

On Representable Ordered Residuated Semigroups

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Abstract

We show that the equational theory of representable lattice-ordered residuated semigroups is not finitely axiomatizable. We apply this result to the problem of completeness of substructural logics.

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1 Introduction

Residuated algebras have been extensively investigated in the literature, partly because of their connection to substructural logics. A residuated algebra is *representable* if it is isomorphic to a family of binary relations and the operations are interpreted as “natural” operations on binary relations — see the precise definition below. An important line of research is to determine precisely which classes of representable residuated algebras have finitely axiomatizable equational or quasiequational theories, since such axiomatizability results yield weak or strong completeness results for substructural logics. Indeed, families of binary relations as semantics for substructural logics have been proposed by various researchers, e.g., for the Lambek calculus (LC) by van Benthem and for relevance logics by Dunn and Maddux. Completeness results of this kind include the completeness of the LC [AM94] and the completeness of the relevance logic with mingle RM [Ma]. We address a similar problem here by looking at the problem of expanding the similarity type of LC with (static) disjunction.

In the remainder of this section we define representable algebras and recall a finite axiomatization of representable lower semilattice-ordered residuated semigroups. In the next section, we look at the possibility of extending the similarity type by including join as well. Finally, we will look at the implication of the main result to substructural logics.

Definition 1.1 A representable relation algebra (an RRA) is $\mathfrak{A} = (A, +, \cdot, -, ;, \smile, 1', 0, 1)$ such that A consists of subsets of an equivalence relation W and $+$ is union, \cdot is intersection, $-$ is set difference, $0 = \emptyset$, $1 = W$ and

$$\begin{aligned}x ; y &= \{(u, v) \in W : \exists w((u, w) \in x \ \& \ (w, v) \in y)\} \\x \smile &= \{(u, v) \in W : (v, u) \in x\} \\1' &= \{(u, v) \in W : u = v\}\end{aligned}$$

Let τ be a set of operations definable in RRA. We denote the class of subreducts of RRA to the signature τ as $R(\tau)$.

Note that RRAs are residuated: the following definition defines the right- and left residuals of $;$

$$x \setminus y = -(x \smile ; -y) \text{ and } x / y = -(-x ; y \smile)$$

and we can define converse negation as well: $\sim x = -(x^\smile)$. The extension of these operations in an RRA with unit W is as follows:

$$x \setminus y = \{(u, v) \in W : \forall w((w, u) \in x \Rightarrow (w, v) \in y)\} \quad (1)$$

$$x / y = \{(u, v) \in W : \forall w((v, w) \in y \Rightarrow (u, w) \in x)\} \quad (2)$$

$$\sim x = \{(u, v) \in W : (v, u) \notin x\} \quad (3)$$

We recall from [AM94] that representable lower semilattice-ordered residuated semigroups are finitely axiomatizable.

Theorem 1.2 *The equational and quasiequational theories of $\mathbf{R}(\cdot, ;, \setminus, /)$ are finitely axiomatizable.*

Indeed, we showed in [AM94] that the following set of quasiequations axiomatizes the class $\mathbf{R}(\cdot, ;, \setminus, /)$:

- \cdot is a (lower) semilattice
- $;$ is associative and monotone w.r.t. \cdot :

$$(x \cdot x') ; (y \cdot y') \leq x ; y \quad (4)$$

- if $x \leq y$, then $x \setminus y$ behaves similarly to a unit element for $;$:

$$z \leq z ; ((x \cdot y) \setminus y) \text{ and } z \leq ((x \cdot y) \setminus y) ; z \quad (5)$$

and similarly for $/$:

$$z \leq z ; (y / (x \cdot y)) \text{ and } z \leq (y / (x \cdot y)) ; z \quad (6)$$

- \setminus and $/$ are the right and left residuals of $;$:

$$x ; y \leq z \iff y \leq x \setminus z \iff x \leq z / y \quad (7)$$

Pratt [Pr90] observed that the quasiequations 7 can be replaced by the following equations:

$$x \setminus (y \cdot y') \leq x \setminus y \quad (8)$$

$$x ; (x \setminus y) \leq y \leq x \setminus (x ; y) \quad (9)$$

and the corresponding equations for $/$. Indeed, assuming $x ; y \leq z$, we have $x \setminus (x ; y) = x \setminus (x ; y \cdot z) \leq x \setminus z$ by 8, and hence $y \leq x \setminus z$ by the second part of 9. If we assume $y \leq x \setminus z$, then $x ; y \leq x ; (x \setminus z)$ by monotonicity, and hence $x ; y \leq z$ by the first part of 9. Since 8 and 9 are easily seen to be valid in $\mathbf{R}(\cdot, ;, \setminus, /)$, we are done.

