3 Basics of Modal Logic

Modal logic subsumes propositional logic. It allows us to speak about not just one world (a specific valuation of atomic formulas) but about "possible worlds" (different valuations of atomic formulas) and their relationships (whether a possible world is an "alternative" of another one).

3.1 Syntax

Formulas are built up from atomic formulas using the connectives $\wedge$ and $\neg$ as in the case of propositional logic, but we have an additional unary connective $\Box$:

- $\Box\varphi$ is a formula for any formula φ.

The intuition is that $\Box\varphi$ holds at the current world if φ holds in all alternative worlds — see the semantics below. We will use the abbreviation $\Diamond\varphi$ for $\neg\Box\neg\varphi$.

Recall that $\mathrm{Sf}(\rho)$ denotes the set of *subformulas* of a formula ρ. $\mathrm{Sf}(\rho)$ can be defined inductively:

- $\mathrm{Sf}(p) = \{p\}$ for atomic p,
- $\mathrm{Sf}(\neg\varphi) = \mathrm{Sf}(\varphi) \cup \{\neg\varphi\}$,
- $\mathrm{Sf}(\varphi \wedge \psi) = \mathrm{Sf}(\varphi) \cup \mathrm{Sf}(\psi) \cup \{\varphi \wedge \psi\}$,
- $\mathrm{Sf}(\Box\varphi) = \mathrm{Sf}(\varphi) \cup \{\Box\varphi\}$.

Exercise 3.1 Show that a formula φ has at most $|\varphi|$ many subformulas, where $|\varphi|$ stands for the number of characters occurring in φ (as a string). **Hint: use formula induction.**

3.2 Semantics

A *(Kripke) frame* is a pair $\mathcal{F} = (S, R)$, where the set of *possible worlds* S is a non-empty set and the *accessibility relation* R is a binary relation on S (i.e., $R \subseteq S \times S$).

A *model* on a frame $\mathcal{F} = (S, R)$ is a triple $\mathcal{M} = (S, R, V)$ with $V : P \to \mathcal{P}(S)$. Hence V assigns to each atomic formula $p \in P$ a subset $V(p)$ of S — those worlds at which p is "true".

Let $\mathcal{M} = (S, R, V)$ be a model and $s \in S$ be a world; we define ρ *is true in* $\mathcal{M} = (S, R, V)$ *at* s, in symbols

$$(\mathcal{M}, s) \models \rho$$

as follows.

$$\begin{array}{lll} (\mathcal{M}, s) \models p & \text{iff} & s \in V(p) \\ (\mathcal{M}, s) \models \neg\varphi & \text{iff} & \text{not } (\mathcal{M}, s) \models \varphi \quad (\text{i.e., } (\mathcal{M}, s) \not\models \varphi) \\ (\mathcal{M}, s) \models (\varphi \wedge \psi) & \text{iff} & (\mathcal{M}, s) \models \varphi \text{ and } (\mathcal{M}, s) \models \psi \\ (\mathcal{M}, s) \models \Box\varphi & \text{iff} & \text{for all } t \in S \text{ such that } sRt, \text{ we have } (\mathcal{M}, t) \models \varphi \end{array}$$

Note that

$$(\mathcal{M}, s) \models \Diamond\varphi \text{ iff there exists } t \in S \text{ with } sRt \text{ and } (\mathcal{M}, t) \models \varphi.$$

Prove it!

A formula φ is *true in model* $\mathcal{M} = (S, R, V)$, denoted $\mathcal{M} \models \varphi$, if it is true at all worlds in $\mathcal{M}$ (i.e., $(\mathcal{M}, s) \models \varphi$ for all $s \in S$). Note that the meaning of a formula in a model (i.e., those possible worlds that satisfy the formula) is determined by the meaning of its subformulas (and the accessibility relation).

A formula φ is *valid in frame* $\mathcal{F} = (R, S)$, denoted $\mathcal{F} \models \varphi$, if it is true in all models $\mathcal{M} = (S, R, V)$ based on $\mathcal{F}$.

If $\mathcal{C}$ is a class of frames (respectively, models), then φ is *valid* (respectively, *true*) *in* $\mathcal{C}$, denoted as $\mathcal{C} \models \varphi$, if φ is valid (respectively, true) in all elements of $\mathcal{C}$. Truth and validity are defined for sets of formulas in the obvious way.

Exercise 3.2 Design a procedure that determines whether a given formula is true in a given finite model.

Different applications require different versions of modal logic. For instance, we can require that the set of possible worlds is a certain fixed set (e.g., the set of natural numbers) and that the accessibility relation satisfies certain conditions (e.g., transitivity). In particular, the modal logic of all frames is denoted by **K** and the modal logic of all frames with equivalence accessibility relation by **S5**.

Consider the following *frame conditions*:

reflexivity: $\forall s\; sRs$

symmetry: $\forall s, t(sRt \to tRs)$

transitivity: $\forall s, t, u[(sRt \land tRu) \to sRu]$.

Now look at these formulas:

refl: $\Box\varphi \to \varphi$

symm: $\varphi \to \Box\Diamond\varphi$

trans: $\Box\varphi \to \Box\Box\varphi$.

Example 3.1 Show that in transitive frames the formula $\Box\varphi \to \Box\Box\varphi$ is valid.

Solution: Let $\mathcal{F} = (S, R)$ be an arbitrary transitive frame, i.e., for every x, y and z, we have that xRy and yRz imply xRz. Let $\mathcal{M} = (S, R, V)$ be a model on $\mathcal{F}$ defined by the valuation $V : P \to \mathcal{P}(S)$. We have to prove that, for every world $s \in S$, we have

$$(\mathcal{M}, s) \models \Box\varphi \to \Box\Box\varphi.$$

Take a world $s \in S$ and assume that $(\mathcal{M}, s) \models \Box\varphi$. We have to show that $(\mathcal{M}, s) \models \Box\Box\varphi$, which means that, for every $t \in S$, we have

$$sRt \text{ implies } (\mathcal{M}, t) \models \Box\varphi$$

or, in other words, that for every t and u in S, we have

$$sRt \text{ implies } (tRu \text{ implies } (\mathcal{M}, u) \models \varphi).$$

So, suppose sRt. Then if tRu, we have sRu by transitivity. Then we have $(\mathcal{M}, u) \models \varphi$, since $(\mathcal{M}, s) \models \Box\varphi$ by assumption.

Example 3.2 Show that the formula $\Box\varphi \to \Box\Box\varphi$ is not valid in non-transitive frames.

Solution: Let $\mathcal{F} = (S, R)$ be an arbitrary frame where R is not transitive, say, we have $s, t, u \in S$ such that sRt and tRu but not sRu. Let p be a propositional atom and V be a valuation such that $u \notin V(p)$ but $v \in V(p)$ for all $v \in S$ such that sRv. If we let $\mathcal{M} = (S, R, V)$, then $(\mathcal{M}, s) \models \Box\varphi$ but $(\mathcal{M}, s) \not\models \Box\Box\varphi$.

Exercise 3.3 Show that each frame condition above is defined by the corresponding formula. That is, show that for any frame, the accessibility relation is reflexive, symmetric, and transitive if and only if, respectively, *refl*, *symm*, and *trans* is valid in the frame.