to check that the class of representable τ -algebras are closed under subalgebras, direct products and ultraproducts (they are pseudo-axiomatizable). Hence they form a quasivariety. Our main result is the following.

Theorem 2.3 Let τ be a similarity type such that $\{\mathbf{d}, ;\} \subseteq \tau \subseteq \{\mathbf{d}, \mathbf{r}, \mathbf{a}, ;, 1', 0\}$. The class of representable τ -algebras is not finitely axiomatizable in first-order logic.

Proof: We will define domain algebras \mathfrak{A}_n in Definition 3.3 and show that

1. the τ -reduct of \mathfrak{A}_n is not representable, Corollary 3.10,
2. non-principal ultraproducts of \mathfrak{A}_n are representable, Theorem 4.5.

By Łoś theorem, the result follows. ■

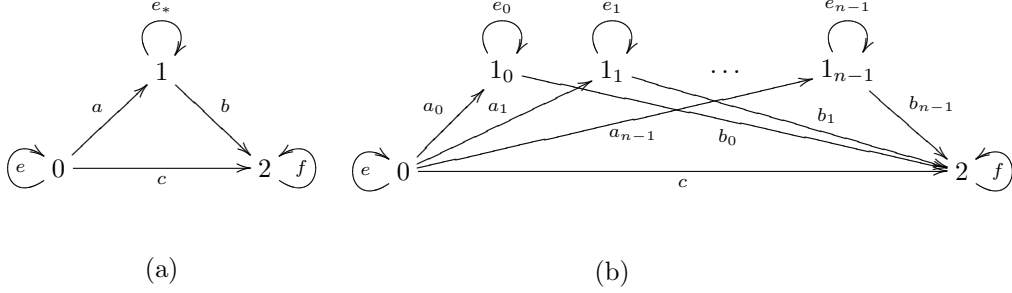


Figure 1: Representations of \mathfrak{C} and \mathfrak{C}_n

3 Non-representability

In this section, we define domain algebras \mathfrak{A}_n and show that they are not representable.

Proposition 3.1 *Let \mathfrak{A} be a domain algebra and let $x, y \in A$.*

1. *If $e = r(x); d(y)$, then e is a domain element and $x; y = x; e; y$.*
2. *$d(x; y) = d(x); d(x; y)$ and $d(d(x)) = d(x)$.*

Definition 3.2 *A domain algebra \mathfrak{A} with a set of domain elements D is said to be loose if it includes an element 0 such that, for all $x, y \in A$,*

1. $0; x = x; 0 = d(0) = r(0) = 0$
2. $x; y \in D \iff (x; y = 0 \text{ or both } x, y \in D)$.

Next we define, for every $n \in \omega$, a domain algebra \mathfrak{A}_n . These algebras are formally defined in Definition 3.3, but first we give a rough outline of how they may be constructed. Start with the representable domain algebra \mathfrak{C} with elements $\{0, e, f, e_*, a, b, c\}$ where the set D of domain elements is $\{0, e, f, e_*\}$. The domain and range operators are defined by $d(\delta) = r(\delta) = \delta$ for $\delta \in D$, $d(a) = d(c) = e$, $d(b) = e_* = r(a)$ and $r(b) = r(c) = f$. Composition is defined for domain elements by $\delta; \delta' = 0$ if $\delta \neq \delta'$ else δ , $d(x); x = x = x; r(x)$ (all x) and $a; b = c$, and all other compositions are zero. A representation of \mathfrak{C} over the base set $\{0, 1, 2\}$ maps the non-zero elements of \mathfrak{C} to singleton sets of ordered pairs (i, j) where $i \leq j < 3$ as shown in Figure 1(a). Secondly, we ‘split’ \mathfrak{C} to obtain a representable domain algebra \mathfrak{C}_n , by replacing the elements e_* , a and b by $\{e_i : i < n\}$, $\{a_i : i < n\}$ and $\{b_i : i < n\}$, respectively, leaving the operators unchanged, except $d(e_i) = r(e_i) = e_i$, $d(a_i) = e$, $r(b_i) = f$, $r(a_i) = e_i = d(b_i)$, and $a_i; b_j = 0$ if $i \neq j$ and $a_i; b_i = c$ for $i < n$. A representation of \mathfrak{C}_n can be obtained from the representation shown in Figure 1(a) of \mathfrak{C} by replacing the base point 1 by n points $1_0, 1_1, \dots, 1_{n-1}$, as shown in Figure 1(b). Finally, we obtain the non-representable domain algebra \mathfrak{A}_n by further splitting the elements e_i, a_i, b_i so that for $x \in \{e_i, a_i, b_i\}$ we replace x by $\{x^{01}, x^{10}, x^{11}\}$, and we replace c by $\{c_i : i < n\}$. (If we were concerned only with signatures not using the antidomain operator, we could get by with just the elements $\{x^{01}, x^{11}\}$ for $x \in \{e_i, a_i, b_i\}$.) Again, the domain and range operators are mostly unchanged, but $d(c_i) = e$, $r(c_i) = f$, $r(a_i^{\lambda, \mu}) = e_i^{\lambda, \mu} = d(b_i^{\lambda, \mu})$. For composition, we refine the composition of \mathfrak{C}_n by (3) below. Crucially, since our signature does not include boolean meet or join, there is no requirement that $c_{i-1} + c_i = a_i^{11}; e_i^{01}; b_i^{11} + a_i^{11}; e_i^{10}; b_i^{11} = a_i^{11}; e_i^{11}; b_i^{11} = c_i$. Now we give the formal definition.