In the next section, we look at the possibility of extending the similarity type by including join as well.

2 Main result

We observed in [AM94] that the quasiequational theory of representable distributive lattice-ordered residuated semigroups is not finitely axiomatizable. We strengthen this result below.

Theorem 2.1 *The equational and quasiequational theories of $\mathbf{R}(+, \cdot, ;, \setminus, /)$ are not finitely axiomatizable. The same holds if we expand the similarity type by any set of operations definable in RRA.*

Proof: Maddux defines non-representable, finite, integral, symmetric relation algebras \mathfrak{A}_n (for $n \in \omega$) whose ultraproduct is representable [Ma89]. Andréka shows that already the $\{+, \cdot, ;\}$ -reduct of \mathfrak{A}_n is not representable, whence τ -subreducts of RRA such that $\{+, \cdot, ;\} \subseteq \tau$ are not finitely axiomatizable, cf. [An91]. Hence the quasivariety $R(+, \cdot, ;, \setminus, /)$ is not finitely axiomatizable. Here we show that the non-representability of the $\{+, \cdot, ;, \setminus, /\}$ -reduct \mathfrak{B}_n of \mathfrak{A}_n is witnessed by an equation.

We recall that \mathfrak{A}_n has the following atoms: identity $1'$, q_i for $1 \leq i \leq m$, and p_j for $1 \leq j \leq n$ with $m = 3 \cdot n!$. Composition is defined such that $q_{i+1} \leq p_1 ; q_i$ for every $i < m$ and $0 = q_i \cdot 1'$; $q_j = q_r \cdot q_s$; $q_t = p_l \cdot p_l$ for every $1 \leq i \neq j \leq m$, $1 \leq r, s, t \leq m$, and $1 \leq l \leq n$. Furthermore, $q_m \setminus q_m = 1'$. It follows that a representation of the $\{+, \cdot, ;\}$ -reduct of \mathfrak{A}_n would require a coloring of the edges of a graph U with $|U| \geq 3 \cdot n!$ using the colors p_i (for $1 \leq i \leq n$) but without monochromatic triangles — an impossible task. See Figure 1, where $U = \{u_1, \dots, u_m\}$ and every dotted arrow should have a color p_i for some $1 \leq i \leq n$.

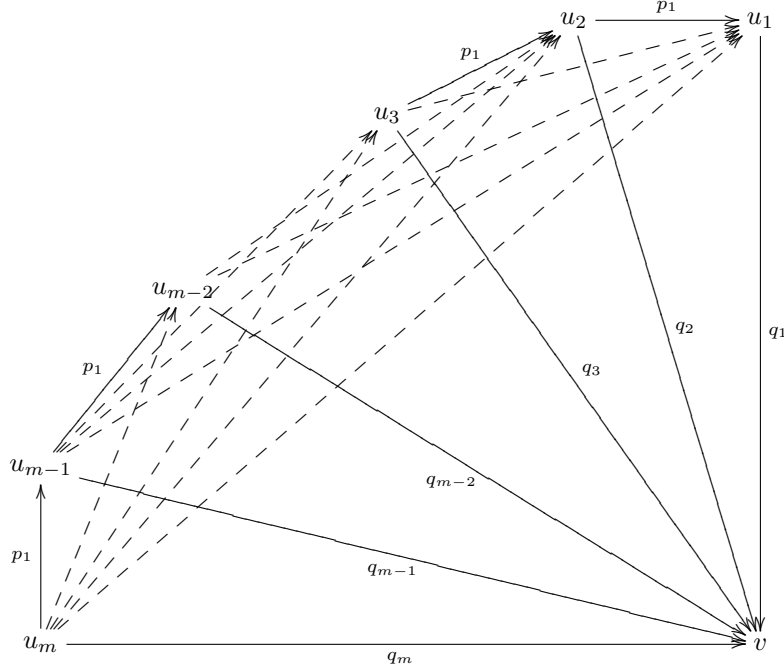


Figure 1: The reason for non-representability

Below we use the abbreviation x^{i+1} that is recursively defined by $x^{i+1} = x ; x^i$ with the convention that $x^0 ; y = y$ for every term y . Let x_i, y_j and z_0 be distinct variables for $1 \leq i \leq n$ and $1 \leq j \leq m$ and let Z stand for $\sum\{x_i : 1 \leq i \leq n\} + \sum\{y_j : 1 \leq j \leq m\} + (y_m \setminus y_m)$. We define the equation e_n for $n \in \omega$ as

