UNDECIDABILITY OF FIRST-ORDER INTUITIONISTIC AND MODAL LOGICS WITH TWO VARIABLES

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Abstract. We prove that the two-variable fragment of first-order intuitionistic logic is undecidable, even without constants and equality. We also show that the two-variable fragment of a quantified modal logic L with expanding first-order domains is undecidable whenever there is a Kripke frame for L with a point having infinitely many successors (such are, in particular, the first-order extensions of practically all standard modal logics like **K**, **K4**, **GL**, **S4**, **S5**, **K4**.**1**, **S4**.**2**, **GL**.**3**, etc.). For many quantified modal logics, including those in the standard nomenclature above, even the monadic two-variable fragments turn out to be undecidable.

§1. Introduction. Ever since the undecidability of first-order classical logic became known [5], there has been a continuing interest in establishing the 'borderline' between its decidable and undecidable fragments; see [2] for a detailed exposition. One approach to this classification problem is to consider fragments with finitely many individual variables. The borderline here goes between two and three: the two-variable fragment of classical first-order logic is decidable [23], while with three variables it becomes undecidable [26], even without constants and equality. (Decidable and undecidable extensions of the two-variable fragment with some natural 'built-in' predicates were considered in [10].)

As classical first-order logic can be reduced to intuitionistic first-order logic by Gödel's double negation translation (see, e.g., [27]), the three-variable fragment of the latter is also undecidable. On the other hand, according to results of Bull [3], Mints [22] and Ono [24], the one-variable fragment (which is equivalent to propositional intuitionistic modal logic **MIPC** in the same way as the one-variable fragment of classical logic is equivalent to propositional modal logic **S5**) is decidable. Gabbay and Shehtman [9] proved undecidability of the two-variable fragment of first-order intuitionistic logic extended with the axiom

$$\forall x\ (P(x) \vee q) \to \forall x\ P(x) \vee q,$$

known as the *constant domain principle.* However, the question whether the two-variable fragment of first-order intuitionistic logic itself is decidable has remained open.

Here we show that the two-variable fragment of first-order intuitionistic logic is *undecidable*, even without constants and equality.

Our proof uses a simple reduction of an infinite tiling problem. As is well-known, such a tiling problem can be easily encoded in the three-variable fragment of classical first-order logic (see, e.g., [8]). Our reduction is based on the observation that the third variable can be used in a very restricted way, only as a kind of 'stack' for substitutions. This view on substitutions originates in the algebraic approach to first-order logics [12].

Intuitionistic first-order logic can be embedded into quantified modal logic **S4** with expanding first-order domains using the Gödel translation which prefixes the necessity operator to every subformula of a first-order intuitionistic formula. This shows that the two-variable fragment of quantified **S4** with expanding domains is undecidable as well. We generalise this result and prove the undecidability of the two-variable fragment of any quantified modal logic L with expanding domains whenever there is a Kripke frame for L with a point having infinitely many successors. This answers an open question from [9], where the same result for first-order modal logics with *constant* domains was obtained. We then show how Kripke's idea from [18] can be used to prove that actually the *monadic* two-variable fragments of many quantified modal logics with expanding domains are undecidable.

§2. Two-variable first-order intuitionistic logic. The alphabet of first-order intuitionistic logic **QInt** (without function symbols, constants and equality) consists of predicate symbols $P, Q, \dots$ of arbitrary finite arity, countably many individual variables $x, y, \dots$, propositional connectives $\wedge$, $\vee$, $\to$ and $\bot$ ('falsehood'), and quantifiers $\forall$ and $\exists$. Formulas are defined in the usual way.

First-order intuitionistic logic **QInt** can be given syntactically by removing the double negation principle (or other equivalent principles) from a (suitable) axiomatic system for classical logic; see, e.g., [27]. Here we only need the definition of **QInt** via its Kripke semantics. A *first-order intuitionistic Kripke model*[1] is a tuple

$$\mathfrak{M} = (\mathfrak{F}, \Delta, \delta, I),$$

where

- $\mathfrak{F} = (W, \leq)$ is an intuitionistic Kripke frame—i.e., $\leq$ is a partial order on $W \neq \emptyset$,
- δ is a function associating with every $w \in W$ a set $\delta(w) \subseteq \Delta$, called the *domain* of w, in such a way that $\delta(u) \subseteq \delta(v)$ whenever $u \leq v$, for $u, v \in W$,

[1] For other equivalent definitions see, e.g., [19, 6, 28].

