DL-Lite with Temporalised Concepts, Rigid Axioms and Roles

A. Artale,[1] R. Kontchakov,[2] V. Ryzhikov,[1] and M. Zakharyaschev[2]

[1] KRDB Research Centre
Free University of Bozen-Bolzano
I-39100 Bolzano, Italy
lastname@inf.unibz.it

[2] School of Comp. Science and Inf. Sys.
Birkbeck College
London WC1E 7HX, UK
{roman,michael}@dcs.bbk.ac.uk

Abstract. We investigate the temporal extension of the description logic $DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ with the *until* operator on concepts, rigid (time-independent) and local (time-dependent) roles, and rigid TBox axioms. Using an embedding into the one-variable fragment of first-order temporal logic and the quasimodel technique, we prove that (i) the satisfiability problem for the resulting logic is PSPACE-complete, and that (ii) by weakening *until* to *sometime in the future* we obtain an NP-complete logic, which matches the complexities of the propositional linear-time temporal logics with the corresponding temporal operators.

1   Introduction

Numerous temporal extensions of various description logics (DLs) have been constructed and investigated since 1993, when K. Schild published his seminal paper [19]. (We refer the reader to the monograph [15] and survey papers [3, 9, 17], where the history of the development of both interval- and point-based temporalised DLs is discussed in full detail.) There are various ways of introducing a temporal dimension in a DL. Temporal operators can be used as constructs for concepts, roles, TBox and ABox axioms—such concepts, roles or axioms are called *temporalised*. Alternatively, one may declare that a certain concept, role or axiom is *rigid* in the sense that its interpretation does not change in time. A number of complexity results have been obtained for different combinations of temporal operators and DLs. For instance, the following is known for combinations of $\mathcal{ALC}$ with the linear-time temporal logic $\mathcal{LTL}$: the satisfiability problem for the temporal $\mathcal{ALC}$ is

- *undecidable* if temporalised concepts together with rigid axioms and roles are allowed in the language is enough); see [15] and references therein;
- 2ExpTime-complete if the language allows rigid concepts and roles with temporalised axioms [10];
- ExpSpace-complete if the language allows temporalised concepts and axioms (but no rigid or temporalised roles) [15];
- ExpTime-complete if the language allows only temporalised concepts and rigid axioms (but no rigid or temporalised roles) [19, 4].

In other words, as long as one wants to express the temporal behaviour of only axioms and concepts (but not roles), the resulting combination is likely to be decidable. As soon as the combination allows reasoning about the temporal behaviour of binary relations, it becomes undecidable, unless we limit the means to describe the temporal behaviour of concepts. Furthermore, we notice that a better computational behaviour is exhibited in cases where rigid axioms are used instead of more general temporalised ones.

In this paper, we are interested in the scenario where axioms are rigid, concepts are temporalised and roles may be rigid or local (i.e., may change arbitrarily). To regain decidability in this case, one has to restrict either the temporal [8] or the DL component [7]. A decidable (in fact, 2ExpTime-complete) logic $\mathbf{S5}_{\mathcal{ALCQI}}$ [8] is obtained by combining the modal logic $\mathbf{S5}$ with $\mathcal{ALCQI}$. This approach weakens the temporal dimension to the much simpler $\mathbf{S5}$, but can nevertheless represent rigid concepts and roles and allows one to state that concept and role memberships change in time (but without discriminating between changes in the past and future).

Temporal extensions of 'weak' DLs from the recently introduced *DL-Lite* and $\mathcal{EL}$ families with rigid roles and temporalised axioms and concepts were investigated in [7]. It was shown that even in this case the resulting temporal DLs turn out to be very complex: ExpSpace-complete for tractable $DL\text{-}Lite_{horn}^{\mathcal{N}}$ and undecidable for tractable $\mathcal{EL}$. An inspection of the lower bound proofs reveals, however, that they do not go through without the use of temporal and Boolean operators on TBox axioms. To find out the complexity of temporal *DL-Lite* logics without these constructs is the main aim of this paper.

Our most expressive *DL-Lite* logic $DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ [2] features non-qualified number restrictions and role inclusion axioms (with limited interaction), full Booleans on concepts as well as some other constructs. In $DL\text{-}Lite_{bool}^{(\mathcal{RN})}$, the satisfiability problem is NP-complete for combined complexity, while instance checking is in AC^0 for data complexity. We also consider the fragment $DL\text{-}Lite_{core}^{(\mathcal{RN})}$ of $DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ with primitive concept inclusion axioms, for which satisfiability is NLogSpace-complete for combined complexity and answering positive existential queries is in AC^0 for data complexity. (Because of this low data complexity of query answering, *DL-Lite* logics form the basis of OWL 2 QL, one of the three profiles of OWL 2; see `http://www.w3.org/TR/owl2-profiles/`.)

We consider two temporal extensions, $T_{\Diamond}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ and $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$, of $DL\text{-}Lite_{bool}^{(\mathcal{RN})}$. Both logics weaken $TDL\text{-}Lite_{bool}$ of [7] by allowing only rigid axioms; the temporalised concepts of $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ can be built using temporal operators *until* $\mathcal{U}$, *next-time* $\bigcirc$ and their derivatives, while in $T_{\Diamond}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ they are limited to *sometime in the future* $\Diamond$ and *always in the future* $\Box$. We show that the satisfiability problem is NP-complete for $T_{\Diamond}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ and PSpace-complete for $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$, which matches the complexity of the component logics. (Note, however, that they are not simple fusions of their components.) The lower bounds hold also for the *core* fragments of $T_{\Diamond}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ and $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$.

2 Temporal *DL-Lite* Logics

We begin by defining temporal extensions $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ and $T_{\Diamond}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ of the description logic $DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ [1, 2], which, in turn, extends the original $DL\text{-}Lite_{\sqcap,\mathcal{F}}$ language [11–13] with full Booleans over concepts as well as cardinality restrictions on roles and role inclusion axioms with limited interaction.

The language of $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ contains *object names* $a_0, a_1, \ldots$, *concept names* $A_0, A_1, \ldots$, *local role names* $P_0, P_1, \ldots$ and *rigid role names* $G_0, G_1, \ldots$; *role names* S, *roles* R, *basic concepts* B and *concepts* C are defined as follows:

$$
\begin{aligned}
S &::= P_i \mid G_i, & R &::= S \mid S^-, \\
B &::= \bot \mid A_i \mid {\geq q\,R}, \\
C &::= B \mid \neg C \mid C_1 \sqcap C_2 \mid {\geq q\,R.C} \mid C_1\,\mathcal{U}\,C_2,
\end{aligned}
$$

where $q \geq 1$ is a natural number. The language $T_{\Diamond}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ is a proper sub-language of $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ in which the until operator $\mathcal{U}$ can occur only in concepts of the form $\top\,\mathcal{U}\,C$, where $\top = \neg\bot$. As usual in temporal logic, we denote $\top\,\mathcal{U}\,C$ by $\Diamond C$, and also write $\Box C$ for $\neg\Diamond\neg C$ and $\bigcirc C$ for $\bot\,\mathcal{U}\,C$ (so, $\bigcirc$ is a $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ concept construct). Other standard abbreviations we use are as follows: $C_1 \sqcup C_2 = \neg(\neg C_1 \sqcap \neg C_2)$, $\exists R = ({\geq 1\,R})$ and $\leq q\,R = \neg({\geq q+1\,R})$.

A $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ *TBox* $\mathcal{T}$ is a finite set of *concept inclusions*, *role inclusions*, and *role disjointness, irreflexivity and reflexivity constraints* of the form:

$$
C_1 \sqsubseteq C_2, \qquad R_1 \sqsubseteq R_2, \qquad \mathsf{Dis}(R_1, R_2), \qquad \mathsf{Irr}(S) \qquad \text{and} \qquad \mathsf{Ref}(S).
$$

We write $inv(R)$ for S^- if $R = S$, and for S if $R = S^-$. Denote by $\sqsubseteq_{\mathcal{T}}^*$ the reflexive and transitive closure of $\{(R, R'), (inv(R), inv(R')) \mid R \sqsubseteq R' \in \mathcal{T}\}$. Say that R' is a *proper sub-role of* R in $\mathcal{T}$ if $R' \sqsubseteq_{\mathcal{T}}^* R$ and $R \not\sqsubseteq_{\mathcal{T}}^* R'$. The following syntactic conditions, limiting the interaction between number restrictions and role inclusions, are imposed on $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ TBoxes $\mathcal{T}$ (cf. [18, 2]):

(inter) if R has a proper sub-role in $\mathcal{T}$ then $\mathcal{T}$ contains no *negative occurrences*[1] of number restrictions $\geq q\,R$ or $\geq q\,inv(R)$ with $q \geq 2$;

(exists) $\mathcal{T}$ may contain only positive occurrences of $\geq q\,R.C$, and if $\geq q\,R.C$ occurs in $\mathcal{T}$ then $\mathcal{T}$ does not contain negative occurrences of $\geq q'\,R$ or $\geq q'\,inv(R)$, for $q' \geq 2$.

It follows that no TBox can contain both a functionality constraint $\geq 2\,R \sqsubseteq \bot$ and an occurrence of $\geq q\,R.C$, for some $q \geq 1$ and some role R. (These conditions are required for NP-completeness of satisfiability in $DL\text{-}Lite_{bool}^{(\mathcal{RN})}$.)

