Axiomatisation of Functional Dependencies in Incomplete Relations

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Abstract

Incomplete relations are relations which contain null values, whose meaning is "value is at present unknown". Such relations give rise to two types of functional dependency (FD). The first type, called the strong FD (SFD), is satisfied in an incomplete relation if for all possible worlds of this relation the FD is satisfied in the standard way. The second type, called the weak FD (WFD), is satisfied in an incomplete relation if there exists a possible world of this relation in which the FD is satisfied in the standard way. We exhibit a sound and complete axiom system for both strong and weak FDs, which takes into account the interaction between SFDs and WFDs. A interesting feature of the combined axiom system is that it is not k-ary for any natural number $k \geq 0$. We show that the combined implication problem for SFDs and WFDs can be solved in time polynomial in the size of the input set of FDs. Finally, we show that Armstrong relations exist for SFDs and WFDs.

Keywords: incomplete relations; functional dependencies; axiom system; Armstrong relations

1 Introduction

It is very rare in practice that all the information in a database is complete. This has lead researchers to extend the relational database model [CODD79, LIEN82, ATZE86, IMIE89] so as to handle incomplete information. Codd's original proposal [CODD79] suggested the addition to the database domains of an unmarked null value, whose meaning is "value at present unknown", which we denote by *unk*. We call such relations, whose tuples may contain the null value *unk*, *incomplete relations*. Following Codd's proposal, incomplete information is represented in SQL by using *unk* as a distinguished null value [DATE93].

Functional Dependencies (or simply FDs) are by far the most common integrity constraint in the real world [ULLM88, ATZE93] and the notion of key (derived from a given set of FDs) [CODD79] is fundamental to the relational model. A sound and complete axiom system for FDs holding in (complete) relations was first given in [ARMS74] (see also [ULLM88, ATZE93]) and is known as *Armstrong's axiom system*. When considering the satisfaction of FDs in incomplete relations the transitivity rule is no longer sound. Thus, Lien [LIEN82] and Atzeni and Morfuni [ATZE86] extended FDs so as to deal with missing information, suggesting the interpretation of "inapplicable" or "nonexistent" and "no information", respectively, of the unmarked null value. A sound and complete axiom system for FDs holding in incomplete relations was obtained by dropping the transitivity rule and adding the union and decomposition rules [LIEN82, ATZE86]; we shall refer to this axiom system as *Lien and Atzeni's axiom system*.

The semantics of an incomplete relation, r, over a relation schema R, are defined using the possible worlds approach. The set of *possible worlds* relative to r, denoted by POSS(r), is the set of all relations that emanate from all possible substitutions of occurrences of *unk* in r by nonnull values in the database domains. We distinguish between two types of FD. A *strong functional dependency* over R (or simply a SFD) is a statement of the form $\Box(X \rightarrow Y)$ and a *weak functional dependency* over R (or simply a WFD) is a statement of the form $\Diamond(X \rightarrow Y)$, where $\Box$ and $\Diamond$ can be viewed as the modal operators necessarily and possibly, respectively. Satisfaction of SFDs and WFDs is defined as follows. An incomplete relation r over R satisfies the SFD $\Box(X \rightarrow Y)$ if and only if all possible worlds, $s \in POSS(r)$, satisfy the FD, $X \rightarrow Y$, on using the standard definition of an FD holding in the complete relation, s [ULLM88, ATZE93]. On the other hand, r over R satisfies the WFD $\Diamond(X \rightarrow Y)$ if and only if there exists a possible world, $s \in POSS(r)$, that satisfies the FD, $X \rightarrow Y$, on using the standard definition of an FD holding in the complete relation s.

Both strong and weak FDs arise in the real world. For example, assume an incomplete relation, r, recording information about students, the departments they study in and the heads of these departments. Furthermore, assume a SFD asserting that a student belongs to one department and a WFD asserting that a department has one head. The incomplete relation, say r, shown in Table 1, can be seen to satisfy the SFD $\Box(STUD \rightarrow DEPT)$ and the WFD $\Diamond(DEPT \rightarrow HEAD)$.

STUD	DEPT	HEAD
Iris	Computing	*unk*
unk	Computing	*unk*

Table 1: The student relation

The SFD implies that whenever one or more occurrences of *unk* are updated (or replaced) by nonnull values we need *not* recheck the satisfaction of the SFD, since the fact that a student belongs to one department holds in all possible worlds. On the other hand, the WFD implies that whenever one or more occurrences of *unk* are updated by nonnull values we *do* need to recheck the satisfaction of the WFD, since in the resulting possible world a department may have more than one head in which case the FD is violated, giving rise to inconsistency. Thus the distinction between SFDs and WFDs has an effect on the efficiency of constraint checking after updates have been made to the database.

Armstrong's axiom system is sound and complete for SFDs and Lien and Atzeni's axiom system is sound and complete for WFDs. The problem we investigate is what happens if we allow both SFDs and WFDs to coexist as constraints over a relation schema. We exhibit a sound and complete axiom system for both strong and weak FDs which takes into account the interaction between SFDs and WFDs. An *interesting feature* of the combined axiom system is that it is not k-ary for any natural number $k \geq 0$ [CASA84], that is, we cannot in general put a bound on the number of WFDs and SFDs used in order to derive a new SFD or WFD. We also show that the combined implication problem for SFDs and WFDs can be solved in time polynomial in the size of the input set of FDs.

Armstrong relations [FAGI82, BEER84] are relations, which satisfy all and only those FDs which are logically implied by a given set of FDs. The existence of Armstrong relations for FDs holding in complete relations was shown in [ARMS74]. Such relations have been shown to be important in the process of database design [MANN86]. Herein, we show the interesting result that Armstrong relations exist for SFDs and WFDs also.

The layout of the rest of the paper is as follows. In Section 2 we formalise incomplete relations and define a partial order in the set of tuples of such relations. In Section 3 we define the notion of FDs and their satisfaction in the context of incomplete relations. In Section 4 we extend the *chase procedure*, defined over complete relations [MAIE79, HONE82, ATZE93], to incomplete relations. In order to establish our results we make extensive use of the chase procedure as a theorem proving tool. In Section 5 we present a sound and complete axiom system for FDs in the context of incomplete relations. In Section 6 we construct an Armstrong relation for such FDs. Finally, in Section 7 we give our concluding remarks.

2 Relations that model incomplete information

In this section we extend relation schemas and relations so as to model incomplete information.

We use the notation |S| to denote the cardinality of a set S. If S is a subset of T we write S $\subseteq$ T and if S is a proper subset of T we write S $\subset$ T. At times we denote the singleton $\{$A$\}$ simply by A, and the union of two sets S, T, i. e. S $\cup$ T, simply by ST. We make use of the two index sets, I $= \{1, 2, \ldots, n\}$ and J $= \{1, 2, \ldots, m\}$. We will refer to the cardinality of some standard encoding [GARE79] of S as the *size* of S.

As usual upper case letters from the end of the alphabet will be used to denote sets of attributes while letters from the beginning of the alphabet will be used to denote single attributes.

Definition 2.1 (Relation schema and relation) A *relation schema* R is a finite set of attributes which we denote by schema(R); we denote the cardinality of R by type(R).

We assume a countably infinite domain of constants, *Dom*, containing two distinguished constants *unk* and *inc*, denoting the null values "unknown" and "inconsistent", respectively.

A type(R)-tuple (or simply a tuple whenever type(R) is understood from context) is a total mapping from schema(R) into *Dom* such that $\forall A_i \in$ schema(R), t$(A_i) \in$ *Dom*. A relation over R is a finite set of type(R)-tuples.

From now on we *let R be a relation schema and r be a relation over R.* In addition, we let REL(R) denote the countably infinite set of relations over R.

We note that we have actually included two types of null value in our formalism: *unk* and *inc*. The inclusion of *unk* was motivated in the introduction and the inclusion of *inc* is motivated by the fact that it allows us to easily detect unwanted inconsistency.

