

# Representable Semilattice-Ordered Monoids

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ABSTRACT. We show that no finite set of first-order axioms can define the class of representable semilattice-ordered monoids.

## 1. Introduction

*Relation Algebra* is an algebraic abstraction of the study of binary relations. Tarski showed that Relation Algebra can act as a vehicle for set theory and hence all of mathematics, cf. [20]. He also showed that the class of relation algebras (RA) and representable relation algebras (RRA) are equational varieties [19]. Monk proved that RRA is indeed a canonical variety (reported in [15, Theorem 2.12]). On the other hand, relation algebras are badly behaved in a number of ways: RRA cannot be defined by finitely many axioms [17]; the equational theories of RA and RRA are not decidable [18, 20, 3]; RRA cannot be defined by any set of canonical equations [13] nor by any set of equations using only finitely many variables [14]; the problem of telling whether a finite relation algebra is representable is an undecidable problem [10]. An important line of research is to restrict the signature of relation algebras and study the behaviour of the corresponding representation class. We are interested to see if the representation class is well behaved in some way: does the class form an equational variety or, if not, a quasi-variety; is its equational theory decidable; is it finitely axiomatisable; etc.?

The signature of relation algebra is that of boolean algebra (BA)  $\{0, 1, +, \cdot, -\}$  together with extra operators  $\{1', \smile, ;\}$ . If  $S \subseteq \{0, 1, +, \cdot, -, 1', \smile, ;\}$  then the *representation class*  $R(S)$  consists of those algebras  $\mathcal{A}$  in this signature for which there is an injection  $h : \mathcal{A} \rightarrow \wp(U)$  (the power set of  $U$ ) where  $U \subseteq D \times D$  (some base set  $D$ ) which respects those boolean operations in  $S$ , and

$$\begin{aligned} h(1') &= \{(d, d) : d \in D\} \\ h(a \smile) &= \{(d, d') : (d', d) \in h(a)\} \\ h(a; b) &= \{(d_1, d_2) : (\exists d_3 \in D)(d_1, d_3) \in h(a) \wedge (d_3, d_2) \in h(b)\} \end{aligned}$$

whenever  $1', \smile, ; \in S$ . An algebra in a signature  $S$  is *representable* iff it belongs to  $R(S)$ .

So the idea is to remove some of the operations for relation algebras and study the behaviour of the corresponding representation class. For any signature  $S \subseteq \{0, 1, +, \cdot, -, 1', \smile, ;\}$  it is trivial that  $R(S)$  is closed under isomorphic copies and subalgebras. It is easy to check that  $R(S)$  is closed under direct products (just take the disjoint union of the representations of the component algebras). It can also be shown that  $R(S)$  is a pseudo-elementary class which is therefore closed under ultraproducts. It follows that  $R(S)$  is a quasi-variety, in all cases.

The main focus of this paper is to investigate whether  $R(S)$  can be defined by a finite number of axioms, for certain signatures  $S$ . Thus, if we restrict to just the boolean operations then a representation is just a field of sets in which

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Signature	Finitely Axiomatisable?	Citation
$\{., ;\}$	yes	[7]
$\{+, ;\}$	no	[1]
$\{., \bar{\phantom{x}}\}$	no	[5]
$S \supseteq \{., +, ;\}$	no	[2, theorem 4]
$S \supseteq \{., \bar{\phantom{x}}, ;\}$	no	[12, 8]
$\{., 1', ;\}$	no	Theorem 2

FIGURE 1. Finite Axiomatisability of Representation Classes.

the boolean operations are interpreted as set-theoretic union and complement. By Stone’s theorem, the axioms for boolean algebra define this representation class. But this may be going too far: we cannot express very much about binary relations in the signature of boolean algebra. So we want to remove as few operations as possible from RA and see if we can achieve better logical and computational behaviour.

The case where we include all the boolean operations and see how many of the other RA operations can be included, while preserving finite axiomatisability, is solved. In the presence of the boolean operations, composition is not finitely axiomatisable, while converse and the identity constant are, see [4]. Thus, a signature between BA and RA has a finitely axiomatisable representation class iff the signature does not contain composition.

But composition is the most interesting of the extra RA-operations. So researchers have also studied the case where we take out or weaken some of the boolean operations and see if we can include composition and perhaps other non-boolean operations, subject to finite axiomatisability. Figure 1 summarises some of the main results. See also [16] for a survey of recent results in this field.

Our main interest in this paper is algebras in the signature  $\{., 1', ;\}$ , called *semilattice-ordered monoids*. We do not consider signatures containing negation, so our signatures are *positive*, but note that with negation and either  $\cdot$  or  $+$  in the signature, the signature is capable of expressing all the boolean operations, a case we have already dealt with.

## 2. Ordered monoids and semilattice-ordered monoids

In this paper we study two related signatures very close to the border of finite axiomatisability. They are ordered monoids and semilattice-ordered monoids, and their signatures are  $\{\leq, 1', ;\}$  and  $\{., 1', ;\}$ , respectively. It is easy to check (cf. [7]) that every proper subsignature of either of these yields a finitely axiomatisable representation class (where  $\leq$  is represented as  $\underline{\leq}$ ).

The signature  $\{., 1', ;\}$  has been the object of some interest.<sup>1</sup> The problem of whether this signature is finitely axiomatisable was raised by B. Schein, see [16, question 5.1]. The main result in the current paper is that the representation class  $R(\{., 1', ;\})$  is not finitely axiomatisable (theorem 2).

### DEFINITION 1.

- (1) A semilattice-ordered monoid (SOM) is an algebra in the signature  $(., 1', ;)$  satisfying
  - $\cdot$  defines a (lower) semilattice
  - $1'$  is an identity for  $;$ , i.e.,  $1'x = x; 1' = x$
  - $;$  is associative
  - $;$  is monotonic with respect to  $\leq$ , i.e.,  $x \leq y$  implies  $xz \leq yz$  and  $zx \leq zy$ , where  $a \leq b$  abbreviates  $a \cdot b = a$ .

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<sup>1</sup>The colloquial term for this language is the “Jerry Fragment”, named after Jerry Seligman who studied it because of its connection to channel algebras from situation theory.

- (2) A representation of an algebra  $\mathcal{A}$  of this type is a map  $h : \mathcal{A} \rightarrow \wp(D \times D)$ , for some base set  $D$ , such that
- $h(a) = h(b) \iff a = b$
  - $h(a \cdot b) = h(a) \cap h(b)$
  - $h(1') = \{(d, d) : d \in D\}$
  - $h(a; b) = h(a)|h(b) = \{(e, d) \in D \times D : (\exists f \in D)(e, f) \in h(a), (f, d) \in h(b)\}$

We write ‘ $\mathcal{A} \in \text{SOM}$ ’ or ‘ $\mathcal{A}$  is a SOM’ to mean ‘ $\mathcal{A}$  is a semilattice-ordered monoid’. Not every semilattice-ordered monoid is representable, as we will see. The question is: can you add finitely many axioms to those defining a semilattice-ordered monoid so as to exactly define the representable semilattice-ordered monoids? The answer is negative:

**THEOREM 2.** *There is no finite set of first-order axioms in the signature  $\{\cdot, 1', ;\}$  defining the class of representable semilattice-ordered monoids.*

This is our main result, the proof is delayed until section 6.

**COROLLARY 3.**  *$R(\{\cdot, 1', ;\})$  is not a variety.*

*Proof.* The equational theory of  $R(\{\cdot, 1', ;\})$  is finitely axiomatisable [6]. Then, assuming that  $R(\{\cdot, 1', ;\})$  was a variety, there would be a first-order sentence  $\varphi$  axiomatising it. By theorem 2, for every finite set  $\Psi$  of equations valid in  $R(\{\cdot, 1', ;\})$ ,  $\Psi \cup \{\neg\varphi\}$  would be satisfiable. Then, by compactness,  $\Phi \cup \{\neg\varphi\}$  would be satisfiable, where  $\Phi$  is the set of equations valid in  $R(\{\cdot, 1', ;\})$ . Since  $\{\varphi\}$  and  $\Phi$  have the same models, this is a contradiction.  $\square$

Next we define ordered monoids that are closely related to SOM.

**DEFINITION 4.**

- (1) An ordered monoid  $\mathcal{A} = (A, \leq, 1', ;)$  is a structure where
  - $\leq$  is a reflexive partial order on  $A$ ,
  - $;$  is an associative binary operation over  $A$ ,
  - $1' \in A$  is the identity for  $;$ ,
  - if  $a \leq b$ , then  $a; c \leq b; c$  and  $c; a \leq c; b$ , for any  $a, b, c \in A$ .
- (2) Write  $a < b$  if  $a \leq b$  but  $b \not\leq a$ . Clearly  $<$  is a strict partial order.
- (3) An ordered monoid representation is an injection  $h : A \rightarrow \wp(D \times D)$  (some base set  $D$ ) such that
  - (a)  $a \leq b \iff h(a) \subseteq h(b)$ ,
  - (b)  $h(1') = \{(d, d) : d \in D\}$ ,
  - (c)  $h(a; b) = h(a)|h(b)$ .
- (4) If  $h$  is an ordered monoid representation of  $\mathcal{A}$  satisfying
  - (a) for all  $x, y \in D$ , there is  $a \in A$  such that, for all  $a' \in A$ , we have  $(x, y) \in h(a') \iff a \leq a'$ , and
  - (b) for all  $a \in A$ , there is  $(x, y) \in h(a)$  such that, for all  $b \in A$ , we have  $(x, y) \in h(b) \iff a \leq b$
then we call  $h$  a principal ordered monoid representation of  $\mathcal{A}$ .

Henceforth we will identify an algebra with its domain, so we may write  $1' \in \mathcal{A}$  instead of  $1' \in A$ , above.

The meet operation ‘ $\cdot$ ’ defines a partial order in a SOM as follows:  $a \leq b \iff a \cdot b = a$ . Thus, from an arbitrary SOM we can define an ordered monoid. The converse is false: a partial order  $\leq$  may not define a semilattice, because a pair of elements might not have any lower bound and, even if they do, there may not be a unique infimum. Nevertheless, there is a Priestley kind of duality between ordered monoids and SOMs.

Let  $\mathcal{A}$  be an ordered monoid. A *downset*  $L$  is a subset of (the domain of)  $\mathcal{A}$  such that if  $a \in L$  and  $b \leq a$  then  $b \in L$ . For  $a \in \mathcal{A}$  the set  $a \downarrow =_{\text{def}} \{b \in \mathcal{A} : b \leq a\}$  is called a *principal downset*. We also write  $S \downarrow =_{\text{def}} \{a \in \mathcal{A} : (\exists s \in S)a \leq s\}$  for any subset  $S$  of the domain of  $\mathcal{A}$ .

**DEFINITION 5.** Let  $\mathcal{A}$  be an ordered monoid. We define a SOM-type algebra  $\mathcal{A}'$  from  $\mathcal{A}$  as follows. The domain of  $\mathcal{A}'$  consists of all downsets of  $\mathcal{A}$ . The meet operation is defined by intersection:  $L \cdot L' =_{\text{def}} L \cap L'$ . It is clear that  $L \cdot L'$  is a downset and that  $\cdot$  defines a semilattice of downsets. The identity of  $\mathcal{A}'$  is  $(1') \downarrow$ , and composition is defined by  $L; L' =_{\text{def}} \{l; l' : l \in L, l' \in L'\} \downarrow$ , for any downsets  $L, L'$ .

It is easy to check that this definition of composition of downsets is associative and that  $(1') \downarrow$  acts as the identity over downsets. Trivially, composition is monotonic with respect to set inclusion. Hence,

**LEMMA 6.** If  $\mathcal{A}$  is an ordered monoid, then  $\mathcal{A}'$  is a SOM.

Furthermore, for all  $a, b \in \mathcal{A}$  we have

$$(a; b) \downarrow = a \downarrow; b \downarrow$$

We define a theory  $\Sigma$  in a two-sorted language. The intended model of this theory will consist of a pair  $(\mathcal{A}, \mathcal{A}')$ , where  $\mathcal{A}'$  is the SOM defined above. A model for  $\Sigma$  will be of the form  $(A, B)$ . Elements of the first sort ( $A$ ) will be of the type of ordered monoids and elements of the second sort will be of the type of SOMs. The signature of  $\Sigma$  includes, as well as the signature of ordered monoids for the first sort and the signature of SOMs for the second sort, a binary predicate  $\varepsilon$  whose first argument is of sort one and whose second argument is of sort two. We define  $\Sigma$  by first giving an informal description of a formula, then the formal definition.

**DEFINITION 7.** Let  $\Sigma$  be the following set of first-order formulas.

- (1) ‘ $A$  is an ordered monoid’ (see definition 4)
- (2) ‘The elements of  $B$  correspond to downsets of  $A$ ’:

$$(\forall b \in B)(\forall a, a' \in A)(a \varepsilon b \wedge a' \leq a \rightarrow a' \varepsilon b)$$

- (3) ‘ $A$  is dense in  $B$ ’:

$$(\forall b, b' \in B)(b \cdot b' \neq b \leftrightarrow (\exists a \in A)a \varepsilon b \wedge a \not\varepsilon b')$$

- (4) ‘The meet operation of  $B$  corresponds to intersection of downsets over  $A$ ’:

$$(\forall b, b' \in B)(\forall a \in A)(a \varepsilon b \cdot b' \leftrightarrow a \varepsilon b \wedge a \varepsilon b')$$

- (5) ‘The identity of  $A$  corresponds to the identity of  $B$ ’:

$$(\forall a \in A)(a \leq 1'_A \leftrightarrow a \varepsilon 1'_B)$$

- (6) ‘Composition in  $A$  corresponds to composition in  $B$ ’:

$$(\forall b_1, b_2 \in B)(\forall a \in A)(a \varepsilon b_1; b_2 \leftrightarrow (\exists a_1, a_2 \in A)a \leq a_1; a_2 \wedge a_1 \varepsilon b_1 \wedge a_2 \varepsilon b_2)$$

**LEMMA 8.** Let  $\mathcal{A}$  be an ordered monoid. The pair  $(\mathcal{A}, \mathcal{A}')$  is a model of  $\Sigma$ .

*Proof.* Just interpret  $\varepsilon$  as membership. □

**LEMMA 9.** Let  $\mathcal{A}$  be an ordered monoid with a principal representation and let  $\mathcal{M} = (A, B)$  be a model of the theory  $\Sigma$  defined above. Then  $B$  is a representable SOM.

*Proof.* Let  $f$  be a principal ordered monoid representation of  $\mathcal{A}$  over the base set  $D$ . Define  $g : B \rightarrow \wp(D \times D)$  by  $g(b) = \bigcup \{f(a) : \mathcal{M} \models a \varepsilon b\}$ . We claim that  $g$  is a SOM representation.

Let  $b, b' \in \mathcal{B}$  and let  $b \neq b'$ , wlog  $b \not\leq b'$ . By definition 7(3) there is  $a \in \mathcal{A}$  such that  $\mathcal{M} \models a \varepsilon b \wedge a \not\varepsilon b'$ . By the definition of principal ordered monoid representation (definition 4(4b)) there is  $(x, y)$  such that  $(x, y) \in f(a') \Leftrightarrow a' \geq a$ . Since  $\mathcal{M} \models a \varepsilon b$ , we have  $(x, y) \in f(a) \subseteq g(b)$ . Also  $(x, y) \in g(b') = \bigcup \{f(a') : a' \varepsilon b'\}$  is impossible; indeed  $a \not\varepsilon b'$ , so by definition 7(2) we have  $(x, y) \in g(b) \setminus g(b')$ . Hence  $g$  is 1–1.

Next observe that  $g(b \cdot b') = g(b) \cap g(b')$  by definition 7(4) and definition 4(4a), and  $g(1'_{\mathcal{B}}) = f(1'_{\mathcal{A}}) = \{(d, d) : d \in D\}$  by definition 7(5). Finally,  $g(b_1; b_2) = \bigcup \{f(a) : \mathcal{M} \models a \varepsilon b_1; b_2\} = \bigcup \{f(a_1; a_2) : \mathcal{M} \models a_1 \varepsilon b_1 \wedge a_2 \varepsilon b_2\}$  by definition 7(6), and the latter is equal to  $\bigcup \{f(a_1) : \mathcal{M} \models a_1 \varepsilon b_1\} \cup \bigcup \{f(a_2) : \mathcal{M} \models a_2 \varepsilon b_2\} = g(b_1) | g(b_2)$ .  $\square$

### 3. Characterising representability

Next we define networks and a representation game using networks.

**DEFINITION 10.** A network  $N$  over an ordered monoid  $\mathcal{A}$  is a partial map  $N : D \times D \rightarrow \mathcal{A}$  for some set of nodes  $D$ . We say that  $N$  is consistent if, for all  $d, e, f \in D$ ,

**Ref1:**  $N$  is defined on  $(d, d)$

**Trans:** if  $N$  is defined on  $(d, e)$  and on  $(e, f)$ , then it is defined on  $(d, f)$

**Id:**  $N(d, e) \leq 1'$  iff  $d = e$

**Comp:**  $N(d, f) \leq N(d, e); N(e, f)$ .

We denote the set of nodes  $D$  by  $\text{nodes}(N)$ .

