Two partial orders on the set of antichains

Jason Crampton and George Loizou

Department of Computer Science, Birkbeck College, University of London, Malet Street, London, WC1E 7HX, England

e-mail: ccram01@dcs.bbk.ac.uk

September 19, 2000

Abstract

Given a poset X, we define two partial orders on the set of antichains of X. We prove that the two resulting posets $\langle \mathcal{A}(X), \preccurlyeq \rangle$ and $\langle \mathcal{A}(X), \preccurlyeq' \rangle$ are lattices which are isomorphic to the lattice of order ideals of X, $\langle \mathcal{I}(X), \subseteq \rangle$. We also establish the meet and join operations of the two lattices.

1 Preliminaries

The reader is assumed to be familiar with the basic concepts of partial orders and lattices. Throughout this paper, X will denote a poset with partial order $\leqslant$.

Definition 1.1 *$\alpha \subseteq X$ is an* antichain *if for all $a, b \in Y$, $a \leqslant b$ implies $a = b$. We denote the set of antichains by $\mathcal{A}(X)$. We define the following orders on $\mathcal{A}(X)$. For all $\alpha, \beta \in \mathcal{A}(X)$,*

- *$\alpha \preccurlyeq \beta$ if, and only if, for all $a \in \alpha$ there exists $b \in \beta$ such that $a \leqslant b$.*
- *$\alpha \preccurlyeq' \beta$ if, and only if, for all $b \in \beta$ there exists $a \in \alpha$ such that $a \leqslant b$.*

Definition 1.2 *Given $Y \subseteq X$, $y \in Y$ is a* maximal element in Y *if for all $z \in Y$, $y \leqslant z$ implies $y = z$. We denote the set of maximal elements in Y by $\overline{Y}$. Similarly $y \in Y$ is a* minimal element in Y *if for all $z \in Y$, $z \leqslant y$ implies $y = z$. We denote the set of minimal elements in Y by $\underline{Y}$.*

Remark 1.1 *For all $\alpha \subseteq X$, $a \in \alpha$,*

$$\overline{\alpha} \subseteq \alpha \tag{1}$$

$$\text{there exists } a' \in \overline{\alpha} \text{ such that } a \leqslant a', \tag{2}$$

$$\overline{\alpha} \in \mathcal{A}(X). \tag{3}$$

The (trivial) proofs of the above remarks follow immediately from Definition 1.2, and are left as an exercise for the interested reader. Analogous remarks hold for $\underline{\alpha}$

Definition 1.3 *Let $\langle X_1, \leqslant_1 \rangle$ and $\langle X_2, \leqslant_2 \rangle$ be two posets. Then $f : X_1 \to X_2$ is*

- *an* order-preserving function *if $x \leqslant_1 y$ implies $f(x) \leqslant_2 f(y)$,*
- *an* order-embedding *if $x \leqslant_1 y$ if, and only if, $f(x) \leqslant_2 f(y)$.*

If f is an order-embedding we will write $f : X_1 \hookrightarrow X_2$.

Definition 1.4 *Let X be a poset. If $f : X \hookrightarrow L$ where L is a complete lattice, then we say that L is a* completion *of X.*

Theorem 1.1 *Two lattices L_1 and L_2 are isomorphic if, and only if, there is a bijection $f : L_1 \to L_2$ such that both f and f^{-1} are order-preserving.*

2 Results

Lemma 2.1 *$\preccurlyeq$ and $\preccurlyeq'$ are partial orders.*

Proof: Clearly $\preccurlyeq$ and $\preccurlyeq'$ are reflexive and transitive.

- $\preccurlyeq$ is anti-symmetric. We proceed by contradiction. Suppose $\alpha, \beta \in \mathcal{A}(X)$ and $\alpha \preccurlyeq \beta$, $\beta \preccurlyeq \alpha$, but $\alpha \neq \beta$. Without loss of generality we can choose $a \in \alpha$ such that $a \notin \beta$. Since $\alpha \preccurlyeq \beta$, there exists $b \in \beta$ such that $a < b$. Furthermore, $b \notin \alpha$ since $\alpha \in \mathcal{A}(X)$ and hence contains no chain. Therefore, there exists $z \in \alpha$ such that $b < z$ since $\beta \preccurlyeq \alpha$. Therefore, we have $a < b < z$ with $a, z \in \alpha$, but, since $\alpha \in \mathcal{A}(X)$ we have a contradiction.