Definition 2.2 (Projection) The *projection* of a type(R)-tuple t onto a set of attributes Y $\subseteq$ schema(R), denoted by t[Y], is the restriction of t to Y. The projection of a relation r over R onto Y, denoted by π_Y(r), is defined by π_Y(r) $= \{$t[Y] $\mid$ t $\in$ r$\}$.

Definition 2.3 (Complete and consistent relations and databases) A type(R)-tuple is said to be *complete* if $\forall A_i \in$ schema(R), t$[A_i] \neq$ *unk* and t$[A_i] \neq$ *inc*, i. e. t$[A_i]$ is a nonnull value, otherwise t is said to be *incomplete*; t is said to be *inconsistent* if $\exists A_i \in$ schema(R), such that t$[A_i] =$ *inc*, otherwise t is said to be *consistent*.

A relation r over R is said to be *complete* if $\forall t \in r$, t is complete, otherwise r is said to be *incomplete* (when no confusion arises we use relation to mean incomplete relation). A relation r over R is said to be *inconsistent* if $\exists t \in r$ such that t is inconsistent, otherwise r is said to be *consistent*.

Definition 2.4 (Less informative constants and tuples) Let r be a relation over R. We define a partial order in *Dom*, denoted by $\sqsubseteq$, as follows:

$$u \sqsubseteq v \text{ if and only if } u = v \text{ or } u = unk \text{ or } v = inc, \text{ where } u, v \in Dom.$$

We extend $\sqsubseteq$ to be a partial order in the set of type(R)-tuples as follows: where t_1 and t_2 are type(R)-tuples, t_1 is *less informative than* t_2 (or equivalently t_2 is *more informative than* t_1), written $t_1 \sqsubseteq t_2$, if $\forall A_i \in$ schema(R), $t_1[A_i] \sqsubseteq t_2[A_i]$.

We observe that the set of all type(R)-tuples is a *complete lattice* having no infinite chains [DAVE90], whose bottom element is <*unk*, ..., *unk*> and whose top element is <*inc*, ..., *inc*>.

We next define the *join* operator [DAVE90] of this complete lattice of tuples.

Definition 2.5 (The join operator) The *join* operator, denoted by $\sqcup$, is a mapping from an ordered pair (v_1, v_2) in $Dom \times Dom$ to a single value in Dom as follows: $v_1 \sqcup v_2$ is the least upper bound of v_1 and v_2 with respect to $\sqsubseteq$. We extend $\sqcup$ to be a mapping from an ordered pair, (t_1, t_2), of type(R)-tuples to a single type(R)-tuple as follows: $t_1 \sqcup t_2$ = t, where t is a type(R)-tuple and $\forall A_i \in$ schema(R), t$[A_i] = t_1[A_i] \sqcup t_2[A_i]$.

It can easily be verified that $t_1 \sqcup t_2$ returns the least upper bound of t_1 and t_2, namely the join operator realises the lattice theoretic join.

Definition 2.6 (The set of possible worlds of a relation) The set of all *possible worlds* relative to a relation r over R, denoted by POSS(r), is defined by

$$\begin{aligned} POSS(r) \quad = \quad & \{s \mid s \text{ is a relation over R and there exists a total and onto mapping} \\ & f : r \to s \text{ such that } \forall t \in r, t \sqsubseteq f(t) \text{ and } f(t) \text{ is complete}\}. \end{aligned}$$

Proposition 2.1 A relation r over R is inconsistent if and only if POSS(r) = $\emptyset$. $\Box$

Hereafter we assume that *unless otherwise stated relations are consistent.*

3 Functional dependencies in relations which may be incomplete

We now define Functional Dependencies (FDs) and their satisfaction in the context of incomplete information.

Definition 3.1 (Functional dependency) A *functional dependency* over R (or simply an FD) is a statement of the form X $\to$ Y, where X, Y $\subseteq$ schema(R). We call an FD of the form X $\to$ Y, where Y $\subseteq$ X, a *trivial* FD, and let FD(R) be the set of all sets of FDs over R.

Definition 3.2 (Strong and weak functional dependencies) Let X $\to$ Y be an FD over R. A *strong functional dependency* over R (or simply a SFD) is a statement of the form $\Box$(X $\to$ Y) and a *weak functional dependency* over R (or simply a WFD) is a statement of the form $\Diamond$(X $\to$ Y).

We will refer to an FD which is either a SFD $\Box$(X $\to$ Y) or a WFD $\Diamond$(X $\to$ Y) simply as the FD X $\to$ Y. When referring to an FD X $\to$ Y more than once in a statement and we want *all* occurrences to be either a SFD or a WFD we will write ?(X $\to$ Y) instead of X $\to$ Y.

From now on we will assume that *F is a set of FDs over R.* Furthermore, we will assume that $\Box$(F) is the set of all SFDs in F and $\Diamond$(F) is the set of all WFDs in F.

Definition 3.3 (Satisfaction of an FD) A SFD $\Box$(X $\to$ Y) is satisfied in a relation r over R, denoted by r $\models \Box$(X $\to$ Y), if and only if POSS(r) $\neq \emptyset$ and $\forall s \in$ POSS(r), $\forall t_1, t_2 \in$ s, if t_1[X] = t_2[X], then t_1[Y] = t_2[Y].

A WFD $\Diamond$(X $\to$ Y) is satisfied in a relation r over R, denoted by r $\models \Diamond$(X $\to$ Y), if and only if $\exists s \in$ POSS(r) such that $\forall t_1, t_2 \in$ s, if t_1[X] = t_2[X], then t_1[Y] = t_2[Y].

We note that the definition of satisfaction of an FD in a relation reduces to the standard definition of the satisfaction of an FD when the relation is complete [ULLM88] (in that case there exists exactly one s $\in$ POSS(r) and $\forall$s $\in$ POSS(r) is equivalent to $\exists$s $\in$ POSS(r)). Furthermore, it follows from the above definition that r must be consistent in order to satisfy an FD. The reason for this is that we assume that only consistent relations are stored in the database. A more liberal definition would only insist that π_{XY}(r) be consistent in order for X $\rightarrow$ Y to be satisfied.

We observe that $\Box$ and $\Diamond$ can be viewed as representing the modal operators, necessarily and possibly, respectively, of a normal system of propositional modal logic [CHEL80]. Furthermore, by the above definition we can view POSS(r) as the set of possible worlds relative to r and thus r $\models \Box$(X $\rightarrow$ Y) if and only if for all possible worlds s $\in$ POSS(r), s $\models$ X $\rightarrow$ Y, and r $\models \Diamond$(X $\rightarrow$ Y) if and only if there exists a possible world s $\in$ POSS(r) such that s $\models$ X $\rightarrow$ Y.

The next lemma, which gives a syntactic characterisation of satisfaction of an FD, follows from Definition 3.3 (cf. Lemmas 6.1 and 6.2 in [ATZE93]).

Lemma 3.1 The following two statements, where r is a consistent relation over R and X, Y $\subseteq$ schema(R), are true:

1. r $\models \Box$(X $\rightarrow$ Y) if and only if $\forall t_1, t_2 \in$ r, if t_1[X] $\sqcup$ t_2[X] is consistent, then t_1[Y$-$X] and t_2[Y$-$X] are complete and t_1[Y$-$X] = t_2[Y$-$X].
2. r $\models \Diamond$(X $\rightarrow$ Y) if and only if $\forall t_1, t_2 \in$ r, if t_1[X] and t_2[X] are complete and t_1[X] = t_2[X], then t_1[Y] $\sqcup$ t_2[Y] is consistent. $\Box$

The next lemma, which follows directly from Lemma 3.1, shows some semantic properties of FD satisfaction.