We say that  $M$  extends  $N$ , and we write  $M \supseteq N$ , if  $\text{nodes}(M) \supseteq \text{nodes}(N)$  and  $M \upharpoonright_{\text{nodes}(N) \times \text{nodes}(N)} = N$ . Given a sequence of (consistent) networks  $N_0 \subseteq N_1 \subseteq \dots$ , we define the limit of the sequence  $\bigcup_i N_i$  to be the network  $N$  with  $\text{nodes}(N) = \bigcup_i \text{nodes}(N_i)$  and labelling  $N(x, y) = N_i(x, y)$  for any  $i$  such that  $N_i(x, y)$  is defined (since the  $N_i$ s are nested, this is well defined) and if there is no such  $i$  then  $N(x, y)$  is undefined. Clearly this is also a (consistent) network. If  $N_i : i \in I$  are all (consistent) networks and  $i \neq j \rightarrow \text{nodes}(N_i) \cap \text{nodes}(N_j) = \emptyset$ , then define  $N = \bigcup_{i \in I} N_i$  to be the network such that  $\text{nodes}(N) = \bigcup_{i \in I} \text{nodes}(N_i)$  and  $N(x, y) = N_i(x, y)$  if  $x, y \in \text{nodes}(N_i)$  but  $N(x, y)$  is undefined if there is no such  $i$ . Again, this is a (consistent) network.

Next we define a representation game.

**DEFINITION 11.** The game  $G(\mathcal{A})$  on  $\mathcal{A}$  has two players  $\forall$  (universal) and  $\exists$  (existential). A play of the game  $G(\mathcal{A})$  consists of a countably infinite sequence of networks  $N_0 \subseteq N_1 \subseteq \dots$ . In round zero,  $\forall$  picks any  $\alpha \in \mathcal{A}$ .  $\exists$  must respond with a consistent network  $N_0$  containing nodes  $x, y$  such that  $N_0(x, y) = \alpha$ . In a later round (round  $t + 1$ ) suppose  $N_t$  has just been played.  $\forall$  picks nodes  $x, y \in N_t$  such that  $N_t$  is defined on  $(x, y)$ , and he picks any  $\alpha, \beta \in \mathcal{A}$  such that  $\alpha; \beta \geq N_t(x, y)$ . We denote this move as  $(N_t, x, y, \alpha, \beta)$ .  $\exists$  must respond with a consistent network  $N_{t+1}$  extending  $N_t$  containing a node  $z$  where  $N_{t+1}(x, z) \leq \alpha$  and  $N_{t+1}(z, y) \leq \beta$ . If  $\exists$  fails to provide the required network in any round, she loses the game. If she succeeds in every round, she wins.

For  $n \in \mathbb{N}$  the game  $G_n(\mathcal{A})$  is similar, but the game ends after round  $n$ . A play of this game is  $N_0 \subseteq N_1 \subseteq \dots \subseteq N_n$ . If in any of these rounds  $\exists$  fails to respond with a suitable network she loses, otherwise she wins.

**PROPOSITION 12** ([11]). Let  $\mathcal{A}$  be an ordered monoid.

- (1) If  $\mathcal{A}$  has a principal representation, then  $\exists$  has a winning strategy in  $G(\mathcal{A})$ .
- (2) If  $\mathcal{A}$  is countable and  $\exists$  has a winning strategy in  $G(\mathcal{A})$ , then  $\mathcal{A}$  has a principal representation.
- (3) If  $\exists$  has a winning strategy in  $G_n(\mathcal{A})$  for all  $n \in \mathbb{N}$ , then  $\mathcal{A}$  is elementarily equivalent to a countable algebra  $\mathcal{B}$  for which  $\exists$  has a winning strategy in  $G(\mathcal{B})$ .

- (4) For all  $n \in \mathbb{N}$ , there is a first-order formula  $\phi_n$  in the language of ordered monoids such that, for all ordered monoids  $\mathcal{B}$ , we have  $\mathcal{B} \models \phi_n$  if and only if  $\exists$  has a winning strategy in  $G_n(\mathcal{B})$ .

*Proof.* (Sketch.)

- (1) Let  $h$  be a principal representation of  $\mathcal{A}$  with domain  $D$ . For her winning strategy,  $\exists$  maintains a map  $' : \text{nodes}(N) \rightarrow D$ , where  $N$  is any network played in the game, such that if  $N(x, y)$  is defined, then it is the generator of the filter of elements holding on  $(x', y')$  in the representation, i.e.,  $a \geq N(x, y) \iff (x', y') \in h(a)$  for any  $a \in \mathcal{A}$ .
- (2) Suppose  $\mathcal{A}$  is countable and  $\exists$  has a winning strategy. In a play of  $G(\mathcal{A})$  suppose  $\forall$  plays  $\alpha$  in the initial round and then plays all possible moves eventually, i.e., if  $N_t$  occurs in the play and  $\beta; \gamma \geq N_t(x, y)$  then at some later round of the game ( $s > t$ )  $\forall$  plays  $(N_s, x, y, \beta, \gamma)$ . Since  $\mathcal{A}$  is countable, this can be scheduled. Let  $\exists$  use her winning strategy in such a game and let  $N_\alpha$  be the limit of the game. By renaming the nodes, if necessary, we can suppose that  $\text{nodes}(N_\alpha) \cap \text{nodes}(N_{\alpha'}) = \emptyset$ , whenever  $\alpha \neq \alpha'$ . Now let  $N = \bigcup_{\alpha \in \mathcal{A}} N_\alpha$ . For a principal representation of  $\mathcal{A}$  over the domain  $\text{nodes}(N)$ , let  $h(a) = \{(x, y) : N(x, y) \text{ is defined and } N(x, y) \leq a\}$ .
- (3) See [11, Proposition 10.14(2)].
- (4) See [11, Theorem 10.12 and Proposition 10.13].

□

#### 4. Non-representability

Let us return to our main problem whether the class of representable SOM is finitely axiomatizable.

A standard way of proving non-finite axiomatisability is to use an ultraproduct construction to show that the complement of the class of algebras is not closed under ultraproducts. So, for example, the representation class  $R(\{\cdot, 1', \sim, ;\})$  was shown to be non-finitely axiomatisable by constructing a sequence of non-representable algebras (so-called *rainbow algebras*) with a classically representable ultraproduct [12]. The signature for this class is very similar to that of SOM, but our attempts to modify this construction to obtain a non-finite axiomatisability result for SOM have failed. But if we could establish that converse is finitely axiomatisable over  $\{\cdot, 1', ;\}$  (as is the case where all boolean operations are present [4]), then non-finite axiomatisability for representable SOM would follow. Alas this is not case, as corollary 14 shows.

**THEOREM 13.** *Let  $\mathcal{A}$  be a SOM, and assume that  $\mathcal{A}$  has a minimal (with respect to the ordering  $\leq$ ) zero element  $0$  such that*

- (1)  $0; a = a; 0 = 0$ ,
- (2)  $1'$  is a minimal non-zero element,
- (3)  $a; b \leq 1' \rightarrow (a \leq 1' \vee b \leq 1')$ .

*Then  $\mathcal{A}$  is representable.*

Observe that the last two conditions are not sound over representable SOMs.

*Proof.* Suppose  $\mathcal{A}$  is countable and satisfies the conditions. We'll define a winning strategy for  $\exists$  in a modified version  $G'(\mathcal{A})$  of the game  $G(\mathcal{A})$  (see definitions 10, 11). In the initial round of  $G'(\mathcal{A})$ ,  $\forall$  can pick any element of  $\mathcal{A} \setminus \{0\}$  (so the modification is that he is not allowed to pick zero). In each round  $\exists$  must play a consistent network  $N$  satisfying:

$$\mathbf{Zero:} \quad N(m, n) \neq 0,$$

for any nodes  $m, n \in \text{nodes}(N)$ . In all other respects the game  $G'(\mathcal{A})$  is the same as  $G(\mathcal{A})$ .

In the initial round of  $G'(\mathcal{A})$ ,  $\forall$  chooses any  $a \in \mathcal{A} \setminus \{0\}$ . If  $a = 1'$  then  $\exists$  plays a network  $N_0$  with just one node  $n$  and labelling  $N_0(n, n) = 1'$ . Otherwise ( $a \not\leq 1'$ ),  $\exists$  lets  $N_0$  have two nodes  $m, n$  and labelling  $N_0(m, m) = N_0(n, n) = 1'$ ,  $N_0(m, n) = a$  (the edge  $(n, m)$  is unlabelled).

In a later round suppose the network  $N$  has just been played.  $\forall$  chooses nodes  $m, n \in N$  and elements  $a, b \in \mathcal{A}$  such that  $a; b \geq N(m, n)$ . Since  $N(m, n) \neq 0$  it follows that  $a \neq 0$  and  $b \neq 0$ .  $\exists$  must define a network  $M \supseteq N$  containing a node  $l$  such that  $M(m, l) \leq a$  and  $M(l, n) \leq b$ . If  $a = 1'$  she can let  $M = N$  and let  $l = m$  and if  $b = 1'$  she lets  $M = N$  and  $l = n$ . Assuming that neither  $a \leq 1'$  nor  $b \leq 1'$  she lets  $M$  have exactly one new node  $l$  not occurring as a node of  $N$  and defines labels of edges incident with  $l$  by

$$\begin{aligned} M(l, l) &= 1' \\ M(u, l) &= N(u, m); a \\ M(l, u) &= b; N(n, u) \end{aligned}$$

where  $u \in N$  is arbitrary. (We use the convention that  $N(u, m); a$  is undefined if  $N(u, m)$  is undefined.) Let us check that  $M$  is a consistent network. Network conditions Refl and Trans are easy to check. For network condition Comp take any  $u, v \in \text{nodes}(N)$ . We have

$$\begin{aligned} M(u, l); M(l, v) &= (N(u, m); a); (b; N(n, v)) \\ &\geq N(u, m); N(m, n); N(n, v) \\ &\geq N(u, v) = M(u, v) \\ M(u, v); M(v, l) &= N(u, v); (N(v, m); a) \\ &\geq N(u, m); a = M(u, l) \end{aligned}$$

and similarly  $M(l, u); M(u, v) \geq M(l, v)$ . To show that the remaining conditions Id and Zero for networks are true in  $M$ , we'll show that  $M(u, l), M(l, u) \not\leq 1'$  for  $u \in \text{nodes}(N)$ . So, for contradiction, suppose  $M(u, l) \leq 1'$  for some  $u \in \text{nodes}(N)$ . Then  $M(u, l) = N(u, m); a \leq 1'$ . So, by condition 3 of the theorem, either  $N(u, m) \leq 1'$  or  $a \leq 1'$ . We are assuming  $a \not\leq 1'$ . If  $N(u, m) \leq 1'$  then  $u = m$ , but then  $M(u, l) = M(m, l) = a \not\leq 1'$ . Thus  $M(u, l) \leq 1'$  is impossible. Similarly, we cannot have  $M(l, u) \leq 1'$  for any  $u \in \text{nodes}(N)$ . Thus  $M$  is a network, whence  $\exists$  has a winning strategy for  $G'(\mathcal{A})$ .

Now let  $N$  be the limiting network in a play of  $G'(\mathcal{A})$  where  $\exists$  uses her winning strategy and  $\forall$  plays all possible moves — i.e., if  $N_i$  is played at some point and  $m, n \in \text{nodes}(N_i)$  and  $a; b \geq N_i(m, n)$ , then at some point  $\forall$  will pick  $m, n$  and  $a, b$  for his move. Since each network occurring in the play is finite and  $\mathcal{A}$  is countable, it is possible to schedule all these moves in the play. As in proposition 12, a representation  $h$  of  $\mathcal{A}$  with base  $\text{nodes}(N)$  can now be defined by

$$h(a) = \{(m, n) \in \text{nodes}(N) \times \text{nodes}(N) : N(m, n) \leq a\}.$$

In fact, this defines a *complete* representation, where arbitrary infima are preserved.

Finally, if  $\mathcal{B}$  is a not necessarily countable SOM satisfying the condition then let  $\mathcal{A} \prec \mathcal{B}$  be a countable elementary subalgebra. Then  $\mathcal{A}$  also satisfies the condition and by the above  $\mathcal{A}$  is representable. Since the class of representable SOMs is a quasi-variety, it is closed under elementary equivalence, hence  $\mathcal{B}$  is also representable.  $\square$

A generalisation of the theorem to algebras where  $1'$  need not be minimal can be obtained, but the axioms, definitions and proof are slightly more complicated.

**COROLLARY 14.**  $R(\{\cdot, 1', \smile, ;, \})$  is not finitely axiomatisable over  $R(\{\cdot, 1', ;, \})$ .

*Proof.* Suppose, for contradiction, that  $R(\{\cdot, 1', \smile, ;\})$  is defined by an axiomatisation  $\Theta$  of  $R(\{\cdot, 1', ;\})$  together with a finite set of axioms. Without loss we can suppose that this finite set consists of a single first-order axiom  $\phi$ . So our supposition is that for any  $\{\cdot, 1', \smile, ;\}$ -algebra  $\mathcal{B}$  we have  $\mathcal{B} \in R(\{\cdot, 1', \smile, ;\}) \Leftrightarrow \mathcal{B} \models \Theta \cup \{\phi\}$ .

Now let  $\mathcal{A}_n$  be the rainbow algebra of [12, section 3.1]. This is a finite relation algebra.<sup>2</sup> The identity of  $\mathcal{A}_n$  is minimal non-zero, for  $x, y \in \mathcal{A}_n$  we have  $x; y \leq 1' \rightarrow (x \leq 1' \vee y \leq 1')$ , and all the other conditions of theorem 13 hold. Hence,  $\mathcal{A}_n \in R(\{\cdot, 1', ;\})$ . By [12, lemma 3.1],  $\mathcal{A}_n \notin R(\{\cdot, 1', \smile, ;\})$ . But by [12, lemma 3.4], for any non-principal ultraproduct  $\mathcal{A}$  of the  $\mathcal{A}_n$ s we have  $\mathcal{A} \in R(\{\cdot, 1', \smile, ;\})$ .

By our assumption,  $\mathcal{A} \models \Theta \cup \{\phi\}$ . By Łoś' theorem,  $\mathcal{A}_n \models \phi$  for “many” values of  $n$ . Since  $\mathcal{A}_n \in R(\{\cdot, 1', ;\})$ , we have  $\mathcal{A}_n \models \Theta$ , for all  $n$ . But then, for “many”  $n$ , we have  $\mathcal{A}_n \models \Theta \cup \{\phi\}$  so  $\mathcal{A}_n \in R(\{\cdot, 1', \smile, ;\})$ , contradicting [12, lemma 3.1].  $\square$

We are about to define a sequence of unrepresentable algebras  $\mathcal{A}_n : n \geq 1$  with a representable ultraproduct. This will allow us to prove the non-finite axiomatisability of the representation class. We have to admit that the construction of the algebras is a little complex and the proof of representability of the ultraproduct is fearsome. Let us try to explain how we arrived at such a construction.

When we looked at the signature  $\{\cdot, 1', ;\}$  we first attempted to prove that  $R(\{\cdot, 1', ;\})$  was finitely axiomatisable. We wrote down a small number of axioms — mostly quite obvious axioms like monotonicity and associativity, but also some extra axioms like  $e, e' \leq 1' \rightarrow e; e' = e \cdot e'$  and  $e \leq 1' \wedge e; x \geq y \rightarrow e; y \geq y$  — and attempted to prove that these axioms were sufficient for representability. Now, if a SOM does not contain any elements  $\bar{f}, f \not\leq 1'$  such that  $\bar{f}; f \leq 1'$  then it is easy to prove that such an algebra satisfying this set of axioms must be representable. You just build an antisymmetric representation of the algebra, step by step, cf. theorem 13. Without such “functional elements”, the identity is not playing an important role and the representability of an algebra reduces to its representability for the signature  $\{\cdot, ;\}$ .

We were even able to prove representability if the algebra contained elements  $\bar{f}_i, f_i : i \in I$  with  $\bar{f}_i; f_i \leq 1'$  or  $f_i; \bar{f}_i \leq 1'$  provided that the following condition holds: if there is an element  $b$  with  $b \leq s$  for some product  $s$  of elements from  $\{f_i, \bar{f}_i : i \in I\}$ , then  $b$  is equal to such a product. Very roughly, we could build clusters of points related by invertible functions and relate clusters to each other by antisymmetric relations as in the previous paragraph. But we could not extend our proof of representability to cover the case where elements such as  $b$  exist.

The failure of our proof led us to construct an unrepresentable semilattice-ordered monoid whose elements included  $f, \bar{f}, g, \bar{g}, b$  where  $\bar{f}; f \leq 1', g; \bar{g} \leq 1'$  and  $b < f; g$  but  $b$  is not equal to any product of  $\{f, \bar{f}, g, \bar{g}\}$ . It also includes elements  $a, c$  and satisfies  $a; b \geq c$ . Our example is essentially  $\mathcal{A}'_1$  of definition 21, below. Because we have  $\bar{f}; f \leq 1'$  and  $f; \bar{f} \geq 1'$ , then if a representation of the algebra were to exist, whenever an edge  $(x, y)$  belonged to the representation of  $f$  it would follow that  $(y, x)$  belonged to the representation of  $\bar{f}$  and vice versa, and a similar property would hold for  $g, \bar{g}$ . We can define a sort of converse operator on products of elements from  $\{f, \bar{f}, g, \bar{g}\}$  by reversing the product and adding/removing the overline. Products of elements from  $\{f, \bar{f}, g, \bar{g}\}$  are *reversible* in the sense that whenever an edge belonged to the representation of such a product, the reverse edge must belong to the representation of the converse. Part of the problem, in attempting to represent the example we constructed, seems to be that  $b$  is also reversible — if  $(x, y)$  belonged to the representation of  $b$  then  $(y, x)$  would belong to the representation of  $\bar{g}; f$  — but  $\bar{g}; f$  is not a true converse to  $b$ .

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<sup>2</sup>We will slightly abuse notation for the sake of brevity. If  $S' \subseteq S$  and  $\mathcal{A}$  is an  $S$ -algebra, then by  $\mathcal{A} \in R(S')$  we mean that the  $S'$ -reduct of  $\mathcal{A}$  is in  $R(S')$ .