Definition 3.3 *Let $n \in \omega$. Let $\mathfrak{D}_n = (D_n, \cdot, -)$ be the boolean algebra generated by the following set of atoms $At(\mathfrak{D}_n) = \{e, f, e_i^{01}, e_i^{10} : i < n\}$. Let \leq denote the usual ordering on \mathfrak{D}_n and $+$ boolean disjunction. We denote the top and bottom elements of \mathfrak{D}_n by $1'$ and 0 , respectively, and define $e_i^{11} = e_i^{01} + e_i^{10}$, for $i < n$.*

Let $\mathfrak{A}_n = (A_n, ;, \mathbf{d}, \mathbf{r}, \mathbf{a}, 1')$ be an algebra with elements

$$D_n \cup \{a_i^{01}, a_i^{10}, a_i^{11}, b_i^{01}, b_i^{10}, b_i^{11}, c_i : i < n\}$$

with operations defined below. For $x \in \{e, a, b\}$ and $i < n$ the element x_i^{00} denotes 0. The set of domain elements of \mathfrak{A}_n is D_n . The domain and range operators are defined by

$$\mathbf{d}(x) = \mathbf{r}(x) = x \quad \text{if } x \in D_n$$

and for non-domain elements x

$$\mathbf{d}(x) = \begin{cases} e & \text{if } x \in \{a_i^{01}, a_i^{10}, a_i^{11}, c_i : i < n\} \\ e_i^{\lambda, \mu} & \text{if } x = b_i^{\lambda, \mu} \end{cases}$$

$$\mathbf{r}(x) = \begin{cases} f & \text{if } x \in \{b_i^{01}, b_i^{10}, b_i^{11}, c_i : i < n\} \\ e_i^{\lambda, \mu} & \text{if } x = a_i^{\lambda, \mu} \end{cases}$$

for any $i < n$ and $(\lambda, \mu) \in \{(0, 1), (1, 0), (1, 1)\}$. Composition is defined first for domain elements:

$$d ; d' = d \cdot d' \quad \text{for all domain elements } d \text{ and } d'.$$

Of course, we will have

$$x = \mathbf{d}(x) ; x = x ; \mathbf{r}(x) \text{ and } 0 = 0 ; x = x ; 0 \quad \text{for all } x.$$

In addition, we require

$$\begin{aligned} a_i^{\lambda, \mu} ; e_i^{\nu, \pi} &= a_i^{\lambda \cdot \nu, \mu \cdot \pi} \\ e_i^{\lambda, \mu} ; b_i^{\nu, \pi} &= b_i^{\lambda \cdot \nu, \mu \cdot \pi} \\ a_i^{\lambda, \mu} ; b_i^{\nu, \pi} &= \begin{cases} c_i & \text{if } \lambda \cdot \nu = 1 \\ c_{i-1} & \text{if } \lambda \cdot \nu = 0 \text{ and } \mu \cdot \pi = 1 \\ 0 & \text{if } \lambda \cdot \nu = \mu \cdot \pi = 0 \end{cases} \end{aligned} \quad (3)$$

for $\lambda, \mu, \nu, \pi \in \{0, 1\}$ and $i < n$, $-$ is modulo n . All other compositions are defined to be zero.

The antidomain operation \mathbf{a} is defined by taking the complement of \mathbf{d} in \mathfrak{D}_n : $\mathbf{a}(x) = -\mathbf{d}(x)$. The antirange operation is already definable in this signature, the antirange of x is $\mathbf{a}(\mathbf{r}(x))$. The boolean connectives of \mathfrak{D}_n can be recovered by using \mathbf{a} as complement and $;$ as conjunction.

Proposition 3.4 For each $n \in \omega$, \mathfrak{A}_n is a loose domain algebra. Furthermore, if $x, y \in A_n$ and neither x nor y is a domain element, then there is an atom e of \mathfrak{D}_n such that $x ; e ; y = x ; y$.

Definition 3.5 Let \mathfrak{A} be a domain algebra, and let D be the set of domain elements of \mathfrak{A} . Define a binary relation \sqsubseteq on \mathfrak{A} by

$$a \sqsubseteq b \quad \text{iff} \quad a = e_0 ; b ; e_1 \text{ for some } e_0, e_1 \in D.$$

Let $a \sqsubset b$ iff $a \sqsubseteq b$ and $a \neq b$. For any $a \in A$, a^\dagger denotes $\{b \in A : a \sqsubseteq b\}$.

We define another the binary relation \preceq on \mathfrak{A} by letting

$$a \preceq b \quad \text{iff} \quad a = e_0 ; u_0 ; e_1 ; u_1 ; \dots ; u_{n-1} ; e_n \text{ and } b = u_0 ; u_1 ; \dots ; u_{n-1}$$

for some natural number n , domain elements e_0, e_1, \dots, e_n and elements u_0, \dots, u_{n-1} . We will write $a \prec b$ if $a \preceq b$ and $a \neq b$.

Clearly $a \sqsubseteq b$ implies $a \preceq b$, but the converse fails, as we will see. The relation $a \sqsubseteq b$ is equivalent to an equation $a = \mathbf{d}(a) ; b ; \mathbf{r}(a)$, but the relation $a \preceq b$ seems to make essential use of existential quantification that cannot (easily) be eliminated. Note that \sqsubseteq coincides with the boolean ordering \leq on D . If \mathfrak{A} is a loose domain algebra, then \preceq also coincides with the boolean ordering on D .

Proposition 3.6 *Let \mathfrak{A} be a domain algebra.*

1. *The relation \sqsubseteq is reflexive and transitive, but \preceq generally is not monotone w.r.t. \sqsubseteq .*
2. *If e is a domain element and $a \sqsubseteq e$, then a is a domain element.*
3. *For any domain element e , we have $\mathbf{d}(x; e; y) \leq \mathbf{d}(x; y)$.*
4. *If $a \preceq b$, then $\mathbf{d}(a) \leq \mathbf{d}(b)$ and $\mathbf{r}(a) \leq \mathbf{r}(b)$.*
5. *\preceq is reflexive and \preceq is monotone w.r.t. \preceq*

$$\text{if } a^- \preceq a^+ \text{ and } b^- \preceq b^+, \text{ then } a^-; b^- \preceq a^+; b^+$$

but generally is not transitive.

6. *If $a \preceq b \sqsubseteq c$, then $a \preceq c$.*
7. *Let \mathcal{M} be a representation of \mathfrak{A} . If $a \preceq b$, then $a^{\mathcal{M}} \subseteq b^{\mathcal{M}}$.*

Now let \mathfrak{A} be a loose domain algebra.