An *ABox* $\mathcal{A}$ consists of assertions of the form:

$$
\bigcirc^n B(a), \quad \Box B(a), \quad \bigcirc^n S(a,b), \quad \Box S(a,b), \quad \bigcirc^n \neg S(a,b) \quad \text{and} \quad \Box\neg S(a,b),
$$

---

[1] An occurrence of a concept on the right-hand (left-hand) side of a concept inclusion is called *negative* if it is in the scope of an odd (even) number of negations $\neg$; otherwise the occurrence is called *positive*.

where B is a basic concept, S a (local or rigid) role name, a,b object names and $\bigcirc^n$ denotes the sequence of n *next-time operators* $\bigcirc$, for $n \geq 0$ (inverse roles could also be allowed in the ABoxes, but they are just syntactic sugar). The TBox and ABox together form the *knowledge base* (KB) $\mathcal{K} = (\mathcal{T}, \mathcal{A})$.

A $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ *interpretation* $\mathcal{I}$ is a function on natural numbers $\mathbb{N}$:

$$\mathcal{I}(n) \;=\; \left(\Delta^{\mathcal{I}}, a_0^{\mathcal{I}}, \ldots, A_0^{\mathcal{I}(n)}, \ldots, P_0^{\mathcal{I}(n)}, \ldots, G_0^{\mathcal{I}(n)}, \ldots\right),$$

where $\Delta^{\mathcal{I}}$ is a non-empty set, the *domain of* $\mathcal{I}$, $a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, $A_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}}$ and $P_i^{\mathcal{I}(n)}, G_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, for all i and all $n \in \mathbb{N}$. Furthermore, $a_i^{\mathcal{I}} \neq a_j^{\mathcal{I}}$ for $i \neq j$ (which means that we adopt the *unique name Assumption*) and $G_i^{\mathcal{I}(n)} = G_i^{\mathcal{I}(m)}$, for all $n, m \in \mathbb{N}$. The role and concept constructs are interpreted in $\mathcal{I}$ as follows: for each moment of time $n \in \mathbb{N}$,

$$(S^-)^{\mathcal{I}(n)} = \left\{(y,x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (x,y) \in S^{\mathcal{I}(n)}\right\}, \qquad \perp^{\mathcal{I}(n)} = \emptyset,$$

$$(\geq q\,R.C)^{\mathcal{I}(n)} = \left\{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \in C^{\mathcal{I}} \mid (x,y) \in R^{\mathcal{I}(n)}\} \geq q\right\}, \quad (\neg C)^{\mathcal{I}(n)} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}(n)},$$

$$(C \sqcap D)^{\mathcal{I}(n)} = C^{\mathcal{I}(n)} \cap D^{\mathcal{I}(n)}, \qquad\qquad (\geq q\,R)^{\mathcal{I}(n)} = (\geq q\,R.\neg\perp)^{\mathcal{I}(n)},$$

$$(C\,\mathcal{U}\,D)^{\mathcal{I}(n)} = \bigcup_{k>n}\left(D^{\mathcal{I}(k)} \cap \bigcap_{n<m<k} C^{\mathcal{I}(m)}\right),$$

where $\sharp X$ is the cardinality of X. The satisfaction relation $\models$ is defined as follows:

$$\mathcal{I} \models C_1 \sqsubseteq C_2 \quad\text{iff}\quad C_1^{\mathcal{I}(n)} \subseteq C_2^{\mathcal{I}(n)} \text{ for all } n \geq 0,$$

$$\mathcal{I} \models R_1 \sqsubseteq R_2 \quad\text{iff}\quad R_1^{\mathcal{I}(n)} \subseteq R_2^{\mathcal{I}(n)} \text{ for all } n \geq 0,$$

$$\mathcal{I} \models \mathsf{Dis}(R_1, R_2) \quad\text{iff}\quad \forall n \geq 0\,(R_1^{\mathcal{I}(n)} \cap R_2^{\mathcal{I}(n)} = \emptyset) \ \ (R_1 \text{ and } R_2 \text{ are } disjoint),$$

$$\mathcal{I} \models \mathsf{Irr}(S) \quad\text{iff}\quad \forall x \in \Delta^{\mathcal{I}} \,\forall n \geq 0\,(x,x) \notin S^{\mathcal{I}(n)} \ \ (S \text{ is } irreflexive),$$

$$\mathcal{I} \models \mathsf{Ref}(S) \quad\text{iff}\quad \forall x \in \Delta^{\mathcal{I}} \,\forall n \geq 0\,(x,x) \in S^{\mathcal{I}(n)} \ \ (S \text{ is } reflexive),$$

$$\mathcal{I} \models \bigcirc^n B(a) \ \text{ iff } \ a^{\mathcal{I}} \in B^{\mathcal{I}(n)}, \qquad \mathcal{I} \models \Box B(a) \ \text{ iff } \ \forall n > 0 \ a^{\mathcal{I}} \in B^{\mathcal{I}(n)},$$

$$\mathcal{I} \models \bigcirc^n S(a,b) \ \text{ iff } \ (a^{\mathcal{I}}, b^{\mathcal{I}}) \in S^{\mathcal{I}(n)}, \qquad \mathcal{I} \models \Box S(a,b) \ \text{ iff } \ \forall n > 0 \,(a^{\mathcal{I}}, b^{\mathcal{I}}) \in S^{\mathcal{I}(n)},$$

$$\mathcal{I} \models \bigcirc^n \neg S(a,b) \ \text{ iff } \ (a^{\mathcal{I}}, b^{\mathcal{I}}) \notin S^{\mathcal{I}(n)}, \quad \mathcal{I} \models \Box \neg S(a,b) \ \text{ iff } \ \forall n > 0 \,(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin S^{\mathcal{I}(n)}.$$

We say that $\mathcal{I}$ is a *model* of a KB $\mathcal{K}$ if $\mathcal{I} \models \alpha$ for all α in $\mathcal{K}$; in this case we also write $\mathcal{I} \models \mathcal{K}$. A concept A (role R) is *satisfiable w.r.t.* $\mathcal{K}$ if there are a model $\mathcal{I}$ of $\mathcal{K}$ and $n \geq 0$ such that $A^{\mathcal{I}(n)} \neq \emptyset$ ($R^{\mathcal{I}(n)} \neq \emptyset$). Note that role symmetry $\mathsf{Sym}(S)$ and asymmetry $\mathsf{Asym}(S)$ constraints are syntactic sugar in this language: they can be equivalently replaced with $S^- \sqsubseteq S$ and $\mathsf{Dis}(S, S^-)$, respectively.

It should be noted that $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ is not a simple fusion of $DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ and $\mathcal{LTL}$. Indeed, consider $\mathcal{K} = (\{\Diamond \exists R^- \sqsubseteq \perp, \exists R \sqsubseteq \Diamond \exists R\}, \{\exists R(a)\})$. Clearly, $\mathcal{K}$ is not satisfiable in $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$. However, it is satisfiable both in $DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ (if we substitute the temporal concepts by fresh $DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ concepts) and in $\mathcal{LTL}$ (by substituting $\exists R$ concepts with fresh atomic propositions).

3 $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ and First-Order Temporal Logic

For a $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, let $ob(\mathcal{A})$ be the set of all object names occurring in $\mathcal{A}$. Let $role^{\pm}(\mathcal{K})$ be the set of rigid and local role names, together

with their inverses, occurring in $\mathcal{K}$ and $grole^\pm(\mathcal{K})$ its subset of rigid roles. For $R \in role^\pm(\mathcal{K})$, let $Q_\mathcal{K}^R$ be the set of natural numbers containing 1 and all the numerical parameters q for which $\geq q\,R$ or $\geq q\,R.C$ occurs in $\mathcal{K}$.

With every object name $a \in ob(\mathcal{A})$ we associate the individual constant a of $\mathcal{QTL}^1$, the one-variable fragment of first-order temporal logic over $(\mathbb{N}, <)$, and with every concept name A the unary predicate $A(x)$ from the signature of $\mathcal{QTL}^1$. For each $R \in role^\pm(\mathcal{K})$, we also introduce $|Q_\mathcal{K}^R|$ fresh unary predicates $E_q R(x)$, for $q \in Q_\mathcal{K}^R$. Intuitively, for each $n \geq 0$, $E_1 R(x)$ and $E_1 R^-(x)$ represent the domain and range of R at moment n (i.e., $E_1 R(x)$ and $E_1 R^-(x)$ are interpreted by the sets of points with *at least one* R-successor and *at least one* R-predecessor at moment n, respectively), while $E_q R(x)$ and $E_q R^-(x)$ represent the sets of points with *at least q distinct* R-successors and *at least q distinct* R-predecessors at moment n.