Straight after this discovery we switched from trying to find a finite number of axioms to define  $R(\{\cdot, 1', ;\})$  to proving that no such finite set of axioms exists. Our aim was to define an unrepresentable ordered monoid  $\mathcal{A}_n$  for each finite  $n > 0$ , and show that  $\mathcal{A}'_n$  is an unrepresentable SOM (definition 5 above explains how the SOM  $\mathcal{A}'_n$  is constructed from the ordered monoid  $\mathcal{A}_n$ ). Later we will show that an algebra which is elementarily equivalent to a non-principal ultraproduct of the  $\mathcal{A}_n$ s has a principal representation, and then we'll be able to deduce our main result — the class of representable SOMs is not finitely axiomatisable.

So we tried to devise an unrepresentable SOM, but we wanted to define it so that the assumption that the SOM is representable only led to a contradiction if more than  $n$  points in a claimed representation were considered, for  $n \in \mathbb{N}$ . This helped us prove the representability of a non-principal ultraproduct of these unrepresentable SOMs.

In order to do this we had to generalise our construction so that more and more points had to be considered, in a hypothetical representation of the algebra, to obtain a contradiction. This generalisation is given below. Here we have  $b \leq f; g; f; \dots; g$  (a product of  $2n$  elements) and we have replaced  $c$  by  $k_{f;g;f;\dots;g}$ . Additional elements have also been introduced, for reasons we discuss later.

The construction is based on strings over an alphabet including  $\bar{f}, f, \bar{g}, g, a, b$  and some other characters, and composition is essentially determined by concatenation. This is an easy way of ensuring the associativity of our algebras. The alphabet also includes characters  $k_s$ . Without these extra characters the algebras could be proved nonrepresentable by considering only five points in a claimed representation, as we'll explain in more detail below (see remark 24).

**4.1. Strings and normal forms.** For any set  $A$ , we let  $A^*$  denote the set of all strings over  $A$ . For any string  $s$ ,  $|s|$  denotes the length of the string, and for  $k \in \mathbb{N}$ ,  $s^k$  denotes the string obtained by concatenating  $k$  copies of  $s$  together; by convention we let  $s^0$  denote  $\Lambda$ , the empty string. Let  $F = \{f, \bar{f}, g, \bar{g}\}$ . For any string  $s \in F^*$ ,  $\bar{s}$  is the string obtained from  $s$  by replacing all occurrences of  $f, \bar{f}, g, \bar{g}$  respectively by  $\bar{f}, f, \bar{g}, g$ , and reversing the order of the string.

Fix  $n \geq 1$ . Let  $S_n$  be the set of strings over the alphabet

$$\Sigma_n = \{a, b, f, \bar{f}, g, \bar{g}\} \cup \{k_s : s \text{ is an initial segment (i.s.) of } (fg)^n\}$$

So our alphabet has  $6 + (2n + 1)$  characters. When reading the following definition, bear in mind that the empty string  $\Lambda$  will play the role of the identity in our algebras.

**DEFINITION 15.**

- We define a binary relation  $\prec$  over  $S_n$ .

$$\begin{aligned} \Lambda &\prec f\bar{f} & (I) \\ \Lambda &\prec \bar{g}g \\ \Lambda &\prec \bar{f}f \prec \Lambda \\ \Lambda &\prec g\bar{g} \prec \Lambda \end{aligned}$$

$$\begin{aligned} b &\prec (fg)^n & (II) \\ (fg)^{n-1}f &\prec \bar{b}\bar{g} \\ g(fg)^{n-1} &\prec \bar{f}b \end{aligned}$$

$$\begin{aligned} k_\Lambda &\prec a & (III) \\ k_{(fg)^n} &\prec ab \end{aligned}$$

$$k_\Lambda b \prec k_{(fg)^n} \bar{g}g \quad (IV)$$

$$\begin{aligned} k_s \phi &\prec k_s \phi & \phi \in \{f, g\}, s\phi \text{ is an i.s. of } (fg)^n & (V) \\ k_s &\prec k_s \bar{\phi} & \phi \in \{f, g\}, s\phi \text{ is an i.s. of } (fg)^n \end{aligned}$$

- If  $\tau$  is one of the reduction listed above, say  $\tau = (t' \succ t)$ , then we may write  $st'u \xrightarrow{\tau} stu$  and we say that  $\tau$  is a reduction from  $st'u$  to  $stu$ .
- We define binary relations  $\leq_k$  over  $S_n$ , for  $k \in \mathbb{N}$ .  
 $\leq_0$  is defined to be equality.  
 $\leq_1 =_{\text{def}} \{(stu, st'u) : t \prec t', s, t, u \in S_n\}$ .  
 $\leq_{k+1} =_{\text{def}} \leq_k \mid \leq_1$ .  
 Now define  $\leq = \bigcup \{\leq_k : k \in \mathbb{N}\}$  — the reflexive transitive closure of  $\leq_1$ .
- We write  $s \equiv t$  iff  $s \leq t$  and  $t \leq s$ , and we write  $s < t$  if  $s \leq t$  but  $t \not\leq s$ .

Care is needed when using the  $x \xrightarrow{\tau} y$  notation as  $x$  and  $\tau$  do not determine  $y$  (e.g. let  $x = (fg)^n (fg)^n$  and let  $\tau = ((fg)^n \succ b)$ ). Also,  $x$  and  $y$  do not determine  $\tau$  (e.g. let  $x = f\bar{f}f$ ,  $y = f$  then  $\tau = (f\bar{f} \succ \Lambda)$  and  $\tau = (\bar{f}f \succ \Lambda)$  are both possible).

**LEMMA 16.** *The following equivalences are true in  $(S_n, \leq)$ .*

$$\begin{aligned} \bar{b}\bar{g} &\equiv (fg)^{n-1}f \\ \bar{f}b &\equiv g(fg)^{n-1} \\ k_s f &\equiv k_s f & sf \text{ is an i.s. of } (fg)^n \\ k_t &\equiv k_t \bar{g} & tg \text{ is an i.s. of } (fg)^n \\ k_{(fg)^n} \bar{g} &\equiv k_s t & stg = (fg)^n \end{aligned}$$

*Proof.* Here are the proofs.

$$\begin{aligned} \bar{b}\bar{g} &\geq (fg)^{n-1}f \equiv (fg)^{n-1}f g \bar{g} \geq \bar{b}\bar{g} \\ \bar{f}b &\geq g(fg)^{n-1} \equiv \bar{f}f g (fg)^{n-1} \geq \bar{f}b \\ k_s f &\geq k_s f \equiv k_s f \bar{f}f \geq k_s f \\ k_t &\equiv k_t \bar{g} \geq k_t \bar{g} \geq k_t \\ k_{(fg)^n} \bar{g} &\equiv k_{(fg)^n} \bar{g} g \bar{g} \geq k_\Lambda b \bar{g} \equiv k_\Lambda (fg)^{n-1}f \\ &\geq k_s t \geq k_{(fg)^{n-1}f} \equiv k_{(fg)^n} \bar{g} \end{aligned}$$

□

Thus, many distinct strings can be equivalent. Next we define a *normal form*  $\text{nf}(x)$  equivalent to a given string  $x$ , and then show that  $\text{nf}(x)$  determines  $x$  up to equivalence (lemma 20).

**DEFINITION 17.** *Define  $\text{nf}(x)$  from  $x$  by repeatedly performing one of the replacements below until no more replacements are possible. At any stage, if there*

is more than one possible replacement that can be performed then (for now) use a fixed but arbitrary order to choose which one to select. See corollary 19 below.

	Replace	By	Condition
1	$ff$	$\Lambda$	
2	$g\bar{g}$	$\Lambda$	
3	$b\bar{g}$	$(fg)^{n-1}f$	
4	$\bar{f}b$	$g(fg)^{n-1}$	
5	$k_s t$	$k_\Lambda s t$	$st = (fg)^{n-1}f, s \neq \Lambda$
6	$k_{sf}$	$k_s f$	$sf$ is an i.s. of $(fg)^n$
7	$k_{sf}g\bar{g}$	$k_s f$	$sf$ is an i.s. of $(fg)^n$

**LEMMA 18** (Commutating replacements). *Let  $x, y, z \in S_n$  and suppose  $\sigma, \tau$  are replacements listed in definition 17 such that  $x \xrightarrow{\sigma} y$  and  $x \xrightarrow{\tau} z$ . Then there are two replacements  $\sigma', \tau'$  and  $w \in S_n$  such that  $y \xrightarrow{\sigma'} w$  and  $z \xrightarrow{\tau'} w$  where  $\sigma', \tau'$  are either the identity replacement or belong to the list of replacements in definition 17 above.*

*Proof.* Let  $\sigma = (s \succ s')$ ,  $\tau = (t \succ t')$ . So  $x$  contains substrings  $s, t$  (possibly  $x$  contains more than one substring  $s$ , but a unique occurrence of  $s$  is replaced using  $\sigma$ ). If these two substrings are disjoint in  $x$  then clearly there is  $w$  such that  $y \xrightarrow{\tau} w$  and  $z \xrightarrow{\sigma} w$ , so we can take  $\tau' = \tau$  and  $\sigma' = \sigma$ .

If  $y = z$  we can let  $w = y$  and both of  $\sigma'$  and  $\tau'$  are the identity replacement.

That leaves the case where  $s, t$  are overlapping but not equal substrings of  $x$ . There are just two possibilities. It could be that  $s = b\bar{g}$  and  $t = \bar{f}b$  (or the other way round). In this case,  $x = x_1\bar{f}b\bar{g}x_2$  (some  $x_1, x_2$ ),  $y = x_1\bar{f}(fg)^{n-1}fx_2$  and  $z = x_1g(fg)^{n-1}\bar{g}x_2$ . The required  $w$  is  $x_1g(fg)^{n-2}fx_2$ . The other case is where  $s = k_{uf}$  and  $t = k_{uf}v$ , where  $ufv = (fg)^{n-1}f$ , (or the other way round), so  $x = x_1k_{uf}vx_2$ ,  $y = x_1k_{uf}vx_2$  and  $z = x_1k_\Lambda(fg)^{n-1}fx_2$ . In this case we can let  $w = z$ .  $\square$

Hence,

**COROLLARY 19.** *When computing the normal form  $\text{nf}(x)$  of any  $x \in S_n$  (definition 17) it does not matter what order is chosen to perform the replacements. That is, every  $x \in S_n$  has an equivalent unique normal form  $\text{nf}(x)$  not dependent on the chosen order for the replacements.*

Furthermore,

**LEMMA 20.** *Let  $x, y \in S_n$ . The following are equivalent.*

- $x \equiv y$
- $\text{nf}(x) = \text{nf}(y)$

*Proof.* Clearly  $x \equiv \text{nf}(x)$  (use lemma 16) so the second statement implies the first. Conversely, suppose  $x \equiv y$ . We prove that  $\text{nf}(x) = \text{nf}(y)$  by induction on the length ( $k$ ) of the chain  $x \geq_k y$ . If  $k = 0$  then  $x = y$  so trivially  $\text{nf}(x) = \text{nf}(y)$ . Assume that  $x \equiv y$  and  $x \geq_k y$  implies  $\text{nf}(x) = \text{nf}(y)$ . Now suppose  $x \geq_1 x' \geq_k y$ , say  $x \xrightarrow{\tau} x'$  for some reduction  $\tau$  from definition 15, and  $x \equiv y$ . Since  $x' \equiv y$  it follows inductively that  $\text{nf}(x') = \text{nf}(y)$ . If  $\tau$  is any of the following reductions then, by corollary 19,  $\text{nf}(x) = \text{nf}(x')$  and hence  $\text{nf}(x) = \text{nf}(y)$ , as required:  $\bar{f}f \succ 1'$ ,  $1' \succ \bar{f}f$ ,  $g\bar{g} \succ 1'$ ,  $1' \succ g\bar{g}$ ,  $b\bar{g} \succ (fg)^{n-1}f$ ,  $\bar{f}b \succ g(fg)^{n-1}$ ,  $k_{sf} \succ k_s f$  or  $k_{tg}\bar{g} \succ k_t$ , where  $sf, tg$  are arbitrary initial segments of  $(fg)^n$ . Also, if  $s\phi t = (fg)^{n-1}f$  and if  $\tau$  satisfies  $k_s\phi t \xrightarrow{\tau} k_{s\phi}t$ , then by the fifth replacement listed in definition 17 and by corollary 19 we have  $\text{nf}(x) = \text{nf}(x')$  so, again,  $\text{nf}(x) = \text{nf}(y)$ . That leaves the following possibilities for  $\tau$ : (i)  $(fg)^n \succ b$ , (ii)  $a \succ k_\Lambda$ , (iii)  $ab \succ k_{(fg)^n}$ , (iv)  $k_{(fg)^n}\bar{g}g \succ k_\Lambda b$ , (v)  $k_{tg} \succ k_{tg}$  or (vi)  $k_{sf}\bar{f} \succ k_s$ . For the first alternative, if there is a string  $\gamma \equiv \bar{g}$  immediately to the right of  $(fg)^n$  or a string  $\phi \equiv \bar{f}$  immediately to

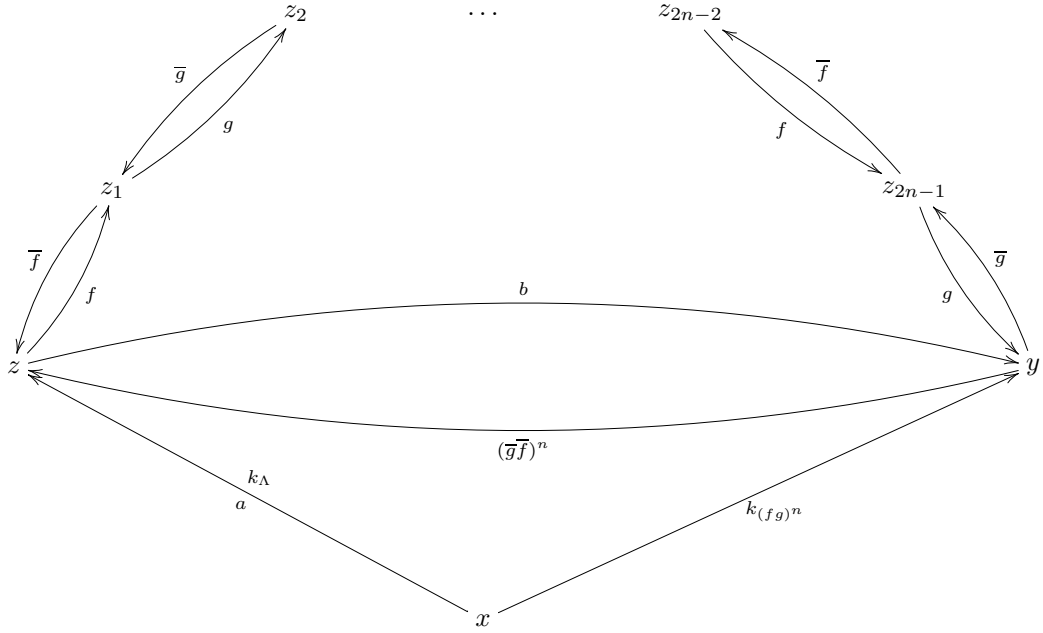


FIGURE 2.

the left of  $(fg)^n$ , then  $\text{nf}(x) = \text{nf}(x')$ ; if there is no such string then  $x > x'$ , contrary to assumption. The second and third alternatives are impossible, since they each imply  $x \not\equiv x'$ . For the fourth alternative, if there is a string  $\gamma \equiv \bar{g}$  immediately to the right of  $k_{(fg)^n}\bar{g}g$ , then  $\text{nf}(x) = \text{nf}(x')$  as required, else  $x \not\equiv x'$  contrary to assumption. For the fifth alternative, if there is a string  $\gamma$  immediately to the right of  $k_t g$  with either  $\gamma \equiv \bar{g}$  or  $tg\gamma \equiv (fg)^{n-1}f$ , then  $\text{nf}(x) = \text{nf}(x')$ , else  $x \not\equiv x'$ . And for the sixth alternative, if there is a string  $\phi \equiv f$  immediately to the right of  $k_{sf}\bar{f}$ , then  $\text{nf}(x) = \text{nf}(x')$ , else  $x \not\equiv x'$ . In each case we either derive a contradiction by showing  $x \not\equiv x'$  or we deduce that  $\text{nf}(x) = \text{nf}(y)$  as required.  $\square$

Also note that  $k_\Lambda$  is in normal form. By definition 15, we have  $k_\Lambda \leq a$  (item (3)), and  $k_\Lambda \leq k_{(fg)^n}(\bar{g}\bar{f})^n$  (item (5)).

**4.2. Non-representable algebras.** We are in a position to define non-representable algebras for every  $1 \leq n \in \mathbb{N}$ .

**DEFINITION 21.** Let  $n \geq 1$ . Let  $A_n = \{\text{nf}(x) : x \in S_n\}$ . For  $x, y \in A_n$ , let  $x; y = \text{nf}(xy)$ , i.e., we take the normal form of the concatenation of  $x$  and  $y$ . Let  $\mathcal{A}_n = (A_n, \leq, 1', ;)$ , where  $1'$  is the empty string  $\Lambda$ .