Lemma 3.2 The following statements, where r is a relation over R and X, Y $\subseteq$ schema(R), are true:

1. If Y $\subseteq$ X, then $\forall$r $\in$ REL(R), if r is consistent, then r $\models$ X $\rightarrow$ Y.
2. r $\models$?(X $\rightarrow$ Y) if and only if r $\models$?(X $\rightarrow$ Y$-$X).
3. If r $\models$?(X $\rightarrow$ Y), then r $\models$?(XW $\rightarrow$ YW), where W $\subseteq$ schema(R).
4. r $\models$?(X $\rightarrow$ YZ) if and only if r $\models$?(X $\rightarrow$ Y) and r $\models$?(X $\rightarrow$ Z).
5. If r $\models \Box$(X $\rightarrow$ Y) and r $\models \Box$(Y $\rightarrow$ Z), then r $\models \Box$(X $\rightarrow$ Z).
6. If r $\models \Box$(X $\rightarrow$ Y) and r $\models \Diamond$(Y $\rightarrow$ Z) or if r $\models \Diamond$(X $\rightarrow$ Y) and r $\models \Box$(Y $\rightarrow$ Z), then r $\models \Diamond$(X $\rightarrow$ Z).
7. If r $\models \Box$(X $\rightarrow$ Y), then r $\models \Diamond$(X $\rightarrow$ Y). $\Box$

As a result of Lemma 3.2, without loss of generality, we will assume from now on, where appropriate, that all the FDs in F are nontrivial and of the form X $\rightarrow$ A, where A is a single attribute.

The following lemma shows an interesting situation when the satisfaction of a particular set of FDs implies the satisfaction of an additional SFD. It leads to an unavoidable inference rule of the axiomatisation, which is presented in Section 5, namely strengthening (FD9).

Lemma 3.3 If r $\models \Box$(X $\rightarrow$ Y), r $\models \Diamond$(Y $\rightarrow$ W) and r $\models \Box$(XW $\rightarrow$ Z), with W $\cap$ Z = $\emptyset$, then r $\models \Box$(X $\rightarrow$ Z), provided the following condition is satisfied:

$\exists$ a sequence $\{A_i\}$ $(i \in \mathrm{I})$, with $\cup_{i \in I} A_i = \mathrm{X} \cap \mathrm{Y}$, such that $\forall i \in \mathrm{I}$,
$\exists V_i \subseteq$ schema(R) such that $T_i = (\mathrm{Y}-\mathrm{X}) \cup \{A_j \mid A_j \in \mathrm{X} \cap \mathrm{Y}$ and $j < i\}$,
$\mathrm{r} \models \Diamond(T_i \rightarrow V_i)$ and $\exists \mathrm{V} \subseteq \mathrm{XY}V_i$ such that $\mathrm{r} \models \Box(\mathrm{V}-A_i \rightarrow A_i)$.

Proof. Suppose that the if part of the lemma is satisfied but $\mathrm{r} \not\models \Box(\mathrm{X} \rightarrow \mathrm{Z})$. Then by Lemma 3.1 part (1) $\exists t_1, t_2 \in \mathrm{r}$ such that t_1 and t_2 are distinct, $t_1[\mathrm{X}] \sqcup t_2[\mathrm{X}]$ is consistent but either $t_1[\mathrm{Z}-\mathrm{X}]$ is incomplete or $t_2[\mathrm{Z}-\mathrm{X}]$ is incomplete or $t_1[\mathrm{Z}-\mathrm{X}] \neq t_2[\mathrm{Z}-\mathrm{X}]$. By Lemma 3.1 part (1) it follows that $t_1[\mathrm{Y}-\mathrm{X}]$ and $t_2[\mathrm{Y}-\mathrm{X}]$ are complete and that $t_1[\mathrm{Y}-\mathrm{X}] = t_2[\mathrm{Y}-\mathrm{X}]$, since $\mathrm{r} \models \Box(\mathrm{X} \rightarrow \mathrm{Y})$. Thus $t_1[\mathrm{XY}] \sqcup t_2[\mathrm{XY}]$ is consistent. We can now deduce that $t_1[\mathrm{Y}]$ and $t_2[\mathrm{Y}]$ are complete with $t_1[\mathrm{Y}] = t_2[\mathrm{Y}]$ by inspecting the above condition. In addition, by Lemma 3.1 part (2) it follows that $t_1[\mathrm{W}] \sqcup t_2[\mathrm{W}]$ is consistent, since $\mathrm{r} \models \Diamond(\mathrm{Y} \rightarrow \mathrm{W})$. We can also conclude, by Lemma 3.1 part (1), that $t_1[\mathrm{Z}-\mathrm{X}]$ and $t_2[\mathrm{Z}-\mathrm{X}]$ are complete with $t_1[\mathrm{Z}-\mathrm{X}] = t_2[\mathrm{Z}-\mathrm{X}]$, since $\mathrm{r} \models \Box(\mathrm{XW} \rightarrow \mathrm{Z})$ and $\mathrm{W} \cap \mathrm{Z} = \emptyset$. The result now follows, since we have derived a contradiction of our assumption that $\mathrm{r} \not\models \Box(\mathrm{X} \rightarrow \mathrm{Z})$. $\Box$

The following lemma shows another interesting situation when the satisfaction of two WFDs and one SFD implies the satisfaction of an additional WFD.

Lemma 3.4 If $\mathrm{r} \models \Diamond(\mathrm{X} \rightarrow \mathrm{Y})$, $\mathrm{r} \models \Box(\mathrm{XY} \rightarrow \mathrm{W})$ and $\mathrm{r} \models \Diamond(\mathrm{XW} \rightarrow \mathrm{Z})$, with $\mathrm{W} \cap \mathrm{Y} = \emptyset$, then $\mathrm{r} \models \Diamond(\mathrm{X} \rightarrow \mathrm{Z})$.

We now generalise the definition of satisfaction of a single FD to the satisfaction of a set of FDs.

Definition 3.4 (Satisfaction of a set of FDs) A set of FDs F over R is satisfied in a relation r over R, denoted by $\mathrm{r} \models \mathrm{F}$, if and only if $\forall\ ?(\mathrm{X} \rightarrow \mathrm{Y}) \in \mathrm{F}$, $\mathrm{r} \models\ ?(\mathrm{X} \rightarrow \mathrm{Y})$.

4 The Chase Procedure for Functional Dependencies in Incomplete Relations

We extend the standard chase procedure [MAIE79, ATZE93] to incomplete relations as a tool to prove the completeness of the axiom system presented in Section 5.

Let $\{0, 1\}$ be two distinct nonnull values in *Dom* and let $\mathrm{X} \subseteq$ schema(R). Furthermore, let u_1, u_2 and u_3 be three tuples over R such that $\forall \mathrm{A} \in \mathrm{X}$, $u_1[\mathrm{A}] = unk$ and $\forall \mathrm{A} \in$ schema(R) $-$ X, $u_1[\mathrm{A}] = 1$, $\forall \mathrm{A} \in \mathrm{X}$, $u_2[\mathrm{A}] = 0$ and $\forall \mathrm{A} \in$ schema(R) $-$ X, $u_2[\mathrm{A}] = 1$, and finally $\forall \mathrm{A} \in$ schema(R), $u_3[\mathrm{A}] = 0$. The tuples u_1, u_2 and u_3 are shown in Table 2. Furthermore, we let $T_1(\mathrm{X}) = \{u_1, u_3\}$ and $T_2(\mathrm{X}) = \{u_2, u_3\}$ be relations over R.

tuple	X	schema(R)$-$X
u_1	$unk \ldots unk$	$1 \ldots 1$
u_2	$0 \ldots 0$	$1 \ldots 1$
u_3	$0 \ldots 0$	$0 \ldots 0$

Table 2: The tuples u_1, u_2 and u_3

We now define two *chase rules* (or simply rules), which are applied to a relation $\{t_1, t_2\}$ over R.

SFD rule: if $\Box(\mathrm{W} \rightarrow \mathrm{B}) \in \mathrm{F}$ and $t_1[\mathrm{W}] \sqcup t_2[\mathrm{W}]$ is consistent, then $t_1[\mathrm{B}], t_2[\mathrm{B}] := 0$.

WFD rule: if $\Diamond$(W $\rightarrow$ B) $\in$ F and t_1[W] = t_2[W] but $t_1[B] \sqcup t_2[B]$ is inconsistent, then t_1[B], t_2[B] := *unk*.