Let us consider first the sub-language of $T_\mathcal{U}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ without qualified number restrictions and role constraints; we denote it by $T_\mathcal{U}DL\text{-}Lite_{bool}^{(\mathcal{RN})^-}$. Without loss of generality, we will assume that $Q_\mathcal{K}^R \subseteq Q_\mathcal{K}^{R'}$ whenever $R \sqsubseteq_\mathcal{T}^* R'$ (for if this is not the case we can always add the missing numbers to $Q_\mathcal{K}^{R'}$ by introducing fictitious concept inclusions of the form $\bot \sqsubseteq\, \geq q\,R'$). By induction on the construction of a $T_\mathcal{U}DL\text{-}Lite_{bool}^{(\mathcal{RN})^-}$ concept C we define the $\mathcal{QTL}^1$- formula C^*:

$$\bot^* = \bot, \qquad\qquad (A)^* = A(x),$$
$$(\geq q\,R)^* = E_q R(x), \qquad\qquad (\neg C)^* = \neg C^*(x),$$
$$(C_1 \sqcap C_2)^* = C_1^*(x) \wedge C_2^*(x), \qquad (C_1\,\mathcal{U}\,C_2)^* = C_1^*(x)\,\mathcal{U}\,C_2^*(x),$$

and then extend this translation to $T_\mathcal{U}DL\text{-}Lite_{bool}^{(\mathcal{RN})^-}$ TBoxes $\mathcal{T}$ by taking:

$$\mathcal{T}^* = \bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}} \Box^+ \forall x\left(C_1^*(x) \to C_2^*(x)\right)$$
$$\wedge \bigwedge_{\substack{R \sqsubseteq R' \in \mathcal{T}\ \text{or} \\ inv(R) \sqsubseteq inv(R') \in \mathcal{T}}} \bigwedge_{q \in Q_\mathcal{K}^R} \Box^+ \forall x\left(E_q R(x) \to E_q R'(x)\right),$$

where $\Box^+ \varphi = \varphi \wedge \Box\varphi$. For $R \in role^\pm(\mathcal{K})$, we need two $\mathcal{QTL}^1$-sentences:

$$\varepsilon_R = \exists x\,E_1 R(x) \;\to\; \exists x\,inv(E_1 R)(x), \tag{1}$$

$$\delta_R = \bigwedge_{\substack{q,q' \in Q_\mathcal{K}^R,\ \ q'>q \\ q'>q''>q\ \text{for no}\ q'' \in Q_\mathcal{K}^R}} \forall x\left(E_{q'} R(x) \to E_q R(x)\right), \tag{2}$$

where $inv(E_1 R)$ is the predicate $E_1 S^-(x)$ if $R = S$ and $E_1 S(x)$ if $R = S^-$, for a role name S. Sentence (1) says that if the domain of R is non-empty then its range is non-empty either; the meaning of (2) should be obvious.

Now we define 'temporal slices' of the ABox $\mathcal{A}$.[2] Denote by $N_\mathcal{A}$ the maximum n with $\bigcirc^n B(a) \in \mathcal{A}$, $\bigcirc^n S(a, b) \in \mathcal{A}$ or $\bigcirc^n \neg S(a, b) \in \mathcal{A}$ (or 0 if there are no such

---

[2] We slightly abuse notation and, for $R \in role^\pm(\mathcal{K})$, write $\bigcirc^n R(a_i, a_j) \in \mathcal{A}$ to indicate that $\bigcirc^n S(a_i, a_j) \in \mathcal{A}$ if $R = S$, or $\bigcirc^n S(a_j, a_i) \in \mathcal{A}$ if $R = S^-$, where S is a (local or rigid) role name; similarly for $\Box R(a_i, a_j) \in \mathcal{A}$.

assertions in $\mathcal{A}$). For a *rigid* role $R \in role^{\pm}(\mathcal{K})$, we take:

$$\mathcal{A}_R^{\square} = \big\{ R(a,b) \mid \bigcirc^n R'(a,b) \in \mathcal{A} \text{ or } \square R'(a,b) \in \mathcal{A}, \text{ and } R' \sqsubseteq_{\mathcal{T}}^* R \big\}$$

and $\mathcal{A}_R^n = \mathcal{A}_R^{\square}$, for all n, $0 \leq n \leq N_{\mathcal{A}}$. For a *local* role $R \in role^{\pm}(\mathcal{K})$, we take:

$$\begin{aligned}
\mathcal{A}_R^{\square} &= \big\{ R(a,b) \mid \square R'(a,b) \in \mathcal{A}, \text{ for } R' \sqsubseteq_{\mathcal{T}}^* R \big\} \\
&\quad \cup \big\{ R(a,b) \mid R'(a,b) \in \mathcal{A}_{R'}^{\square}, \text{ for } R' \sqsubseteq_{\mathcal{T}}^* R \big\}, \\
\mathcal{A}_R^n &= \big\{ R(a,b) \mid \bigcirc^n R'(a,b) \in \mathcal{A}, \text{ for } R' \sqsubseteq_{\mathcal{T}}^* R \big\} \\
&\quad \cup \big\{ R(a,b) \mid R'(a,b) \in \mathcal{A}_{R'}^{\square}, \text{ for } R' \sqsubseteq_{\mathcal{T}}^* R \\
&\qquad\qquad\qquad \text{and either } R' \in grole^{\pm}(\mathcal{K}) \text{ or } n > 0 \big\}.
\end{aligned}$$

We also set $\mathcal{A}^{\square} = \bigcup_{R \in role^{\pm}(\mathcal{K})} A_R^{\square}$ and $\mathcal{A}^n = \bigcup_{R \in role^{\pm}(\mathcal{K})} A_R^n$, for $0 \leq n \leq N_{\mathcal{A}}$. The $\mathcal{QTL}^1$ translation of the ABox $\mathcal{A}$ is defined as follows:

$$\begin{aligned}
\mathcal{A}^* = \bigwedge_{\bigcirc^n B(a) \in \mathcal{A}} \bigcirc^n B^*(a) \ \wedge \ \bigwedge_{R(a,b) \in \mathcal{A}^n} \bigcirc^n E_{q_{R,a,\mathcal{A}^n}} R(a) \ \wedge \ \bigwedge_{R(a,b) \in \mathcal{A}^{\square}} \square E_{q_{R,a,\mathcal{A}^{\square}}} R(a) \\
\wedge \bigwedge_{\bigcirc^n \neg S(a,b) \in \mathcal{A}} (\bigcirc^n \neg S(a,b))^{\perp} \ \wedge \ \bigwedge_{\square \neg S(a,b) \in \mathcal{A}} (\square \neg S(a,b))^{\perp},
\end{aligned}$$

where, for a role R, an ABox $\mathcal{A}'$ and $a \in ob(\mathcal{A}')$,

$$q_{R,a,\mathcal{A}'} = \max(\{0\} \cup \{q \in Q_{\mathcal{K}}^R \mid R(a,a_i) \in \mathcal{A}', 1 \leq i \leq q \ \& \ a_{i_1} \neq a_{i_2} \text{ if } i_1 \neq i_2\}),$$

and $(\bigcirc^n \neg S(a,b))^{\perp} = \perp$ if $S(a,b) \in \mathcal{A}^n$ and $\top$ otherwise, and $(\square \neg S(a,b))^{\perp} = \perp$ if $S(a,b) \in \mathcal{A}^{\square}$ or $S(a,b) \in \mathcal{A}^n$, for $0 < n \leq N_{\mathcal{A}}$, and $\top$ otherwise. Finally, let

$$\mathcal{K}^{\ddagger} = \mathcal{T}^* \wedge \bigwedge_{R \in role^{\pm}(\mathcal{K})} \square^+ (\varepsilon_R \wedge \delta_R) \ \wedge \bigwedge_{T \in grole^{\pm}(\mathcal{K})} \bigwedge_{q \in Q_{\mathcal{K}}^T} \square^+ \forall x \, \big(E_q T(x) \leftrightarrow \square E_q T(x) \big) \ \wedge \ \mathcal{A}^*.$$

Observe that the length of $\mathcal{K}^{\ddagger}$ is linear in length of $\mathcal{K}$. It can be shown in a way similar to [7, Theorem 2 and Corollary 3]) that we have:

Lemma 1. *A* $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})^-}$ *KB* $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ *is satisfiable iff the* $\mathcal{QTL}^1$*-sentence* $\mathcal{K}^{\ddagger}$ *is satisfiable.*

Proof. ($\Leftarrow$) Let $\mathfrak{M}$ be a first-order temporal model with a *countable* domain D and let $(\mathfrak{M}, 0) \models \mathcal{K}^{\ddagger}$ (if $\mathcal{K}^{\ddagger}$ is satisfiable then such an $\mathfrak{M}$ clearly exists). We denote the interpretations of unary predicates P in $\mathfrak{M}$ at moment n by $P^{\mathfrak{M},n}$ and the interpretations of constants a in $\mathfrak{M}$ by $a^{\mathfrak{M}}$ (without loss of generality we assume that the $a^{\mathfrak{M}}$ are all distinct). We are going to construct a model $\mathcal{I}$ of $\mathcal{K}$ based on the domain $\Delta = \bigcup_{m=0}^{\infty} W_m$, where

$$W_0 = \big\{ a^{\mathfrak{M}} \mid a \in ob(\mathcal{A}) \big\} \subseteq D \qquad \text{and} \qquad W_{m+1} = W_m \cup (D \times \mathbb{N} \times \{m\}).$$

The interpretations of object names in $\mathcal{I}$ are given by their interpretations in $\mathfrak{M}$: $a^{\mathcal{I}(n)} = a^{\mathfrak{M}} \in W_0$. The interpretations $A^{\mathcal{I}(n)}$ of concept names A in $\mathcal{I}$ are

set to be $A^{\mathcal{I}(n)} = \{w \in \Delta \mid (\mathfrak{M}, n) \models A^*[cp(w)]\}$, where $cp \colon \Delta \to D$ is defined by taking:

$$cp(w) = \begin{cases} w, & \text{if } w \in W_0, \\ d, & \text{if } w = (d, k, m) \in D \times \mathbb{N} \times \mathbb{N}. \end{cases}$$

We call w a *copy* of $cp(w)$.