**LEMMA 22.** For  $n \geq 1$ ,  $\mathcal{A}_n$  is an ordered monoid.

*Proof.* Use lemma 18.  $\square$

**THEOREM 23.**  $\mathcal{A}'_n$  is an unrepresentable SOM.

*Proof.* By lemmas 6 and 22,  $\mathcal{A}'_n$  is a SOM. Suppose for contradiction that there is a SOM representation  $h : \mathcal{A}'_n \rightarrow \wp(D \times D)$  (see figure 2). Note that  $\emptyset$  is a downset of  $\mathcal{A}'_n$ , so it is the least element of  $\mathcal{A}'_n$ .

$$\begin{aligned}
\emptyset \neq k_{(fg)^n} \downarrow &\Rightarrow (\exists x, y \in D)(x, y) \in h(k_{(fg)^n} \downarrow) \setminus h(\emptyset) \\
k_{(fg)^n} \downarrow \leq (a; b) \downarrow &\Rightarrow (\exists z \in D)(x, z) \in h(a \downarrow) \wedge (z, y) \in h(b \downarrow) \\
b \downarrow \leq (fg)^n \downarrow &\Rightarrow (\exists z_1, \dots, \exists z_{2n-1})(z, z_1) \in h(f \downarrow), (z_1, z_2) \in h(g \downarrow), \\
&\quad \dots, (z_{2n-1}, y) \in h(g \downarrow) \\
f \downarrow; \bar{f} \downarrow \geq 1' \downarrow &\Rightarrow \exists w ((z, w) \in h(f \downarrow) \wedge (w, z) \in \bar{f} \downarrow) \\
&\Rightarrow (w, z_1) \in h((\bar{f}; f) \downarrow) = h(1' \downarrow) \\
&\Rightarrow w = z_1, (z_1, z) \in h(\bar{f} \downarrow) \\
\text{Similarly:} &\quad (z_3, z_2) \in h(\bar{f} \downarrow), \dots, (z_{2n-1}, z_{2n-2}) \in h(\bar{f} \downarrow) \\
\text{Similarly:} &\quad (z_2, z_1) \in h(\bar{g} \downarrow), \dots, (y, z_{2n-1}) \in h(\bar{g} \downarrow) \\
\text{Putting together:} &\quad (y, z) \in h((\bar{g}; \bar{f}; \dots; \bar{g}; \bar{f}) \downarrow) = h((\bar{g}\bar{f})^n \downarrow) \\
&\Rightarrow (x, z) \in h((k_{(fg)^n}; (\bar{g}\bar{f})^n) \downarrow) \cap h(a \downarrow) \\
&\Rightarrow (x, z) \in h((k_{(fg)^n}; (\bar{g}\bar{f})^n) \downarrow \cdot a \downarrow) = h(k_\Lambda \downarrow) \\
&\Rightarrow (x, y) \in h((k_\Lambda; b) \downarrow) \cap h(k_{(fg)^n} \downarrow) = h((k_\Lambda; b) \downarrow \cdot k_{(fg)^n} \downarrow) \\
&\Rightarrow (x, y) \in h(\emptyset)
\end{aligned}$$

The third to last line uses the fact that  $k_\Lambda$  is the greatest lower bound of  $k_{(fg)^n}; (\bar{g}\bar{f})^n$  and  $a$  in  $\mathcal{A}_n$ , and the last line uses the fact that  $k_\Lambda; b$  and  $k_{(fg)^n}$  have no lower bound in  $\mathcal{A}_n$ . This is a contradiction, so  $\mathcal{A}'_n$  is not representable.  $\square$

Note that one can write a quasi-equation that is valid in representable SOMs but fails in  $\mathcal{A}'_n$ . The antecedent would describe that part of the algebra  $\mathcal{A}'_n$  that we used in the above argument ( $k_{(fg)^n} \downarrow \leq (a; b) \downarrow \wedge b \downarrow \leq (fg)^n \downarrow \wedge \dots$ ) and the consequent would be  $k_{(fg)^n} \downarrow \leq (k_\Lambda; b) \downarrow$ .

**REMARK 24.** *We can now explain, in outline, why we needed to include the characters  $k_s$  in our alphabet. A simplified version, say  $\mathcal{B}_n$ , of  $\mathcal{A}_n$  could be defined without these characters but with a character  $c \prec a; b$  replacing  $k_{(fg)^n}$ , and with appropriate alterations to definition 15 above, that is we delete all reductions involving characters  $k_s$  (items (III), (IV) and (V) in definition 15) and include reductions  $c \prec ab$ ,  $ab \prec c\bar{g}g$ ,  $ab \prec cf\bar{f}$  instead. The problem here is that it would be “too easy” to prove that the algebra  $\mathcal{B}'_n$  was not representable. Very roughly, if a representation did exist then it would contain points  $(x, y)$  in the representation of  $c \leq a; b \leq a; (fg)^{n-1}; f; g$ , so there would be points  $z, z_1, z_2$  with  $(x, z), (z, z_1), (z_1, z_2)$  and  $(z_2, y)$ , respectively, in the representations of  $a, (fg)^{n-1}, f$  and  $g$ . Using  $\bar{f}; f, g; \bar{g} \leq 1'$  we could show that  $(y, z_2), (z_2, z_1)$  belonged, respectively, to the representations of  $\bar{g}$  and  $\bar{f}$  and so  $(y, z_1)$  would belong to the representation of  $\bar{g}; \bar{f}$ . But then  $(x, z_1)$  would belong to the representations of  $a(fg)^{n-1}$  and  $c\bar{g}\bar{f}$ . Yet these two elements have no lower bound in this modified version of the algebra so we could derive a contradiction. The problem is that we could prove the non-representability of the algebra by considering only five nodes in a claimed representation. With this construction we would be unable to prove that an ultraproduct of our algebras was representable. We add the character  $k_{(fg)^{n-1}}$  precisely to act as a lower bound of  $a(fg)^{n-1}$  and  $c\bar{g}\bar{f}$  and we replace  $c$  by  $k_{(fg)^n}$ .*

Once we have  $k_{(fg)^{n-1}} \leq a(fg)^{n-1}, k_{(fg)^n} \bar{g}\bar{f}$  in our alphabet, a similar argument persuades us to include  $k_{(fg)^{n-2}} \leq a(fg)^{n-2}, k_{(fg)^{n-1}} \bar{g}\bar{f}$  in our alphabet too and, in the same way, characters  $k_{(fg)^i} : i \leq n$ . Characters of the form  $k_{(fg)^i} f : i < n$  are not strictly necessary (note that  $k_{(fg)^i} f \equiv k_{(fg)^i}$ ), but are also included for consistency of notation.

**PROBLEM 25.** *It follows from lemma 9 and theorem 23 that the ordered monoid  $\mathcal{A}_n$  has no principal representation. Does it have an ordered monoid representation?*

## 5. Towards representability

Our next aim is to show that a non-principal ultraproduct of the  $\mathcal{A}_n$  ( $n \in \mathbb{N}$ ) has a principal representation. In the next section, we will describe a winning strategy for  $\exists$  on  $G_k(\mathcal{A}_n)$  for  $k < 2^n$ . It will follow that  $\exists$  has a winning strategy in the infinite game on the ultraproduct. To describe the winning strategy, we need more on normal forms.

**5.1. More on normal forms.** Next we state further properties of the system  $(S_n, \leq)$ . These results are rather technical, and we postpone the proofs to an appendix.

**LEMMA 26.** *Let  $x, y \in (F \cup \{b\})^*$  and  $x \geq y$ . Then either  $\text{nf}(x) \xrightarrow{\psi \bar{\psi}^{\succ} 1'} z \geq \text{nf}(y)$  (some  $z$  and some  $\psi \in \{f, \bar{g}\}$ ) or  $\text{nf}(y)$  is obtained from  $\text{nf}(x)$  by a sequence of reductions  $(fg)^n \succ b$ .*

**LEMMA 27.** *Let  $x, y, z \in (F \cup \{b\})^*$ . Suppose  $x = \text{nf}(x)$  and  $yz = \text{nf}(yz)$  and that  $y, z$  are minimal subject to  $yz \geq x$  (i.e., if  $y' \leq y$ ,  $z' \leq z$  and  $y'z' \geq x$  then  $y' \equiv y$  and  $z' \equiv z$ ). Then either*

- $y = y_0s$ ,  $z = tz_0$ , for some  $y_0, z_0, s, t$  where  $st = (fg)^{in}$  for some  $i \in \mathbb{N}$ , and  $x = y_0b^iz_0$ , or
- $y = y_0\phi$ ,  $z = \bar{\phi}z_0$  and  $y_0z_0 \geq x$ , for some  $\phi \in F$  and some  $y_0, z_0$ .

**LEMMA 28.** *Let  $e \geq 1'$  and  $\rho \in \mathcal{A}_n$ .*

- (1) *Either  $k_{(fg)^n}e \geq k_\Lambda b$  or  $e \equiv 1'$  (and not both).*
- (2) *Either  $k_\Lambda be \geq k_{(fg)^n}$  or  $e \equiv 1'$ .*
- (3) *If  $\gamma \geq \rho k_{(fg)^n}$  but  $\gamma \not\geq \rho k_\Lambda b$ , then  $\gamma \equiv \rho^+ k_{(fg)^n}$  for some  $\rho^+ \geq \rho$ .*
- (4) *If  $\gamma \geq \rho k_\Lambda b$  but  $\gamma \not\geq \rho k_{(fg)^n}$ , then  $\gamma \equiv \rho^+ k_\Lambda b$  for some  $\rho^+ \geq \rho$ .*

The next step is to define a right and a left quotient  $x/\phi$  and  $\phi \setminus x$ , for  $x \in \mathcal{A}_n$  and  $\phi \in F$ . We concentrate on the right quotient as the left quotient is easier. We intend to define  $x/\phi$  as a minimal solution  $z$  of  $z; \phi \geq x$ , i.e., in such a way that  $\forall y (y \geq x/\phi \leftrightarrow y; \phi \geq x)$ . Unfortunately this is not possible: there are two inequivalent minimal solutions,  $z = k_{(fg)^n}$  and  $z = k_\Lambda b$ , to the inequality  $z; \bar{g} \geq k_{(fg)^n} \bar{g}$ . So we have to define  $x/\phi$  as a *subset* of (rather than a single element of)  $\mathcal{A}_n$ . In one type of case, essentially variants of  $(k_{(fg)^n} \bar{g})/\bar{g} = \{k_{(fg)^n}, k_\Lambda b\}$ ,  $x/\phi$  is a set with two elements, and in all other cases  $x/\phi$  is a singleton set. We may identify this singleton with the element of  $\mathcal{A}_n$  it contains.

**DEFINITION 29.** *Let  $x \in \mathcal{A}_n$ . Define  $x]$  and  $[x$  from  $x$  as follows.*

- *If  $x = x_0(fg)^n$ , for some  $x_0 \in \mathcal{A}_n$ , then  $x] = x_0b$ , else  $x] = x$ .*
- *If  $x = (fg)^n x_0$ , for some  $x_0 \in \mathcal{A}_n$ , then  $[x = bx_0$ , else  $[x = x$ .*

Note that both  $x]$  and  $[x$  are in normal form if  $x$  is in normal form, hence  $x], [x \in \mathcal{A}_n$ .

**DEFINITION 30.** *Let  $x \in \mathcal{A}_n$  and let  $\phi \in F$ . We define  $x/\phi$  as follows.*

- (I) *If  $\phi \in \{\bar{f}, g\}$ , then  $x/\phi = \{\text{nf}(x\bar{\phi})\}$  (which we write as  $x/\phi = \text{nf}(x\bar{\phi})$ ).*
- (II) *Let  $\phi = f$ .*
  - (a) *If  $x = x_0f$  (some  $x_0 \in \mathcal{A}_n$ ), then  $x/f = x_0$ .*
  - (b) *Else ( $x \neq x_0f$  for any  $x_0 \in \mathcal{A}_n$ )  $x/f = \text{nf}(x\bar{f})$ .*
- (III) *Let  $\phi = \bar{g}$ .*
  - (a) *If  $x = x_0k_\Lambda(fg)^{n-1}f$ , for some  $x_0 \in \mathcal{A}_n$ , then  $x/\bar{g} = \{x_0k_\Lambda b, x_0k_{(fg)^n}\}$ .*
  - (b) *If  $x = x_0\bar{g}$ , some  $x_0 \in \mathcal{A}_n$ , then  $x/\bar{g} = x_0$ .*
  - (c) *If  $x = x_0k_s f$ , some  $x_0 \in \mathcal{A}_n$  and some initial segment  $sf$  of  $(fg)^n$ , then  $x/\bar{g} = x_0k_s fg$ .*

(d) Else  $(x \neq x_0 k_\Lambda (fg)^{n-1} f, x_0 \bar{g}, x_0 k_s f \text{ for any } x_0, s) x/\bar{g} = (xg)$ .

Note that  $x/\phi$  consists of strings in normal form, hence elements of  $A_n$ . Next we state without proof two rather easy lemmas.

**LEMMA 31.** *Let  $x, z \in A_n$  and  $\phi \in F$ . If  $z \in x/\phi$  then  $z; \phi \geq x$  and  $z \leq x; \bar{\phi}$ .*

**LEMMA 32.** *Let  $x \in A_n$  and let  $\phi \in F$ . Then there is  $z \in (x; \phi)/\phi$  with  $z \leq x$ .*

The following theorem summarizes the main features of the right quotient; the proof is in the appendix.

**THEOREM 33.** *Let  $x, y \in A_n$  and  $\phi \in F$ .*

- (1)  $x \leq y \Rightarrow (\forall z \in y/\phi)(\exists z' \in x/\phi)z' \leq z$ .
- (2)  $y; \phi \geq x \Leftrightarrow (\exists z \in x/\phi)z \leq y$ .
- (3)  $(\forall z \in y/\phi)(\exists z' \in (x; y)/\phi)z' \leq x; z$ .

Next we define the left quotient.

**DEFINITION 34.** *Let  $x \in A_n$  and  $\phi \in F$ . The left quotient  $\phi \setminus x$  is defined similarly, but this time we find that  $\phi \setminus x$  is always a singleton, so we let  $\phi \setminus x \in A_n$  (rather than a subset of  $A_n$ ).*

- (I) *If  $\phi \in \{f, \bar{g}\}$ , then  $\phi \setminus x = \text{nf}(\bar{\phi}x)$ .*
- (II) *If  $x = gx_0$  (some  $x_0 \in A_n$ ), then  $g \setminus x = x_0$ , else  $(x \neq gx_0, \text{ any } x_0)$   
 $g \setminus x = \bar{g}x$ .*
- (III) *If  $x = \bar{f}x_0$  (some  $x_0 \in A_n$ ), then  $\bar{f} \setminus x = x_0$ , else  $(x \neq \bar{f}x_0, \text{ any } x_0)$   
 $\bar{f} \setminus x = \lfloor (fx)$ .*

Note that the last two cases here are simpler than the corresponding cases in the definition of right quotient. This is because there are no reductions of the form  $\alpha k_s \succ \beta$ , for any  $\alpha, s, \beta$ . Hence the following theorem has a simpler form than theorem 33.

**THEOREM 35.** *Let  $x, y \in A_n$  and  $\phi \in F$ .*

- (1)  $x \leq y \Rightarrow \phi \setminus x \leq \phi \setminus y$ .
- (2)  $\phi; y \geq x \Leftrightarrow y \geq \phi \setminus x$ .
- (3)  $\phi \setminus (x; y) \leq (\phi \setminus x); y$ .

We state the following easy lemma without proof:

**LEMMA 36.**

$$\bar{\phi}; x \geq \phi \setminus x.$$

**5.2. A strategy.** Fix  $n > 0$  and fix  $m > 2^n$ . We aim to show that  $\exists$  has a winning strategy in the game  $G_n(\mathcal{A}_m)$  (see theorem 47 below).

First we define short elements — the intuition is that an edge  $(x, y)$  labelled by a short element, a “short” edge, will always have a converse edge  $(y, x)$ . Thus,  $\exists$  has to choose the label for these edges very carefully, e.g.,  $N(x, y); N(y, x) \geq 1'$ .  $\exists$  will not label the converse edge for “long” edges, and this will make the move easier. The “length” of an element is defined relative to the number of remaining rounds of the play.

**DEFINITION 37.** *Let  $k > 0$ .*

- *An element  $x \in \mathcal{A}_m$  is called  $k$ -short if there are  $X_i, Y_i \in \mathcal{A}_m : i < k$  such that*

$$x \leq X_0; Y_0; X_1; Y_1; \dots; X_{k-1}; Y_{k-1}$$

*and  $\bar{X}_i; X_i = Y_i; \bar{Y}_i = 1'$ , for each  $i < k$ . Observe, for  $X \neq 1'$ , that  $\bar{X}; X = 1' \iff X \in \{f, \bar{g}\}^*$  and  $Y; \bar{Y} = 1' \iff Y \in \{\bar{f}, g\}^*$ .*

- *We say that  $x$  is  $k^-$ -short if  $x; Y$  is  $k$ -short for some  $Y$  with  $Y; \bar{Y} = 1'$ .*

- $x$  is called  $\bar{k}$ -short if  $X; x$  is  $k$ -short for some  $X$  with  $\overline{X}; X = 1'$ .

For example, in  $\mathcal{A}_m$ , the element  $b$  is  $m$ -short because  $b \leq (fg)^m$  (take each  $X_i = f$  and  $Y_i = g$ ) but  $b$  is not  $(m-1)$ -short, nor is it  $m^-$ -short or  $\bar{m}$ -short. Indeed,

$$s \text{ is } k\text{-short (some } k < m) \Rightarrow s \in F^*.$$

Note that if an element is  $k$ -short,  $k^-$ -short, or  $\bar{k}$ -short, then it is  $l$  short for every  $l \geq k$ . Any member of  $F \cup \{\Lambda\}$  is 1-short,  $\bar{f}$  and  $g$  are  $\bar{1}$ -short, and  $\bar{g}$  and  $f$  are  $1^-$ -short.