- $\preccurlyeq'$ is anti-symmetric. Suppose $\alpha, \beta \in \mathcal{A}(X)$ and $\alpha \preccurlyeq' \beta$, $\beta \preccurlyeq' \alpha$, but $\alpha \neq \beta$. Without loss of generality we can choose $a \in \alpha$ such that $a \notin \beta$. Since $\beta \preccurlyeq' \alpha$, there exists $b \in \beta$ such that $b < a$. Furthermore, $b \notin \alpha$ since $\alpha \in \mathcal{A}(X)$ and hence contains no chain. Therefore, there exists $z \in \alpha$ such that $b < z$ since $\alpha \preccurlyeq' \beta$. Therefore, we have $z < b < a$ with $a, z \in \alpha$, but, since $\alpha \in \mathcal{A}(X)$ we have a contradiction.

$\blacksquare$

Lemma 2.2 *Let $f : \mathcal{P}(\rightarrow)\mathcal{A}(X)$, $g : \mathcal{P}(\rightarrow)\mathcal{I}(X)$, $f' : \mathcal{P}(\rightarrow)\mathcal{A}(X)$ and $g' : \mathcal{P}(\rightarrow)\mathcal{F}(X)$ be defined as follows.*

$$f(\alpha) = \overline{\alpha} \quad \textit{and} \quad g(\alpha) = \downarrow\alpha$$
$$f'(\alpha) = \underline{\alpha} \quad \textit{and} \quad g'(\alpha) = \uparrow\alpha$$

Then f, g, f' and g' are well-defined functions.

Proof: We will only prove that f' and g' are well-defined, the proof that f and g are well-defined can be found in [1].

- f' is well-defined. The proof proceeds by contradiction.)(Note that is equivalent to proving that for all $Y \subseteq X$, $\underline{Y}$ is unique.) Suppose that $f'(\alpha) = \beta_1$, $f'(\alpha) = \beta_2$ with $\beta_1 \neq \beta_2$. Then without loss of generality we can choose $b \in \beta_1 \setminus \beta_2$, and since $\beta_1 \subseteq \alpha$, $b \in \alpha$. Therefore, by (2') there exists $b_2 \in \underline{\alpha}$ such that $b_2 \leqslant b_1$. Now, by (3'), $\beta_1 \in \mathcal{A}(X)$ and hence $b_2 \notin \beta_1$ (otherwise there is a chain $\{b_1, b_2\} \subseteq \beta_1$). Therefore, there exists $b_3 \in \beta_1$ such that $b_3 < b_2$ (since $b_2 \in \alpha$. Therefore, we have $b_3 < b_2 < b_1$ with $b_1, b_3 \in \beta_1$ which is a contradiction.

- The proof also proceeds by contradiction. (Note that it is equivalent to proving that for any $Y \subseteq X$, $\uparrow Y$ is unique.) Suppose $g(\alpha) = \beta_1$, $g(\alpha) = \beta_2$, and $\beta_1 \neq \beta_2$. Then without loss of generality we can choose $b \in \beta_1 \setminus \beta_2$. Now $\alpha \subseteq \beta_1$, $\alpha \subseteq \beta_2$, and hence $b_1 \notin \alpha$. Therefore, by definition β_2, there exists $a \in \alpha$ such that $a < b_1$. Now we have $a \in \beta_2$ and $b_1 \notin \beta_2$. In other words, β_2 is not an order filter.