It remains to define $S^{\mathcal{I}(n)}$ for each role S in $\mathcal{K}$ and $n \in \mathbb{N}$. Let us first consider a minimal role S for which $R \sqsubseteq^*_{\mathcal{T}} S$ implies $S \sqsubseteq^*_{\mathcal{T}} R$, for every $R \in role^{\pm}(\mathcal{K})$. Let $[S] = \{R \in role^{\pm}(\mathcal{K}) \mid R \sqsubseteq^*_{\mathcal{T}} S \text{ and } S \sqsubseteq^*_{\mathcal{T}} R\}$. Consider first the case when $[S] \cap grole^{\pm}(\mathcal{K}) = \emptyset$ (i.e., S is not equivalent to a rigid role). Fix some $n \in \mathbb{N}$. We set $S^{\mathcal{I}(n)} = \bigcup_{m=0}^{\infty} S^{n,m}$, where $S^{n,m} \subseteq W_m \times W_m$ are defined inductively (on $m \geq 0$) as follows. For the basis of induction, set

$$S^{n,0} = \{(a^{\mathfrak{M}}, b^{\mathfrak{M}}) \in W_0 \times W_0 \mid S(a, b) \in \mathcal{A}^n, n \leq N_{\mathcal{A}}, \text{ or } S(a, b) \in \mathcal{A}^{\square}, n > 0\}.$$

Suppose the $S^{n,m}$ have been defined. Given $R = S$ or S^-, the *required R-rank* $r^n(R, d)$ *of* $d \in D$ *at moment* n is $\max(\{0\} \cup \{q \in Q^R_{\mathcal{K}} \mid (\mathfrak{M}, n) \models E_q R[d]\})$. By (2), if $r^n(R, d) = q$ then, for every $q' \in Q^R_{\mathcal{K}}$, we have $(\mathfrak{M}, n) \models E_{q'} R[d]$ whenever $q' \leq q$ and $(\mathfrak{M}, n) \models \neg E_{q'} R[d]$ whenever $q < q'$. The *actual R-rank* $r^n_m(R, w)$ *of* $w \in \Delta$ *at moment* n *and step* m is $\sharp\{w' \in W_m \mid (w, w') \in R^{n,m}\}$, where $R^{n,m} = S^{n,m}$ if $R = S$ and $R^{n,m} = \{(w, w') \mid (w', w) \in S^{n,m}\}$ if $R = S^-$. It will follow from our construction that $r^n_m(S, w) \leq r^n(S, cp(w))$, for all $w \in W_m$ (we leave the easy inductive proof to the reader). Consider now the two sets of *defects* $\Lambda^{n,m}_S$ and $\Lambda^{n,m}_{S^-}$ in $S^{n,m}$, where

$$\Lambda^{n,m}_R = \{w \in W_m \mid r^n_m(R, w) < r^n(R, cp(w))\}.$$

The purpose of $\Lambda^{n,m}_R$ is to identify those 'defective' points $w \in W_m$ from which precisely $r^n(R, cp(w))$ distinct R-arrows should start (according to $\mathfrak{M}$), but some arrows are missing (only $r^n_m(R, w)$ many arrows exist). To 'cure' these defects, we need a pool $F_m \subseteq W_{m+1} \setminus W_m$ of witnesses that can be used at step m of the unravelling construction: more precisely, it contains, for each role R, a countably infinite supply of points w (*witnesses for R*) such that $cp(w) = d$ and $(\mathfrak{M}, n) \models inv(E_1 R)[d]$, provided that $(\mathfrak{M}, n) \models \exists x \, (E_1 R(x) \vee inv(E_1 R)(x))$ (by (1), either both $E_1 R(x)$ and $inv(E_1 R)(x)$ are empty or both are non-empty). It should be emphasised that this set will be the same for all roles R and all moments of time n. We extend $S^{n,m}$ to $S^{n,m+1}$ according to the following rules:

$(\Lambda^{n,m}_S)$ If $w \in \Lambda^{n,m}_S$, then let $q = r^n(S, d) - r^n_m(S, w)$ and $d = cp(w)$. We have $(\mathfrak{M}, n) \models E_{q'} S[d]$ for some $q' \in Q^S_{\mathcal{K}}$. By (2), $(\mathfrak{M}, n) \models E_1 S[d]$. In this case we take q *fresh witnesses* $w_1, \ldots, w_q \in F_m$ for S, remove them from F_m and then add the pairs (w, w_i), $1 \leq i \leq q$, to $S^{n,m+1}$; we also add the pairs (w_i, w), $1 \leq i \leq q$, to $S^{n,m+1}$ if $S^- \in [S]$.

$(\Lambda^{n,m}_{S^-})$ The mirror image of $(\Lambda^{n,m}_S)$.

In a way similar to [2, Section 5.3] one can show that, for all $n \geq 0$ and $R = S, S^-$

$$(\mathfrak{M}, n) \models E_q R[cp(w)] \quad \text{iff} \quad w \in (\geq q\, R)^{\mathcal{I}(n)}. \tag{3}$$

Once we have defined $S^{\mathcal{I}(n)}$ for S, we set $(S')^{\mathcal{I}(n)} = S^{\mathcal{I}(n)}$ for all $S' \in [S]$ and $n \geq 0$ (observe that $\mathcal{A}_S^\square = \mathcal{A}_{S'}^\square$ and $\mathcal{A}_S^n = \mathcal{A}_{S'}^n$, for all n, if $S' \in [S]$). If $[S] \cap grole^\pm(\mathcal{K}) \neq \emptyset$, then we perform the above unravelling procedure at moment 0 and set $S^{\mathcal{I}(n)} = S^{\mathcal{I}(0)}$ for all $n > 0$, and $(S')^{\mathcal{I}(n)} = S^{\mathcal{I}(n)}$, for all $S' \in [S]$ and $n \geq 0$ (observe that $\mathcal{A}_S^\square = \mathcal{A}_{S'}^\square = \mathcal{A}_S^n = \mathcal{A}_{S'}^n$, for all n, if $S' \in [S]$).

Suppose now that S has a proper sub-role and we have already defined the $R^{\mathcal{I}(n)}$ for all such proper sub-roles R. For S with $[S] \cap grole^\pm(\mathcal{K}) = \emptyset$, the unravelling procedure is analogous to the one described above: the basis of induction is defined as above; then, at every step, we first expand $S^{n,m}$ with the pairs that belong to its proper sub-roles, i.e., $\bigcup_{R \sqsubseteq_{\mathcal{T}}^* S, S \not\sqsubseteq_{\mathcal{T}}^* R} R^{n,m}$, and only after that start curing the defects (remember that a point in F_m can never be used twice as a witness). As the actual rank may be greater than the required rank, we have the following: for all $n \geq 0$ and $R = S, S^-$,

$$\text{if } (\mathfrak{M}, n) \models E_q R[cp(w)] \quad \text{then} \quad w \in (\geq q\, R)^{\mathcal{I}(n)}. \tag{4}$$

If $[S] \cap grole^\pm(\mathcal{K}) \neq \emptyset$ then we perform the unravelling procedure only for the moment 0 and, at each step, first expand $S^{0,m}$ with the pairs that belong to its proper sub-roles at any moment of time (i.e., $\bigcup_{R \sqsubseteq_{\mathcal{T}} S, S \not\sqsubseteq_{\mathcal{T}} R} \bigcup_{n=0}^\infty R^{n,m}$) and only after that cure the defects (yet again a point in F_m can never be used twice as a witness). Finally, we set $S^{\mathcal{I}(n)} = S^{\mathcal{I}(0)}$, for all $n > 0$, and $(S')^{\mathcal{I}(n)} = S^{\mathcal{I}(n)}$, for all $S' \in [S]$ and $n \geq 0$.

It remains to show that the constructed interpretation $\mathcal{I}$ is indeed a model of $\mathcal{K}$. It follows from the construction that $\mathcal{I} \models R_1 \sqsubseteq R_2$, for each $R_1 \sqsubseteq R_2 \in \mathcal{T}$. It also follows that (3) holds for every role without proper sub-roles and, for a role that has proper sub-roles, (4) is enough in view of **(inter)**: by induction on the structure of concepts, one can show that, for each concept inclusion $C_1 \sqsubseteq C_2 \in \mathcal{T}$, we have $\mathcal{I} \models C_1 \sqsubseteq C_2$ whenever $(\mathfrak{M}, n) \models \forall x\, (C_1^*(x) \to C_2^*(x))$, for all $n \geq 0$. Thus, $\mathcal{I} \models \mathcal{T}$. We also have $\mathcal{I} \models \mathcal{A}$ and thus $\mathcal{I} \models \mathcal{K}$.

For $(\Rightarrow)$ we refer the reader to [7, Theorem 2 and Corollary 3]. ❏

Denote by $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{\mathcal{N}}$ the fragment of $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})^-}$ such that (i) it has no rigid roles, (ii) its TBoxes contain only concept inclusions and (iii) its ABoxes contain only assertions of the form $\bigcirc^n B(a)$ and $\square B(a)$.