8. *If e is a domain element and $e \preceq f$, then f is a domain element.*

Let $2 < k < \omega$. A sequence $(\alpha_0, \dots, \alpha_{k-1})$ of elements, where for each $i < k-1$, we have $\alpha_i \prec \alpha_{i+1}$, is called a *k-chain from α_0 to α_{k-1}* or simply a *chain*. The length of such a chain is k . A *k-cycle* is a *k-chain* $(\alpha_0, \dots, \alpha_{k-1})$ such that $\alpha_{k-1} \prec \alpha_0$.

Lemma 3.7 *If a domain algebra has a cycle, then it is not representable.*

Proof: In a representation \mathcal{M} , $a \preceq b$ implies $a^{\mathcal{M}} \subseteq b^{\mathcal{M}}$, as we saw. Hence, if $(a_0, a_1, \dots, a_{k-1})$ is a cycle, then $a_0^{\mathcal{M}} = a_1^{\mathcal{M}} = \dots = a_{k-1}^{\mathcal{M}}$. Since $a_0 \neq a_{k-1}$, this contradicts the faithfulness of the representation. ■

It might be instructive to check the definitions of \sqsubseteq and \preceq in \mathfrak{A}_n .

Lemma 3.8 *Let $2 \leq n < \omega$.*

1. *For every $x \in A_n$, we have $0 \sqsubseteq x \sqsubseteq x$.*
2. *Whenever $\lambda \leq \nu$ and $\mu \leq \pi$, we have*

$$e_i^{\lambda, \mu} \sqsubseteq e_i^{\nu, \pi} \quad a_i^{\lambda, \mu} \sqsubseteq a_i^{\nu, \pi} \quad \text{and} \quad b_i^{\lambda, \mu} \sqsubseteq b_i^{\nu, \pi}$$

and no other pairs of distinct elements are related by \sqsubseteq .

3. *The binary relation \preceq in \mathfrak{A}_n is*

$$\sqsubseteq \cup \{(c_i, c_{i+1}), (c_{n-1}, c_0) : i < n-1\}$$

Proof: The first part holds since $0 = 0; x; 0$ and $x = \mathbf{d}(x); x; \mathbf{r}(x)$. The second part is easily verified. For the third part, we know that $x \sqsubseteq y$ implies $x \preceq y$ and

$$c_i = a_{i+1}^{01}; b_{i+1}^{01} = (a_{i+1}^{11}; e_{i+1}^{01}); (e_{i+1}^{01}; b_{i+1}^{11}) = a_{i+1}^{11}; e_{i+1}^{01}; b_{i+1}^{11} \prec a_{i+1}^{11}; b_{i+1}^{11} = c_{i+1}$$

(all $i < n$, addition modulo n), so $c_0 \prec c_1 \prec \dots \prec c_{n-1} \prec c_0$.

Conversely, suppose $x \prec y$, i.e., there are non-domain elements u_0, u_1, \dots, u_{k-1} (some k) and domain elements e_0, e_1, \dots, e_k such that $y = u_0; u_1; \dots; u_{k-1}$ and $x = e_0; u_0; e_1; \dots; u_{k-1}; e_k$ and $x \neq y$. If $k \leq 1$, then $x \sqsubseteq y$. If $k \geq 2$, then y is the product of at least two non-domain elements. By looseness, either $y = 0$ (so $x = y = 0$, but of course $0 \sqsubseteq 0$) or y is not a domain element. In \mathfrak{A}_n this can only happen if $k = 2$ and the product is of the form $a_i^{\lambda, \mu}; b_i^{\nu, \pi}$ (some $i < n$, some $\lambda, \mu, \nu, \pi < 2$). Therefore if $x \prec y$ and $x \not\sqsubseteq y$, then $x = a_i^{\lambda, \mu}; e_i^{\rho, \sigma}; b_i^{\nu, \pi}$ and $y = a_i^{\lambda, \mu}; b_i^{\nu, \pi}$. Given that $x \neq 0$ and $x \neq y$, we must have $x = c_{i-1}$ and $y = c_i$, as required. ■

Corollary 3.9 *Let $2 \leq n < \omega$. \mathfrak{A}_n has an n -cycle but no k -cycle for $k < n$.*

Hence, by Proposition 3.4 and Lemma 3.7 and Corollary 3.9, we have the following.

Corollary 3.10 *Let τ be a signature containing $\{\mathbf{d}, ;\}$. For each $2 \leq n < \omega$, the τ -reduct of \mathfrak{A}_n is not representable.*

4 Representability

In this section, we show that any non-principal ultraproduct \mathfrak{A} of \mathfrak{A}_n over ω is representable.

Game for representability of domain algebras

First we work out the details for domain algebras without the antidomain operation, and show the necessary modifications for including antidomain later.

Let $\mathfrak{A} = (A, \mathbf{d}, \mathbf{r}, ;, \mathbf{1}')$ be a domain algebra with a set D of domain elements. A *network* $N = (N^1, N^2)$ over \mathfrak{A} consists of a finite set N^1 (of nodes) and a map $N^2 : (N^1 \times N^1) \rightarrow \wp(A)$ satisfying the following *coherence* conditions:

- C1** if $i \neq j$, then $N^2(i, j) \cap D = \emptyset$
- C2** $\mathbf{d}(N^2(i, j)) \subseteq N^2(i, i)$ and $\mathbf{r}(N^2(i, j)) \subseteq N^2(j, j)$
- C3** if $a \in N^2(i, j)$ and $a \sqsubseteq b$, then $b \in N^2(i, j)$
- C4** $N^2(i, j); N^2(j, k) \subseteq N^2(i, k)$

for all $i, j, k \in N^1$.

A network $N = (N^1, N^2)$ over \mathfrak{A} is *saturated* if it satisfies

- S1** if $\mathbf{d}(a) \in N^2(i, i)$, then $a \in N^2(i, l)$ for some l
- S2** if $\mathbf{r}(a) \in N^2(j, j)$, then $a \in N^2(l, j)$ for some l
- S3** if $a; b \in N^2(i, j)$, then $a \in N^2(i, k)$ and $b \in N^2(k, j)$ for some k

for all $i, j \in N^1$ and $a, b \in A$.