- I is a function associating with every $w \in W$ a classical first-order structure

$$I(w) = (\Delta, P^w, Q^w, \dots),$$

- the truth of predicates is preserved along the accessibility relation $\leq$, that is, for every predicate symbol P and all $u, v \in W$, if $u \leq v$ then $P^u \subseteq P^v$.

An *assignment in* Δ is a function $\mathfrak{a}$ from the set of individual variables to Δ. The *truth-relation* $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ (or simply $w \models^{\mathfrak{a}} \varphi$, if understood) is defined as follows:

- $w \models^{\mathfrak{a}} P(x_1, \dots, x_n)$ iff $P^w\big(\mathfrak{a}(x_1), \dots, \mathfrak{a}(x_n)\big)$,
- $w \models^{\mathfrak{a}} \psi \wedge \chi$ iff $w \models^{\mathfrak{a}} \psi$ and $w \models^{\mathfrak{a}} \chi$,
- $w \models^{\mathfrak{a}} \psi \vee \chi$ iff $w \models^{\mathfrak{a}} \psi$ or $w \models^{\mathfrak{a}} \chi$,
- $w \models^{\mathfrak{a}} \psi \to \chi$ iff $v \models^{\mathfrak{a}} \psi$ implies $v \models^{\mathfrak{a}} \chi$ for all $v \geq w$,
- $w \not\models^{\mathfrak{a}} \bot$,
- $w \models^{\mathfrak{a}} \forall x \psi$ iff $v \models^{\mathfrak{b}} \psi$ for every $v \geq w$ and every assignment $\mathfrak{b}$ in Δ such that $\mathfrak{b}(x) \in \delta(v)$ and $\mathfrak{a}(y) = \mathfrak{b}(y)$ for all variables $y \neq x$,
- $w \models^{\mathfrak{a}} \exists x \psi$ iff $w \models^{\mathfrak{b}} \psi$ for an assignment $\mathfrak{b}$ in Δ such that $\mathfrak{b}(x) \in \delta(w)$ and $\mathfrak{a}(y) = \mathfrak{b}(y)$ for all variables $y \neq x$.

We say that a formula φ is *true in* $\mathfrak{M}$ if $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ holds for every world $w \in W$ and every assignment $\mathfrak{a}$ in Δ such that $\mathfrak{a}(x) \in \delta(w)$ for all individual variables x.

First-order intuitionistic logic **QInt** is the set of all formulas that are true in all first-order intuitionistic Kripke models. We denote by **QInt(2)** the two-variable fragment of **QInt**, that is, the collection of those formulas from **QInt** that contain only two (bound or free) individual variables.

Our main result is the following:

THEOREM 1. **QInt(2)** *is undecidable.*

PROOF. The following $\mathbb{N} \times \mathbb{N}$ *tiling problem* is known to be undecidable [1]: given a finite set T of *tile types* that are four-tuples of colours

$$t = (\mathit{left}(t), \mathit{right}(t), \mathit{up}(t), \mathit{down}(t)),$$

decide whether T tiles the grid $\mathbb{N} \times \mathbb{N}$ in the sense that there exists a function (called a *tiling*) τ from $\mathbb{N} \times \mathbb{N}$ to T such that, for all $i, j \in \mathbb{N}$,

$$\mathit{up}(\tau(i,j)) = \mathit{down}(\tau(i,j+1)) \quad \text{and} \quad \mathit{right}(\tau(i,j)) = \mathit{left}(\tau(i+1,j)).$$

We reduce this tiling problem to the complement of **QInt(2)**, that is, to the set of two-variable formulas that are refutable in some first-order intuitionistic Kripke models.