Lemma 2. *Given a $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ KB $\mathcal{K}$, one can construct (in polynomial time) an equisatisfiable $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{\mathcal{N}}$ KB $\mathcal{K}'$.*

Proof. As a first step, one can construct a $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})^-}$ KB such that it is equisatisfiable with $\mathcal{K}$. The proof is based on the forest model property (cf. [2, Remark 5.15]) and is similar to that of [2, Lemma 5.17]. It shows how to get rid of qualified number restrictions, role disjointness, reflexivity and irreflexivity constraints. So, without loss of generality we may assume that $\mathcal{K}$ is a $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})^-}$ KB.

Next, we set $\mathcal{K}' = (\{\top \sqsubseteq \bot\}, \emptyset)$ if $\bigcirc^n \neg S(a, b) \in \mathcal{A}$ and $S(a, b) \in \mathcal{A}^n$, or $\square \neg S(a, b) \in \mathcal{A}$ and $S(a, b) \in \mathcal{A}^\square$ or $S(a, b) \in \mathcal{A}^n$, $0 < n \leq N_{\mathcal{A}}$. Otherwise, let $\mathcal{A}_0$ be the part of $\mathcal{A}$ with assertions of the form $\bigcirc^n B(a)$ and $\square B(a)$ only

and let $\mathcal{T}_0$ be the part of $\mathcal{T}$ that contains no role inclusion axioms. Consider $\mathcal{K}' = (\mathcal{T}_0 \cup \mathcal{T}'', \mathcal{A}_0 \cup \mathcal{A}'')$, where

$$\mathcal{T}'' = \big\{ \square(\geq q\, T) \sqsubseteq (\geq q\, T),\ (\geq q\, T) \sqsubseteq \square(\geq q\, T) \mid q \in Q^T_{\mathcal{K}}, T \in grole^{\pm}(\mathcal{K}) \big\}$$
$$\cup \big\{ (\geq q\, R) \sqsubseteq (\geq q\, R') \mid q \in Q^R_{\mathcal{K}}, R \sqsubseteq R' \in \mathcal{T} \text{ or } inv(R) \sqsubseteq inv(R') \in \mathcal{T} \big\},$$
$$\mathcal{A}'' = \big\{ \bigcirc^n (\geq q_{R,a,\mathcal{A}^n}\, R)(a) \mid R(a,b) \in \mathcal{A}^n \big\} \cup \big\{ \square(\geq q_{R,a,\mathcal{A}^\square}\, R)(a) \mid R(a,b) \in \mathcal{A}^\square \big\}.$$

Clearly, $\mathcal{K}^{\ddagger} = (\mathcal{K}')^{\ddagger}$. The claim follows immediately from Lemma 1. $\qquad\qquad \square$

4 Satisfiability of $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ KBs is PSpace-complete

It follows from Lemma 2 that satisfiability of $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ KBs is reducible to satisfiability of $T^0_{\mathcal{U}}DL\text{-}Lite_{bool}^{\mathcal{N}}$ KBs. Our plan is as follows. First, we define a notion of *quasimodel* for a $T^0_{\mathcal{U}}DL\text{-}Lite_{bool}^{\mathcal{N}}$ KB and prove that such a KB is satisfiable iff there exists a quasimodel for it. Then we show that if there is a quasimodel for a KB then there exists an *ultimately periodic quasimodel* such that both the length of the prefix and the length of the period are exponential in the length of the KB. The existence of such a quasimodel can be checked in non-deterministic polynomial space, which together with Lemma 2 provides us with a PSpace upper complexity bound for satisfiability in $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$; the matching lower bound follows from the complexity of $\mathcal{LTL}$.

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a $T^0_{\mathcal{U}}DL\text{-}Lite_{bool}^{\mathcal{N}}$ KB. We assume $ob(\mathcal{A}) \neq \emptyset$. Denote by $ev(\mathcal{K})$ the set of all concepts of the form $C\,\mathcal{U}\,D$ occurring in $\mathcal{K}$. We introduce, for every $C\,\mathcal{U}\,D \in ev(\mathcal{K})$, a fresh concept name $F_{C\mathcal{U}D}$, the *surrogate* of $C\,\mathcal{U}\,D$, and then, for a concept C, denote by $\overline{C}$ the result of replacing each $C'\,\mathcal{U}\,D'$ in C, which is not in the scope of another $\mathcal{U}$, with the surrogate $F_{C'\mathcal{U}D'}$. For a $T^0_{\mathcal{U}}DL\text{-}Lite_{bool}^{\mathcal{N}}$ TBox $\mathcal{T}$, denote by $\overline{\mathcal{T}}$ the $DL\text{-}Lite_{bool}^{\mathcal{N}}$ TBox obtained by replacing every concept C in $\mathcal{T}$ with $\overline{C}$.

Let $cl(\mathcal{K})$ be the closure under negation of all concepts occurring in $\mathcal{T}$ together with the $\exists R$, for $R \in role^{\pm}(\mathcal{K})$, and the B, for $\bigcirc^n B(a) \in \mathcal{A}$ or $\square B(a) \in \mathcal{A}$. A *type* for $\mathcal{K}$ is a subset $\mathbf{t}$ of $cl(\mathcal{K})$ such that

- $C \sqcap D \in \mathbf{t}$ iff $C, D \in \mathbf{t}$, for every $C \sqcap D \in cl(\mathcal{K})$;
- $\neg C \in \mathbf{t}$ iff $C \notin \mathbf{t}$, for every $C \in cl(\mathcal{K})$.

A type $\mathbf{t}$ for $\mathcal{K}$ is *realisable* if the concept $\sqcap_{C \in \mathbf{t}} \overline{C}$ is satisfiable w.r.t. $\overline{\mathcal{T}}$.

A function r mapping $\mathbb{N}$ to types for $\mathcal{K}$ is called a *coherent* and *saturated run* for $\mathcal{K}$ if the following conditions are satisfied:

(real) $r(i)$ is realisable, for every $i \geq 0$,

(coh) for all $0 \leq j < i$ and $C\,\mathcal{U}\,D \in ev(\mathcal{K})$, if $D \in r(i)$ and $C \in r(k)$, for all k, $j < k < i$, then $C\,\mathcal{U}\,D \in r(j)$;

(sat) for all $i \geq 0$ and $C\,\mathcal{U}\,D \in ev(\mathcal{K})$, if $C\,\mathcal{U}\,D \in r(i)$ then there is $j > i$ such that $D \in r(j)$ and $C \in r(k)$ for all k, $i < k < j$.

Given a model $\mathcal{I}$ of $\mathcal{K}$ and $d \in \Delta^{\mathcal{I}}$, let $run_{\mathcal{I}}(d)\colon i \mapsto \{C \in cl(\mathcal{K}) \mid d \in C^{\mathcal{I}(i)}\}$. Clearly, each $run_{\mathcal{I}}(d)$ is a coherent and saturated run for $\mathcal{K}$. For a run r and a finite sequence $s = (s(0), \dots, s(n))$ of types for $\mathcal{K}$, we denote by s^k the k repetitions of s and let

$$r^{<i} = (r(0), \dots, r(i-1)), \qquad s^{\omega} = (s(0), \dots, s(n), s(0), \dots, s(n), \dots),$$
$$r^{\geq i} = (r(i), r(i+1), \dots), \qquad s \cdot r = (s(0), \dots, s(n), r(0), r(1), \dots).$$

A *witness* for $\mathcal{K}$ is a pair of the form (r, Ξ), where r is a coherent and saturated run for $\mathcal{K}$, $\Xi \subseteq \mathbb{N}$ and $|\Xi| \leq 1$. A *quasimodel* for $\mathcal{K}$ is a quadruple $\mathfrak{Q} = \langle W, K, K_0, L \rangle$, where W is a set of witnesses for $\mathcal{K}$ and K, K_0, L are natural numbers with $0 \leq K < K_0 < L$ such that:

(run) $W = \{(r_a, \emptyset) \mid a \in ob(\mathcal{A})\} \cup \{(r_R, \{i_R\}) \mid R \in \Omega\}$, for some $\Omega \subseteq role^{\pm}(\mathcal{K})$;
(rep) $r_R(K) = r_R(K_0)$, for each $(r_R, \{i_R\}) \in W$;
(obj) if $\bigcirc^n B(a) \in \mathcal{A}$ then $B \in r_a(n)$; if $\square B(a) \in \mathcal{A}$ then $B \in r_a(i)$ for all $i > 0$;
(role) for all $i \geq 0$ and $R \in role^{\pm}(\mathcal{K})$, if $\exists R^- \in r(i)$, for some $(r, \Xi) \in W$, then $(r_R, \{i_R\}) \in W$, $\exists R \in r_R(i_R)$ and either $i \leq i_R < K$ or $K \leq i_R < L$.