The set of domain elements of any domain algebra together with the composition operator forms a lower semilattice. In any network (N^1, N^2) and any node $i \in N^1$, the set $D \cap N^2(i, i)$ is a filter, i.e., for any domain elements e, e' if $e, e' \in N^2(i, i)$, then $e; e' \in N^2(i, i)$, and if $e \in N^2(i, i)$ and $e; e' = e$, then $e' \in N^2(i, i)$, by (C3). Recall that for each domain element $e \in D$, $e^\uparrow = \{e' \in D : e; e' = e\}$, the principal filter generated by e . A network (N^1, N^2) is *singular* if for every $i \in N^1$, there is $\delta_i \in D$ such that $N^2(i, i) \cap D = \delta_i^\uparrow$.

Let $N = (N^1, N^2)$ and $M = (M^1, M^2)$ be networks. We write $N \subseteq M$ if $N^1 \subseteq M^1$ and for all $i, j \in N^1$, we have $N^2(i, j) \subseteq M^2(i, j)$. We sometimes drop the superscripts and write N for the network, the set of nodes and the map, distinguishing cases by context, though we may write $\text{nodes}(N)$ to denote the set N^1 of nodes of N . If N is a singular network, we may write δ_i^N for the minimal domain element of $N(i, i)$. If the context is clear, we may omit the superscript N in δ_i^N . Note that δ_i is the minimal domain element of $N(i, i)$, but it need not be minimal within the set of all non-zero domain elements.

The two player game $G_t(\mathfrak{A})$ has t rounds. In the initial round, player \forall (male) picks a set $\{\alpha, \beta\}$ of distinct elements of \mathfrak{A} . Player \exists (female) responds with either α or β . In the former case, she has to prove that there is a pair of nodes witnessing α but not β , in the latter case the other way round — this suffices to prove that the representation she constructs is faithful. Without loss of generality, suppose she picks α . In the initial round, \exists has to define a network N_0 with nodes 0 and 1 such that $\alpha \in N_0(0, 1)$ but $\beta \notin N_0(0, 1)$. Of course, she identifies 0 with 1 precisely when α is a domain element. Suppose $0 < i < t$ and a network N_{i-1} has just been played. \forall can choose from the following types of moves.

Domain move He can demand to see a domain witness. He picks $j \in \text{nodes}(N_{i-1})$ and an element $a \in A$ such that $\mathbf{d}(a) \in N_{i-1}(j, j)$. Such a move is denoted (j, a) . Then \exists has to play a network $N_i \supseteq N_{i-1}$, such that $a \in N_i(j, k)$ for some $k \in \text{nodes}(N_i)$.

Range move He can demand to see a range witness. This is completely symmetric to the domain move.

Composition move He can demand to see a composition witness. He picks $j, k \in \text{nodes}(N_{i-1})$ and $a, b \in N_{i-1}(j, k)$. Such a move is denoted (j, k, a, b) . \exists has to play a network $N_i \supseteq N_{i-1}$ where there is $l \in \text{nodes}(N_i)$ such that $a \in N_i(j, l)$ and $b \in N_i(l, k)$.

If at any stage \exists fails to define the required network or if a network is played such that $\beta \in N(0, 1)$, then \forall wins. Otherwise \exists wins.

Lemma 4.1 *Let \mathfrak{A} be a finite or countable domain algebra. If \exists has a winning strategy in $G_\omega(\mathfrak{A})$, then \mathfrak{A} is representable.*

Proof: Let us assume that \forall picked $\{\alpha, \beta\}$ in the initial round of the game and \exists responded with α . Since \exists has a winning strategy, there is a saturated network N such that $\alpha \in N(0, 1)$ but $\beta \notin N(0, 1)$. It is easy to see that saturated networks define representable algebras. Hence there is a representable, homomorphic image \mathfrak{M} of \mathfrak{A} such that $\alpha^{\mathfrak{M}} \neq \beta^{\mathfrak{M}}$. The homomorphism h is given by

$$h(a) = \{(j, k) \in N \times N : a \in N(j, k)\}$$

for every $a \in A$.

Assume that the players repeat the game for every pair $\{\alpha, \beta\}$ of distinct elements of A and \exists applies her winning strategy in each of these games. It follows that \mathfrak{A} can be isomorphically embedded into the product of representable algebras, which is again a representable algebra. ■

Lemma 4.2 *Let \mathfrak{A} be a loose domain algebra (see Definition 3.2) and t be a natural number. If for every $x \in A$ there is no cycle of length less than or equal to $6t$, then \exists has a winning strategy in $G_t(\mathfrak{A})$.*

Proof: Suppose that no such cycle exists. We describe a winning strategy for \exists . Let \forall play $\{\alpha, \beta\}$ in the initial round. If there is no chain from α to β of length less than or equal to $3t$, then \exists plays α . Otherwise there can be no chain from β to α of length less than or equal to $3t$, and \exists plays β in this case. Assume the former case holds, \exists plays α . Then she defines N_0 by $N_0(0, 1) = \alpha^\uparrow$, $N_0(0, 0) = \mathbf{d}(\alpha)^\uparrow$, $N_0(1, 1) = \mathbf{r}(\alpha)^\uparrow$ and $N_0(1, 0) = \emptyset$ if α is not a domain element, or by identifying 0 with 1 and $N_0(0, 0) = \alpha^\uparrow$ if $\alpha = \mathbf{d}(\alpha)$. In either case, N_0 is singular and we have $\alpha \in \alpha^\uparrow = N_0(0, 1)$.

We will prove, by induction over the round i , that

1. N_i is a singular network and
2. if $\gamma \in N_i(0, 1)$, then $\gamma \in N_0(0, 1)$ or there is a m -chain from α to γ such that $m \leq 3i$.

Suppose $0 < i < n$ and a singular network N_{i-1} has just been played.