We now give the pseudo-code of an algorithm designated, CHASE(i, F), which, given $i \in \{1, 2\}$ and a set of FDs F over R, applies the SFD and WFD rules to the tuples of T_i(X) as long as possible and returns the resulting relation over R. CHASE(i, F) will also denote the relation resulting from the ensuing Algorithm 1.

Algorithm 1 (CHASE(i, F))

```
1. begin
2.   Result := T_i(X);
3.   Tmp := ∅;
4.   while Tmp ≠ Result do
5.     Tmp := Result;
6.     Apply the SFD rule or the WFD rule to Result;
7.   end while
8.   return Result;
9. end.
```

Results analogous to Lemmas 4.1, 4.2 and 4.3 for FDs without nulls can be found in [MAIE79, ATZE93].

Lemma 4.1 Algorithm 1 terminates.

Proof. Algorithm 1 terminates, since no new values are generated during the execution of line 6 of Algorithm 1 implying that the algorithm must terminate after executing the while loop beginning at line 4 a finite number of times. $\square$

The next lemma is immediate from the definitions of the SFD and WFD rules.

Lemma 4.2 The relation CHASE(i, F) is consistent, where $i \in \{1, 2\}$. $\square$

Lemma 4.3 The relation CHASE(i, F) is unique, where $i \in \{1, 2\}$.

Proof. Let $\hat{r_i} = \{\hat{u_i}, \hat{u_3}\}$ be intermediate states of CHASE(i, F) = $\{t_i, t_3\}$, where $i \in \{1, 2\}$, during the execution of Algorithm 1. We call an execution of line 6 in Algorithm 1 a *chase step*, and we say that the chase step *applies* the SFD rule or the WFD rule to the relation $\hat{r_i}$ for the FD ?(W $\rightarrow$ B). We need two technical definitions before we can proceed.

- An attribute B $\in$ schema(R) is *S-equatable* if for some FD $\Box$(W $\rightarrow$ B) $\in$ F, $\hat{u_i}$[W] $\sqcup$ $\hat{u_3}$[W] is consistent.
- An attribute B $\in$ schema(R) is *S-equatable* if for some FD $\Box$(W $\rightarrow$ B) $\in$ F, $\forall$A $\in$ W, A is either S-equatable or W-equatable.
- An attribute B $\in$ schema(R) is *W-equatable* if for some FD $\Diamond$(W $\rightarrow$ B) $\in$ F, $\hat{u_i}$[W] = $\hat{u_3}$[W].
- An attribute B $\in$ schema(R) is *W-equatable* if for some FD $\Diamond$(W $\rightarrow$ B) $\in$ F, $\forall$A $\in$ W, A is S-equatable.

It follows from the above definitions that both S-equatability and W-equatability are preserved under chase steps disregarding the order in which the chase steps are applied to $\hat{r_i}$. Thus if B is S-equatable then for some state $\hat{r_i}$ of CHASE(i, F) the SFD rule is applied to $\hat{r_i}$ for $\Box$(W $\rightarrow$ B), resulting in t_i[B] = 0. In addition, if B is both W-equatable and S-equatable then it is possible that at some state of $\hat{r_i}$, prior to the said state, the WFD rule is applied for $\Diamond$(W $\rightarrow$ B), resulting in $\hat{u}_i$[B] = *unk*. In this case the application of this WFD rule is redundant. Moreover, if B is W-equatable but not S-equatable then for some state of $\hat{r_i}$ the WFD rule is applied for $\Diamond$(W $\rightarrow$ B) resulting in t_i[B] = *unk*.

The result follows, since for an attribute B $\in$ schema(R), t_i[B] = 0 if and only if B is S-equatable and t_i[B] = *unk* if and only if B is W-equatable but B is not S-equatable. $\Box$

The next theorem is immediate from Algorithm 1 on using Lemma 3.1.

Theorem 4.4 $\forall$?(W $\rightarrow$ B) $\in$ F, CHASE(i, F) $\models$?(W $\rightarrow$ B), where $i \in \{1, 2\}$. $\Box$

The next lemma can be proved by inspecting Algorithm 1 and the definitions of the SFD and WFD rules, since line 6 can be executed at most 2(type(R)) times, i. e. at most 2(type(R)) chase steps can be applied to T_i(X), where F is a set of FDs over R and $i \in \{1, 2\}$ (cf. [AHO79, HONE82]).

Lemma 4.5 CHASE(i, F) can be computed in time polynomial in the size of F, where $i \in \{1, 2\}$. $\Box$

5 An Axiom System for Functional Dependencies in Incomplete Relations

Herein we exhibit a sound and complete axiom system for FDs, which may be either weak or strong.

Definition 5.1 (Logical implication and inference for FDs) A set of FDs F over R logically implies an FD ?(X $\rightarrow$ Y) over R, denoted by F $\models$?(X $\rightarrow$ Y), if and only if $\forall$r $\in$ REL(R), if r $\models$ F, then r $\models$?(X $\rightarrow$ Y).

Given an axiom system ?(X $\rightarrow$ Y) is inferred from F, denoted by F $\vdash$?(X $\rightarrow$ Y), if there exists a proof of ?(X $\rightarrow$ Y) from F using the given axiom system.

Definition 5.2 (Sound and complete axiom system) Let F be a set of FDs and ?(X $\rightarrow$ Y) be a single FD both over R. An axiom system is *sound* if F $\vdash$?(X $\rightarrow$ Y) implies F $\models$?(X $\rightarrow$ Y). An axiom system is *complete* if F $\models$?(X $\rightarrow$ Y) implies F $\vdash$?(X $\rightarrow$ Y).

We now present our axiom system for FDs, where F is a set of FDs over R. The inference rules FD1, FD2 and FD5 constitute Armstrong's axiom system. The inference rules FD1, FD2, FD3 and FD4 constitute Lien and Atzeni's axiom system. The inference rules FD6, FD7 and FD8 are mixed rules that allow us to derive WFDs and finally the inference rule FD9 is a mixed rule that allows us to derive SFDs.

FD1 Reflexivity: if Y $\subseteq$ X $\subseteq$ schema(R), then F $\vdash$ X $\rightarrow$ Y.

FD2 Augmentation: if F $\vdash$?(X $\rightarrow$ Y) and W $\subseteq$ schema(R), then F $\vdash$?(XW $\rightarrow$ YW).

FD3 Union: if F $\vdash$?(X $\rightarrow$ Y) and F $\vdash$?(X $\rightarrow$ Z), then F $\vdash$?(X $\rightarrow$ YZ)

FD4 Decomposition: if F $\vdash$?(X $\rightarrow$ YZ), then F $\vdash$?(X $\rightarrow$ Y) and F $\vdash$?(X $\rightarrow$ Z).

FD5 Strong transitivity: if $F \vdash \Box(X \rightarrow Y)$ and $F \vdash \Box(Y \rightarrow Z)$, then $F \vdash \Box(X \rightarrow Z)$.

FD6 Weak transitivity: if $F \vdash \Box(X \rightarrow Y)$ and $F \vdash \Diamond(Y \rightarrow Z)$ or if $F \vdash \Diamond(X \rightarrow Y)$ and $F \vdash \Box(Y \rightarrow Z)$, then $F \vdash \Diamond(X \rightarrow Z)$.

FD7 Chaining: if $F \vdash \Diamond(X \rightarrow Y)$, $F \vdash \Box(XY \rightarrow W)$ and $F \vdash \Diamond(XW \rightarrow Z)$, with $W \cap Y = \emptyset$, then $F \vdash \Diamond(X \rightarrow Z)$.

FD8 Weakening: if $F \vdash \Box(X \rightarrow Y)$, then $F \vdash \Diamond(X \rightarrow Y)$.