Theorem 1. *A $T^0_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{bool}$ KB $\mathcal{K}$ is satisfiable iff there is a quasimodel $\mathfrak{Q} = \langle W, K, K_0, L \rangle$ for $\mathcal{K}$ with $L \leq N_{\mathcal{A}} + 2^{|cl(\mathcal{K})| \cdot |role^{\pm}(\mathcal{K})|} \cdot (|role^{\pm}(\mathcal{K})| + 1) + 2^{|cl(\mathcal{K})|} + 2$.*

Proof. Suppose $\mathcal{I} \models \mathcal{K}$. For $m \geq 0$, let

$$\mathbf{F}^m = \{R \in role^{\pm}(\mathcal{K}) \mid \text{ there is } i \geq m \text{ with } R^{\mathcal{I}(i)} \neq \emptyset\}.$$

Lemma 3. *For all $n, v \geq 0$, there exists m such that $n \leq m \leq n + v \cdot |\mathbf{F}^0|$ and, for every role $R \in \mathbf{F}^0$, either $R \in \mathbf{F}^{m+v+1}$ or $R \notin \mathbf{F}^{m+1}$.*

Proof. If a role R is non-empty infinitely often then $R \in \mathbf{F}^{m+v+1}$, for any m. So we have to consider only those roles that are non-empty finitely many times. Let $\mathbf{FG} = \{R \in role^{\pm}(\mathcal{K}) \mid \text{ there is } i \geq 0 \text{ with } R \notin \mathbf{F}^i\}$. For $R \in \mathbf{FG} \cap \mathbf{F}^0$, let $i_R = \min\{i \mid R \notin \mathbf{F}^{i+1}\}$ (i.e., i_R is the last moment when R is non-empty). If $\max\{i_R \mid R \in \mathbf{FG}\} \leq n + v \cdot |\mathbf{F}^0|$, we take $m = \max(\{n\} \cup \{i_R \mid R \in \mathbf{FG}\})$. Clearly, $\mathbf{FG} \cap \mathbf{F}^{m+1} = \emptyset$ (so all roles in $\mathbf{FG}$ are empty after m). Otherwise, $\mathbf{FG} \cap \mathbf{F}^0 \neq \emptyset$ and without loss of generality we may assume that $\mathbf{FG} \cap \mathbf{F}^0 = \{R_1, \dots, R_s\}$ and $i_{R_1} \leq i_{R_2} \leq \cdots \leq i_{R_s}$. If $i_{R_1} > n + v$, we take $m = n$; then $\mathbf{FG} \cap \mathbf{F}^0 \subseteq \mathbf{F}^{m+v+1}$ (all roles in $\mathbf{FG} \cap \mathbf{F}^0$ are non-empty after $m+v$). Otherwise, $i_{R_1} \leq n + v$ and $i_{R_s} > n + v \cdot |\mathbf{F}^0|$, whence $i_{R_s} - i_{R_1} > (v-1) \cdot |\mathbf{F}^0|$. Let j_0 be the smallest j, $1 \leq j < s$, such that $i_{R_j} \geq n$ and $i_{R_{j+1}} - i_{R_j} > v$ (it exists as $s \leq |\mathbf{F}^0|$), and let $m = i_{R_{j_0}}$. We then clearly have $R_1, \dots, R_{j_0} \notin \mathbf{F}^{m+1}$ and $R_{j_0+1}, \dots, R_s \in \mathbf{F}^{m+v+1}$. $\qquad \Box$

Let $V = 2^{|cl(\mathcal{K})| \cdot |role^{\pm}(\mathcal{K})|}$. By Lemma 3, there is M, $N_{\mathcal{A}} \leq M \leq N_{\mathcal{A}} + V \cdot |\mathbf{F}^0|$, such that, for every role $R \in \mathbf{F}^0$, either $R \in \mathbf{F}^{M+V+1}$ or $R \notin \mathbf{F}^{M+1}$. We set

$$i_R = \begin{cases} \min\{i \geq M + V + 1 \mid R^{\mathcal{I}(i)} \neq \emptyset\}, & R \in \mathbf{F}^{M+V+1}, \\ \max\{i \mid R^{\mathcal{I}(i)} \neq \emptyset\}, & R \in \mathbf{F}^0 \setminus \mathbf{F}^{M+1}. \end{cases}$$

Clearly, for each $R \in \mathbf{F}^0$, either $i_R \leq M$ or $i_R \geq M + V + 1$. We fix some $d_R \in (\exists R)^{\mathcal{I}(i_R)}$ and set $r_R = run_{\mathcal{I}}(d_R)$, for each $R \in \mathbf{F}^0$. For $a \in ob(\mathcal{A})$, set $r_a = run_{\mathcal{I}}(a^{\mathcal{I}})$. Let

$$W = \big\{(r_a, \emptyset) \mid a \in ob(\mathcal{A})\big\} \cup \big\{(r_R, \{i_R\}) \mid R \in \mathbf{F}^0\big\}.$$

Clearly, **(run)** and **(obj)** hold for W. Also, $\exists R^- \in r(i)$ iff $\exists R \in r_R(i_R)$ and $(r_R, \{i_R\}) \in W$, for all $(r, \Xi) \in W$ and $i \geq 0$.

Observe that there are K, K_0 such that $M < K < K_0 \leq M + V + 1$ and $r_R(K) = r_R(K_0)$, for all $(r_R, \{i_R\}) \in W$. Let $L = K_0 + T + 1$. Next, for each $(r_R, \{i_R\}) \in W$ with $i_R \geq L$, we construct a new witness $(r'_R, \{i'_R\})$ such that $\exists R \in r'_R(i'_R)$ and $i'_R < L$: we remove every part $(r_R(n), \ldots, r_R(n'))$ of the run r_R such that $r_R(n) = r_R(n' + 1)$, for $K_0 < n < n' < i_R$. Let $(r'_R, \{i'_R\})$ be the result of this operation. It should be clear that r'_R is a coherent and saturated run for $\mathcal{K}$ and, as there are only T different types for $\mathcal{K}$, we have $i'_R < L$. Denote by W' the resulting set of witnesses. It is as an exercise for the reader to check that $\mathfrak{Q} = \langle W', K, K_0, L \rangle$ is a quasimodel for $\mathcal{K}$.

$(\Leftarrow)$ Let $\mathfrak{Q} = \langle W, K, K_0, L \rangle$ be a quasimodel for $\mathcal{K}$. We construct a model for $\mathcal{K}^{\ddagger}$ which, by Theorem 1, will show that $\mathcal{K}$ is satisfiable. Let

$$\mathfrak{R} = \big\{r_a \mid (r_a, \emptyset) \in W\big\} \cup$$
$$\big\{\big(r_R^{<K} \cdot (r_R(K), \ldots, r_R(K_0 - 1))^i \cdot r_R^{\geq K_0}\big)^{\geq j} \mid (r_R, \{i_R\}) \in W, \ i > 0, \ j \geq 0\big\}.$$

Clearly, each $r \in \mathfrak{R}$ is a coherent and saturated run for $\mathcal{K}$. Moreover, if we have $(r_R, \{i_R\}) \in W$ and $i_R < K$ then, for all $i, 0 \leq i \leq i_R$, there is $r' \in \mathfrak{R}$ with $\exists R \in r'(i)$. And if $(r_R, \{i_R\}) \in W$ and $i_R \geq K$ then, for all $i \geq 0$, there is $r' \in \mathfrak{R}$ with $\exists R \in r'(i)$. As follows from **(role)**, for each $R \in \Omega$, we have $R^- \in \Omega$ and either $i_R \geq K$ and $i_{R^-} \geq K$ or $i_R = i_{R^-} < K$. So, for all $i \geq 0$ and $r \in \mathfrak{R}$,

if $\exists R^- \in r(i)$ then there is $r' \in \mathfrak{R}$ such that $\exists R \in r'(i)$.

We construct a first-order temporal model $\mathfrak{M}$ based on the domain $D = \mathfrak{R}$ by taking $a^{\mathfrak{M}} = r_a$, for each $a \in ob(\mathcal{A})$, and $(B^*)^{\mathfrak{M},i} = \{r \in \mathfrak{R} \mid B \in r(i)\}$, for each $B \in cl(\mathcal{K})$ and $i \geq 0$. It should be clear that $(\mathfrak{M}, 0) \models \mathcal{K}^{\ddagger}$. $\qquad \square$

Theorem 2. *If there is a quasimodel $\mathfrak{Q} = \langle W, K, K_0, L \rangle$ for $\mathcal{K}$ then there is an ultimately periodic quasimodel $\mathfrak{Q}' = \langle W', K, K_0, L' \rangle$ for $\mathcal{K}$, that is, there are $L' \leq L + 2^N$ and $P \leq N \cdot 2^N$, where $N = |W| \cdot |cl(\mathcal{K})|$, such that $r'(i + P) = r'(i)$, for all $i \geq L'$ and $(r', \Xi') \in W'$.*

Proof. The proof is a straightforward modification of the standard $\mathcal{LTL}$ construction (see, e.g., [14]) with the set of propositions being $cl(\mathcal{K}) \times W$. $\qquad \square$

It follows from Lemma 2 and Theorems 1 and 2 that we have the following:

Theorem 3. *Satisfiability of $T_{\mathcal{U}}DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ KBs is* PSPACE-*complete.*

5 Satisfiability of $T_\diamond DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ KBs is NP-complete

We notice that Lemma 2 holds for $T_\diamond DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ and its respective fragment $T_\diamond^0 DL\text{-}Lite_{bool}^{\mathcal{N}}$. Thus, to prove that satisfiability of $T_\diamond DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ KBs is in NP, it is enough to consider $T_\diamond^0 DL\text{-}Lite_{bool}^{\mathcal{N}}$ KBs. We proceed as in Section 4. First we prove that a $T_\diamond^0 DL\text{-}Lite_{bool}^{\mathcal{N}}$ KB is satisfiable iff there exists a quasimodel for it. Then we show that if there is a quasimodel for $\mathcal{K}$ then there exists an ultimately periodic quasimodel for $\mathcal{K}$ such that both the length of the prefix and the length of the period are *polynomial* in the length of $\mathcal{K}$. As the existence of such a quasimodel can be checked in non-deterministic polynomial time, we obtain the NP upper bound. The matching lower bound will be shown for a sublogic $T_\diamond DL\text{-}Lite_{core}$ of $T_\diamond DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ with rather primitive concept inclusions.