Domain move Assume \forall plays (j, a) , where $\mathbf{d}(a) \in N_{i-1}(j, j)$. Recall that δ_j denotes the minimal domain element of $N_{i-1}(j, j)$, so $\delta_j \leq \mathbf{d}(a)$. If there is a witness $l \in N_{i-1}$ such that $a \in N_{i-1}(j, l)$, then she lets $N_i = N_{i-1}$, so assume that there is no such witness in N_{i-1} . Then \exists plays the singular network N_i defined as follows. She lets $\text{nodes}(N_i) = \text{nodes}(N_{i-1}) \cup \{l\}$, for some new node $l \notin \text{nodes}(N_{i-1})$. For edges not incident with the new node l , the labelling in N_i is the same as in N_{i-1} (hence the minimal domain label δ_k is the same in N_i as in N_{i-1} , for $k \in N_{i-1}$). \exists lets $\delta_l = \mathbf{r}(\delta_j ; a)$ and

$$\begin{aligned} N_i(l, l) &= \delta_l^\uparrow \\ N_i(j', l) &= \{x : (\exists u \in N_{i-1}(j', j))u ; a ; \delta_l \preceq x\} \\ N_i(l, j') &= \emptyset \end{aligned}$$

for $j' \in \text{nodes}(N_{i-1})$. See Figure 2.

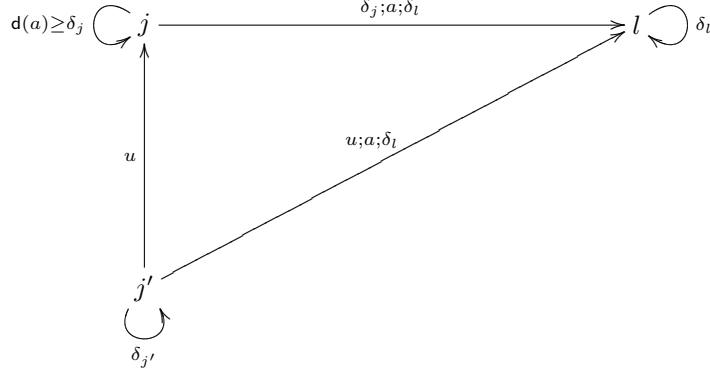


Figure 2: Domain move.

Range move Range moves are handled symmetrically.

Composition move Assume \forall plays (j, k, a, b) where $a; b \in N_{i-1}(j, k)$. If there is already a witness $l \in N_{i-1}$ such that $a \in N_{i-1}(j, l)$ and $b \in N_{i-1}(l, k)$, then \exists lets $N_i = N_{i-1}$. Assume there is no such $l \in N_{i-1}$, it follows that neither a nor b is a domain element. Then \exists plays the singular network N_i where $\text{nodes}(N_i) = \text{nodes}(N_{i-1}) \cup \{l\}$ (some $l \notin \text{nodes}(N_{i-1})$), with minimal domain element $\delta_l = r(\delta_j; a); d(b; \delta_k)$ and the labelling is defined by

$$\begin{aligned} N_i(l, l) &= \delta_l^\uparrow \\ N_i(j', l) &= \{x : (\exists u \in N_{i-1}(j', j))u; a; \delta_l \preceq x\} \\ N_i(l, k') &= \{y : (\exists v \in N_{i-1}(k, k'))\delta_l; b; v \preceq y\} \\ N_i(j', k') &= N_{i-1}(j', k') \cup \{z : (\exists x \in N_i(j', l))(\exists y \in N_i(l, k'))x; y \preceq z\} \end{aligned}$$

for $j', k' \in \text{nodes}(N_{i-1})$.

See Figure 3. We claim, for any $j', k' \in N_{i-1}$, that there are no new domain elements included in $N_i(j', k') \setminus N_{i-1}(j', k')$. To prove this claim, suppose e is a domain element in $N_i(j', k') \setminus N_{i-1}(j', k')$. Then $e \succcurlyeq x; y$ for some $x \in N_i(j', l)$ and $y \in N_i(l, k')$, hence there are $u \in N_{i-1}(j', j)$ and $v \in N_{i-1}(k, k')$ such that $u; a; \delta_l \preceq x$ and $\delta_l; b; v \preceq y$. By looseness and Proposition 3.6(8) and (3), we have $x, y, u, a, b, v \in D$, contradicting our assumption that no witness node exists in N_{i-1} . Hence there are no new domain elements in $N_i(j', k')$, as claimed.

By considering the case where $j' = k'$ we see, by the claim, that there are no new domain elements in $N_i(j', j') \setminus N_{i-1}(j', j')$, so $\delta_{j'}$ remains unchanged in N_i , and our notation is not as ambiguous as it might appear to be.

We have to prove that N_i as defined above is indeed a network. It is clearly singular. (C3) is clear, since we defined all non-empty labels of edges of N_i to be ‘upward closed’ (recall from Proposition 3.6(6) that $a \preceq b \sqsubseteq c$ implies $a \preceq c$). (C1) holds for edges of N_i not incident with the new node l by the claim, and it holds for edges $(j', l), (l, k')$ by looseness. For (C4), consider a domain move. Since labels of edges not incident with l do not change from N_{i-1} to N_i and since $N_i(l, k') = \emptyset$ for $k' \in N_{i-1}$, we need only check that $N_i(j^*, j'); N_i(j', l) \subseteq N_i(j^*, l)$, for $j^*, j' \in N_{i-1}$. See Figure 4. Suppose $v \in N_i(j^*, j')$ and $x \in N_i(j', l)$. Then $x \succcurlyeq u; a; \delta_l$, for some $u \in N_{i-1}(j', j)$. By Proposition 3.6(5), $v; x \succcurlyeq v; u; a; \delta_l$, and since $v; u \in N_{i-1}(j^*, j)$, we have $v; x \in N_i(j^*, l)$, as required. For a composition move (j, k, a, b) , the proof that $N_i(j^*, j'); N_i(j', l) \subseteq N_i(j^*, l)$ and $N_i(l, k'); N_i(k', k^*) \subseteq N_i(l, k^*)$ is the same as above. It remains to show that $N_i(j', l); N_i(l, k') \subseteq N_i(j', k')$ for all $j', k' \in N_{i-1}$. But this follows easily from the definition of $N_i(j', k')$: if $x \in N_i(j', l)$ and $y \in N_i(l, k')$, then $x; y \in N_i(j', k')$, since $x; y \preceq x; y$. This proves (C4).