**LEMMA 38.** *For any  $k \in \mathbb{N}$  and any  $x \in \mathcal{A}_m$  we have that  $x$  is  $k$ -short iff  $x]$  is  $k$ -short iff  $\lfloor x$  is  $k$ -short.*

*Proof.* Recall from definition 29 that  $x]$  is either identical to  $x$  or it is defined from  $x$  by replacing a final segment  $(fg)^m$  by  $b$ . Now, for any  $s \in F^*$  we have  $b \leq s \iff (fg)^m \leq s$ , so the lemma follows from this.  $\square$

We omit the proof of the following easy lemma.

**LEMMA 39.** *Let  $\phi \in F$  satisfy  $\bar{\phi}; \phi = 1'$  (so  $\phi \in \{f, \bar{g}\}$ ) and  $x, s \in \mathcal{A}_m$ . Let  $0 < k < m$ .*

- (1)  $x$  is  $k$ -short iff  $x; \bar{\phi}$  is  $k$ -short iff  $\phi; x$  is  $k$ -short.
- (2)  $x$  is  $\bar{k}$ -short iff  $x; \bar{\phi}$  is  $\bar{k}$ -short.
- (3)  $x$  is  $k^-$ -short iff  $x; \phi$  is  $k^-$ -short.
- (4)  $x$  is  $\bar{k}$ -short iff  $\bar{\phi}; x$  is  $\bar{k}$ -short.
- (5) If  $s; x$  is  $k$ -short (respectively  $\bar{k}$ -short), then  $x$  is  $k$ -short ( $\bar{k}$ -short).
- (6) If  $x; s$  is  $k$ -short (respectively  $k^-$ -short), then  $x$  is  $k$ -short ( $k^-$ -short).

The strategy is designed to ensure that if there are  $r$  rounds left in the game, then the current network is  $2^r$ -good, as defined next.

**DEFINITION 40.** *Let  $N$  be a partial map  $(D \times D) \rightarrow \mathcal{A}_m$  (some set of nodes  $D$ ) and let  $0 < k < m$ . Say  $N$  is  $k$ -good (respectively  $k^-$ -good,  $\bar{k}$ -good) if, for all  $x, y, z \in N$  and all  $\alpha, \beta \in \mathcal{A}_m$ ,*

- (A)  $N$  is a consistent network,
- (B) if  $N(x, y)$  is  $k$ -short ( $k^-$ -short,  $\bar{k}$ -short) then  $N(y, x)$  is defined,
- (C) if
  - (a)  $N(x, y), N(y, x)$  are both defined and  $\alpha, \beta$  are minimal solutions of  $N(x, y) \leq \alpha; \beta$  (i.e.  $\alpha_0 \leq \alpha \wedge \beta_0 \leq \beta \wedge \alpha_0; \beta_0 \geq N(x, y) \rightarrow (\alpha \equiv \alpha_0 \wedge \beta \equiv \beta_0)$ ),
  - (b) it is not the case that  $\alpha = \alpha' \phi, \beta = \bar{\phi} \beta'$  and  $\alpha'; \beta' \geq N(x, y)$  (for any  $\alpha', \beta' \in \mathcal{A}_m$  and  $\phi \in F$ ),
  - (c) it is not the case that  $\alpha = \alpha' b, \beta = \bar{g} \beta'$  nor is it the case that  $\alpha = \alpha' \bar{f}, \beta = b \beta'$  (for any  $\alpha', \beta' \in \mathcal{A}_m$ ),
then there is  $w \in N$  such that  $N(x, w) = \alpha$  and  $N(w, y) = \beta$ .

Clearly, if  $N$  is  $k$ -good then  $N$  is  $l$ -good for all  $l \leq k$ .

Note that if  $\alpha, \beta \in \mathcal{A}_m$  are as in condition C above, then  $\alpha; \beta = \text{nf}(\alpha\beta) = \alpha\beta$ . The next few lemmas include the technical results needed to show that these conditions can be preserved from one round of the game to the next. We will define a number of networks  $(N^0, N^1, N^2, N^3, N^4)$  and throughout we will adopt the following convention.

**CONVENTION.** Let  $N$  be a network,  $x, y \in N$ ,  $\alpha \in \mathcal{A}_m$  and  $\phi \in F$ . When we write  $\beta = \alpha; N(x, y)$ , if  $N(x, y)$  is undefined then  $\beta$  is also undefined. Similarly, if we write  $\beta = \text{nf}(\alpha N(x, y))$ ,  $\beta = N(x, y); \alpha$ ,  $\beta = N(x, y)/\phi$ ,  $\beta = \phi \setminus N(x, y)$ , etc., if  $N(x, y)$  is undefined then  $\beta$  is undefined.

**LEMMA 41.** *Let  $1 \leq k < m$ . Suppose  $N$  is  $k$ -good,  $x, y \in N$ ,  $\alpha; \beta \geq N(x, y)$ ,  $N(y, x)$  is not defined, and neither  $\alpha$  nor  $\beta$  is  $k$ -short. Then there is a  $k$ -good extension  $N^0 \supseteq N$  such that there is  $z$  with  $N^0(x, z) = \alpha$  and  $N^0(z, y) = \beta$ .*

*Indeed, if the witness  $z$  is missing from  $N$ , we define  $N^0 = N^0(N, x, y, \alpha, \beta)$  as follows. Let  $N^0$  have one node,  $z$ , in addition to those of  $N$ . To define the labelling of edges incident with  $z$ , let  $w \in N$  be arbitrary.*

$$\begin{aligned} N^0(z, z) &= 1' \\ N^0(w, z) &= N(w, x); \alpha \\ N^0(z, w) &= \beta; N(y, w) \end{aligned}$$

*Proof.* Consistency of  $N^0$  (condition A) is easily checked, using the consistency of  $N$  (for condition Id note that if  $\gamma; \delta \leq 1'$ , then  $\gamma$  and  $\delta$  are 1-short). Condition B is trivial since  $N^0$  has no  $k$ -short edges incident with the new node  $z$  (use lemma 39). Condition C holds because  $N^0(w, z)$  and  $N^0(z, w)$  cannot both be defined, else by our convention  $N(w, x)$  and  $N(y, w)$  would be defined, but we are assuming that  $N(y, x)$  is not defined.  $\square$

Next we deal with the cases where either  $\alpha$  or  $\beta$  is  $k$ -short. To handle the case where  $\alpha$  is  $k$ -short we consider, in the next lemma,  $\forall$ -moves of the form  $(N, x, y, \phi, \beta)$  for  $\phi \in F$ , i.e., we suppose  $\alpha = \phi$ . The response turns out to be independent of  $\beta$ . We then have to re-use this lemma  $|\alpha|$  times to handle more general moves where  $\alpha$  is  $k$ -short.

Lemma 43 is to help us with  $\forall$ -moves where  $\beta$  is  $k$ -short. Note, however, that the lemmas are not exactly symmetrical: the response to  $N(x, y, \alpha, \phi)$  does depend on  $\alpha$ . To compensate we use the assumption that  $N(y, x)$  is not defined in lemma 43, whereas this assumption is not needed in lemma 42. To deal with the case where  $\beta$  is  $k$ -short and  $N(y, x)$  is defined we need an extra lemma (see  $N^3$  in lemma 44).

**LEMMA 42.** *Let  $1 \leq k < m$  and let  $N$  be a network. Let  $x, y \in N$  be such that  $N(x, y)$  is defined, and assume  $\phi \in F$ . Then there is  $N^1 \supseteq N$  and  $z$  such that  $N(x, z) = \phi$ . Furthermore,*

- (1) *if  $N$  is  $\bar{k}$ -good and  $\phi \in \{\bar{f}, g\}$ , then  $N^1$  is  $\bar{k}$ -good;*
- (2) *if  $N$  is  $k$ -good and  $\phi \in \{f, \bar{g}\}$  then  $N^1$  is  $k$ -good.*

*Indeed, if the required witness is missing from  $N$ , we can define  $N^1 = N^1(N, x, \phi)$  with one extra node,  $z$  say, and labelling of edges incident with  $z$  given by*

$$\begin{aligned} N^1(z, z) &= 1' \\ N^1(v, z) &= N(v, x); \phi \\ N^1(z, v) &= \begin{cases} \phi \setminus N(x, v) & \text{if } N(y, v) \text{ is defined} \\ \phi; N(x, v) & \text{otherwise.} \end{cases} \end{aligned}$$

Note, in the definition of  $N^1(z, v)$ , that if  $N(y, v)$  is defined, then  $N(x, v)$  must also be defined, since we are assuming that  $N(x, y)$  is defined. Since

$$\bar{\phi}; N(x, v) \geq \phi \setminus N(x, v)$$

(cf. lemma 36), whichever alternative is used in the definition of  $N^1(z, v)$  we have

$$N^1(z, v) \geq \phi \setminus N(x, v).$$

So, by theorem 35,

$$\phi; N^1(z, v) \geq N(x, v).$$

*Proof.* We prove 1 (2 is easier). Suppose  $N$  is  $\bar{k}$ -good and the required witness is missing from  $N$ . We must show that  $N^1$  is also  $\bar{k}$ -good. First we check that  $N^1$

is a consistent network (condition A). Conditions Reff and Trans are obvious. For Comp, let  $u, v \in N$  be arbitrary.

$$\begin{aligned} N^1(u, v); N^1(v, z) &= N(u, v); N(v, x); \phi \\ &\geq N(u, x); \phi \\ &= N^1(u, z). \end{aligned}$$

Also, by theorem 35,

$$\begin{aligned} N^1(u, z); N^1(z, v) &\geq N(u, x); \phi; (\phi \setminus N(x, v)) \\ &\geq N(u, x); N(x, v) \\ &\geq N^1(u, v). \end{aligned}$$

We must also check that  $N^1(z, u); N^1(u, v) \geq N^1(z, v)$  (whenever  $N^1(z, u), N^1(u, v)$  are both defined). If  $N(y, u)$  is undefined then

$$\begin{aligned} N^1(z, u); N^1(u, v) &= \overline{\phi}; N(x, u); N(u, v) \\ &\geq \overline{\phi}; N(x, v) \\ &= N^1(z, v). \end{aligned}$$

If  $N(z, u), N(u, v)$  are defined and  $N(y, u)$  is also defined, then by transitivity of the labelling of  $N$ ,  $N(y, v)$  is also defined, so by theorem 35,

$$\begin{aligned} N^1(z, u); N^1(u, v) &= (\phi \setminus N(x, u)); N(u, v) \\ &\geq \phi \setminus (N(x, u); N(u, v)) \\ &\geq \phi \setminus N(x, v) \\ &= N^1(z, v). \end{aligned}$$

For Id, we have  $N^1(v, z) \neq 1'$ , since  $\phi \in \{\overline{f}, g\}$ . Also  $N^1(z, v) = 1'$  would imply  $N(x, v) = \phi$  (cf. definition 34), but we assumed that  $N$  contained no node  $w$  satisfying  $N(x, w) = \phi$ .

We check condition B. If  $N^1(u, z) = N(u, x); \phi$  is  $\bar{k}$ -short, then since  $\phi \in \{\overline{f}, g\}$  and by lemma 39(2),  $N(u, x)$  is  $\bar{k}$ -short. Since  $N$  is  $\bar{k}$ -good,  $N(x, u)$  is defined and hence  $N^1(z, u)$  is also defined. This shows that if  $N^1(u, z)$  is  $\bar{k}$ -short then  $N^1(z, u)$  is defined. Now suppose  $N^1(z, u)$  is  $\bar{k}$ -short. We must show that  $N^1(u, z)$  is defined. By assumption we have  $\phi \in \{\overline{f}, g\}$ . So if  $N^1(z, u)$  is  $\bar{k}$ -short then  $N(x, u) \leq \phi$ ;  $N^1(z, u)$  is also  $\bar{k}$ -short, by lemma 39. Inductively,  $N(u, x)$  must be defined and hence  $N^1(u, z)$  is also defined. This proves condition B.

For condition C, suppose  $N^1(u, z), N^1(z, u)$  are both defined (hence  $N(u, x)$  and  $N(x, u)$  are both defined),  $N^1(u, z) = N(u, x); \phi \leq \alpha; \beta$  and  $\alpha, \beta$  satisfy all the hypotheses of condition C. We claim that

$$\alpha; \beta = \alpha\beta$$

(i.e., relative product coincides with concatenation). To prove this, suppose not. There must be a non-empty sequence of replacements (see definition 17) taking us from  $\alpha\beta$  to  $\text{nf}(\alpha\beta)$ . Note that  $\alpha, \beta \in (F \cup \{b\})^*$  (since  $\alpha; \beta; N(z, u) \geq 1'$ ) so these replacements must be 1, 2, 3 or 4 from definition 17. The first replacement cannot be confined to  $\alpha$  (or to  $\beta$ ) since  $\text{nf}(\alpha) = \alpha$  (and  $\text{nf}(\beta) = \beta$ ). Thus the first replacement must replace the concatenation of a non-empty final segment of  $\alpha$  and a non-empty initial segment of  $\beta$ . But each of the three possible cases is ruled out by the stipulations for  $\alpha, \beta$  in condition C:  $b\overline{g} \rightarrow (fg)^{n-1}f$ ,  $\overline{f}b \rightarrow g(fg)^{n-1}$  and  $\psi\overline{\psi} \rightarrow 1'$ . This proves the claim.

Hence we can apply lemma 27. Since it is not the case that  $\alpha = \alpha'\psi$ ,  $\beta = \overline{\psi}\beta'$  and  $\alpha'\beta' \geq N^1(u, z)$ , that lemma tells us that there are  $\alpha_0, \beta_0, s, t$  and  $i \in \mathbb{N}$  with  $st = (fg)^{im}$ ,  $\alpha = \alpha_0s$ ,  $\beta = t\beta_0$  and  $N^1(u, z) = \text{nf}(N(u, x)\phi) = \alpha_0b^i\beta_0$ .

If  $\beta = 1'$  then the required witness is  $z$ . Now assume  $\beta \neq 1'$ . There are four cases according to the reductions used in computing  $N^1(u, z) = \text{nf}(N(u, x)\phi)$  from  $N(u, x)\phi$ .

- If  $N^1(u, z) = N(u, x); \phi = N(u, x)\phi$  then, by the above,  $N(u, x)\phi = \alpha_0 b^i \beta_0$ . Since  $\phi \neq b$ ,  $\beta_0 \neq 1'$  and  $\beta_0 = \beta_1 \phi$  (for some  $\beta_1$ ) and  $N(u, x) = \alpha_0 b^i \beta_1 \leq \alpha_0 s; t\beta_1$ , where  $st = (fg)^{im}$ . We have  $\alpha_0 s = \text{nf}(\alpha_0 s)$ ,  $t\beta_1 = \text{nf}(t\beta_1)$ , and, since  $st = (fg)^{im}$ , it is not the case that  $\alpha_0 s = \alpha_2 \psi$ ,  $t\beta_1 = \bar{\psi}\beta_2$  (any  $\alpha_2, \beta_2$ ), so by condition C for  $N$ , there is  $w \in N$  with  $N(u, w) \leq \alpha_0 s$ ,  $N(w, x) \leq t\beta_1$  and so  $N^1(w, z) \leq t\beta_1; \phi = t\beta_0 = \beta$ , and  $w$  is the witness we need in  $N^1$ .
- If  $N(u, x) = \rho\bar{\phi}$  and  $\bar{\phi}; \phi = 1'$  then  $N^1(u, z) = \text{nf}(\rho\bar{\phi}\phi) = \rho$ . Then  $\alpha; \beta \geq \rho$  implies  $\alpha; \beta\bar{\phi} \geq \rho\bar{\phi} = N(u, x)$ . It is not difficult to show that the requirements for condition C are still met by  $\alpha$  and  $\text{nf}(\beta\bar{\phi}) = \beta\bar{\phi}$ , so inductively there is  $w \in N$  with  $N(u, w) \leq \alpha$ ,  $N(w, x) \leq \beta\bar{\phi}$ . Hence  $N^1(w, z) \leq \beta; \bar{\phi}; \phi = \beta$  and  $w$  is the required witness in  $N^1$ .
- If  $N(u, x) = \rho b$  (some  $\rho$ ) and  $\phi = \bar{g}$  then  $N^1(u, z) = \text{nf}(\rho b\bar{g}) = \rho(fg)^{n-1}f$ . So  $\alpha; \beta; g \geq N^1(u, z); g = \rho(fg)^n \geq N(u, x)$ . It is easy to check that the requirements for condition C are met by  $\alpha, \beta g$ , so inductively there is  $w \in N$  with  $N(u, w) \leq \alpha$  and  $N(w, x) \leq \beta g$ . Hence  $N^1(w, z) \leq \beta; g; \bar{g} = \beta$  and  $w$  is the required witness.
- The case where  $N(u, x) = \rho b$  and  $\phi = f$  is similar.

Also, consider the case where  $N^1(u, z), N^1(z, u)$  are defined and  $N^1(z, u) \leq \alpha; \beta$ . If  $N^1(z, u) = \bar{\phi}; N(x, u)$  then this is entirely similar to the previous case. So suppose  $N(y, u)$  is defined and  $N^1(z, u) = \phi \setminus N(x, u) \leq \alpha; \beta$  where  $\alpha\beta = \text{nf}(\alpha\beta)$ , and it is not the case that  $\alpha = \alpha'\psi$ ,  $\beta = \bar{\psi}\beta'$  and  $\alpha'\beta' \geq N^1(z, u)$  (any  $\alpha', \beta'$  and  $\psi \in F$ ). Since  $N^1(z, u) = \phi \setminus N(x, u) \leq \alpha\beta$  we have  $N(x, u) \leq \text{nf}(\phi\alpha); \beta = \text{nf}(\phi\alpha\beta)$  so, inductively, there is  $w \in N$  such that  $N(x, w) \leq \text{nf}(\phi\alpha)$ , and  $N(w, u) \leq \beta$ . Since  $N(y, w) \leq N(y, u); N(u, x); N(x, w)$  is defined, we have  $N^1(z, w) = \phi \setminus N(x, w) \leq \phi \setminus (\phi; \alpha) \leq \alpha$  (by theorem 35), so  $w$  is the required witness in  $N^1$ . This proves condition C.