To this end, given a finite set T of tile types, define a formula ψ_T to be the conjunction the following sentences (1)–(6):

$$\forall x \bigvee_{t\in T}\Big(P_t(x) \wedge \bigwedge_{t'\neq t}(P_{t'}(x) \to \bot)\Big), \tag{1}$$

$$\bigwedge_{right(t)\neq left(t')} \forall x\forall y\ \big(succ_H(x,y) \wedge P_t(x) \wedge P_{t'}(y) \to \bot\big), \tag{2}$$

$$\bigwedge_{up(t)\neq down(t')} \forall x\forall y\ \big(succ_V(x,y) \wedge P_t(x) \wedge P_{t'}(y) \to \bot\big), \tag{3}$$

$$\forall x\exists y\ succ_H(x,y)\ \wedge\ \forall x\exists y\ succ_V(x,y), \tag{4}$$

$$\forall x\forall y\ \big(succ_V(x,y) \vee (succ_V(x,y) \to \bot)\big), \tag{5}$$

$$\forall x\forall y\ \big[succ_V(x,y) \wedge \exists x\,(D(x) \wedge succ_H(y,x)) \to \\ \forall y\ \big(succ_H(x,y) \to \forall x\,(D(x) \to succ_V(y,x))\big)\big]. \tag{6}$$

Now, let

$$\varphi_T\ = \psi_T \to \exists x\,(D(x) \to \bot).$$

We claim that

$$\varphi_T \notin \mathbf{QInt(2)} \qquad \text{iff} \qquad T \text{ tiles } \mathbb{N}\times\mathbb{N}.$$

Suppose first that $\varphi_T \notin \mathbf{QInt(2)}$, that is, there exist a first-order intuitionistic Kripke model $\mathfrak{M} = ((W,\leq),\Delta,\delta,I)$ and some $w \in W$ such that $(\mathfrak{M},w) \models \psi_T$ and

$$(\mathfrak{M},w) \not\models \exists x\,(D(x) \to \bot). \tag{7}$$

We prove that $I(w)$ satisfies the following property:

$$\forall a,b,c \in \delta(w) \\ \big(succ_H^w(a,b) \wedge succ_V^w(a,c) \to \exists d \in \delta(w)\,(succ_H^w(c,d) \wedge succ_V^w(b,d)\big). \tag{8}$$

Indeed, let $a,b,c \in \delta(w)$ be such that $succ_H^w(a,b)$ and $succ_V^w(a,c)$. By (4), there is $d \in \delta(w)$ such that $succ_H^w(c,d)$. We show that $succ_V^w(b,d)$ holds as well. To this end, observe that, by (7), there is $u \geq w$ with $D^u(d)$. As the truth of predicates is preserved along the accessibility relation, we have $succ_H^u(a,b)$, $succ_V^u(a,c)$ and $succ_H^u(c,d)$. So, by (6), we obtain $succ_V^u(b,d)$. Finally, $succ_V^w(b,d)$ follows by (5).

Now, by (4) and (8), there exist $a_{i,j} \in \delta(w)$ $(i,j \in \mathbb{N})$ such that $succ_H^w(a_{i,j},a_{i+1,j})$ and $succ_V^w(a_{i,j},a_{i,j+1})$ hold for all $i,j \in \mathbb{N}$. So, by (1)–(3), the function τ defined by taking

$$\tau(i,j) = t \qquad \text{iff} \qquad P_t^w(a_{i,j})$$

is a tiling of $\mathbb{N}\times\mathbb{N}$.

Conversely, suppose that there is a tiling $\tau : \mathbb{N} \times \mathbb{N} \to T$. We define a first-order intuitionistic Kripke model $\mathfrak{M} = ((W, \leq), \Delta, \delta, I)$ refuting φ_T as follows:

- $W = \{w_0\} \cup (\mathbb{N} \times \mathbb{N})$ and $\leq$ is the reflexive closure of $\{w_0\} \times (\mathbb{N} \times \mathbb{N})$,
- $\Delta = \mathbb{N} \times \mathbb{N}$,
- for every $w \in W$, $\delta(w) = \Delta$, $I(w) = (\Delta, succ_H^w, succ_V^w, D^w, P_t^w)_{t \in T}$, where
  - $succ_H^w = \{ \big((i,j),\ (i+1,j)\big) \mid (i,j) \in \Delta\}$,
  - $succ_V^w = \{ \big((i,j),\ (i,j+1)\big) \mid (i,j) \in \Delta\}$,
  - $D^{w_0} = \emptyset$ and $D^w = \{w\}$ whenever $w \neq w_0$ and
  - $P_t^w = \{(i,j) \in \Delta \mid \tau(i,j) = t\}$ for every $t \in T$.