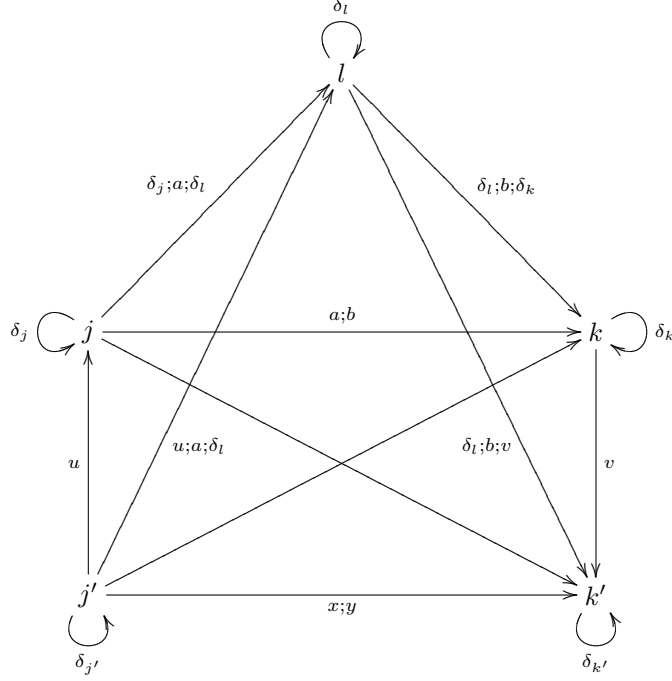


Figure 3: Composition move.

We check (C2) first for domain moves, range moves and then composition moves. Let \forall pick the node $j \in N_{i-1}$ and an element a such that $\mathbf{d}(a) \geq \delta_j$ but there is no node $l \in N_{i-1}$ with $a \in N_{i-1}(j, l)$. \exists plays $N_i \supseteq N_{i-1}$ with a new node l . Let $j' \in N_{i-1}$. If $x \in N_i(j', l)$, then $u; a; \delta_l \preceq x$ for some $u \in N_{i-1}(j', j)$. By Proposition 3.6(4) and (3), $\mathbf{d}(x) \geq \mathbf{d}(u; a; \delta_l) \geq \mathbf{d}(u; \delta_j; a; \delta_l) = \mathbf{d}(u; \delta_j; a; r(\delta_j; a)) = \mathbf{d}(u; \delta_j; a) = \mathbf{d}(u; \delta_j; \mathbf{d}(a)) \geq \delta_{j'}$, the last inequality holding by (C2) on N_{i-1} . Similarly, $r(x) \geq r(u; a; \delta_l) = \delta_l$. So $\mathbf{d}(x) \in N_i(j', j')$ and $r(x) \in N_i(l, l)$, as required. Range moves are similar.

For composition moves, suppose $x \in N_i(j', l)$, so $u; a; \delta_l \preceq x$ for some $u \in N_{i-1}(j', j)$. This time, $\delta_l = r(\delta_j; a); \mathbf{d}(b; \delta_k)$. As before, $\mathbf{d}(x) \geq \mathbf{d}(u; a; \delta_l) \geq \mathbf{d}(u; \delta_j; a; r(\delta_j; a); \mathbf{d}(b; \delta_k)) = \mathbf{d}(u; \delta_j; a; \mathbf{d}(b; \delta_k))$. Inductively, $\delta_j \leq \mathbf{d}(a; b; \delta_k) = \mathbf{d}(a; \mathbf{d}(b; \delta_k))$, hence $\mathbf{d}(x) \geq \mathbf{d}(u; \delta_j) \geq \delta_{j'}$. And $r(x) \geq r(u; a; \delta_l) = \delta_l$, so again $\mathbf{d}(x) \in N_i(j', j')$ and $r(x) \in N_i(l, l)$, verifying (C2). Similarly, (C2) can be verified for each $y \in N_i(l, k')$ where $k' \in N_{i-1}$. Finally, we check (C2) on edges not incident with l . Suppose $z \in N_i(j', k') \setminus N_{i-1}(j', k')$. Then $z \succeq x; y$ for some $x \in N_i(j', l)$ and $y \in N_i(l, k')$. Now $r(x), \mathbf{d}(y) \geq \delta_l$ so $\mathbf{d}(z) \geq \mathbf{d}(x; y) \geq \mathbf{d}(x; \delta_l; y) = \mathbf{d}(x; \delta_l) \geq \delta_{j'}$ and similarly $r(z) \geq \delta_{k'}$, so (C2) holds. This proves that N_i is a network.

We now check the other inductive condition for N_i , that $\gamma \in N_i(0, 1) \setminus N_0(0, 1)$ implies the existence of a chain of length at most $3i$ from α to γ . We show that in each round i of the game, if a new element z is included in the label $N_i(0, 1)$, then there are elements $z_1 \preceq z_2 \preceq z_3 \preceq z$ with $z_1 \in N_{i-1}(0, 1)$. Observe that the only moves of \exists that add additional elements to the label of $(0, 1)$ are composition moves. In round $i > 0$, if \forall plays (j, k, a, b) where $a; b \in N_{i-1}(j, k)$, then the only new elements z included in the label $N_i(0, 1)$ satisfy $x; y \preceq z$ for some $x \in N_i(0, l)$ and $y \in N_i(l, 1)$. Since $x \in N_i(0, l)$, there is $u \in N_{i-1}(0, j)$ such that $u; a; \delta_l \preceq x$ and similarly there is $v \in N_{i-1}(k, 1)$ such that $\delta_l; b; v \preceq y$. Hence $u; \delta_j; a; b; \delta_k; v = u; \delta_j; a; \delta_l; b; \delta_k; v \preceq u; a; \delta_l; b; v \preceq x; y \preceq z$ and $u; \delta_j; a; b; \delta_k; v \in N_{i-1}(0, 1)$. Hence there is a chain from α to z of length at most $3i$.

