2 Basics of Propositional Logic

In this section, we recall the basics of propositional logic in a nutshell. Some of the examples are from [Ho91] W. Hodges, *Logic*, Penguin Books, 1991.

2.1 Syntax

Syntax defines the language of the logic, i.e., determines the set of well-formed expressions.

The set F of formulas is defined recursively as follows. Let P be a countably infinite set, the set of *atomic formulas*. A formula φ is either

- an atomic formula $\varphi \in P$,
- a *conjunction* $(\psi \land \rho)$ of formulas ψ and ρ, or
- a *negation* $\neg\psi$ of a formula ψ,

and nothing else is a formula. A *subformula* of a formula φ is a well-formed formula that is a substring of φ. The set of subformulas of a set of formulas Γ is denoted by $\mathrm{Sf}(\Gamma)$; if Γ is a singleton $\{\varphi\}$, then we use the notation $\mathrm{Sf}(\varphi)$.

Exercise 2.1 Compute an upper bound for the number of subformulas of an arbitrary formula.

We will use the following abbreviations.

- *Disjunction*: $(\varphi \lor \psi) =_{\text{def}} \neg(\neg\varphi \land \neg\psi)$.
- *Implication*: $(\varphi \to \psi) =_{\text{def}} (\neg\varphi \lor \psi)$.
- *Equivalence*: $(\varphi \leftrightarrow \psi) =_{\text{def}} ((\varphi \to \psi) \land (\psi \to \varphi))$.
- *True*: $\top =_{\text{def}} (p \lor \neg p)$, for an arbitrary $p \in P$.
- *False*: $\bot =_{\text{def}} \neg\top$.

2.2 Semantics

Semantics is to give meaning to formulas, in this case, the *truth values* 0 and 1 ('false' and 'true', respectively).

A *valuation* v is a function $P \to 2$, i.e., for every $p \in P$, we have either $v(p) = 0$ or $v(p) = 1$. The valuation v is extended to a *truth function*, also denoted by v, in a recursive way: for formulas φ and ψ,

- $v(\neg\varphi) = 1 - v(\varphi)$,
- $v(\varphi \land \psi) = v(\varphi) \cdot v(\psi)$.

Thus the meaning of a formula is determined by the meaning of its atomic subformulas — this is called *compositionality*.

Exercise 2.2 The meaning of the other connectives $(\lor, \to, \leftrightarrow, \top, \bot)$ is determined by their definition above. Compute it.

We say that a formula 'φ is true under the valuation v' or 'v satisfies φ' if $v(\varphi) = 1$.

A set Γ of formulas is

- *valid*, denoted as $\models \Gamma$, if every element of Γ is true under all valuations,
- *satisfiable* if there is a valuation v such that each element of Γ is true under v,
- *unsatisfiable* if it is not satisfiable.

Given a set Γ of formulas and a formula φ, we say that φ *is a semantical consequence of* Γ, denoted as $\Gamma \models \varphi$, if the set $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable (i.e., if every valuation satisfying Γ also satisfies φ). In case $\Gamma = \{\psi\}$, we will write $\psi \models \varphi$ instead of $\{\psi\} \models \varphi$. Two formulas ψ and φ are *semantically equivalent* if we have both $\psi \models \varphi$ and $\varphi \models \psi$. Show that this is equivalent to $\models \psi \leftrightarrow \varphi$.

Lemma 2.3 *$\Gamma \models \varphi$ iff, for every valuation v, if v satisfies Γ, then v satisfies φ as well.*

Proof: Assume that $\Gamma \models \varphi$. Let v be an arbitrary valuation satisfying (all elements of) Γ. By the definition of semantic consequence, v cannot satisfy $\neg\varphi$ (otherwise $\Gamma \cup \{\neg\varphi\}$ would be satisfiable). Hence v satisfies φ as well.

For the other direction assume that, for every valuation v, if v satisfies Γ, then v satisfies φ as well. This implies that we cannot find a valuation that would satisfy both Γ and $\neg\varphi$. ∎

Exercise 2.4 How many non-equivalent formulas there are using n many atomic formulas?

It is worth noting the following properties of the semantical consequence relation:

Extension: If $X \models \varphi$, then $X \cup Y \models \varphi$.

Repetition: If $\varphi \in X$, then $X \models \varphi$.

Cut: If $X \models \varphi$ and $X \cup \{\varphi\} \models \psi$, then $X \models \psi$.

Transitivity: If $\varphi \models \psi$ and $\psi \models \chi$, then $\varphi \models \chi$.

Example 2.5 [Ho91] Is the following argument valid?

> If Higgins was born in Bristol, then Higgins is not a Cockney. Higgins is either a Cockney or an impersonator. Higgins is not an impersonator. Therefore Higgins was born in Bristol.

Solution:

- $\{b \to \neg c, c \vee i, \neg i\} \models b$.
- The argument is not valid, since the valuation v defined as

$$v(\neg i) = v(c) = v(\neg b) = 1$$

satisfies $\{b \to \neg c, c \vee i, \neg i, \neg b\}$.

2.3 Semantic Tableaux

Next we see a technique that we will use to determine whether a finite set Γ of formulas is satisfiable.

1. Rewrite each element of Γ using the connectives $\neg$ and $\wedge$ only. (Here we use that $\{\neg, \wedge\}$ is a complete set of connectives.)
2. "Push" each occurrence of $\neg$ inside as far as possible. (Use the equivalence $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$.) Hence we can assume that $\neg$ occurs only in front of propositional atoms.
3. Apply the following rules as long as possible:

   Rule $\wedge$: If a branch contains $\varphi \wedge \psi$, then add both φ and ψ to the branch.

   Rule $\vee$: If a branch contains $\varphi \vee \psi$, then create two extensions of the current branch: one by adding φ and another by adding ψ.