It is straightforward to check that $(\mathfrak{M}, w_0) \not\models \varphi_T$. ⊣

It may be worth noting that in fact we have proved a statement somewhat more general than Theorem 1. Call a first-order intuitionistic Kripke model $\big((W, \leq), \Delta, \delta, I\big)$ an *infinite fan* if

- $W = \{w_0\} \cup V$ is countably infinite and $\leq$ is the reflexive closure of $\{w_0\} \times V$,
- Δ is countably infinite and $\delta(w) = \Delta$, for all $w \in W$.

Now let Σ be a set of two-variable formulas such that $\mathbf{QInt(2)} \subseteq \Sigma$ and all formulas in Σ are true in all infinite fans. Then Σ is undecidable.

§3. Two-variable first-order modal logics with expanding domains. The alphabet of (constant and equality free) first-order modal logics consists of predicate symbols $P, Q, \ldots$ of arbitrary finite arity, countably many individual variables $x, y, \ldots$, (classical) propositional connectives $\wedge$ and $\neg$, quantifier $\forall$, and the necessity operator $\Box$ (with $\vee$, $\to$, $\exists$ and the possibility operator $\Diamond$ defined as standard abbreviations, e.g., $\Diamond ::= \neg\Box\neg$). First-order modal formulas are defined in the usual way, in particular, if φ is a formula then so is $\Box\varphi$.

A *first-order Kripke model with expanding domains* is a tuple

$$\mathfrak{M} = (\mathfrak{F}, \Delta, \delta, I),$$

where

- $\mathfrak{F} = (W, R)$ is a modal frame—i.e., R is a binary relation on $W \neq \emptyset$,
- $\delta(u) \subseteq \delta(v) \subseteq \Delta$ whenever uRv, for $u, v \in W$,
- I is a function associating with every $w \in W$ a classical first-order structure
$$I(w) = (\Delta, P^w, Q^w, \ldots).$$

An *assignment in* Δ is a function $\mathfrak{a}$ from the set of individual variables to Δ. The *truth-relation* $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ (or simply $w \models^{\mathfrak{a}} \varphi$) is defined as follows:

- $w \models^{\mathfrak{a}} P(x_1, \ldots, x_n)$ iff $P^w\big(\mathfrak{a}(x_1), \ldots, \mathfrak{a}(x_n)\big)$,

- $w \models^{\mathfrak{a}} \psi \wedge \chi$ iff $w \models^{\mathfrak{a}} \psi$ and $w \models^{\mathfrak{a}} \chi$,
- $w \models^{\mathfrak{a}} \neg\varphi$ iff $w \not\models^{\mathfrak{a}} \varphi$,
- $w \models^{\mathfrak{a}} \Box\psi$ iff $v \models^{\mathfrak{a}} \psi$ for every $v \in W$ with wRv,
- $w \models^{\mathfrak{a}} \forall x\psi$ iff $w \models^{\mathfrak{b}} \psi$ for all assignments $\mathfrak{b}$ in Δ such that $\mathfrak{b}(x) \in \delta(w)$ and $\mathfrak{a}(y) = \mathfrak{b}(y)$ for all variables $y \neq x$.

We say that a formula φ is *true in* $\mathfrak{M}$ if $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ holds for every world $w \in W$ and every assignment $\mathfrak{a}$ in Δ such that $\mathfrak{a}(x) \in \delta(w)$ for all individual variables x.

Given a propositional modal logic L, denote by $\mathbf{Q}^e L$ the set of all formulas that are true in every first-order Kripke model $\mathfrak{M} = (\mathfrak{F}, \Delta, \delta, I)$ with expanding domains such that $\mathfrak{F}$ is a frame for L (i.e., validates all formulas in L). Standard examples are $\mathbf{Q}^e\mathbf{K}$ with arbitrary frames, $\mathbf{Q}^e\mathbf{K4}$ with transitive frames, $\mathbf{Q}^e\mathbf{S4}$ with quasi-ordered frames, and $\mathbf{Q}^e\mathbf{GL}$ with quasi-ordered Noetherian frames.

We say that a formula φ is $\mathbf{Q}^e L$-satisfiable if $\neg\varphi \notin \mathbf{Q}^e L$.