FD9 Strengthening: if $F \vdash \Box(X \rightarrow Y)$, $F \vdash \Diamond(Y \rightarrow W)$ and $F \vdash \Box(XW \rightarrow Z)$, with $W \cap Z = \emptyset$, then $F \vdash \Box(X \rightarrow Z)$, provided the following condition is satisfied:

$\exists$ a sequence $\{A_i\}$ $(i \in I)$, with $\cup_{i \in I} A_i = X \cap Y$, such that $\forall i \in I$,
$\exists V_i \subseteq$ schema(R) such that $T_i = (Y - X) \cup \{A_j \mid A_j \in X \cap Y \text{ and } j < i\}$,
$F \vdash \Diamond(T_i \rightarrow V_i)$ and $\exists V \subseteq XYV_i$ such that $F \vdash \Box(V - A_i \rightarrow A_i)$.

We observe that when $X \cap Y = \emptyset$, the above condition in the strengthening rule (FD9) is vacuously satisfied. We also note that as a consequence of weakening (FD8) $\Box(X)^+ \subseteq \Diamond(X)^+$ holds (see Definition 5.3). Furthermore, FD9 cannot be derived from FD1 to FD8, since it can be verified that, for example, $\Box(A \rightarrow D)$ cannot be inferred from the set of FDs $F = \{\Box(A \rightarrow B), \Diamond(B \rightarrow C), \Box(AC \rightarrow D)\}$ without using FD9. On the other hand, we observe that $F \vdash \Diamond(A \rightarrow D)$ holds on using the weak transitivity rule together with the augmentation rule. Thus, if $F = \{\Box(X \rightarrow Y), \Diamond(Y \rightarrow W), \Box(XW \rightarrow Z)\}$ then we can infer $F \vdash \Diamond(X \rightarrow Z)$ on using the weak transitivity rule together with the augmentation rule. It follows that by using FD9 we strengthen our inference capability in the sense that we can infer the SFD $\Box(X \rightarrow Z)$, whereas the WFD $\Diamond(X \rightarrow Z)$ can be inferred without the use of FD9.

As a further example of strengthening let $G = \{\Box(ABC \rightarrow D), \Box(ABCG \rightarrow E), \Box(ACF \rightarrow B), \Box(ABFH \rightarrow C), \Diamond(BCD \rightarrow G), \Diamond(D \rightarrow F), \Diamond(DB \rightarrow H)\}$. Then it can be verified that $G \vdash \Box(ABC \rightarrow E)$ on using strengthening with the total ordering $B < C$.

The next lemma follows from Lemmas 3.2, 3.3 and 3.4.

Lemma 5.1 The axiom system comprising inference rules FD1 to FD9 is sound for FDs. $\Box$

Definition 5.3 (Closure of a set of FDs) The closure of a set of FDs F over R with respect to FD1-FD9, denoted by F^+, is given by $F^+ = \{?(X \rightarrow Y) \mid F \vdash ?(X \rightarrow Y)\}$.

The closure of a set of attributes $X \subseteq$ schema(R), with respect to FD1-FD9 and a set of FDs F over R, denoted by $?(X)^+_F$ (or simply $?(X)^+$ if F is understood from context), is given by $?(X)^+ = \cup \{Y \mid \exists\, ?(X \rightarrow Y) \in F^+\}$; as usual *all* occurrences of ? are either $\Diamond$ or $\Box$.

We observe that $\Box(X)^+ \subseteq \Diamond(X)^+$.

Lemma 5.2 Let CHASE(1, F) $= \{t_1, t_3\}$, where t_1 originated from u_1 and t_3 originated from u_3. Then $A \in \Box(X)^+$ if and only if either $A \in X$ or $t_1[A] = 0$.

Proof. If: When $A \in X$ the result is immediate by reflexivity, so suppose that $t_1[A] = 0$. We prove the result by induction on the minimum number, say k, of chase steps, applied in the computation of CHASE(1, F), which are needed to obtain $t_1[A] = 0$.

(*Basis*): If $k = 1$ then $\Box(Y \rightarrow A) \in F$, with $Y \subseteq X$, and thus $A \in \Box(X)^+$ holds by augmentation and decomposition.

(*Induction*): Assume that the result holds when the minimum number of chase steps, applied in the computation of CHASE(1, F), in order to obtain $t_1[\mathrm{A}] = 0$ is k, where $k \geq 1$; we then need to prove that the result holds when the minimum number of chase steps applied is $k + 1$.

It follows that the last chase step is applied using the FD $\Box(\mathrm{W} \rightarrow \mathrm{A}) \in \mathrm{F}$ in order to obtain CHASE(1, F), where $\mathrm{A} \notin \mathrm{X}$. Therefore $t_1[\mathrm{W}] \sqcup t_3[\mathrm{W}]$ is consistent. We assume that $\mathrm{W} \not\subseteq \mathrm{X}$, otherwise the result follows as in the basis step.

Let $Z_1 \subseteq (\mathrm{W} - \mathrm{X})$ be the set of attributes such that $\forall \mathrm{B} \in Z_1$, $t_1[\mathrm{B}] = 0$ and let $Z_2 = (\mathrm{W} - \mathrm{X}) - Z_1$ be the set of all attributes such that $\forall \mathrm{B} \in Z_2$, $t_1[\mathrm{B}] = unk$. By inductive hypothesis $Z_1 \subseteq \Box(\mathrm{X})^+$. Now, if $Z_2 = \emptyset$, then the result follows by strong transitivity with respect to $\Box(\mathrm{X} \rightarrow \mathrm{W})$ and $\Box(\mathrm{W} \rightarrow \mathrm{A})$. So suppose that $Z_2 \neq \emptyset$.

Let $\mathrm{Y} \subseteq \mathrm{schema(R)} - \{\mathrm{A}\}$ be the maximal set of attributes such that $\forall \mathrm{B} \in \mathrm{Y}$, $t_1[\mathrm{B}] = 0$ ($Z_1 \subseteq \mathrm{Y}$ holds). Thus by inductive hypothesis $\mathrm{Y} \subseteq \Box(\mathrm{X})^+$. Furthermore, $Z_2 \subseteq \Diamond(\mathrm{Y})^+$ by the definition of the WFD rule on using augmentation, decomposition and union. It follows that $\mathrm{W} - \mathrm{X} \subseteq \Diamond(\mathrm{Y})^+$ by reflexivity and union. Moreover, $\mathrm{A} \in \Box(\mathrm{XW})^+$, since $\Box(\mathrm{W} \rightarrow \mathrm{A}) \in \mathrm{F}$. Now, let $\mathrm{V} = (\mathrm{Y} \cap \mathrm{X})$.

We invoke a further induction on $|\mathrm{V}|$ to show that the condition specified in the strengthening rule is satisfied. The basis step, when $|\mathrm{V}| = 0$, is vacuously true. So assume that $\mathrm{V} = \{A_1, A_2, \ldots, A_n\}$, with $n \geq 1$. Let $T_n = \mathrm{Y} - A_n$ and let $\mathrm{Z} \subseteq \mathrm{schema(R)}$ be the maximal set of attributes such that $\forall \mathrm{B} \in \mathrm{Z}$, $t_1[\mathrm{B}] = unk$. Then by the definition of the WFD rule on using the if part of Lemma 5.3 it follows that for some set of attributes $V_n \subseteq \mathrm{Z}$, it is true that $V_n = \Diamond(T_n)^+ - T_n A_n$. Furthermore, $A_n \in \Box(\mathrm{XY}V_n - A_n)^+$ by the definition of the SFD rule, since $t_1[A_n] = 0$. The result now follows by strengthening on using the inductive hypothesis of this further induction.

Only if: Suppose that $\mathrm{A} \in \Box(\mathrm{X})^+$ and thus $\mathrm{F} \vdash \Box(\mathrm{X} \rightarrow \mathrm{A})$. We prove the result by induction on the minimum number, say k, of inference rules used in the derivation of $\Box(\mathrm{X} \rightarrow \mathrm{A})$ from F.

(*Basis*): If $k = 0$ then $\Box(\mathrm{X} \rightarrow \mathrm{A}) \in \mathrm{F}$, with $\mathrm{A} \notin \mathrm{X}$, since we have assumed that F does not contain any trivial FDs. The result then follows by a single application of the SFD rule for $\Box(\mathrm{X} \rightarrow \mathrm{A})$ resulting in $t_1[\mathrm{A}] = 0$.