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a $T_\diamond^0 DL\text{-}Lite_{bool}^{\mathcal{N}}$ KB. We say that a type $\mathbf{t}$ for $\mathcal{K}$ is *stutter-invariant* if $\neg\diamond C \in \mathbf{t}$ implies $\neg C \in \mathbf{t}$, for each $\diamond C \in ev(\mathcal{K})$. A *quasimodel* for $\mathcal{K}$ is a triple $\mathfrak{Q} = \langle W, K, L \rangle$, where W is a set of witnesses for $\mathcal{K}$ and K, L are natural numbers with $0 \leq K \leq L$ such that they satisfy **(run)**, **(obj)**, **(role)** and the following condition

(stuttr) $r(K)$ and the $r(i)$, for $i \geq L$, are stutter-invariant for each $(r, \Xi) \in W$.

Theorem 4. *A $T_\diamond^0 DL\text{-}Lite_{bool}^{\mathcal{N}}$ KB $\mathcal{K}$ is satisfiable iff there is a quasimodel $\mathfrak{Q} = \langle W, K, L \rangle$ for $\mathcal{K}$ such that $L \leq N_{\mathcal{A}} + |ev(\mathcal{K})| \cdot (|role^{\pm}(\mathcal{K})| + 2) + 3$.*

Proof. $(\Rightarrow)$ Suppose $\mathcal{I} \models \mathcal{K}$. Let $V = |ev(\mathcal{K})|$. By Lemma 3, there exists M with $N_{\mathcal{A}} \leq M \leq N_{\mathcal{A}} + V \cdot |\mathbf{F}^0|$ such that, for every role $R \in \mathbf{F}^0$, either $R \in \mathbf{F}^K$ or $R \notin \mathbf{F}^{M+1}$, where $K = M + V + 1$. We then set

$$i_R = \begin{cases} \min\{i \geq K \mid R^{\mathcal{I}(i)} \neq \emptyset\}, & R \in \mathbf{F}^K, \\ \max\{i \mid R^{\mathcal{I}(i)} \neq \emptyset\}, & R \in \mathbf{F}^0 \setminus \mathbf{F}^{M+1}. \end{cases}$$

Clearly, for $R \in \mathbf{F}^0$, either $i_R \leq M$ or $i_R \geq K$. For each $R \in \mathbf{F}^0$, we fix some $d_R \in (\exists R)^{\mathcal{I}(i_R)}$ and set $r_R = run_{\mathcal{I}}(d_R)$. For $a \in ob(\mathcal{A})$, set $r_a = run_{\mathcal{I}}(a^{\mathcal{I}})$. Let

$$W = \big\{(r_a, \emptyset) \mid a \in ob(\mathcal{A})\big\} \cup \big\{(r_R, \{i_R\}) \mid R \in \mathbf{F}^0\big\}.$$

Clearly, **(run)** and **(obj)** hold. Also, we have $\exists R^- \in r(i)$ iff $\exists R \in r_R(i_R)$ and $(r_R, \{i_R\}) \in W$, for all $(r, \Xi) \in W$ and $i \geq 0$.

We now transform W by expanding and pruning runs in such a way that the $r(i)$ are never thrown out for $(r, \Xi) \in W$ and $i \in \Xi$.

Lemma 4. *For each coherent and saturated run r,*

$$\big|\{i \mid r(i) \text{ is not stutter-invariant}\}\big| \leq \big|ev(\mathcal{K})\big|.$$

Proof. Suppose that there are $0 \leq i_1 < \cdots < i_n$ such that $n > |ev(\mathcal{K})|$ and $r(i_1), \ldots, r(i_n)$ are not stutter-invariant, i.e., there are $\diamond C_j \in ev(\mathcal{K})$ with $\neg\diamond C_j, C_j \in r(i_j)$. Then there is $\diamond C \in ev(\mathcal{K})$ such that $\neg\diamond C, C \in r(i_j), r(i_{j'})$

for some $0 \leq i_j < i_{j'}$. As $C \in r(i_{j'})$, we have, by **(coh)**, $\Diamond C \in r(i_j)$, contrary to $\neg \Diamond C \in r(i_j)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ❏

Step 1. By Lemma 4, for each $(r, \Xi) \in W$, there is j_r, $M < j_r \leq K$, such that $r(j_r)$ is stutter-invariant. Set

$$r' = r^{<j_r} \cdot r(j_r)^{K-j_r} \cdot r^{\geq j_r},$$
$$\Xi' = \{i \mid i \in \Xi, \ i \leq j_r\} \cup \{i + K - j_r \mid i \in \Xi, \ i > j_r\}.$$

Clearly, r' is a coherent and saturated run. Denote by W' the set of all (r', Ξ') constructed as above. Then, for each $(r', \Xi') \in W'$, $r'(K)$ is stutter-invariant. It is easy to see that, for each $R \in \mathbf{F}^0$, $(r'_R, \{i'_R\}) \in W'$ and either $i'_R \leq M$ or $i'_R \geq K$.

Step 2. For $(r', \Xi') \in W'$, let $\Xi^0 = \{i > K \mid r'(i) \text{ is not stutter-invariant}\}$. By Lemma 4, $|\Xi^0| \leq |ev(\mathcal{K})|$. If $\Xi^0 \cup \Xi' \neq \emptyset$, we prune the run r' by removing all stutter-invariant $r'(i)$ with $K < i < \max(\Xi^0 \cup \Xi')$. The resulting function r'' is a coherent and saturated run for $\mathcal{K}$. Set

$$\Xi'' = \{i \mid i \in \Xi', \ i \leq K\} \cup \{K + \sharp\{j \in \Xi^0 \cup \Xi' \mid j \leq i\} \mid i \in \Xi', \ i > K\}.$$

Let W'' be the set of all witnesses (r'', Ξ'') constructed as above and $L = K + V + 2$. Clearly, for each $(r'', \Xi'') \in W''$, all the types $r''(i)$ are stutter-invariant, for $i \geq L$. Thus, **(stuttr)** holds. It is easy to see that, for each $R \in \mathbf{F}^0$, we have $(r''_R, \{i''_R\}) \in W''$ and $K \leq i''_R < L$ if $R \in \mathbf{F}^K$, and $i''_R \leq M$ if $R \notin \mathbf{F}^{M+1}$. So **(role)** holds as well. It is readily seen now that $\mathfrak{Q} = \langle W'', K, L \rangle$ is as required.

($\Leftarrow$) Let $\mathfrak{Q} = \langle W, K, L \rangle$ be a quasimodel for $\mathcal{K}$. We construct a model for $\mathcal{K}^{\ddagger}$ which, by Theorem 1, will show that $\mathcal{K}$ is satisfiable. Let

$$\mathfrak{R} \quad = \quad \left\{ r_a \mid (r_a, \emptyset) \in W \right\} \quad \cup \quad \left\{ r_R^{\geq i} \mid (r_R, \{i_R\}) \in W, \ 0 \leq i \leq i_R \right\} \quad \cup$$
$$\left\{ r_R^{<K} \cdot (r_R(K))^{i-i_R} \cdot r_R^{\geq K} \mid (r_R, \{i_R\}) \in W, \ i > i_R \geq K \right\}.$$

Clearly, each $r \in \mathfrak{R}$ is a coherent and saturated run for $\mathcal{K}$. Moreover, if we have $(r_R, \{i_R\}) \in W$ and $i_R < K$ then, for all i, $0 \leq i \leq i_R$, there is $r' \in \mathfrak{R}$ with $\exists R \in r'(i)$. And if $(r_R, \{i_R\}) \in W$ and $i_R \geq K$ then, for all $i \geq 0$, there is $r' \in \mathfrak{R}$ with $\exists R \in r'(i)$. As follows from **(role)**, for each $R \in \Omega$, we have $R^- \in \Omega$ and either $i_R \geq K$ and $i_{R^-} \geq K$ or $i_R = i_{R^-} < K$. So, for all $i \geq 0$ and $r \in \mathfrak{R}$,

$$\text{if } \exists R^- \in r(i) \text{ then there is } r' \in \mathfrak{R} \text{ such that } \exists R \in r'(i).$$

We construct a first-order temporal model $\mathfrak{M}$ based on the domain $D = \mathfrak{R}$ by taking $a^{\mathfrak{M}} = r_a$, for each $a \in ob(\mathcal{A})$, and $(B^*)^{\mathfrak{M}, i} = \{r \in \mathfrak{R} \mid B \in r(i)\}$, for each $B \in cl(\mathcal{K})$ and $i \geq 0$. It should be clear that $(\mathfrak{M}, 0) \models \mathcal{K}^{\ddagger}$. $\qquad\qquad$ ❏

Theorem 5. *If there is a quasimodel $\mathfrak{Q} = \langle W, K, L \rangle$ for $\mathcal{K}$ then there is an ultimately periodic quasimodel $\mathfrak{Q}' = \langle W', K, L \rangle$, that is, there is $P \leq |ev(\mathcal{K})|$ such that $r'(i + P) = r'(i)$, for all $i > L$ and $(r', \Xi') \in W'$.*

Proof. We begin the proof with the following observation:

Lemma 5. *Let r be a coherent and saturated run for $\mathcal{K}$ and let $l \geq 0$ be such that every $r(i)$, $i \geq l$, is stutter-invariant. Then there are $i_1, \ldots, i_{|ev(\mathcal{K})|} \geq l$ such that $r' = r^{\leq l} \cdot \left(r(i_1) \cdot \ldots \cdot r(i_{|ev(\mathcal{K})|}) \right)^\omega$ is a coherent and saturated run for $\mathcal{K}$.*

Proof. First we show that

$$r(l) \cap ev(\mathcal{K}) = r(j) \cap ev(\mathcal{K}), \quad \text{for all } j > l. \tag{5}$$

Suppose that there is $j > l$ and $\Diamond C \in r(l)$ such that $\Diamond C \notin r(j)$. As $r(j)$ is stutter-invariant, $C \notin r(j)$ and, by **(coh)**, $\Diamond C \notin r(j-1)$. By repeating this argument sufficiently many times, we obtain $\Diamond C \notin r(l)$, contrary to our assumption. The converse direction—i.e., for each $j > l$, if $\Diamond C \in r(j)$ then $\Diamond C \in r(l)$—follows from **(coh)**.