As is well-known (see, e.g., [25]), intuitionistic first-order logic can be embedded into $\mathbf{Q}^e\mathbf{S4}$ by using the *Gödel translation* T which prefixes $\Box$ to every subformula of an intuitionistic formula. Namely, for every intuitionistic formula φ,

$$\varphi \in \mathbf{QInt} \quad \text{iff} \quad \mathsf{T}(\varphi) \in \mathbf{Q}^e\mathbf{S4}.$$

So, by Theorem 1, the two-variable fragment of $\mathbf{Q}^e\mathbf{S4}$ is undecidable as well.

Our next result is a generalisation of both this statement and the results from [9] on first-order modal logics with *constant* domains.

Say that a Kripke frame (W, R) *contains a point with infinitely many successors* if there exist a point $w \in W$ and an infinite subset $V \subseteq W$ such that wRv holds for every $v \in V$.

THEOREM 2. *Let L be any propositional modal logic having a Kripke frame that contains a point with infinitely many successors. Then the two-variable fragment of $\mathbf{Q}^e L$ is undecidable.*

PROOF. We reduce the $\mathbb{N}\times\mathbb{N}$ tiling problem to the satisfiability problem for the two-variable fragment of $\mathbf{Q}^e L$. Given a finite set T of tile types, define χ_T to be the conjunction of the following sentences:

$$\forall x \bigvee_{t\in T} \Big(P_t(x) \wedge \bigwedge_{t'\neq t} \neg P_{t'}(x)\Big),$$

$$\forall x\forall y \ \Big(succ_H(x,y) \to \bigwedge_{right(t)\neq left(t')} \neg\big(P_t(x) \wedge P_{t'}(y)\big)\Big),$$

$$\forall x\forall y \ \Big(succ_V(x,y) \to \bigwedge_{up(t)\neq down(t')} \neg\big(P_t(x) \wedge P_{t'}(y)\big)\Big),$$

$$\forall x \exists y\ succ_H(x,y) \ \wedge\ \forall x \exists y\ succ_V(x,y),$$
$$\forall x \forall y\ \big(succ_H(x,y) \to \Box succ_H(x,y)\big),$$
$$\forall x \forall y\ \big(succ_V(x,y) \to \Box succ_V(x,y)\big),$$
$$\forall x \forall y\ \big(\Diamond succ_V(x,y) \to succ_V(x,y)\big),$$
$$\forall x \Diamond D(x),$$
$$\Box \forall x \forall y\ \big[succ_V(x,y) \wedge \exists x\,(D(x) \wedge succ_H(y,x)) \to \forall y\ \big(succ_H(x,y) \to \forall x\,(D(x) \to succ_V(y,x))\big)\big].$$

An argument analogous to the one proving Theorem 1 shows that

$$\chi_T \text{ is } \mathbf{Q}^e L\text{-satisfiable} \qquad \text{iff} \qquad T \text{ tiles } \mathbb{N}\times\mathbb{N}.$$

Here we only show that χ_T is $\mathbf{Q}^e L$-satisfiable whenever T tiles $\mathbb{N}\times\mathbb{N}$, and leave the other direction to the reader.

Suppose $\tau : \mathbb{N}\times\mathbb{N} \to T$ is a tiling. Take any frame $\mathfrak{F} = (W, R)$ for L that contains a point $w_0 \in W$ such that the set $V = \{w \in W \mid w_0 R w\}$ is infinite. Let f be a surjection from V onto $\mathbb{N}\times\mathbb{N}$. Define a first-order Kripke model $\mathfrak{M} = (\mathfrak{F}, \Delta, \delta, I)$ by taking

- $\Delta = \mathbb{N}\times\mathbb{N}$,
- for every $w \in W$, $\delta(w) = \Delta$, $I(w) = (\Delta, succ_H^w, succ_V^w, D^w, P_t^w)_{t\in T}$, where
  - $succ_H^w = \{\ \big((i,j),(i+1,j)\big) \mid (i,j) \in \Delta\}$,
  - $succ_V^w = \{\ \big((i,j),(i,j+1)\big) \mid (i,j) \in \Delta\}$,
  - if $w \in V$ then $D^w = \{f(w)\}$, otherwise $D^w = \emptyset$, and
  - $P_t^w = \{(i,j) \in \Delta \mid \tau(i,j) = t\}$, for every $t \in T$.