$\blacksquare$

Lemma 2.3 *For all $\alpha \in \mathcal{A}(X)$, $\beta \in \mathcal{I}(X)$, $\gamma \in \mathcal{F}(X)$,*

$$\overline{\downarrow\alpha} = \alpha \quad \textit{and} \quad \downarrow\overline{\beta} = \beta; \qquad \underline{\uparrow\alpha} = \alpha \quad \textit{and} \quad \uparrow\underline{\gamma} = \gamma.$$

Proof: It follows immediately from the definitions of $\overline{\alpha}$, $\downarrow\alpha$, $\underline{\alpha}$ and $\uparrow\alpha$. $\blacksquare$

Theorem 2.1 *$\langle \mathcal{A}(X), \preccurlyeq \rangle$ and $\langle \mathcal{A}(X), \preccurlyeq' \rangle$ are isomorphic to $\langle \mathcal{I}(X), \subseteq \rangle$.*

Proof: By Lemma 2.2 $\phi : \mathcal{I}(X) \rightarrow \mathcal{A}(X)$ and $\psi : \mathcal{A}(X) \rightarrow \mathcal{I}(X)$ where $\phi(\alpha) = \overline{\alpha}$ and $\psi(\alpha) = \downarrow\alpha$ are well-defined. Similarly the $\phi' : \mathcal{F}(X) \rightarrow \mathcal{A}(X)$ and $\psi' : \mathcal{A}(X) \rightarrow \mathcal{F}(X)$ where $\phi'(\alpha) = \underline{\alpha}$ and $\psi'(\alpha) = \uparrow\alpha$ are well-defined.

- ϕ is order-preserving - that is, for $\alpha, \beta \in \mathcal{I}(X)$,

$$\alpha \subseteq \beta \text{ implies } \overline{\alpha} \preccurlyeq \overline{\beta} \tag{4}$$

  Suppose $\alpha \subseteq \beta$. Then $\overline{\alpha} \subseteq \alpha \subseteq \beta$. Hence, if $a \in \overline{\alpha}$ then $a \in \beta$. Therefore, by (2), there exists $b \in \overline{\beta}$ such that $a \leqslant b$. That is $\overline{\alpha} \preccurlyeq \overline{\beta}$.

- $\psi = \phi^{-1}$ is order-preserving - that is, for $\alpha, \beta \in \mathcal{A}(X)$,

$$\alpha \preccurlyeq \beta \text{ implies } \downarrow\alpha \subseteq \downarrow\beta \tag{5}$$

  Suppose $\alpha \preccurlyeq \beta$ and $a \in \downarrow\alpha$. Then there exists $a' \in \alpha$ such that $a \leqslant a'$. Since $\alpha \preccurlyeq \beta$ there exists $b \in \beta$ such that $a \leqslant a' \leqslant b$. Hence $a \in \downarrow\beta$. That is $\downarrow\alpha \subseteq \downarrow\beta$.

- ϕ' is order-preserving - that is, for $\alpha, \beta \in \mathcal{F}(X)$,

$$\alpha \supseteq \beta \text{ implies } \underline{\alpha} \preccurlyeq' \underline{\beta} \tag{6}$$

  Suppose $\alpha \supseteq \beta$. Then $\underline{\beta} \supseteq \beta \supseteq \alpha$. Hence, if $b \in \underline{\beta}$ then $b \in \alpha$. Therefore, there exists $a \in \underline{\alpha}$ such that $a \leqslant b$. That is $\overline{\alpha} \preccurlyeq' \overline{\beta}$.

- $\psi' = \phi'^{-1}$ is order-preserving - that is, for $\alpha, \beta \in \mathcal{A}(X)$,

$$\alpha \preccurlyeq' \beta \text{ implies } \uparrow\alpha \supseteq \uparrow\beta \tag{7}$$

  Suppose $\alpha \preccurlyeq' \beta$ and $b \in \uparrow\beta$. Then there exists $b' \in \alpha$ such that $b' \leqslant b$. Since $\alpha \preccurlyeq' \beta$ there exists $a \in \alpha$ such that $a \leqslant b'$. Hence we have $a' \leqslant b$. Hence $b \in \uparrow\alpha$. That is $\uparrow\alpha \supseteq \uparrow\beta$.