(*Induction*): Assume that the result holds when the minimum number of inference rules used in the derivation of $\Box(\mathrm{X} \rightarrow \mathrm{A})$ from F is k, where $k \geq 0$; we then need to prove that the result holds when the minimum number of inference rules used is $k + 1$.

If the last inference rule used is reflexivity, then $k = 1$ and the result follows since $\mathrm{A} \in \mathrm{X}$. On the other hand, if the last inference rule used is decomposition, then due to our assumption that all the right-hand sides of FDs in F can be singletons, the penultimate inference rule used must be augmentation. Therefore, $\Box(\mathrm{Y} \rightarrow \mathrm{A})$, with $\mathrm{Y} \subset \mathrm{X}$, can be derived using k or less inference rules and the result follows by inductive hypothesis. Now suppose that the last inference rule used is strong transitivity with respect to $\Box(\mathrm{X} \rightarrow \mathrm{Y})$ and $\Box(\mathrm{Y} \rightarrow \mathrm{A})$. Then by inductive hypothesis $\forall \mathrm{B} \in \mathrm{Y}$, $t_1[\mathrm{B}] = 0$. The result that $t_1[\mathrm{A}] = 0$ follows by inductive hypothesis, since $\mathrm{A} \in \Box(\mathrm{Y})^+$ can be derived in k or less inference rules. Finally, suppose that the last inference rule used is strengthening, with Y and W being the sets of attributes specified in the definition of this inference rule. Then by inductive hypothesis it follows that $\forall \mathrm{B} \in \mathrm{Y}$, $t_1[\mathrm{B}] = 0$ and by the WFD rule $\forall \mathrm{B} \in \mathrm{W}$, $t_1[\mathrm{B}] \neq 1$. The result that $t_1[\mathrm{A}] = 0$ follows by the SFD rule since $\mathrm{A} \in \Box(\mathrm{XW})^+$. $\Box$

Lemma 5.3 Let CHASE(2, F) $= \{t_2, t_3\}$, where t_2 originated from u_2 and t_3 originated from u_3. Then $\mathrm{A} \in \Diamond(\mathrm{X})^+$ if and only if $t_2[\mathrm{A}] \neq 1$.

Proof. If: Suppose that $t_2[\mathrm{A}] \neq 1$. We prove the result by induction on the minimum number, say k, of chase steps, applied in the computation of CHASE(2, F), which are needed to obtain $t_2[\mathrm{A}] \neq 1$.

(*Basis*): If $k = 0$ then $\mathrm{A} \in \mathrm{X}$ and $\mathrm{A} \in \Diamond(\mathrm{X})^+$ can be derived using reflexivity.

(*Induction*): Assume that the result holds when the minimum number of chase steps, applied in the computation of CHASE(2, F), in order to obtain $t_2[\mathrm{A}] \neq 1$ is k, where $k \geq 0$; we then need to prove that the result holds when the minimum number of chase steps applied is $k + 1$.

Now suppose that the last chase step is applied using the FD $?(\mathrm{W} \rightarrow \mathrm{A}) \in \mathrm{F}$ in order to obtain CHASE(2, F), where $\mathrm{A} \notin \mathrm{X}$, and thus $t_2[\mathrm{W}] \sqcup t_3[\mathrm{W}]$ is consistent. By inductive hypothesis $\mathrm{W} \subseteq \Diamond(\mathrm{X})^+$ holds. If $\mathrm{W} \subseteq \mathrm{X}$, then the result follows by augmentation, decomposition and weakening when $? = \Box$. So assume that $\mathrm{W} \not\subseteq \mathrm{X}$. Now if $? = \Box$, then the result follows by inductive hypothesis on using the weak transitivity rule. So assume that $? = \Diamond$ and thus $\forall \mathrm{B} \in \mathrm{W}$, $t_2[\mathrm{B}] = t_3[\mathrm{B}] = 0$ and $t_2[\mathrm{A}] = t_3[\mathrm{A}] = unk$. It follows by the SFD rule, strong transitivity, augmentation, decomposition and union that $\mathrm{W} \subseteq \Box(\mathrm{XY})^+$, where $\mathrm{Y} = \Diamond(\mathrm{X})^+ - \mathrm{WA}$. (We illustrate the derivation when W $= \{B_1, B_2\}$, with the knowledge that as a result of the application of at most two SFD rules the equality $t_2[B_1] = t_3[B_1] = t_2[B_2] = t_3[B_2] = 0$ obtains. Without loss of generality suppose that $B_1 \in \Box(\mathrm{XY})^+$ as a result of a SFD $\Box(V_1 \rightarrow B_1) \in \mathrm{F}$, with $V_1 \subseteq \mathrm{XY}$, on using augmentation and decomposition, and $B_2 \in \Box(\mathrm{XY}B_1)^+$ as a result of a SFD $\Box(V_2 \rightarrow B_2) \in \mathrm{F}$, with $V_2 \subseteq \mathrm{XY}B_1$, on using augmentation and decomposition. Moreover, by augmentation and strong transitivity $B_2 \in \Box(\mathrm{XY})^+$ and by union $\mathrm{W} \subseteq \Box(\mathrm{XY})^+$.) Furthermore, $\mathrm{A} \in \Diamond(\mathrm{XW})^+$ by augmentation and decomposition, since $\Diamond(\mathrm{W} \rightarrow \mathrm{A}) \in \mathrm{F}$. The result that $\mathrm{A} \in \Diamond(\mathrm{X})^+$ follows by chaining.

Only if: Suppose that $\mathrm{A} \in \Diamond(\mathrm{X})^+$ and thus $\mathrm{F} \vdash \Diamond(\mathrm{X} \rightarrow \mathrm{A})$. We prove the result by induction on the minimum number, say k, of inference rules used in the derivation of $\Diamond(\mathrm{X} \rightarrow \mathrm{A})$ from F.

(*Basis*): If $k = 0$ then $\Diamond(\mathrm{X} \rightarrow \mathrm{A}) \in \mathrm{F}$, with $\mathrm{A} \notin \mathrm{X}$, since we have assumed that F does not contain any trivial FDs. The result then follows by a single application of the WFD rule for $\Diamond(\mathrm{X} \rightarrow \mathrm{A})$ resulting in $t_2[\mathrm{A}] \neq 1$.

(*Induction*): Assume that the result holds when the minimum number of inference rules used in the derivation of $\Diamond(\mathrm{X} \rightarrow \mathrm{A})$ from F is k, where $k \geq 0$; we then need to prove that the result holds when the minimum number of inference rules used is $k + 1$.

If the last inference rule used is reflexivity, then $k = 1$ and the result follows since $\mathrm{A} \in \mathrm{X}$. On the other hand, if the last inference rule used is decomposition, then due to our assumption that all the right-hand sides of FDs in F can be singletons, the penultimate inference rule used must be augmentation. Therefore, $\Diamond(\mathrm{Y} \rightarrow \mathrm{A})$, with $\mathrm{Y} \subset \mathrm{X}$, can be derived using k or less inference rules and the result follows by inductive hypothesis. Now if the last inference rule is weakening with respect to $\Box(\mathrm{X} \rightarrow \mathrm{A})$, then the result follows by an argument similar to that made in the only if part of Lemma 5.2. So assume that the last inference rule is weak transitivity. There are two cases to consider. Firstly, weak transitivity is used with respect to $\Box(\mathrm{X} \rightarrow \mathrm{Y})$ and $\Diamond(\mathrm{Y} \rightarrow \mathrm{A})$. By an argument similar to that made in the only if part of Lemma 5.2 it follows that $\forall \mathrm{B} \in \mathrm{Y}$, $t_2[\mathrm{B}] = 0$ and the result follows by inductive hypothesis. Secondly, weak transitivity is used with respect to $\Diamond(\mathrm{X} \rightarrow \mathrm{Y})$ and $\Box(\mathrm{Y} \rightarrow \mathrm{A})$. By inductive hypothesis $\forall \mathrm{B} \in \mathrm{Y}$, $t_2[\mathrm{B}] \neq 1$. The result then follows by an argument similar to that made in the only if part of Lemma 5.2. $\Box$

Theorem 5.4 The axiom system comprising inference rules FD1 to FD9 is sound and complete for FDs.