It is straightforward to check that $(\mathfrak{M}, w_0) \models \chi_T$. ⊣

It follows that almost all standard first-order modal logics (such as, e.g., **K**, **K4**, **GL**, **S4**, **S5**, **K4.1**, **S4.2**, **GL.3**, **Grz**) with two variables and expanding domains are undecidable. Note that the proof above also goes through for modal logics with constant domains which were shown to be undecidable in [9] with the help of a more involved reduction. (In fact, satisfiability in models with expanding domains is always reducible to satisfiability in models with constant domains; see, e.g., [8].)

For many modal logics we can draw an even finer borderline between decidable and undecidable. Recall that Kripke [18] showed in fact that the *monadic* fragment of a first-order modal logic $\mathbf{Q}^e L$ is undecidable whenever $L \subseteq \mathbf{S5}$. He used a reduction of the undecidable first-order classical theory of one dyadic predicate R by replacing every atom $R(x,y)$ with the modal *monadic* formula $\Diamond(P(x) \wedge Q(y))$. As was pointed out in [17, pp. 271–272], the same proof actually works for the monadic fragment of any first-order modal logic $\mathbf{Q}^e L$ whenever L has a frame containing a point with infinitely many successors. In [15] Kripke's idea was used to prove that certain monadic two-variable temporal logics with constant domains are not recursively enumerable.

Here we show that a similar trick can be used to prove undecidability of the *monadic two-variable* fragments of many modal logics, both with expanding and constant domains.

THEOREM 3. *Let L be any propositional modal logic with a Kripke frame (W, R) satisfying the following condition:*

($*$) *there are $w_0 \in W$ and two disjoint infinite subsets $V_1, V_2 \subseteq W$ such that $w_0 R v$ for all $v \in V_1$, and $v_1 R v_2$ for all $v_1 \in V_1$, $v_2 \in V_2$.*

Then the monadic two-variable fragment of $\mathbf{Q}^e L$ is undecidable.

PROOF. First, take a fresh monadic predicate symbol Q and replace each subformula $\Box\psi$ of χ_T above with $\Box(\forall x Q(x) \to \psi)$, and each subformula $\Diamond\psi$ of χ_T with $\Diamond(\forall x Q(x) \land \psi)$. Denote the resulting formula by χ_T^Q. Next, take two fresh monadic predicate symbols Q_H, Q_V and replace each occurrence of $succ_H(x', y')$ and $succ_V(x', y')$ (for $x', y' \in \{x, y\}$) in χ_T^Q with $\Diamond(D(x') \land Q_H(y'))$ and $\Diamond(D(x') \land Q_V(y'))$, respectively. Denote the resulting formula by ξ_T. We claim that

$$\xi_T \text{ is } \mathbf{Q}^e L\text{-satisfiable} \qquad \text{iff} \qquad T \text{ tiles } \mathbb{N} \times \mathbb{N}.$$

The argument proving the implication ($\Rightarrow$) is again similar to the one used in Theorem 1 (we simply regard $\Diamond(D(x) \land Q_H(y))$ and $\Diamond(D(x) \land Q_V(y))$ as binary predicates defining the $\mathbb{N} \times \mathbb{N}$ grid).

Now suppose that $\tau : \mathbb{N} \times \mathbb{N} \to T$ is a tiling. Take any frame $\mathfrak{F} = (W, R)$ for L satisfying ($*$), and let f_1 and f_2 be surjections from V_1 and V_2 onto $\mathbb{N} \times \mathbb{N}$, respectively. Define a first-order Kripke model $\mathfrak{M} = (\mathfrak{F}, \Delta, \delta, I)$ by taking

- $\Delta = \mathbb{N} \times \mathbb{N}$,
- for each $w \in W$, $\delta(w) = \Delta$ and $I(w) = (\Delta, D^w, Q_H^w, Q_V^w, Q^w, P_t^w)_{t \in T}$, where
  - if $w \in V_1$ then $Q^w = \Delta$, otherwise $Q^w = \emptyset$,
  - if $w \in V_k$, for $k = 1, 2$, and $f_k(w) = (i, j)$, then $D^w = \{(i, j)\}$, $Q_H^w = \{(i+1, j)\}$, $Q_V^w = \{(i, j+1)\}$,
  - if $w \notin V_1 \cup V_2$, then $D^w = Q_H^w = Q_V^w = \emptyset$,
  - $P_t^w = \{(i, j) \in \Delta \mid \tau(i, j) = t\}$ for every $t \in T$.