$$\sigma_n ; (\alpha_n \cdot \beta_n \cdot \gamma_n \cdot \delta_n) \leq z_0$$

where

$$\sigma_n = y_m \cdot (x_1 ; (y_{m-1} \cdot ((x_1 \cdot x_1 \setminus Z) ; (y_{m-2} \cdot (x_1 \cdot x_1 \setminus Z \cdot (x_1 ; x_1) \setminus Z) ; (\dots))))))$$

$$\alpha_n = \prod\{(x_1^{m-i} ; (y_i \cdot y_j)) \setminus z_0 : 1 \leq i \neq j \leq m\}$$

$$\beta_n = \prod\{(x_1^i ; (x_l \cdot x_l ; x_l) ; y_j) \setminus z_0 : 1 \leq i, j \leq m, 1 \leq l \leq n\}$$

$$\gamma_n = \prod\{(x_1^i ; (y_{m-i} \cdot y_m \setminus y_m ; y_{m-j})) \setminus z_0 : 0 < i < j < m\}$$

$$\delta_n = \prod\{(x_1^i ; (y_{m-i} \cdot y_k ; y_j)) \setminus z_0 : 1 \leq i < m, 1 \leq j, k \leq m\}$$

We claim that e_n

1. fails in \mathfrak{B}_n
2. is valid in representable algebras.

For item 1, let ι be the evaluation of the variables

$$\iota(x_i) = p_i \quad \iota(y_j) = q_j \quad \iota(z_0) = 0$$

Then $\iota(RHS(e_n)) = \iota(z_0) = 0$. On the other hand, $\iota(LHS(e_n)) \neq 0$ because of the following. First note that $\iota(Z) = 1$, since $\iota(y_m \setminus y_m) = q_m \setminus q_m = 1'$. Then we have $\iota(\tau \setminus Z) = 1$ for any term τ . In \mathfrak{B}_n , $q_{i+1} \leq p_1 ; q_i$ for every $1 \leq i < m$, whence $\iota(\sigma_n) \neq 0$. Furthermore, the terms on the left of \setminus in α_n , β_n , γ_n and δ_n evaluate to 0, since the meet of distinct atoms is 0 (for α_n), there are no monochromatic triangles for p_i (for β_n), $q_m \setminus q_m \leq 1'$ (for γ_n) and there are no q -triangles (for δ_n). Hence $\iota(\alpha_n) = \iota(\beta_n) = \iota(\gamma_n) = \iota(\delta_n) = 1$. Thus $\iota(LHS(e_n)) \neq 0$.

For item 2, let $\mathfrak{C} = (C, +, \cdot, ;, \setminus, /)$ be a representable algebra and ι be an arbitrary valuation of the variables. Assume that $(u_m, u) \in \iota(LHS(e_n))$. Then there is v such that $(u_m, v) \in \iota(\sigma_n)$ and $(v, u) \in \iota(\alpha_n \cdot \beta_n \cdot \gamma_n \cdot \delta_n)$. By $(u_m, v) \in \iota(\sigma_n)$, we have $(u_{i+1}, u_i) \in \iota(x_1)$ and $(u_i, v) \in \iota(y_i)$ for every $1 \leq i < m$. See Figure 2.