Proof. Let F be a set of FDs over R and let $?(\mathrm{X} \rightarrow \mathrm{Y}) \in \mathrm{F}$ be an FD over R. On using Lemma 3.2 together with the union and decompostion inference rules we can assume, without loss of generality, that $\mathrm{Y} = \{\mathrm{A}\}$, i.e. that $|\mathrm{Y}| = 1$.

Soundness of the axiom system follows from Lemma 5.1. It remains to show that the axiom system is complete, i. e. that $\mathrm{F} \models ?(\mathrm{X} \rightarrow \mathrm{A})$ implies that $\mathrm{F} \vdash ?(\mathrm{X} \rightarrow \mathrm{A})$. Thus we need to show that $\mathrm{F} \not\vdash ?(\mathrm{X} \rightarrow \mathrm{A})$ implies that $\mathrm{F} \not\models ?(\mathrm{X} \rightarrow \mathrm{A})$. Equivalently, it is sufficient to exhibit a relation r over R such that $\forall\ ?(\mathrm{W} \rightarrow \mathrm{Z}) \in \mathrm{F}$, $\mathrm{r} \models ?(\mathrm{W} \rightarrow \mathrm{Z})$ but $\mathrm{r} \not\models ?(\mathrm{X} \rightarrow \mathrm{A})$. We consider two cases corresponding to $\Box(\mathrm{X} \rightarrow \mathrm{A})$ and $\Diamond(\mathrm{X} \rightarrow \mathrm{A})$.

Firstly, let $\mathrm{r} = \mathrm{CHASE}(1, \mathrm{F}) = \{t_1, t_3\}$. Then, by Theorem 4.4, $\forall\ ?(\mathrm{W} \rightarrow \mathrm{Z}) \in \mathrm{F}$, $\mathrm{r} \models ?(\mathrm{W} \rightarrow \mathrm{Z})$. Furthermore, $\mathrm{A} \notin \Box(\mathrm{X})^+$, since $\mathrm{F} \not\vdash \Box(\mathrm{X} \rightarrow \mathrm{A})$. Therefore, by Lemma 5.2, $t_1[\mathrm{A}] \neq 0$ implying that $\mathrm{r} \not\models \Box(\mathrm{X} \rightarrow \mathrm{A})$, since $\mathrm{A} \notin \mathrm{X}$.

Secondly, let $\mathrm{r} = \mathrm{CHASE}(2, \mathrm{F}) = \{t_2, t_3\}$. Then, again by Theorem 4.4, $\forall\ ?(\mathrm{W} \rightarrow \mathrm{Z}) \in \mathrm{F}$, $\mathrm{r} \models ?(\mathrm{W} \rightarrow \mathrm{Z})$. Furthermore, $\mathrm{A} \notin \Diamond(\mathrm{X})^+$, since $\mathrm{F} \not\vdash \Diamond(\mathrm{X} \rightarrow \mathrm{A})$. Therefore, by Lemma 5.3, $t_2[\mathrm{A}] = 1$ implying that $\mathrm{r} \not\models \Diamond(\mathrm{X} \rightarrow \mathrm{A})$, since $\mathrm{A} \notin \mathrm{X}$. $\Box$

The following theorem (cf. Corollary 5.2 in [CASA84]) shows that we cannot put a bound on the number of FDs that are used in a derivation which uses the strengthening inference rule. This means that for no natural number $k \geq 0$, does there exist a k-ary complete axiomatisation for FDs in the sense of Theorem 5.1 in [CASA84].

Theorem 5.5 For no natural number $k \geq 0$ does there exist a k-ary axiomatisation for FDs.

Proof. Let X = $\{A_1, A_2, \ldots, A_k\}$, with $k > 1$, Y = X$-A_1$ and schema(R) = X $\cup$ {C, D}, where X $\cap$ CD = $\emptyset$. Consider the set of FDs, $\Gamma_k = \{\Box(\mathrm{X}-A_2 \rightarrow A_2), \ldots, \Box(\mathrm{X}-A_k \rightarrow A_k), \Diamond(\mathrm{Y} \rightarrow \mathrm{C}), \Box(\mathrm{YC} \rightarrow \mathrm{D})\}$. We observe that $|\Gamma_k| = k+1$ and that by the soundness of FD9 it follows that $\Gamma_k \models \Box(\mathrm{X} \rightarrow \mathrm{D})$.

It can be verified that for no proper subset G $\subset \Gamma_k$ is it true that G $\models \Box(\mathrm{X} \rightarrow \mathrm{D})$. Now, let Γ be the set of all FDs that can be derived from Γ_k using at most a pair of FDs in Γ_k, that is, Γ is the closure of Γ_k under 2-ary implication. By inspecting Γ_k it can be verified that if G is a set of at most k FDs of Γ and for some FD σ, G $\models \sigma$, then there exist at most two FDs, $\tau_1, \tau_2 \in$ G such that $\{\tau_1, \tau_2\} \models \sigma$. Thus Γ is also closed under *k-ary implication* [CASA84]. The result follows since it can also be verified that Γ is *not* closed under implication, since $\Box(\mathrm{X} \rightarrow \mathrm{D}) \notin \Gamma$. □

The implication problem for FDs is the problem of finding the computational complexity of determining whether F $\models$ X $\rightarrow$ A, given a set of FDs F. It is well known that $\Box(\mathrm{X})^+$ can be computed in time linear in the size of the set of SFDs in F [BEER79] and $\Diamond(\mathrm{X})^+$ can be computed in time linear in the size of the set of WFDs in F [ATZE86]. The next theorem shows that $?(\mathrm{X})^+$ can be computed in time polynomial in the size of the set of FDs F. The result follows from Lemmas 5.2 and 5.3 on using Lemma 4.5.

Theorem 5.6 The implication problem for a set of FDs, F, can be solved in polynomial time in the size of F. □

6 The existence of Armstrong relations

Armstrong relations [FAGI82, BEER84] are relations, which satisfy all and only those FDs which are logically implied by a given set of FDs. The existence of Armstrong relations for FDs holding in complete relations was shown in [ARMS74]. The importance of such relations in the process of database design was discussed in [MANN86]. In this section we show that Armstrong relations also exist for FDs defined over incomplete relations. Our construction is a combination of the disjoint union and agreement sets techniques for constructing Armstrong relations, which were discussed in [FAGI82].

Definition 6.1 Let F^* be the set of all FDs that are logically implied by a set of FDs F over R. An *Armstrong relation* for F is a relation r over R such that r $\models ?(\mathrm{X} \rightarrow \mathrm{Y})$ if and only if $?(\mathrm{X} \rightarrow \mathrm{Y}) \in F^*$. That is, r satisifies the FDs in F^* and no other FDs.

By Theorem 5.4 $F^* = F^+$, so in the proof of the following result we will employ F^+ instead of F^*.

Theorem 6.1 For each set of FDs F over R there exists an Armstrong relation for F.

Proof. Let $\Sigma^\Diamond$ be the set of all WFDs that are *not* members of F^+ and let $\Sigma^\Box$ be the set of all SFDs that are *not* members of F^+ but their weak counterparts are members of F^+. By the reflexivity, union and decomposition rules we can assume, without loss of generality, that $\Sigma^\Diamond$ and $\Sigma^\Box$ contain only nontrivial FDs and that all the FDs in these two sets are of the form $?(\mathrm{X} \rightarrow \mathrm{Y})$, where Y = $?(\mathrm{X})^+ - \mathrm{X}$.