We now have, by Theorem 1.1, $\langle \mathcal{A}(X), \preccurlyeq \rangle \equiv \langle \mathcal{I}(X), \subseteq \rangle$ and $\langle \mathcal{A}(X), \preccurlyeq' \rangle \equiv \langle \mathcal{F}(X), \supseteq \rangle$. Hence, since $\langle \mathcal{I}(X), \subseteq \rangle \equiv \langle \mathcal{F}(X), \supseteq \rangle$ via the mapping $\alpha \mapsto X \setminus \alpha$, we have $\langle \mathcal{A}(X), \preccurlyeq' \rangle \equiv \langle \mathcal{I}(X), \subseteq \rangle$ via the mapping $\alpha \mapsto X \setminus \uparrow\alpha$. ∎

We summarise the relationships between the four lattices in the diagram below.

$$\begin{tikzcd}
\langle \mathcal{I}(X), \subseteq \rangle \arrow[r, "\alpha \mapsto X \setminus \alpha"] \arrow[d, "\alpha \mapsto \overline{\alpha}"'] & \langle \mathcal{F}(X), \supseteq \rangle \arrow[d, "\alpha \mapsto \underline{\alpha}"] \\
\langle \mathcal{A}(X), \preccurlyeq \rangle & \langle \mathcal{A}(X), \preccurlyeq' \rangle \arrow[l, "\alpha \mapsto \overline{X \setminus \uparrow\alpha}"]
\end{tikzcd}$$

Lemma 2.4 *For all* $\alpha, \beta \in \mathcal{A}(X)$

$$\alpha \wedge \beta = \overline{\downarrow\alpha \cap \downarrow\beta} \quad \textit{and} \quad \alpha \vee \beta = \overline{\alpha \cup \beta};$$
$$\alpha \wedge' \beta = \underline{\alpha \cup \beta} \quad \textit{and} \quad \alpha \vee' \beta = \underline{\uparrow\alpha \cap \uparrow\beta}.$$

Proof: We will prove the result for $\wedge'$ and $\vee'$. The proof for $\wedge$ and $\vee$ can be found in [1].

- $\underline{\alpha \cup \beta}$ is a lower bound of α and β. Suppose $x \in \alpha$. Then $x \in \alpha \cup \beta$, and hence there exists $y \in \underline{\alpha \cup \beta}$ such that $y \leqslant x$. There $\alpha \cup \beta \preccurlyeq' \alpha$. Similarly $\alpha \cup \beta \preccurlyeq' \beta$.

- $\underline{\uparrow\alpha \cup \uparrow\beta}$ is an upper bound of α and β. Suppose $x \in \underline{\uparrow\alpha \cap \uparrow\beta}$. Then $x \in \uparrow\alpha \cap \uparrow\beta$ and hence $x \in \uparrow\alpha$ and $x \in \uparrow\beta$. Therefore, there exists $x' \in \alpha$ and $x'' \in \beta$ such that $x' \leqslant x$ and $x'' \leqslant x$. Therefore, $\alpha \preccurlyeq' \underline{\uparrow\alpha \cap \uparrow\beta}$ and $\alpha \preccurlyeq' \underline{\uparrow\alpha \cap \uparrow\beta}$.

- $\underline{\alpha \cap \beta}$ is the greatest lower bound of α and β. Suppose $\gamma \in \mathcal{A}(X)$ and $\gamma \preccurlyeq' \alpha$, $\gamma \preccurlyeq' \beta$. Now $\uparrow\gamma \supseteq \alpha$ and $\uparrow\gamma \supseteq \beta$. Therefore, $\alpha \cup \beta \subseteq \uparrow\gamma$. Hence $\underline{\alpha \cup \beta} \subseteq \uparrow\gamma = \gamma$.

- $\underline{\uparrow\alpha \cup \uparrow\beta}$ is the least upper bound of α and β. Suppose $\gamma \in \mathcal{A}(X)$ and $\alpha \preccurlyeq' \gamma$, $\beta \preccurlyeq' \gamma$. Then $\uparrow\alpha \supseteq \gamma$ and $\uparrow\beta \supseteq \gamma$. Hence $\gamma \subseteq \uparrow\alpha \cap \uparrow\beta$ and $\underline{\uparrow\alpha \cap \uparrow\beta} \preccurlyeq' \underline{\gamma} = \gamma$.

∎

References

[1] J. Crampton and G. Loizou. Embedding a poset in a lattice. Technical Report BBKCS-0001, Birkbeck College, University of London, May 2000.