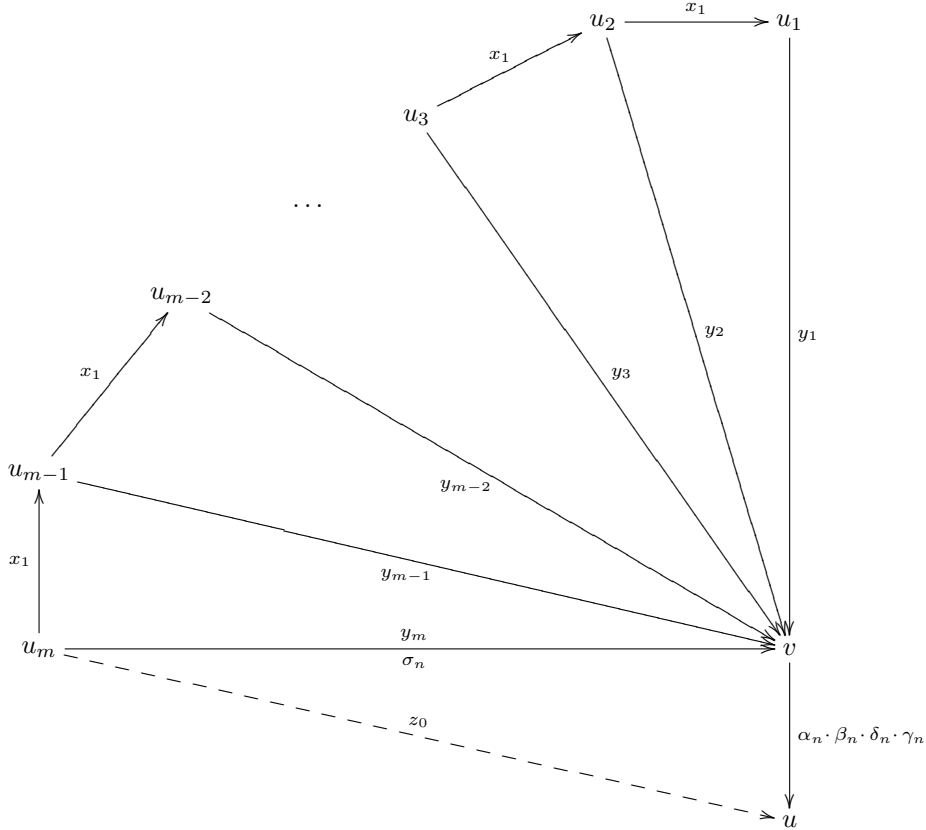


Figure 2: The validity of e_n

First we consider the case where there are $i \neq j$ such that $u_i = u_j$. In this case, we have $(u_i, v) \in \iota(y_i) \cdot \iota(y_j)$ and thus $(u_m, v) \in \iota(x_1^{m-i} ; (y_i \cdot y_j))$. Since $(v, u) \in \iota(\alpha_n)$, we get $(u_m, u) \in \iota(z_0) = \iota(RHS(e_n))$.

Now let us assume that all the u_i s are different. Note that

$$(u_i, u_j) \in \iota(Z) = \iota(\sum \{x_k : 1 \leq k \leq n\} + \sum \{y_l : 1 \leq l \leq m\} + (y_m \setminus y_m))$$

for every $1 \leq i \neq j \leq m$, by $(u_m, v) \in \iota(\sigma_n)$. If there are $i < j$ and k such that $(u_i, u_j) \in \iota(y_k)$, then $(u_m, v) \in \iota(x_1^i; (y_{m-i} \cdot y_k; y_j))$. Hence $(u_m, u) \in \iota(z_0)$ by $(v, u) \in \iota(\delta_n)$, i.e., $(u_m, u) \in \iota(RHS(e_n))$. If there are $i < j$ such that $(u_i, u_j) \in \iota(y_m \setminus y_m)$, then $(u_m, v) \in \iota(x_1^i; (y_{m-i} \cdot y_m \setminus y_m; y_{m-j}))$. Hence $(u_m, u) \in \iota(z_0)$ by $(v, u) \in \iota(\gamma_n)$, i.e., $(u_m, u) \in \iota(RHS(e_n))$. It remains to consider the case where, for all $i < j$, there is l such that $(u_i, u_j) \in \iota(x_l)$. By $m = 3 \cdot n!$, we have that for some $1 \leq i < j < k \leq m$, (u_i, u_j, u_k) is a monochromatic triangle the edges of which are colored with some $\iota(x_l)$. Hence $(u_m, v) \in \iota(x_1^{m-i}; (x_l \cdot x_l; x_l); y_k)$. Then $(u_m, v) \in \iota(z_0)$ for some $1 \leq i \leq m$, by $(v, u) \in \iota(\beta_n)$. Hence $(u_m, u) \in \iota(RHS(e_n))$ in this case as well. Thus e_n is indeed valid in representable algebras. ■

3 Substructural logics

In this section, we explain the connection of the main result to substructural logics.

Lambek calculus In [AM94], we showed that the Lambek calculus with static conjunction is complete w.r.t. relational semantics $\text{Rel}(\text{LC})$ consisting of binary relations, where fusion (\bullet in the original formulation and $;$ in our notation) is interpreted as relation composition, conjunction (\wedge or \cdot) as intersection and \setminus and $/$ as the two residuals of composition. There is one subtle difference between this semantics $\text{Rel}(\text{LC})$ and the subreduct $\text{R}(\cdot, ;, \setminus, /)$ of RRA. Namely, in $\text{Rel}(\text{LC})$, we do not require that the union W of all relations is an equivalence relation. In fact, in the completeness proof, W is irreflexive ($(x, x) \notin W$) and antisymmetric (for $x \neq y$, $(x, y) \in W$ implies $(y, x) \notin W$), and the residuals are defined as in 1 and 2 with W as a parameter. Since W is antisymmetric, so are $x \setminus y$ and x / y , hence 5 and 6 fail in accordance with the absence of the corresponding rules in the original LC [La58]. Thus Theorem 2.1 does not seem to answer the problem of the weak completeness of the LC extended with disjunction. But we claim that essentially the same proof yields the following.