It follows that $\beta \notin N_i(0, 1)$, so \exists wins. \blacksquare

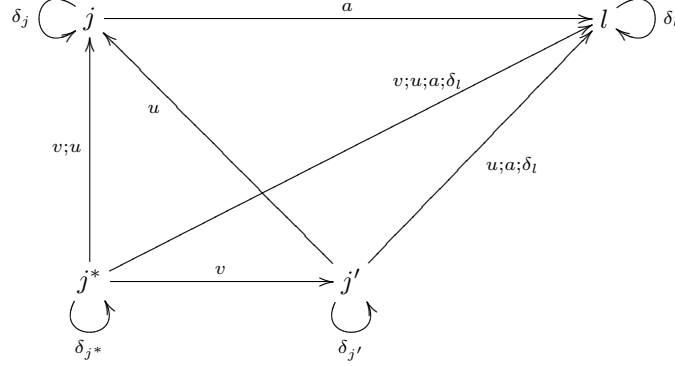


Figure 4: Checking (C4) for domain move.

Game with antidomain

In this section, we describe the necessary modifications of networks and games for dealing with antidomain as well.

Let $\mathfrak{A} = (A, d, r, a, ;, 1')$ be a domain algebra with a set D of domain elements forming the boolean algebra \mathfrak{D} . An additional requirement in the definition of a network is that

$$N(i, i) \cap D \text{ is an ultrafilter of } \mathfrak{D} \quad (4)$$

Note that this extra condition for antidomain networks is necessary and sufficient for the antidomain operation to be properly represented in a saturated antidomain network. Indeed, if N is a saturated antidomain network and $i \in \text{nodes}(N)$, we have

$$a(x) \in N(i, i) \iff d(x) \notin N(i, i) \iff \neg(\exists j \in \text{nodes}(N)) x \in N(i, j)$$

by (4), (C2) and (S1). For antidomain algebras where \mathfrak{D} is a finite boolean algebra, condition (4) is equivalent to

$$N(i, i) \cap D = e^\uparrow \text{ for some atom } e \text{ of } \mathfrak{D} \quad (5)$$

and more generally, a singular antidomain network N is a singular network such that, for every $i \in \text{nodes}(N)$, the element δ_i is an atom of \mathfrak{D} . For the remainder of this section, all networks are antidomain networks (i.e., they satisfy (4)).

The game $G_t^a(\mathfrak{A})$ is almost identical to the previously defined game $G_t(\mathfrak{A})$, the only difference is that the networks played have to be antidomain networks, i.e., there has to be an atom in the label of a reflexive edge. In all other respects, the definition of the game $G_t^a(\mathfrak{A})$ is the same as the definition of $G_t(\mathfrak{A})$. Lemma 4.1 still holds, since the only additional part to show is that a saturated antidomain network is an antidomain algebra representation, and this is true, since each reflexive edge of an antidomain network is labelled by an ultrafilter over the boolean algebra of domain elements. Instead of Lemma 4.2 we have the following.

Lemma 4.3 *Let t be a natural number and let \mathfrak{A} be a loose antidomain algebra with no cycles of length $6t$ or less. Suppose for all non-domain elements $x, y \in A$, there is an atom e of \mathfrak{D} such that $x ; y = x ; e ; y$. Then player \exists has a winning strategy in $G_t^a(\mathfrak{A})$.*

Proof: The winning strategy for \exists is very similar to the one we gave before. This time, in response to a domain, range or composition move in round i , if \exists has to include a new node l in N_i , then she has to let δ_l be an atom of \mathfrak{D} . In response to a domain move (j, a) , she lets δ_l be any atom below $r(\delta_j ; a)$, range moves are similar. In response to a composition move (j, k, a, b) , where neither a nor b are domain elements, she lets δ_l be any atom below $r(\delta_j ; a) ; d(b ; \delta_k)$ such that $\delta_j ; a ; b ; \delta_k = \delta_j ; a ; \delta_l ; b ; \delta_k$. Such an atom δ_l exists, by the assumption in the lemma. In

other respects, the definition of N_i is the same as before. The proofs that N_i is a network and that every element of $N_i(0,1)$ can be reached from an element of $N_0(0,1)$ by a chain of length at most $3i$ are the same as before. ■

Corollary 4.4 *Let n be a natural number. \exists has a winning strategy in the antidomain network game $G_t^a(\mathfrak{A}_{6t})$.*

Proof: By Proposition 3.4, \mathfrak{A}_n is a loose algebra with no cycles of length less than n , and for all non-domain elements $x, y \in A_n$, there is an atom e such that $x;e; y = x;y$. By Lemma 4.3, \exists has a winning strategy in $G_t^a(\mathfrak{A}_{6t})$. ■

Representing the ultraproduct

Theorem 4.5 *If \mathfrak{B} is countable and elementarily equivalent to a non-principal ultraproduct \mathfrak{A} of the \mathfrak{A}_n s, then \mathfrak{B} is a representable domain algebra. Since representable algebras form an elementary class, the ultraproduct \mathfrak{A} is representable.*

Proof: We have seen that \exists has a winning strategy in $G_n(\mathfrak{A}_{6t})$ and $G_n^a(\mathfrak{A}_{6t})$, Lemma 4.2 and Corollary 4.4. \exists can combine her winning strategies in $G_n(\mathfrak{A}_{6t})$ (or in $G_n^a(\mathfrak{A}_{6t})$) to a winning strategy in $G_\omega(\mathfrak{B})$ (or in $G_\omega^a(\mathfrak{A}_{6t})$), for any countable, elementary subalgebra \mathfrak{B} of the non-principal ultraproduct of the \mathfrak{A}_n s. Hence \mathfrak{B} is representable by Lemma 4.1. See [HH02, Theorem 10.12] for more details on how to design a winning strategy on the ultraproduct. ■

5 Extending the similarity type

A natural question from both the theoretical and application point of view is whether the same non-finite axiomatizability holds for larger similarity types. Obvious choices for the extra operations include join $+$, meet \cdot , converse \smile , top 1 and bottom 0 constants and the Kleene star $*$ (reflexive-transitive closure).

Recall from Definition 3.3 that for each $x \in A_n$ we have either $x;x = x$ (for domain element x) or $x;x = 0$. Hence it is easy to extend the signature of these algebras to include the transitive (but not reflexive) closure operation by letting the transitive closure of each element be itself. Thus our non-finite axiomatizability result holds for signatures including the transitive closure operation. But it is not easy to see how to extend the signature of the algebras \mathfrak{A}_n to include the reflexive and transitive closure operation $*$, since for non-domain elements x the natural definition of x^* would be $1' + x$, but our algebras do not include such elements.