Part 2 is proved similarly.  $\square$

**LEMMA 43.** *Let  $N$  be a network,  $x, y \in N$  and  $\alpha; \phi \geq N(x, y)$ , for some  $\alpha \in \mathcal{A}_m$  and  $\phi \in F$ . Suppose that  $N(y, x)$  is not defined. Then there is an extension  $N^2 \supseteq N$  such that there is  $z$  with  $N^2(x, z) \leq \alpha$  and  $N^2(z, y) = \phi$ . Further,*

- (1) *if  $N$  is  $k$ -good and  $\phi \in \{\bar{f}, g\}$  then  $N^2$  is  $k$ -good;*
- (2) *if  $N$  is  $k^-$ -good and  $\phi \in \{f, \bar{g}\}$  then  $N^2$  is  $k^-$ -good.*

*If the required witness is missing from  $N$ , we define  $N^2 = N^2(N, x, \alpha, \phi)$  with exactly one node,  $z$ , in addition to those of  $N$ . For the labelling of edges incident with  $z$ , let  $N^2(z, z) = 1'$ . Let  $w \in N$  be arbitrary.*

$$N^2(z, w) = \phi; N(y, w) \tag{1}$$

Let

$$N^2(w, z) \begin{cases} \in N(w, y)/\phi & \text{if } N(w, x) \text{ is defined} \\ = N(w, y); \bar{\phi} & \text{otherwise.} \end{cases} \tag{2}$$

*With the first alternative in (2) we must write ‘ $\in$ ’ instead of ‘ $=$ ’ because  $N(w, y)/\phi$  is a set, not necessarily a singleton. If it is not a singleton then we have not uniquely defined  $N^2(w, z)$  (yet). If  $N(w, y)/\phi$  has two elements then  $N(w, y) = \rho k_\Lambda (fg)^{m-1} f$  (see definition 30.III(a)) and  $\phi = \bar{g}$ , some  $\rho$ , and*

$$N(w, y)/\bar{g} = (\rho k_\Lambda (fg)^{m-1} f)/\bar{g} = \{\rho k_\Lambda b, \rho k_{(fg)^m}\}.$$

*Note that  $N(w, x); \alpha; \bar{g} \geq N(w, x); N(x, y) \geq N(w, y)$  so by theorem 33 there is  $\gamma \in N(w, y)/\bar{g}$  with  $N(w, x); \alpha \geq \gamma$ . I.e.*

$$N(w, x); \alpha \geq \rho k_{(fg)^m} \quad \text{or} \quad N(w, x); \alpha \geq \rho k_\Lambda b. \tag{3}$$

Let

$$N^2(w, z) = \begin{cases} \rho k_{\Lambda} b & \text{if } N(w, x); \alpha \geq \rho k_{\Lambda} b \\ \rho k_{(fg)^m} & \text{otherwise.} \end{cases} \quad (4)$$

With  $N^2$  thus defined we have  $N^2(x, z) \leq \alpha$  and  $N^2(z, y) = \phi$ .

*Proof.* Again assume that the required witness is missing from  $N$ .

Note that  $N^2(u, z); \phi \geq N^2(u, y)$  in all cases. The proof is roughly similar to the proof of lemma 42, but the proof of consistency is rather more complicated because  $x/\phi$  is not always a singleton set. Suppose  $N$  is  $k$ -good and  $\phi \in \{\bar{f}, g\}$ . We must show that  $N^2$  is also  $k$ -good.

First we check the consistency of  $N^2$ . Trans and Refl are immediate. Condition Id follows from the following observation. If  $1' \geq N^2(z, w) = \phi; N(y, w)$ , then we have either  $N(w, y) \leq \bar{f}$  or  $N(w, y) \leq g$ . In either case, we get  $N(x, w) \leq N(x, y); N(y, w) \leq \alpha; \phi; \bar{\phi} \leq \alpha$ , i.e., the required witness is present in  $N$ . Similarly, if  $N(w, z) = N(w, y); \bar{\phi}$ , then  $N(w, y) \in \{\bar{f}, g\}$ , whence  $N(x, w) \leq \alpha$  as above. Finally, note that if  $N^2(z, w) \in N(w, y)/\phi$ , then  $1' \geq N(w, z)$  is impossible.

Next we check Comp. Let  $u, v \in N$  and suppose  $(u, v)$  is labelled in  $N$ . Then

$$\begin{aligned} & N^2(u, z); N^2(z, v) \\ &= N^2(u, z); (\phi; N(y, v)) \quad \text{by (1)} \\ &\geq N(u, y); N(y, v) \quad \text{since } N^2(u, z); \phi \geq N(u, y) \\ &\geq N^2(u, v). \end{aligned}$$

Similarly,

$$\begin{aligned} N^2(z, u); N^2(u, v) &= \phi; N(y, u); N(u, v) \\ &\geq \phi; N(y, v) \\ &= N^2(z, v). \end{aligned}$$

Finally, we must check that  $N^2(u, v); N^2(v, z) \geq N^2(u, z)$ . If  $(v, x)$  is unlabelled in  $N$  then  $N^2(v, z) = N(v, y); \bar{\phi}$ , by (2). In that case

$$N^2(u, v); N^2(v, z) = N(u, v); N(v, y); \bar{\phi} \geq N(u, y); \bar{\phi} \geq N^2(u, z)$$

by lemma 31. So assume that  $(v, x)$  is labelled in  $N$  (hence  $(u, x)$  is also labelled, since in the current case  $(u, v)$  is labelled). By (2),  $N^2(u, z) \in N(u, y)/\phi$ . If  $N(u, y)/\phi$  is a singleton then  $N^2(u, z) = N(u, y)/\phi$ . Since

$$\begin{aligned} (N(u, v); N^2(v, z)); \phi &= N(u, v); (N^2(v, z); \phi) \\ &\geq N(u, v); N(v, y) \\ &\geq N(u, y) \end{aligned}$$

it follows by theorem 33 that

$$N(u, v); N^2(v, z) \geq N(u, y)/\phi = N^2(u, z).$$

That leaves the situation where  $N(u, y)/\phi$  is not a singleton. In this case  $\phi = \bar{g}$  and there is  $\rho$  such that  $N(u, y) = \rho k_{\Lambda} (fg)^{m-1} f$ . We have

$$N(u, y)/\phi = \{\rho k_{(fg)^m}, \rho k_{\Lambda} b\}.$$

Now  $(N(u, v); N^2(v, z)); \phi \geq N(u, v); N(v, y) \geq N(u, y)$  so by theorem 33 there is  $\gamma \in \{\rho k_{(fg)^m}, \rho k_{\Lambda} b\}$  with

$$N(u, v); N^2(v, z) \geq \gamma. \quad (5)$$

By (2),  $N^2(u, z) \in \{\rho k_{\Lambda} b, \rho k_{(fg)^m}\}$ . If  $N^2(u, z) = \gamma$  we are done. The two remaining possibilities we have to consider (and eliminate) are

- (i)  $N^2(u, z) = \rho k_{(fg)^m}$ ,  $N^2(u, v); N^2(v, z) \geq \rho k_{\Lambda} b$  but  $N^2(u, v); N^2(v, z) \not\geq \rho k_{(fg)^m}$ , and
- (ii)  $N^2(u, z) = \rho k_{\Lambda} b$ ,  $N^2(u, v); N^2(v, z) \geq \rho k_{(fg)^m}$  but  $N^2(u, v); N^2(v, z) \not\geq \rho k_{\Lambda} b$ .

For (i), suppose  $N^2(u, z) = \rho k_{(fg)^m}$ . By (4),  $N(u, x); \alpha \not\geq \rho k_{\Lambda} b$  and, by (3),  $N(u, x); \alpha \geq \rho k_{(fg)^m}$ . By lemma 28,  $N(u, x); \alpha = \rho^+ k_{(fg)^m}$ , for some  $\rho^+ \geq \rho$ . Hence, either  $\alpha = \alpha_0 k_{(fg)^m}$  (some  $\alpha_0$ ), or  $N(u, x) = \rho^+ k_{(fg)^m} \bar{\alpha}$  and  $\bar{\alpha}; \alpha = 1'$ . In the former case,  $N^2(u, v); N^2(v, z) \leq N(u, v); N(v, x); \alpha_0 k_{(fg)^m}$ . Since  $k_{(fg)^m}$  is minimal with respect to  $<$ , this means that  $N^2(u, v); N^2(v, z) \geq \rho k_{\Lambda} b$  is impossible. But we are assuming that  $N^2(u, v); N^2(v, z) \not\geq \rho k_{(fg)^m}$ . This contradicts (5). For the latter case,  $\bar{\alpha}; \alpha = 1'$  so  $\alpha \in \{f, \bar{g}\}^*$  and  $\phi = \bar{g}$ . So  $N(x, y) \leq \alpha; \bar{g} \in \{f, \bar{g}\}^*$  is 1-short. By condition B for  $N$ ,  $N(y, x)$  is defined, contrary to the assumption in the lemma. Thus case (i) cannot happen.

Now consider possibility (ii) and suppose

$$N^2(u, z) = \rho k_{\Lambda} b \text{ but } N^2(u, v); N^2(v, z) \not\geq \rho k_{\Lambda} b.$$

By (5),  $N^2(u, v); N^2(v, z) \geq \rho k_{(fg)^m}$ . By lemma 28,  $N^2(u, v); N^2(v, z) = \rho^+ k_{(fg)^m}$ , for some  $\rho^+ \geq \rho$ . The character  $k_{(fg)^m}$  belongs either to  $N^2(u, v)$  or  $N^2(v, z)$ . In the former case,  $N^2(u, v) = \rho^+ k_{(fg)^m} s$  and  $N^2(v, z) = \bar{s}$  for some  $s$  with  $s; \bar{s} = 1'$ . This means that  $N^2(v, y) \leq N^2(v, z); \bar{g} = \bar{s}; \bar{g}$  is 1-short so  $N^2(y, v)$  is defined. Hence  $N(v, x)$  cannot be defined, else  $N(y, x)$  would be defined against our assumption. If  $N(v, x)$  is undefined, then  $N^2(v, z) = N(v, y); g$ . By lemma 28, this contradicts our assumption that  $N^2(u, v); N^2(v, z) \not\geq \rho k_{\Lambda} b$ .

The other case is where the character  $k_{(fg)^m}$  belongs to  $N^2(v, z)$ . Say  $N(u, v) = \rho_1$ ,  $N^2(v, z) = \rho_2 k_{(fg)^m}; e$  for some  $\rho_1, \rho_2, e$  where  $\rho_1; \rho_2 \geq \rho$  and  $e \geq 1'$ , but  $N^2(u, v); N^2(v, z) \not\geq \rho k_{\Lambda} b$ . By lemma 28,  $e = 1'$ , so  $N^2(v, z) = \rho_2 k_{(fg)^m}$ . Also, by (4),  $N(v, x); \alpha \not\geq \rho_2 k_{\Lambda} b$ . So, by (3),  $N(v, x); \alpha \geq \rho_2 k_{(fg)^m}$  and, by lemma 28,  $N(v, x); \alpha = \rho_2^+ k_{(fg)^m}$ , for some  $\rho_2^+ \geq \rho_2$ . The character  $k_{(fg)^m}$  belongs either to  $N(v, x)$  or  $\alpha$ . In the latter case we must have  $N(u, x); \alpha \not\geq \rho k_{\Lambda} b$ , but this contradicts (4) and the assumption that  $N^2(u, z) = \rho k_{\Lambda} b$ . In the former case we have  $N(v, x) = \rho_2^+ k_{(fg)^m} s$  and  $s; \alpha = 1'$ . Hence  $\alpha \in \{f, \bar{g}\}^*$  is 1-short and  $N(x, y) \leq \alpha \bar{g}$  is also 1-short. By condition B for  $N$ ,  $N(y, x)$  must be defined, contrary to the assumption in the lemma. Thus case (ii) cannot happen.

This proves that  $N^2$  is consistent (condition A). The proofs of conditions B and C for  $N^2$  are similar to the corresponding proofs for lemma 42.  $\square$

In the next lemma we define two extensions  $N^3, N^4$  to  $N$ . Each of them has a witness  $z$  satisfying  $N^3(x, z) = N^4(x, z) = \phi$ , for some specified  $x \in N$  and  $\phi \in F$ .

**LEMMA 44.** *Let  $1 < k < m$ , let  $N$  be a  $k$ -good network,  $\phi \in F$  and  $x \in N$ . Then there are extensions  $N^3$  and  $N^4$  of  $N$  such that  $N^3(x, z) = N^4(x, z) = \phi$  for some  $z$ , and  $N^3$  is  $k^-$ -good and  $N^4$  is  $k$ -good.*

*If the required witness is missing from  $N$ , we define  $N^3 = N^3(N, x, \phi)$  and  $N^4 = N^4(N, x, \phi)$ , each with a single node,  $z$  say, in addition to those of  $N$ . Of course,  $N^3(z, z) = N^4(z, z) = 1'$ . To label other edges incident with  $z$ , let  $u \in N$  be arbitrary.*

$$\begin{aligned} N^3(u, z) &\in N(u, x)/\bar{\phi} \\ N^3(z, u) &= \bar{\phi}; N(x, u) \end{aligned}$$

$$\begin{aligned} N^4(u, z) &= N(u, x); \phi \\ N^4(z, u) &= \bar{\phi}; N(x, u) \end{aligned}$$

*In the definition of  $N^3(u, z)$ , if  $N(u, x)/\bar{\phi}$  is a set with two elements, say  $N^3(u, x)/\bar{\phi} = \{\rho k_{\Lambda} b, \rho k_{(fg)^m}\}$ , for some  $\rho$ , then we let  $N^3(u, z) = \rho k_{\Lambda} b$ .*

*Proof.* Consistency of  $N^3$  and  $N^4$  is easy to check (use theorem 33 for  $N^3$ ).

The proofs of conditions B and C for  $N^3, N^4$  are the same as the proofs of the same conditions in lemma 42.  $\square$

**LEMMA 45.** *Let  $m > k \geq 1$ , let  $N$  be  $k$ -good,  $x, y \in N$  and suppose  $N(x, y), N(y, x)$  are both defined. If  $\alpha; \beta \geq N(x, y)$  then there is a  $(k - 1)$ -good  $N' \supseteq N$  containing a node  $z$  such that  $N'(x, z) \leq \alpha$  and  $N'(z, y) \leq \beta$ .*

*Proof.* We can assume that  $\alpha, \beta$  are minimal, i.e. if  $\alpha' \leq \alpha, \beta' \leq \beta$  and  $\alpha'; \beta' \geq N(x, y)$  then  $\alpha' = \alpha$  and  $\beta' = \beta$ .

If (i)  $\alpha = \alpha_0 b s, \beta = \bar{s} \bar{g} \beta_0$  and  $\alpha_0 b; \bar{g} \beta_0 \geq N(x, y)$ , or (ii)  $\alpha = \alpha_0 \bar{f} s$  and  $\beta = \bar{s} b \beta_0$  and  $\alpha_0 \bar{f}; b \beta_0 \geq N(x, y)$ , for some  $\alpha_0, \beta_0$  and some  $s \in F^*$ , then we say that  $(\alpha, \beta)$  is *dangerous*.

For any character  $c \in \Sigma_m$  define the rank  $\rho(c) \in \mathbb{N}$  by

$$\rho(c) = \begin{cases} 1 & \text{if } c \in F \\ 2m & \text{if } c = b \\ 0 & \text{otherwise.} \end{cases}$$

For any string  $s = s_0 s_1 \dots s_{j-1}$  over  $\Sigma_m$  let the rank of  $s$  be defined by  $\rho(s) = \sum_{i < j} \rho(s_i)$ . We will prove the lemma by strong induction over  $\rho(\alpha) + \rho(\beta)$ .

**Induction hypothesis:** Assume  $N$  is  $k$ -good. Let  $l \in \mathbb{N}$ , and assume that  $N(x, y), N(y, x)$  are both defined. If  $\alpha_1; \beta_1 \geq N(x, y)$ , and  $\rho(\alpha_1) + \rho(\beta_1) < l$ , then there is a  $(k - 1)$ -good  $N' \supseteq N$  containing a node  $z$  such that  $N'(x, z) \leq \alpha_1$  and  $N'(z, y) \leq \beta_1$ . Furthermore, if  $(\alpha_1, \beta_1)$  is not dangerous then there is a  $k$ -good  $N' \supseteq N$  with a node  $z$  such that  $N'(x, z) \leq \alpha_1$  and  $N'(z, y) \leq \beta_1$ .

Now we consider the case where  $\rho(\alpha) + \rho(\beta) = l$ . Suppose that none of the following are true.

- $\alpha = \alpha' \psi, \beta = \bar{\psi} \beta'$  and  $\alpha'; \beta' \geq N(x, y)$ ,
- $\alpha = \alpha' b$  and  $\beta = \bar{g} \beta'$ ,
- $\alpha = \alpha' \bar{f}$  and  $\beta = b \beta'$ ,

for any  $\psi \in F$  and any  $\alpha', \beta'$ . Since, under these conditions,  $(\alpha, \beta)$  is not dangerous we must find a  $k$ -good  $N' \supseteq N$  with a suitable witness node. Well, by condition C of definition 40, there is  $w \in N$  with  $N(x, w) \leq \alpha$  and  $N(w, y) \leq \beta$  so we may let  $N' = N$ . This is  $k$ -good, as required.