Corollary 3.1 *No extension with finitely many axioms of the Lambek calculus with disjunction is weakly complete w.r.t. relational semantics Rel .*

Proof: In this case a representable algebra has the form $\mathfrak{D} = (D, +, \cdot, ;, \setminus, /)$ where D is a collection of binary relations, $+$ is union, \cdot is intersection, $;$ is relation composition as before and

$$\begin{aligned} x \setminus y &= \{(u, v) \in W : \forall w((w, u) \in x \Rightarrow (w, v) \in y)\} \\ x / y &= \{(u, v) \in W : \forall w((v, w) \in y \Rightarrow (u, w) \in x)\} \end{aligned}$$

where $W = \bigcup\{d : d \in D\}$, i.e., W is not necessarily an equivalence relation. An inspection of the proof of Theorem 2.1 reveals that we did not use anywhere that the unit of the representable algebra would be an equivalence relation, and that, in particular, $\iota(y_m \setminus y_m)$ would be reflexive¹. Hence the same argument as above shows that \mathfrak{B}_n is not representable while a non-trivial ultraproduct of \mathfrak{B}_n is representable², and that the equation e_n fails in \mathfrak{B}_n while it is valid in representable algebras. ■

Relevance logic Finally, let us mention an open problem related to relevance logic [AB75, ABD92]. Recall that relevance logics can be soundly interpreted over families of binary relations. Let $\text{R}^{cd}(\cdot, \cdot, +, \setminus, \sim)$ be that subclass of $\text{R}(\cdot, \cdot, +, \setminus, \sim)$ where each algebra is commutative $x; y = y; x$ and dense (or square-increasing) $x \leq x; x$. Let $\mathfrak{A} \in \text{R}^{cd}(\cdot, \cdot, +, \setminus, \sim)$ with unit W and v be a valuation of the propositional atoms into A . We extend v to compound formulas by interpreting

¹We used transitivity of the unit, but that holds for W in this case as well because of composition, and $x \setminus x$ always contains the reflexive edges from W .

²Hence strong completeness fails as well [AM94].

conjunction as intersection, disjunction as union, implication as the residual operation (right and left residuals coincide by commutativity) and relevant negation as converse negation. We define

$$\mathfrak{A} \models \varphi \iff \text{Id} \subseteq v(\varphi)$$

where $\text{Id} = \{(u, v) \in W : u = v\}$. Then relevance logic R is sound w.r.t. $\mathbf{R}^{cd}(\cdot, \cdot, +, \backslash, \sim)$, and relevance logic RM with the mingle axiom is complete w.r.t. the semantics $\mathbf{R}^{cdt}(\cdot, \cdot, +, \backslash, \sim)$, that subclass of $\mathbf{R}^{cd}(\cdot, \cdot, +, \backslash, \sim)$ where each algebra is transitive (or square-decreasing) $x; x \leq x$, see [Ma]. On the other hand, completeness fails without the mingle axiom: the logic of $\mathbf{R}^{cd}(\cdot, \cdot, +, \backslash, \sim)$ is not finitely axiomatizable [Mi09].

The question is whether we could achieve more in the absence of relevant negation. Let R^p denote the fragment of the relevance logic R to the language of fusion, conjunction, disjunction and implication axiomatized by the axioms $A1$ – $A11$, $A14$ and $A15$ and derivation rules $R1$, $R2$ of [RM73]. Obviously, R^p is sound w.r.t. the semantics defined by the commutative and dense algebras $\mathbf{R}^{cd}(\cdot, \cdot, +, \backslash)$.

The algebras \mathfrak{A}_n in the proof of Theorem 2.1 are commutative, but they are not dense, since there are no monochromatic triangles. And this latter fact seems to be crucial in the non-finite axiomatizability proof. Hence we ask the following.

Problem 3.2 Is R^p complete w.r.t. the semantics $\mathbf{R}^{cd}(\cdot, \cdot, +, \backslash)$? Is the equational theory of $\mathbf{R}^{cd}(\cdot, \cdot, +, \backslash)$ finitely axiomatizable?

We note that the quasiequational theory of $\mathbf{R}^{cd}(\cdot, \cdot, +, \backslash)$ is not finitely axiomatizable, since the $\{\cdot, \cdot, +, \backslash\}$ -reduct of the algebras \mathfrak{A}_n from [Mi09] are not representable (while their ultraproduct is).

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