It might be possible to modify the definition of \mathfrak{A}_n to include a definition of $+$ or \cdot , but the resulting algebra is likely to have $c_i \leq c_{i+1} \pmod{n}$ for all $i < n$, and then $c_0 = c_1 = \dots = c_{n-1}$, whence \mathfrak{A}_n would be representable. Instead, for signatures including $+$, we recall the following from [An88].

Theorem 5.1 *Let $\{+,;\} \subseteq \tau \subseteq \{0,1,+,1',\smile,;,*\}$. Then the class $R(\tau)$ of representable τ -algebras is not finitely axiomatizable.*

Our contribution here is to make the fairly trivial observation that there is an obvious way to define domain and range operations for the algebras used in [An88]. Since [An88] is not widely available, we recall the key steps of the proof. For every natural number n , Andr eka constructs an algebra $\mathfrak{A}_n = (A_n, 0, 1, +, 1', \smile, ;, *)$ such that

1. the $\{+,;\}$ -reduct of \mathfrak{A}_n is not representable
2. any non-trivial ultraproduct \mathfrak{A} of \mathfrak{A}_n (for $n \in \omega$) is representable.

Let m be a natural number much bigger than n . Define

$$G = \{a, a'_1, a''_1, \dots, a'_m, a''_m, b, b'_1, b''_1, \dots, b'_m, b''_m, o, i, 0\}$$

Let $(A_n, +)$ be the free upper semilattice generated freely by G under the defining relations:

$$a \leq a'_i + a''_i \quad b \leq b'_i + b''_i \quad 0 + x = x$$

for $1 \leq i \leq m$ and $x \in G$. Let S denote the following set of two-element subsets of A_n :

$$S = \{\{a, b'_1\}\} \cup \{\{a'_i, b''_i\} : 1 \leq i \leq m\} \cup \{\{a''_i, b'_{i+1}\} : 1 \leq i < m\} \cup \{\{a''_m, b\}\}$$

The other operations on A_n are defined as follows.

$$\begin{aligned} 0 &= \emptyset & 1 &= \sum G & 1' &= i & x^\smile &= x \\ 0^* &= 0 & 1'^* &= 1' & x^* &= 1 \text{ if } x \notin \{0, 1'\} \\ 0 ; x &= 0 = x ; 0 & 1' ; x &= x = x ; 1' \\ \text{for } x, y &\notin \{0, 1'\} & x ; y &= \begin{cases} o & \text{if } \{x, y\} \in S \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

1. The quasiequation q_n is defined as

$$\begin{aligned} &\bigwedge_{i=1}^m (x \leq x'_i + x''_i \wedge y \leq y'_i + y''_i) \rightarrow \\ &x ; y \leq x ; y'_1 + \sum_{i=1}^{m-1} (x'_i ; y''_i + x''_i ; y'_{i+1}) + x'_m ; y''_m + x''_m ; y \end{aligned}$$

By an induction on m one can show that q_m is valid in representable algebras. On the other hand, the evaluation ϵ given by

$$\epsilon(x) = a \quad \epsilon(x'_i) = a'_i \quad \epsilon(x''_i) = a''_i \quad \epsilon(y) = b \quad \epsilon(y'_i) = b'_i \quad \epsilon(y''_i) = b''_i$$

falsifies q_m in \mathfrak{A}_n . Since q_m uses only the operations $;$ and $+$, it follows that already the $\{+, ;\}$ -reduct of \mathfrak{A}_n is not representable.

2. By a step-by-step argument one can build a representation of the ultraproduct \mathfrak{A} .

Note that in \mathfrak{A}_n and \mathfrak{A} , one can define the (anti)domain and range operations by letting

$$\begin{aligned} d(0) &= r(0) = 0 & \text{and} & & d(x) &= r(x) = 1' \text{ for } x \neq 0 \\ a(0) &= 1' & \text{and} & & a(x) &= 0 \text{ for } x \neq 0 \end{aligned}$$

Obviously, \mathfrak{A}_n expanded with (anti)domain and/or range remains non-representable, while the representation of the ultraproduct respects both the domain, range and antirange operations. Indeed, if $x \neq 0$, then $x^* = 1$, whence the representation of x is a relation such that both its domain and range contain all elements of the base, thus it is sound to represent $d(x)$ and $r(x)$ as the identity relation and $a(x)$ as the empty relation. Hence we have the following.

Corollary 5.2 *Let $\{+, ;\} \subseteq \tau \subseteq \{0, 1, +, d, r, a, 1', ;, *, \smile\}$. Then the class $R(\tau)$ of representable τ -algebras is not finitely axiomatizable.*

Including meet \cdot into the similarity type does not seem promising either. Let τ be a similarity type such that the elements of τ are definable in representable relation algebras (i.e., using the booleans, composition, converse and identity). Andr eka [An91] shows non-finite axiomatizability for representable algebras of similarity type $\tau \supseteq \{\cdot, +, ;\}$. In [HM07], we defined non-representable algebras of the similarity type $\{\cdot, 1', ;\}$ whose ultraproduct is representable. Since $1'$ is a minimal non-zero element in these algebras, defining domain, range and antidomain operations should not be a problem. Hence we conjecture that representable algebras of the similarity type $\tau \supseteq \{\cdot, d, ;\}$ form a non-finitely axiomatizable quasivariety.

6 Conclusion

Note that representing domain monoids or domain–range monoids is easier than representing antidomain algebras, since the labels on reflexive arrows do not have to be ultrafilters.

Problem 6.1 Let \mathfrak{A} be a domain monoid or domain–range monoid (but do not assume that \mathfrak{A} is loose). If \mathfrak{A} has no cycles, must it be representable?

If so, we can find a very simple, infinite, recursive axiomatisation of the representation class.

Finally we mention the problem of finitely axiomatizing the equational theories of representable algebras. Hollenberg [Ho97] shows finite axiomatizability of the variety generated by representable antidomain algebras. We conjecture that the same can be achieved for domain and domain–range semigroups/monoids and their expansions with lattice operations join and meet.

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