   Rule $\neg$: If a branch contains $\neg\neg\varphi$, then add φ to the branch.

The resulting labelled tree is called the *tableau* for Γ. A branch is *closed* if it contains both φ and $\neg\varphi$ for some formula φ, otherwise it is open.

Lemma 2.6 *A finite set Γ of formulas is satisfiable iff the tableau for Γ contains an open branch.*

Proof: From left to right: use contraposition. Assume that the tableau for Γ contains only closed branches. That is, all our attempts to "break down" the elements of Γ into simpler formulas resulted in contradiction (i.e., we had to include a formula and its negation into the branch). Hence Γ cannot be satisfied.

For the other direction note that an open branch induces a valuation that satisfies Γ. In fact, given an open branch, we can define a valuation v as follows: for every propositional atom p in $\mathrm{Sf}(\Gamma)$, let

$$v(p) = 1 \text{ iff } p \text{ is in the branch.}$$

By using formula induction we can show that, for every formula φ,

$$\text{if } \varphi \text{ occurs in the branch, then } v(\varphi) = 1.$$

Base case:

- If p is in the branch, then $v(p) = 1$ by definition.
- If $\neg p$ is in the branch, then $v(p) = 0$, whence $v(\neg p) = 1$.

Inductive steps:

- If $\psi \wedge \chi$ is in the branch, then both ψ and χ are in the branch. By inductive hypothesis, $v(\psi) = v(\chi) = 1$. Thus $v(\psi \wedge \chi) = 1$.
- If $\psi \vee \chi$ is in the branch, then either ψ or χ is in the branch. By inductive hypothesis, $v(\psi) = 1$ or $v(\chi) = 1$. In either case, $v(\psi \vee \chi) = 1$.

Then, since Γ occurs at the root of the tree, v satisfies Γ. ∎

Example 2.7 [Ho91] Formalise and create the tableau for the following text.

> If cobalt but no nickel is present, a brown colour appears. Nickel and manganese are present. Cobalt is present, but only a green colour appears.

Are these sentences together satisfiable?

Solution:

- We can translate the text as follows: $(c \wedge \neg n) \rightarrow b$, $n \wedge m$, $c \wedge \neg b$. In the last formula, we used $\neg b$, since "only a green colour appears" indicates that no "brown colour appears".
- After re-writing the formulas, we have the formula set

  $$\{n \wedge m, c \wedge \neg b, (\neg c \vee n) \vee b\}$$

  Thus each branch contains $\{n, m, c, \neg b\}$, and we have a fork with a branch containing $\neg c \vee n$ and another branch containing b. The latter branch is closed. The first branch again contains a fork ($\neg c$ and n), where the first branch is closed and the second is open. Hence the valuation $v(c) = v(n) = v(m) = v(\neg b) = 1$ satisfies the formula set. See Figure 1.

Exercise 2.8 [Ho91] Formalise the following text and determine whether the argument is valid.

> The mother will die unless the doctor kills the child. If the doctor kills the child, the doctor will be taking life. If the mother dies, the doctor will be taking life. Therefore either way, the doctor will be taking life.

Exercise 2.9 Formalise the following text and determine whether the argument is valid.

> If the Iraq war is legal, then the world is a better place without Saddam. The world is a better place without Saddam. Therefore the Iraq war is legal.

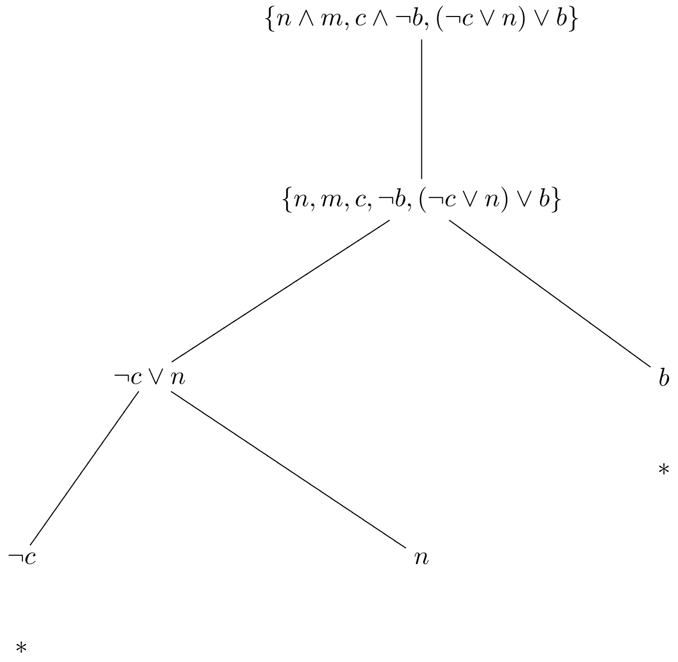

Figure 1: A tableau.

2.4 Decidability

Theorem 2.10 *Both the*

1. *satisfiability and*
2. *validity*

problems for propositional logic are decidable.

Proof: For 1 apply the semantic tableaux technique.

Alternatively apply the following procedure. Given a formula φ, consider the atomic formulas occurring in φ, say, $p_1, \ldots, p_n$. Take all possible valuations of $p_1, \ldots, p_n$ (2^n many of them). For each valuation v check whether φ is true under v (e.g., by using the parse tree of the formula). The formula is satisfiable iff it is true under at least one valuation.

For 2 note that a formula is valid iff its negation is unsatisfiable. ∎

Theorem 2.11 *[Complexity] The complexity of the satisfiability problem is NP (nondeterministic polynomial).*

Indeed, by guessing a valuation v, it can be checked in time polynomial of the length of the formula (the number of characters occurring in the formula) whether it is true under v.