Now let  $\alpha = \alpha' \psi, \beta = \bar{\psi} \beta'$  and  $\alpha'; \beta' \geq N(x, y)$ , for some  $\psi \in F$  and some  $\alpha', \beta'$ . Since  $\rho(\alpha') + \rho(\beta') = \rho(\alpha) - 1 + \rho(\beta) - 1$ , we can use our induction hypothesis. So there is a  $(k - 1)$ -good  $N^+ \supseteq N$  with a node  $w$  such that  $N^+(x, w) \leq \alpha'$  and  $N^+(w, y) \leq \beta'$ . Let  $N' = N^4(N^+, w, \psi)$ , defined in lemma 44. By that lemma  $N'$  is also  $(k - 1)$ -good.

Suppose that  $(\alpha' \psi, \bar{\psi} \beta')$  is not dangerous. Then  $(\alpha', \beta')$  is not dangerous either so by the induction hypothesis there is a  $k$ -good  $N^+ \supseteq N$  with a node  $w$  such that  $N^+(x, w) \leq \alpha'$  and  $N^+(w, y) \leq \beta'$ . As above we let  $N' = N^4(N^+, w, \psi)$  and by lemma 44 this is  $k$ -good.

Let  $\alpha = \alpha' b$  and  $\beta = \bar{g} \beta'$ . Since  $(\alpha' b, \bar{g} \beta')$  is dangerous we are only required to find a  $(k - 1)$ -good  $N^+ \supseteq N$  with the required witness. We have  $\alpha; \beta \equiv \alpha'; (fg)^{m-1} f; \beta' \geq N(x, y)$ . Since  $\rho(\alpha' (fg)^{m-1} f) + \rho(\beta') < \rho(\alpha) + \rho(\beta)$  we can use our induction hypothesis. If  $(\alpha' (fg)^{m-1} f, \beta')$  is not dangerous then there is a  $k$ -good  $N^+ \supseteq N$  with a node  $w$  such that  $N^+(x, w) \leq \alpha' (fg)^{m-1} f$  and  $N^+(w, y) \leq \beta'$ . Let  $N' = N^3(N^+, w, \bar{g})$ . By lemma 44,  $N'$  is  $(k - 1)$ -good and it contains a node  $z$  such that  $N'(x, z) = N^+(x, w) / \bar{g} \leq (\alpha' (fg)^{m-1} f) / \bar{g} \leq \alpha' b = \alpha$  and  $N'(z, y) = \bar{g}; N^+(w, y) \leq \bar{g} \beta' = \beta$ .

Now suppose  $(\alpha' (fg)^{m-1} f, \beta')$  is dangerous. Then  $\beta' = \overline{(fg)^{m-1} f} \beta_0$  and  $\alpha'; \beta_0 \geq N(x, y)$ , for some  $\beta_0$ . By the induction hypothesis there is a  $(k - 1)$ -good  $N_0 \supseteq N$  with a node  $w$  such that  $N_0(x, w) \leq \alpha'$  and  $N_0(w, y) \leq \beta_0$ . Next we make a sequence of  $(k - 1)$ -good extensions  $N_0 \subseteq N_1 \subseteq \dots \subseteq N_{2m-1}$  as follows. Let  $z_0 = w$ , let  $i < 2m - 1$  and suppose we have defined a  $(k - 1)$ -good  $N_i$  containing a node  $z_i$  such that  $N_i(z_0, z_1) = f, N_i(z_1, z_2) = g, \dots, N_i(z_{i-1}, z_i) = \phi$ , where  $\phi$  is the

$i$ th character of  $(fg)^{m-1}f$ . Let  $\phi'$  be the  $(i+1)$ th character of  $(fg)^{m-1}f$ . Let  $N_{i+1} = N^4(N_i, z_i, \phi')$ . By lemma 44  $N_{i+1}$  is  $(k-1)$ -good and it contains a node  $z_{i+1}$  such that  $N_{i+1}(z_i, z_{i+1}) = \phi'$ . Thus we can define a  $(k-1)$ -good  $N_{2m-1} \supseteq N_0$  with a node  $z_{2m-1}$  such that  $N_{2m-1}(u, z_{2m-1}) = N_0(u, z_0); (fg)^{m-1}f$  and  $N_{2m-1}(z_{2m-1}, u) = \overline{(fg)^{m-1}f}; N_0(z_0, u)$ . Finally, let  $N' = N^3(N_{2m-1}, z_{2m-1}, g)$ . Lemma 44 merely tells us that this is  $(k-2)$ -good, but this is not good enough. In fact  $N'$  will be  $(k-1)$ -good. To see this, let  $z \in N'$  satisfy  $N'(z_{2m-1}, z) = g$ . Suppose for contradiction that  $N'$  is not  $(k-1)$ -good. Since  $N_{2m-1}$  is  $(k-1)$ -good, this means that there is  $u \in N_{2m-1}$  such that either (a)  $N'(u, z)$  is  $(k-1)$ -short but  $N'(z, u)$  is not defined, (b)  $N'(z, u)$  is  $(k-1)$ -short but  $N'(u, z)$  is not defined or (c)  $N'(u, z), N'(z, u)$  are both defined but  $N'$  contains no witness for  $\alpha, \beta$  as in condition C of definition 40. Observe that if  $u \in N_0$  then  $N'(u, z) = (N_0(u, z_0); (fg)^{m-1}f)/\bar{g} \leq N_0(u, z_0); ((fg)^{m-1}f/\bar{g}) = N_0(u, z_0); b$  by theorem 33. By lemma 39 this is not  $(k-1)$ -short (since  $k < m$ ,  $b$  is not  $k$ -short). Also  $N'(z, u)$  is not  $(k-1)$ -short if  $u \in N_0$ . So for (a) and (b) we can suppose that  $u \in \{z_1, z_2, \dots, z_{2m-1}\}$ . But then, by construction of  $N'$ , we know that  $N'(z, u)$  is defined (also  $N'(u, z)$  is defined). For (c), if  $N'(u, z), N'(z, u)$  are both defined and  $u \in \{z_1, \dots, z_{2m-1}\}$  and  $\alpha, \beta$  are as in condition C of definition 40 then a suitable witness can be found in  $\{z_1, \dots, z_{2m-1}\}$ . On the other hand, if  $u \in N_0$  then, since  $N'(u, z), N'(z, u)$  are both defined, we must have  $N_0(u, z_0), N_0(z_0, u)$  both defined and  $N'(u, z) = N_0(u, z_0); b$ ,  $N'(z, u) = \overline{(fg)^m}; N_0(z_0, u)$ . If  $\alpha; \beta \geq N'(u, z)$  as in condition C of definition 40 then either  $\alpha = \alpha_0; b$  and  $\beta = \overline{(fg)^m}; \beta_0$ , in which case the required witness can be found by considering  $\alpha_0, \beta_0$  in  $N_0$ , or  $\alpha = \alpha_0; s$  and  $\beta = t$  for some  $s, t$  with  $st = (fg)^m$ , in which case a suitable witness can be found in  $\{z_1, \dots, z_{2m-1}\}$ . The case  $\alpha; \beta \geq N'(z, u)$  is similar. This proves that  $N'$  is in fact  $(k-1)$ -good. We have  $N'(x, z) = N_{2m-1}(x, z_{2m-1})/\bar{g} \leq (\alpha'(fg)^{2m-1}f)/\bar{g} \leq \alpha'b = \alpha$  and  $N'(z, y) = \bar{g}; \overline{(fg)^{m-1}f}\beta_0 \leq \bar{g}\beta' = \beta$ , as required.

The case where  $\alpha = \alpha'\bar{f}$  and  $\beta = b\beta'$  is similar, but instead of using  $N^3$  in the construction of  $N'$  we use  $N^1$  from lemma 42.  $\square$

Putting these together we get

**LEMMA 46.** *Let  $2^r < m$  and  $N$  be  $2^r$ -good. For any  $\forall$ -move  $(N, x, y, \alpha, \beta)$  there is a legal  $\exists$ -response  $N' \supseteq N$  and  $N'$  is  $2^{r-1}$ -good.*

*Proof.* First suppose  $N(y, x)$  is undefined. If neither  $\alpha$  nor  $\beta$  is  $2^{r-1}$ -short, then by lemma 41 there is a  $2^{r-1}$ -good  $N^0 \supseteq N$  making a legal response to the move.

Suppose  $\alpha \leq X_0; Y_0; \dots; X_{k-1}; Y_{k-1}$  is  $2^{r-1}$  short, where  $k \leq 2^{r-1}$  and  $X_i \in \{f, \bar{g}\}^*$ ,  $Y_i \in \{\bar{f}, g\}^*$  for  $i < k$ . Since  $N$  is  $2^r$ -good it is certainly  $\neg 2^r$ -good. Using lemma 42  $|X_0|$  times, there is a  $\neg 2^r$ -good  $N^+ \supseteq N$  with a node  $z_0$  such that  $N^+(x, z_0) \leq X_0$ ,  $N^+(z_0, y) \leq Y_0; X_1; \dots; Y_{k-1}; \beta$ . Since  $N^+$  is  $\neg 2^r$ -good it is certainly  $(2^r - 1)$ -good. Using the same lemma  $|Y_0|$  times again, there is a  $(2^r - 1)$ -good  $N^* \supseteq N^+$  with a node  $z_1$  such that  $N^*(z_1, z_2) \leq Y_0$  and  $N^*(z_2, y) \leq X_1; Y_1; \dots; Y_{k-1}; \beta$ . Continuing like this we obtain a  $2^r - 2^{r-1} = 2^{r-1}$ -good  $N' \supseteq N$  with a node  $z$  such that  $N'(x, z) \leq X_0; Y_0; \dots; Y_{k-1}$  and  $N'(z, y) \leq \beta$ .

Similarly, if  $\beta$  is  $2^{r-1}$ -short then repeated use of lemma 43 shows that there is a  $2^{r-1}$ -good  $N' \supseteq N$  making a legal response to the move.

Now suppose that  $N(y, x)$  is defined. By lemma 45 there is a  $(2^r - 1)$ -good legal response  $N' \supseteq N$  to the move  $(N, x, y, \alpha, \beta)$ . Since  $N'$  is  $(2^r - 1)$ -good and  $r \geq 1$ , it is certainly  $2^{r-1}$ -good.  $\square$

**THEOREM 47.**  $\exists$  has a winning strategy in  $G_n(\mathcal{A}_m)$  for  $2^n < m$ .

*Proof.* We define a strategy for  $\exists$  in  $G_n(\mathcal{A}_m)$ . We will design the strategy so that if  $N_t$  is played in round  $t$  (so there are  $r = n - t$  rounds left in the game) then  $N_t$  is a  $2^r$ -good network.

In the initial round (round zero) let  $\forall$  play  $\gamma$ . If  $\gamma$  is not  $2^n$ -short then let  $N_0$  have exactly two nodes,  $x$  and  $y$  and labelling

$$\begin{aligned} N_0(x, x) &= N_0(y, y) = 1' \\ N_0(x, y) &= \gamma \\ N_0(y, x) &= \text{undefined} \end{aligned}$$

If  $\gamma$  is  $2^n$ -short then  $N_0$  has  $|\gamma| + 1$  nodes  $x_0, x_1, \dots, x_{|\gamma|}$ . Let  $\gamma = \gamma_0 \dots \gamma_l$  (some  $l$ , some  $\gamma_i \in F$  for  $i \leq l$ ) and let

$$N_0(x_i, x_j) = \begin{cases} \frac{\gamma_i \gamma_{i+1} \dots \gamma_{j-1}}{N_0(x_j, x_i)} & \text{if } i \leq j \\ \text{otherwise} & \end{cases}$$

It is easy to check that  $N_0$  is  $2^n$ -good.

By lemma 46, in each round  $\exists$  can play a  $2^r$ -good network if there are  $r$  rounds left in the game. Since a  $2^r$ -good network is certainly consistent she will win the game if she plays this way.  $\square$

## 6. Ultraproducts and non-finite axiomatisability

**THEOREM 48** (Łoś' theorem). *Let  $\{A_i : i \in I\}$  be a set of structures in a signature  $L$  and let  $D$  be an ultrafilter over  $I$ . For each  $i \in I$  let  $\bar{a}_i$  be an  $n$ -tuple of elements of  $A_i$  and let  $\bar{a} = [(\bar{a}_i : i \in I)]_D$ . For any  $L$ -formula  $\phi(\bar{x})$  where  $\bar{x}$  is an  $n$ -tuple of variables,*

$$\Pi_D A_i \models \phi(\bar{a}) \Leftrightarrow \{i \in I : A_i \models \phi(\bar{a}_i)\} \in D.$$

We are ready to prove our main theorem.

*Proof.* (Theorem 2) Consider  $\mathcal{A}_n$  and let  $\mathcal{A}'_n$  be the SOM constructed from  $\mathcal{A}_n$  in definition 5. We saw in lemma 8 that  $(\mathcal{A}_n, \mathcal{A}'_n) \models \Sigma$ , for all  $n \in \mathbb{N}$ , where  $\Sigma$  is the two-sorted theory given in definition 7. Let  $D$  be a non-principal ultrafilter over  $\mathbb{N}$ . Note that every cofinite subset of  $\mathbb{N}$  belongs to  $D$ . By Łoś' theorem  $\Pi_D(\mathcal{A}_i, \mathcal{A}'_i) \cong (\Pi_D \mathcal{A}_i, \Pi_D \mathcal{A}'_i) \models \Sigma$ . For  $2^r < n \exists$  has a winning strategy in  $G_r(\mathcal{A}_n)$  (theorem 47) hence she also has a winning strategy in  $G_r(\mathcal{A}_{n'})$  whenever  $n' \geq n$ . By proposition 12(4) there is a first-order formula  $\sigma_r$  where  $\mathcal{A}_{n'} \models \sigma_r$  iff  $\exists$  has a winning strategy in  $G_r(\mathcal{A}_{n'})$ . By Łoś' theorem,  $\Pi_D \mathcal{A}_i \models \sigma_r$  for each  $r \in \mathbb{N}$ . By proposition 12(4) again,  $\exists$  has a winning strategy in  $G_r(\Pi_D \mathcal{A}_i)$  for all  $r \in \mathbb{N}$ .

By proposition 12(3) there is a countable  $(X, Y)$  with  $(X, Y) \equiv (\Pi_D \mathcal{A}_i, \Pi_D \mathcal{A}'_i)$  such that  $\exists$  has winning strategy in the infinite game  $G(X)$ . By proposition 12(2),  $X$  is an ordered monoid with a principal representation. By elementary equivalence  $(X, Y) \models \Sigma$ , and by lemma 9,  $Y$  is a representable SOM.

Now, each  $\mathcal{A}'_n$  is an unrepresentable SOM (theorem 23) but the non-principal ultraproduct  $\Pi_D \mathcal{A}'_i$  is elementarily equivalent to a representable SOM,  $Y$ . Suppose for contradiction that a finite number of axioms, wlog a single axiom  $\sigma$ , defines the class of representable SOMs. Then  $\mathcal{A}'_n \not\models \sigma$  for all  $n$ , but  $Y \models \sigma$ . This is impossible, by Łoś' theorem.  $\square$

The signature of ordered monoids is slightly less expressive than that of SOMs. In [9] it is proved that the class of all representable monoids is not finitely axiomatisable.

**PROBLEM 49.** *Is the class of ordered monoids with principal representations elementary?*

Note that lemma 9 and theorems 23 and 47 can be used to establish that this class is not finitely axiomatisable.

## Appendix

This appendix contains a technical theorem and some lemmas from the paper (mainly about normal forms) and their proofs.

**LEMMA 26.** *Let  $x, y \in (F \cup \{b\})^*$  and  $x \geq y$ . Then either  $\text{nf}(x) \xrightarrow{\psi\bar{\psi} \succ 1'} z \geq \text{nf}(y)$  (some  $z$  and some  $\psi \in \{f, \bar{g}\}$ ) or  $\text{nf}(y)$  is obtained from  $\text{nf}(x)$  by a sequence of reductions  $(fg)^n \succ b$ .*

*Proof.* All the strings in the lemma belong to  $(F \cup \{b\})^*$  so the only possible reductions are those in (I) and (II) from definition 15, and the only replacements that we can apply from definition 17 are (1)–(4).

Let  $\text{nf}(x) = x \geq y = \text{nf}(y)$ . There is a chain of reductions from  $x$  to  $y$

$$x = x_0 \xrightarrow{\tau_0} x_1 \xrightarrow{\tau_1} \dots \xrightarrow{\tau_{k-1}} x_k = y. \quad (6)$$

Let us assume that there is no redundancy in this chain, in the following sense: there is no proper subsequence  $(\sigma_0, \dots, \sigma_{j-1})$  of  $(\tau_0, \tau_1, \dots, \tau_{k-1})$  such that there is a chain

$$x = x'_0 \xrightarrow{\sigma_0} x'_1 \xrightarrow{\sigma_1} \dots \xrightarrow{\sigma_{j-1}} x'_j = y.$$

**Claim 1:** Let  $i \leq k$  and let  $x_i$  be from (6). If there is no  $j < i$  such that  $\tau_j = (\psi\bar{\psi} \succ 1')$ , where  $\psi \in \{f, \bar{g}\}$ , then  $\text{nf}(x_i) = x_i$ .

To prove claim 1, first note that  $\text{nf}(x_0) = x_0$ . Now let  $i > 0$  and suppose inductively that  $\text{nf}(x_{i-1}) = x_{i-1}$ . Now suppose, for contradiction, that  $\text{nf}(x_i) \neq x_i$ . Since  $x_i \in (F \cup \{b\})^*$  this implies that  $x_i$  contains a substring  $b\bar{g}$ ,  $\bar{f}b$ ,  $g\bar{g}$  or  $\bar{f}f$  (see definition 17).