In the following let CHASE(2, F) = $\{t_2, t_3\}$, where t_2 originated from u_2 and t_3 originated from u_3 and let CHASE(1, F) = $\{t_1, t_3\}$, where t_1 originated from u_1 and t_3 originated from u_3; we recall that u_3 is a tuple containing only zeros. We write t_i(X) instead of t_i to emphasise that T_i(X) was input to the chase procedure, where $i \in \{1, 2\}$. In addition, we denote the tuple resulting from substituting all occurrences of the nonnull value 1 in t_i by a nonnull value $j \in Dom$ by t_i(X, j).

Now let $\Sigma^{\Diamond} = \{\Diamond(X_1 \to Y_1), \ldots, \Diamond(X_n \to Y_n)\}$ and let $\Sigma^{\Box} = \{\Box(W_1 \to Z_1), \ldots, \Box(W_m \to Z_m)\}$. The relation r^{Arm} is defined by

$$r^{Arm} = \{u_3\} \bigcup (\bigcup_{i \in I} t_2(X_i, i)) \bigcup (\bigcup_{j \in J} t_1(W_j, n+j)).$$

We claim that r^{Arm} is an Armstrong relation for F. Thus we need to show that $r^{Arm} \models\ ?(\mathrm{X} \to \mathrm{Y})$ if and only if $?(\mathrm{X} \to \mathrm{Y}) \in F^+$. We consider the two cases when ? = $\Diamond$ and ? = $\Box$ separately. In both cases we use contraposition.

Case1: We show that $r^{Arm} \models \Diamond\ (\mathrm{X} \to \mathrm{Y})$ if and only if $\Diamond(\mathrm{X} \to \mathrm{Y}) \in F^+$.

If: If $r^{Arm} \not\models \Diamond(\mathrm{X} \to \mathrm{Y})$, then $\exists$A $\in$ Y such that $r^{Arm} \not\models \Diamond(\mathrm{X} \to \mathrm{A})$. Thus $\exists i \in$ I such that X = X_i and $\forall$B $\in X_i$, $t_2(X_i, i)[\mathrm{B}] = 0$ and $t_2(X_i, i)[\mathrm{A}] = i$. Assume that $\Diamond(\mathrm{X} \to \mathrm{A}) \in F^+$. A contradiction arises proving the result, since by Theorem 4.4 on using Definition 5.1 it must be the case that $\{u_3, t_2(X_i, i)\} \models \Diamond(\mathrm{X} \to \mathrm{A})$.

Only if: If $\Diamond(\mathrm{X} \to \mathrm{Y}) \notin F^+$, then $\exists$A $\in$ Y such that $\Diamond(\mathrm{X} \to \mathrm{A}) \notin F^+$. Thus by Lemma 5.3 $\exists i \in$ I such that X = X_i and $\forall$B $\in X_i$, $t_2(X_i, i)[\mathrm{B}] = 0$ and $t_2(X_i, i)[\mathrm{A}] = i$. It follows that $r^{Arm} \not\models \Diamond(\mathrm{X} \to \mathrm{Y})$ as required, since $\forall$B $\in$ XA, $u_3[\mathrm{B}] = 0$.

Case2: We show that $r^{Arm} \models \Box\ (\mathrm{X} \to \mathrm{Y})$ if and only if $\Box(\mathrm{X} \to \mathrm{Y}) \in F^+$.

If: If $r^{Arm} \not\models \Box(\mathrm{X} \to \mathrm{Y})$, then $\exists$A $\in$ Y such that $r^{Arm} \not\models \Box(\mathrm{X} \to \mathrm{A})$. Moreover, if $r^{Arm} \not\models \Diamond(\mathrm{X} \to \mathrm{Y})$, then the result follows by the if part of Case1. So assume that $r^{Arm} \models \Diamond(\mathrm{X} \to \mathrm{Y})$. Thus $\exists j \in$ J such that X = W_j and $\forall$B $\in W_j$, $t_1(W_j, n+j)[\mathrm{B}] = 0$ or $t_1(W_j, n+j)[\mathrm{B}] = unk$ and $t_1(W_j, n+j)[\mathrm{A}] \neq 0$. Assume that $\Box(\mathrm{X} \to \mathrm{A}) \in F^+$. A contradiction arises proving the result, since by Theorem 4.4 on using Definition 5.1 it must be the case that $\{u_3, t_1(W_j, n+j)\} \models \Box(\mathrm{X} \to \mathrm{A})$.

Only if: If $\Box(\mathrm{X} \to \mathrm{Y}) \notin F^+$, then $\exists$A $\in$ Y such that $\Box(\mathrm{X} \to \mathrm{A}) \notin F^+$. Moreover, if $\Diamond(\mathrm{X} \to \mathrm{A}) \notin F^+$, then the result follows by the only if part of Case1. So assume that $\Diamond(\mathrm{X} \to \mathrm{A}) \in F^+$. Thus by Lemma 5.2 $\exists j \in$ J such that X = W_j and $\forall$B $\in W_j$, either $t_1(W_j,\ n+j)[\mathrm{B}] = 0$ or $t_1(W_j,\ n+j)[\mathrm{B}] = unk$ and $t_1(W_j,\ n+j)[\mathrm{A}] \neq 0$. It follows that $r^{Arm} \not\models \Box(\mathrm{X} \to \mathrm{Y})$ as required, since $\forall$B $\in$ XA, $u_3[\mathrm{B}] = 0$. $\Box$

Further research could be carried out in this area in order to generalise some of the combinatorical results concerning the size of Armstrong relations (whose size is, in general, exponential) and to construct algorithms that generate such relations [FAGI82, BEER84, MANN86, DEME93].

7 Concluding Remarks

We have presented an in-depth investigation of FDs holding in relations which may be incomplete. In the context of this paper FDs are either weak, in which case there exists a possible world satisfying the FD, or strong, in which case all possible worlds satisfy the FD. We have exhibited an axiom system for such FDs, which was shown in Theorem 5.4 to be sound and complete. We have also shown in Theorem 5.5 that for no natural number $k \geq 0$ does there exist a k-ary complete axiomatisation for such FDs in the sense of [CASA84]. Moreover, we have shown in Theorem 5.6 that the implication problem for such FDs can be solved in time polynomial in the size of the input set of FDs. Finally, we have shown in Theorem 6.1 that Armstrong relations exist for such FDs.

An interesting extension of the formalism presented herein, which copes with inclusion dependencies [CASA84] in the presence of incomplete information, can be found in [LEVE97]. It would be an interesting research topic to extend the results obtained herein to *or-sets* [IMIE89], i. e. allowing, instead of any occurrence of *unk*, a finite set of possible values, one of which is the true value.

Finally, we indicate how the results in [LEVE95] can be extended to show an equivalence between FDs holding in relations and propositional implications holding in a subset of three-valued propositional logic. Let us associate with any FD $X \rightarrow Y$, a logical implication $X \Rightarrow Y$ such that X and Y are each associated with a conjunction of atomic sentences, also denoted by X and Y, respectively. In order to show the said equivalence, let the three truth values of the logic be true, false and possible (or simply poss) and assume that a *valuation* of a conjunction of logical implications can be recursively defined as a mapping from the set {true, false, poss} to type(R) by using Kleene's truth tables for conjunction and implication [RESC69] (cf. [LEVE95]). We can now define satisfaction logically, where V is a valuation, as follows: $?(X \Rightarrow Y)$ is satisfied in V, with $? = \Box$ or $? = \Diamond$, written $V \models ?(X \Rightarrow Y)$, if $V(?(X \Rightarrow Y))$ = true according to the truth table given in Figure 3. Thus, for SFDs the designated truth value is true and for WFDs the designated truth values are true and poss. By using an argument similar to that of Lemma 5.1 and Theorem 5.4, it can be shown that our axiom system, which is sound and complete for FDs, is also sound and complete for the corresponding logical implications that hold in valuations.

$X \Rightarrow Y$	$\Box(X \Rightarrow Y)$	$\Diamond(X \Rightarrow Y)$
true	true	true
false	false	false
poss	false	true

Table 3: The three-valued logic truth table for $\Box$ and $\Diamond$

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