It is not hard to see that for all $w \in \{w_0\} \cup V_1$, all $(i, j), (i', j') \in \Delta$, and all assignments $\mathfrak{a}$ with $\mathfrak{a}(x) = (i, j)$, $\mathfrak{a}(y) = (i', j')$,

$$(\mathfrak{M}, w) \models^{\mathfrak{a}} \Diamond\big(D(x) \land Q_H(y)\big) \qquad \text{iff} \qquad i' = i + 1 \text{ and } j' = j,$$
$$(\mathfrak{M}, w) \models^{\mathfrak{a}} \Diamond\big(D(x) \land Q_V(y)\big) \qquad \text{iff} \qquad i' = i \text{ and } j' = j + 1.$$

It follows that $(\mathfrak{M}, w_0) \models \xi_T$, as required. ⊣

Standard propositional modal logics such as **K**, **K4**, **GL**, **S4**, **S5**, **K4.1**, **S4.2**, **GL.3**, **Grz** all have frames satisfying condition ($*$) of Theorem 3. It follows that the monadic two-variable fragments of these logics with expanding (and so with constant) domains are undecidable.

§4. Discussion. The results obtained above can possibly be generalised in different ways.

It was shown in [21, 20] that the monadic fragment of first-order intuitionistic logic is undecidable, even with a single monadic predicate symbol [7]. One might conjecture that, similarly to the modal case above, the *monadic* fragment of $\mathbf{QInt(2)}$ is undecidable. However, it seems that neither the intuitionistic analogue of Kripke's trick (i.e., substituting $\neg\neg(P(x) \wedge Q(x))$ for $R(x, y)$) nor the more refined technique of [7] are applicable to our proof in a straightforward manner. To define the minimal number of individual variables which makes the monadic fragment of **QInt** undecidable still remains an open problem.

Those who are interested in 'abstract' first-order superintuitionistic and modal logics may find it interesting to consider quantified extensions of *tabular* and *pretabular* logics: each of the former is characterised by a single finite frame, while the latter are not tabular themselves, but all their proper extensions are (for details see, e.g., [4]). We conjecture that the two-variable fragment of the quantified extension of a propositional superintuitionistic or modal logic L is decidable iff L is tabular. For some more details and discussion see [9].

It could also be of interest to generalise the ideas above in order to prove undecidability of the so-called '*restricted*' fragment of two-variable $\mathbf{Q}^e L$. This fragment is equality- and (first-order) substitution-free, that is, all atomic formulas are of the form $P(x, y)$ (so that formulas with atoms like $succ_H(y, x)$ do not belong to this fragment); see [12, 8]. To obtain such a generalisation, one may try to express substitutions with the help of 'abstract' equality predicates, and then postulate some properties of these predicates in the usual algebraic logic way; see [11, 12]. It is worth noting that the restricted fragment of a two-variable first-order extension of a propositional modal logic L with expanding domains is equivalent to the modal product logic of the form $(L \times (\mathbf{S5} \times \mathbf{S5}))^{\mathsf{ex}}$; for definitions and more details see [8, Section 9.1].

Products of propositional modal logics can possibly be used to draw a finer borderline between decidable and undecidable fragments. With the help of a very subtle reduction of the infinite tiling problem, Hirsch and Hodkinson [13] proved that representability is not decidable for finite relation algebras. This result is used in [14] to show that every modal logic between $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$ is undecidable. A simplified version of the reduction from [13] is used in [16] to prove undecidability of the *one-variable* fragment of first-order computational tree logic $\mathbf{CTL}^*$. We conjecture that a similar reduction can prove the undecidability of all logics of the form $(L_1 \times (L_2 \times L_3))^{\mathsf{ex}}$, where L_1, L_2 and L_3 are any Kripke complete propositional modal logics between **K** and **S5**. (On the other hand, the strongest *decidable* fragments of standard first-order modal logics known so far are the *monodic* fragments from [29] which allow applications of modal operators to formulas with at most one free variable only.)

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