Suppose  $x_i = x^0 b\bar{g} x^1$ , for some  $x^0, x^1$ . Since  $\text{nf}(x_{i-1}) = x_{i-1}$ ,  $x_{i-1}$  does not contain a substring  $b\bar{g}$ . So the reduction  $\tau_{i-1}$  must have either replaced a string by something ending with  $b$ , replaced a string by something beginning with  $\bar{g}$  or deleted a string between the  $b$  and the  $\bar{g}$ . There are no reductions of the form  $s \succ \bar{g}t$  (any  $s, t$ ), so the second case is impossible. If a string between  $b$  and  $\bar{g}$  was deleted, then  $\tau_{i-1} = (\psi\bar{\psi} \succ 1')$  for some  $\psi$  and since  $x_{i-1}$  does not contain  $\bar{f}f$  or  $g\bar{g}$  we must have  $\psi \in \{f, \bar{g}\}$ . But the assumption in the claim is that  $\tau_{i-1}$  is not a reduction of this form. There is only one reduction to a string ending with  $b$  and it is  $(fg)^n \succ b$ . But if  $\tau_{i-1} = ((fg)^n \succ b)$  then  $x_{i-1} = x^0 (fg)^n \bar{g} x^1$  so  $\text{nf}(x_{i-1}) \neq x_{i-1}$ , contrary to assumption.

If  $x_i = x^0 g\bar{g} x^1$  (some  $x^0, x^1$ ) then, as  $x_{i-1}$  does not contain  $g\bar{g}$ , we must have one of the following:  $\tau_{i-1} = (s \succ tg)$  or  $\tau_{i-1} = (s \succ \bar{g}t)$  for some  $s, t$ , or  $\tau_{i-1} = 1' \succ g\bar{g}$ , or  $\tau_{i-1}$  deleted a string  $\psi\bar{\psi}$  where  $\psi \in \{f, \bar{g}\}$ . The last alternative (the deletion) can be ruled out using the assumption in the claim. Consulting definition 15, we see that there are no reductions to a string starting with  $\bar{g}$  or a string ending with  $g$ , so this case cannot happen. The ‘reduction’  $1' \succ g\bar{g}$  is ruled out by assumption that there is no redundancy in the chain.

The cases where  $x_i$  contains  $\bar{f}b$  or  $\bar{f}f$  are similar. Thus  $x_i$  does not contain a substring  $b\bar{g}$ ,  $\bar{f}b$ ,  $g\bar{g}$  or  $\bar{f}f$  and so  $\text{nf}(x_i) = x_i$ . This proves claim 1.

Suppose there is  $i < k$  such that  $\tau_i = (\psi\bar{\psi} \succ 1')$  with  $\psi \in \{f, \bar{g}\}$  and let  $i$  be the least such that  $\tau_i$  is of this form. By claim 1,  $x_j$  does not contain  $b\bar{g}$ ,  $g\bar{g}$ ,  $\bar{f}b$  or  $\bar{f}f$ , for each  $j < i$ . We aim to show that  $i$  can be taken to be 0. Suppose  $i > 0$ . We have

$$x_{i-1} \xrightarrow{\tau_{i-1}} x_i \xrightarrow{\psi\bar{\psi} \succ 1'} x_{i+1}.$$

**Claim 2:** If  $x_{i-1} \xrightarrow{\tau_{i-1}} x_i \xrightarrow{\psi\bar{\psi} \succ 1'} x_{i+1}$  then there is  $\tau'_i$  and  $x'_i$  such that  $x_{i-1} \xrightarrow{\psi\bar{\psi} \succ 1'} x'_i \xrightarrow{\tau'_i} x_{i+1}$ .

To prove claim 2 let  $\tau_{i-1} = (s \succ t)$ . If  $t$  and  $\psi\bar{\psi}$  are disjoint substrings of  $x_i$  then clearly we can let  $\tau'_i = \tau_{i-1}$  and reverse the order of the two reductions.

The only way that  $t$  and  $\psi\bar{\psi}$  are not disjoint in  $x_i$  is if  $t$  ends with  $\psi$  or  $t$  starts with  $\bar{\psi}$ . Let  $\psi = f$  (the case  $\psi = \bar{g}$  is similar). The only reductions  $s \succ t$  where  $t$  ends with  $f$  or  $t$  starts with  $\bar{f}$  are  $1' \succ \bar{f}f$  and  $b\bar{g} \succ (fg)^{n-1}f$ . The (overlapping) sequence of reductions  $x_{i-1} \xrightarrow{1' \succ \bar{f}f} x_i \xrightarrow{f\bar{f} \succ 1'} x_{i+1}$  contains a redundancy (since  $x_{i-1} = x_{i+1}$  so we could delete both reductions) contrary to assumption. The case  $\tau_{i-1} = (b\bar{g} \succ (fg)^{n-1}f)$  is contradicted by claim 1. This proves the second claim.

Hence, if there is any  $\tau_i$  in the chain of the form  $\psi\bar{\psi} \succ 1'$ , for  $\psi \in \{f, \bar{g}\}$  then by claim 2 either  $i = 0$  or we can replace the chain (6) by another chain in which the reduction  $\psi\bar{\psi} \succ 1'$  occurs as the  $(i - 1)$ th reduction in the chain. Thus, this reduction can be moved to the front of the chain and so  $x \xrightarrow{\psi\bar{\psi} \succ 1'} z \geq y$  (some  $z$ ). This is the first alternative conclusion in the lemma.

Otherwise, none of the  $\tau_i$  has the form  $\psi\bar{\psi} \succ 1'$ , for  $\psi \in \{f, \bar{g}\}$ . By claim 1,  $\text{nf}(x_i) = x_i$  for each  $x_i$  in chain 6. Hence, the only possibility for each reduction  $\tau_i$  is  $(fg)^n \succ b$  — any other reduction  $x_i \xrightarrow{\tau_i} x_{i+1}$  would either be  $\psi\bar{\psi} \succ 1'$  ( $\psi \in \{f, \bar{g}\}$ ) or it would imply that  $x_i$  contained  $b\bar{g}, \bar{f}b, g\bar{g}$  or  $\bar{f}f$  contradicting  $\text{nf}(x_i) = x_i$ . This gives the second alternative conclusion in the lemma.  $\square$

**LEMMA 27.** *Let  $x, y, z \in (F \cup \{b\})^*$ . Suppose  $x = \text{nf}(x)$  and  $yz = \text{nf}(yz)$  and that  $y, z$  are minimal subject to  $y; z \geq x$  (i.e. if  $y' \leq y, z' \leq z$  and  $y'z' \geq x$  then  $y' \equiv y$  and  $z' \equiv z$ ). Then either*

- $y = y_0s, z = tz_0$ , for some  $y_0, z_0, s, t$  where  $st = (fg)^{in}$  for some  $i \in \mathbb{N}$ , and  $x = y_0b^iz_0$ , or
- $y = y_0\phi, z = \bar{\phi}z_0$  and  $y_0z_0 \geq x$ , for some  $\phi \in F$  and some  $y_0, z_0$ .

*Proof.* Assume the conditions in the lemma. We have  $\text{nf}(yz) \geq \text{nf}(x)$ . By lemma 26, either  $yz \xrightarrow{\psi\bar{\psi} \succ 1'} \gamma \geq x$  (some  $\gamma$  and some  $\psi \in \{f, \bar{g}\}$ ) or  $\text{nf}(x)$  is obtained from  $\text{nf}(yz)$  by a sequence of replacements  $(fg)^n \succ b$ . In the former case, the string  $\psi\bar{\psi}$  in the reduction  $\psi\bar{\psi} \succ 1'$  cannot be confined to  $y$  (or  $z$ ) else  $y$  (or  $z$ ) is not minimal. So we must have  $y = y_0\psi, z = \bar{\psi}z_0$  and  $y_0z_0 \geq x$ , which is the second alternative conclusion in the lemma.

The other case is where  $x$  is obtained from  $yz$  by a sequence of replacements  $(fg)^n \succ b$ . We can group together adjacent substrings  $(fg)^n$  of  $yz$  that are reduced to  $b$ . So  $x$  is obtained from  $yz$  by replacing some substrings  $(fg)^{in}$  by  $b^i$ , where  $i > 0$ . The substrings  $(fg)^{in}$  are not adjacent to each other.

If  $(fg)^{in}$  is replaced by  $b^i$  then  $(fg)^{in}$  cannot be confined to  $y$  (or  $z$ ) else  $y$  (or  $z$ ) is not minimal. Hence  $x$  is obtained from  $yz$  by replacing a single substring  $(fg)^{in}$  by  $b^i$  and this substring must be the concatenation of a non-empty final segment of  $y$  and a non-empty initial segment of  $z$ . This gives the first alternative conclusion in the lemma.  $\square$

**LEMMA 28.** *Let  $e \geq 1'$  and  $\rho \in \mathcal{A}_n$ .*

- (1) *Either  $k_{(fg)^ne} \geq k_{\Lambda}b$  or  $e \equiv 1'$  (and not both).*
- (2) *Either  $k_{\Lambda}be \geq k_{(fg)^n}$  or  $e \equiv 1'$ .*
- (3) *If  $\gamma \geq \rho k_{(fg)^n}$  but  $\gamma \not\geq \rho k_{\Lambda}b$  then  $\gamma \equiv \rho^+ k_{(fg)^n}$  for some  $\rho^+ \geq \rho$ .*
- (4) *If  $\gamma \geq \rho k_{\Lambda}b$  but  $\gamma \not\geq \rho k_{(fg)^n}$  then  $\gamma \equiv \rho^+ k_{\Lambda}b$  for some  $\rho^+ \geq \rho$ .*

*Proof.* For the first part, suppose  $k_{(fg)^ne} \not\geq k_{\Lambda}b$ . Note that  $k_{(fg)^n}f\bar{f} \geq k_{\Lambda}b$  and  $k_{(fg)^n}g\bar{g} \geq k_{\Lambda}b$ . Thus  $e \geq 1'$  must be obtained from the empty string by a series of insertions of  $\bar{f}fs$  and  $g\bar{g}s$ . These insertions are all equivalences, so  $e \equiv 1'$ . The second part is similar.

For the third part let  $\gamma \geq \rho k_{(fg)^n}$ . The only reductions in definition 15 involving a character  $k_s$  are listed as (III), (IV) and (V) and each has the form  $k_s x \succ k_t y$  or  $ax \succ k_t$ , for some  $s, t, x, y$ . Hence if  $x \geq \rho k_s$  then  $x = x^+ k_t y$  or  $x = x^+ a y$ , for some  $x^+ \geq x$ , some  $t, y$  such that  $k_t y \geq k_s$ . In the case under consideration we have  $\gamma = \rho' \kappa$  for some  $\rho' \geq \rho$  and some  $\kappa \geq k_{(fg)^n}$ .

Since  $\kappa \geq k_{(fg)^n}$  there is a chain  $\kappa \xrightarrow{\tau_0} \dots \xrightarrow{\tau_i} \rho k_{(fg)^n}$ . The only possible reductions  $s \xrightarrow{\tau_i} e k_{(fg)^n} e'$  (any  $e, e' \geq 1'$ ) are (a)  $\tau_i = (ab \succ k_{(fg)^n})$ , (b)  $\tau_i = (k_{(fg)^{n-1}} f g \succ k_{(fg)^n})$  or (c)  $\tau_i = (\phi \bar{\phi} \succ 1')$ . Now  $ab \geq k_\Lambda b$  so we can rule out reduction (a). Also,  $k_{(fg)^{n-1}} f g \equiv k_\Lambda (fg)^n \geq k_\Lambda b$  (use lemma 16) so we can rule out reduction (b). Thus all the reductions in the chain are of the type (c), which means that all the strings in the chain have the form  $e k_{(fg)^n} e'$ , so  $\kappa = e k_{(fg)^n} e'$  for some  $e, e' \geq 1'$ . By the first part of this lemma, since  $\kappa \not\geq k_\Lambda b$ , we have  $e' \equiv 1'$ . Thus  $\kappa \equiv e k_{(fg)^n}$ , for some  $e \geq 1'$  and  $\gamma = \rho' e k_{(fg)^n}$ . Let  $\rho^+ = \rho' e$  to get the required result.

Part 4 of the lemma is similar.  $\square$

**THEOREM 33.** *Let  $x, y \in \mathcal{A}_n$  and  $\phi \in F$ .*

- (1)  $x \leq y \Rightarrow x/\phi \leq y/\phi$  — i.e. for each  $z \in y/\phi$  there is  $z' \in x/\phi$  with  $z' \leq z$ .
- (2)  $y; \phi \geq x \Leftrightarrow y \geq x/\phi$  — i.e. there is  $z \in x/\phi$  with  $z \leq y$ .
- (3)  $(x; y)/\phi \leq x; (y/\phi)$  — i.e. for all  $z \in y/\phi$  there is  $z' \in (x; y)/\phi$  with  $z' \leq x; z$ .

*Proof.* 1: First let  $\phi \in \{\bar{f}, g\}$ , so  $x/\phi$  and  $y/\phi$  are defined by case I of definition 30:  $x/\phi = \text{nf}(x\bar{\phi})$  and  $y/\phi = \text{nf}(y\bar{\phi})$ . So if  $x \leq y$  then  $x/\phi \equiv x\bar{\phi} \leq y\bar{\phi} \equiv y/\phi$ .

Now let  $\phi = f$ , so  $x/\phi$  and  $y/\phi$  are defined by case II of definition 30. Suppose  $y \geq_k x$ . We deal with the case  $k = 1$ , i.e.  $x$  is obtained from  $y$  by a single reduction. The general case will then follow by a simple induction over  $k$ . So assume  $y \geq_1 x$ . If  $y \equiv x$  then, by lemma 20,  $y = x$  and so  $y/f = x/f$ . So we assume that  $y < x$ . This eliminates several possibilities for the single reduction  $y \geq_1 x$ . Now we break into the two possibilities in the definition of  $y/f$  (case II). If  $y \neq y_0 f$  (for any  $y_0$ ) then  $y/f = y\bar{f} \geq x\bar{f} \geq z'$ , for all  $z' \in x/f$  by lemma 31. If  $y = y_0 f$  for some  $y_0$ , we see from definition 17 that there are three possibilities. First,  $x = x_0 f$  and  $y_0 \geq x_0$ . Then  $x/f = (x_0 f)/f = x_0$  and  $y/f = (y_0 f)/f = y_0$  and we are done. Second, if  $x = x_0 \phi'$ , for some  $\phi' \leq f$ , then as  $f$  is minimal with respect to  $<$  we have  $\phi' = f$ , so the reduction is in fact an equivalence, contrary to assumption. Finally, any other possible reduction  $\tau$  with  $y_0 f \xrightarrow{\tau} x$  must involve a non-empty final segment of  $y_0$  concatenated with a non-empty initial segment of  $f$ . But any such reduction must be an equivalence. This completes the proof for case II.

Similarly we can prove case III, where  $\phi = \bar{g}$  and  $y \geq_1 x$ . If  $y = y_0 k_\Lambda (fg)^{n-1} f$  (some  $y_0$ ) then  $y/\bar{g} = \{y_0 k_\Lambda b, y_0 k_{(fg)^n}\}$ . Since  $k_\Lambda (fg)^{n-1} f$  is minimal with respect to  $<$ , we have  $x = x_0 k_\Lambda (fg)^{n-1} f$  for some  $x_0 \leq y_0$  and  $x/\bar{g} = \{x_0 k_\Lambda b, x_0 k_{(fg)^n}\}$  and the lemma holds true. Next suppose  $y = y_0 \bar{g}$  for some  $y_0$  and  $y/\bar{g} = y_0$ . From definition 17 we must have  $x = x_0 \bar{g}$  and  $y_0 \geq x_0$ . Then lemma 32 gives us the result. If  $y = y_0 k_s f$ , for some  $y_0$  and some initial segment  $sf$  of  $(fg)^n$  then, as in the proof of case II, we have  $x/\bar{g} \leq y/\bar{g}$ , as required. Finally, suppose  $y \neq y_0 k_\Lambda (fg)^{n-1} f, y_0 \bar{g}, y_0 k_s f$ , for any  $y_0$  and any initial segment  $sf$  of  $(fg)^n$ . Then  $y/\bar{g} = (y; g)$ . Either  $(y; g) = y; g$  or  $y = y_0 (fg)^{n-1} f$ , for some  $y_0$ , and  $(y; g) = y_0 b$ . By lemma 31 there is  $z \in x/\bar{g}$  with  $z \leq xg$ . If  $(y; g) = y; g$  then there is  $z \in x/\bar{g}$  with  $z \leq x; g \leq (y; g) = y/\bar{g}$ , as required. Else  $y = y_0 (fg)^{n-1} f$  (some  $y_0$ ) and, since  $(fg)^{n-1} f$  is minimal with respect to  $<$ , we have  $x = x_0 (fg)^{n-1} f$  for some  $x_0 \leq y_0$ . Hence  $x/\bar{g} = x_0 b \leq y_0 b = y/\bar{g}$ .

2: The right to left implication holds, by lemma 31. For the left to right implication we must show that if  $y; \phi \geq x$  then there is  $z \in x/\phi$  with  $y \geq z$ . By lemma 32,

there is  $z' \in (y; \phi)/\phi$  with  $z' \leq y$ . Since  $x \leq y; \phi$ , the first part of the theorem says that there is  $z \in x/\phi$  with  $z' \geq z$ . So there is  $z \in x/\phi$  with  $z \leq z' \leq y$ .

3: This easily follows by the second part of the theorem. □

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