For each $\Diamond C \in ev(\mathcal{K})$, we can select an i, $i \geq l$, such that $C \in r(i)$ whenever $\Diamond C \in r(l)$. Let $i_1, \ldots, i_{|ev(\mathcal{K})|}$ be all such i. It remains to show that r' is coherent and saturated. For coherency of r', let $C \in r'(i)$, for $i \geq 0$. By **(coh)** for r, we have $\Diamond C \in r'(j)$, for each $0 \leq j < i$ such that $j \leq l$. It remains to consider j with $l < j < i$. It follows that $r'(i) = r(i_k)$, for some $1 \leq k \leq |ev(\mathcal{K})|$, from which, by **(coh)** for r, $\Diamond C \in r(l) = r'(l)$ and, by (5), $\Diamond C \in r'(j)$. For saturation of r', let $\Diamond C \in r'(i)$, for $i \geq 0$. If $\Diamond C \in r(l)$ then $C \in r(i_k)$ for $1 \leq k \leq |ev(\mathcal{K})|$ and, by the construction of r', there is $j > i$ such that $r'(j) = r(i_k)$. Thus $C \in r'(j)$. If $\Diamond C \notin r(l)$ then, by (5), $i < l$, from which $\Diamond C \in r(i)$. By **(sat)** for r, there is $j > i$ with $C \in r(j)$ and, by (5), $j \leq l$. Thus $C \in r(j) = r'(j)$. ❑

Let $P = |ev(\mathcal{K})|$. For each (r, Ξ), we take $r' = r^{\leq L} \cdot (r(i_1) \cdot \ldots r(i_P))^\omega$ provided by Lemma 5. Denote the set of all (r', Ξ) by W'. It follows that $\mathfrak{Q}' = \langle W', K, L \rangle$ is an ultimately periodic quasimodel for $\mathcal{K}$ (with period P). ❑

It is now easy to devise an NP algorithm which can check whether there exists a quasimodel for a $T_\Diamond^0 DL\text{-}Lite_{bool}^{\mathcal{N}}$ KB. By Lemma 2 and Theorems 4 and 5, this means that satisfiability of $T_\Diamond DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ KBs is in NP. We prove the matching lower bound for the fragment $T_\Diamond DL\text{-}Lite_{core}$ of $T_\Diamond DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ that allows only concept inclusions of the form $A_1 \sqsubseteq A_2$, $A_1 \sqsubseteq \neg A_2$, $\Diamond A_1 \sqsubseteq A_2$ or $A_1 \sqsubseteq \Diamond A_2$, where A_1 and A_2 are concept names.

Lemma 6. *The satisfiability problem for $T_\Diamond DL\text{-}Lite_{core}$ KBs is NP-hard.*

Proof. We prove this by reduction of the graph 3-colourability (3-COL) problem, which is formulated as follows: given a graph $G = (V, E)$, decide whether there is an assignment of colours $\{1, 2, 3\}$ to vertices V such that no two vertices $a_i, a_j \in V$ sharing the same edge, $(a_i, a_j) \in E$, have the same colour. Let X_0, X_1, X_2, V, U and A_i, for $A_i \in V$, be concept names and a an object name. Consider the following KB $\mathcal{K}_G$:

$$
\begin{aligned}
&V(a), \quad &&V \sqsubseteq \Diamond A_i, \quad A_i \sqsubseteq X_3, \quad &&\text{for all } A_i \in V, \\
& &&A_i \sqsubseteq \neg A_j, \quad \text{for all } (A_i, A_j) \in E, \\
& &&V \sqsubseteq \neg U, \quad \Diamond X_0 \sqsubseteq U, \quad \Diamond X_1 \sqsubseteq X_0, \quad \Diamond X_2 \sqsubseteq X_1, \quad \Diamond X_3 \sqsubseteq X_2.
\end{aligned}
$$

It is easy to see that $\mathcal{K}_G$ is satisfiable iff G is 3-colourable. ❏

Thus we obtain the following theorem:

Theorem 6. *The satisfiability problem for* $T_\Diamond DL\text{-}Lite_{bool}^{(\mathcal{RN})}$ *KBs is* NP-*complete.*

It is also of interest to note that the fragment $T_\mathcal{U} DL\text{-}Lite_{core}$ of $T_\mathcal{U}^0 DL\text{-}Lite_{bool}^{\mathcal{N}}$ with concept inclusions of the form $A_1 \sqsubseteq A_2$, $A_1 \sqsubseteq \neg A_2$ or $A_1 \sqsubseteq A_2 \mathcal{U} A_3$ (the A_i concept names) turns out to be as complex as the whole logic $T_\mathcal{U} DL\text{-}Lite_{bool}^{(\mathcal{RN})}$:

Theorem 7. *The satisfiability problem for* $T_\mathcal{U} DL\text{-}Lite_{core}$ *KBs is* PSPACE-*hard.*

The proof can be found in the full version of the paper available online at `http://www.dcs.bbk.ac.uk/~roman/`.

6 Conclusions

The obtained complexity results look encouraging in view of possible applications for reasoning about temporal conceptual data models [4]. On the one hand, the logic $DL\text{-}Lite_{bool}^{\mathcal{N}}$ was shown to be adequate for representing different aspects of conceptual models: ISA, disjointness and covering for classes, domain and range of relationships, n-ary relationships, attributes and participation constraints [6]. On the other hand, the approach of [8] shows that rigid axioms and roles with temporalised concepts are enough to capture temporal data models.

The logic $T_\Diamond DL\text{-}Lite_{bool}$ presented in this paper can capture some form of *evolution constraints* [5, 20, 16] thanks to the $\Diamond$ operator. Furthermore, it also captures *snapshot* classes—i.e., classes whose instances do not change over time. However, by restricting the temporal component only to $\Diamond$ and $\Box$, we lose the ability to capture *temporary* entities and relationships whose instances have a limited lifespan. To overcome this limitation, we plan to extend the logics presented here with either past temporal operators or with a special kind of axioms that hold over finite prefix.

References

1. A. Artale, D. Calvanese, R. Kontchakov, and M. Zakharyaschev. *DL-Lite* in the light of first-order logic. In *Proc. of the 22nd AAAI Conf. on Artificial Intelligence*, pages 361–366. AAAI Press, 2007.
2. A. Artale, D. Calvanese, R. Kontchakov, and M. Zakharyaschev. The *DL-Lite* family and relations. Technical Report BBKCS-09-03, School of Computer Science and Information Systems, Birbeck College, London, 2009. Available at `http://www.dcs.bbk.ac.uk/research/techreps/2009/bbkcs-09-03.pdf`.
3. A. Artale and E. Franconi. Temporal description logics. In M. Fisher, D. Gabbay, and L. Vila, editors, *Handbook of Time and Temporal Reasoning in Artificial Intelligence*, pages 375–388. Elsevier, 2005.

4. A. Artale, E. Franconi, F. Wolter, and M. Zakharyaschev. A temporal description logic for reasoning about conceptual schemas and queries. In *Proc. of the 8th Joint European Conf. on Logics in Artificial Intelligence (JELIA)*, volume 2424 of *LNAI*, pages 98–110. Springer, 2002.

5. A. Artale, C. Parent, and S. Spaccapietra. Evolving objects in temporal information systems. *Annals of Mathematics and AI*, 50(1-2):5–38, 2007.

6. A. Artale, D. Calvanese, R. Kontchakov, V. Ryzhikov, and M. Zakharyaschev. Reasoning over extended ER models. In *Proc. of the 26th Int. Conf. on Conceptual Modeling (ER)*, pages 277–292, volume 4801 of LNCS. Springer, 2007.

7. A. Artale, R. Kontchakov, C. Lutz, F. Wolter, and M. Zakharyaschev. Temporalising tractable description logics. In *Proc. of the 14th Int. Symposium on Temporal Representation and Reasoning (TIME)*. IEEE Computer Society, 2007.

8. A. Artale, C. Lutz, and D. Toman. A description logic of change. In *Proc. of Int. Joint Conf. on Artificial Intelligence (IJCAI)*, pages 218–223, 2007.

9. F. Baader, R. Küsters, and F. Wolter. Extensions to description logics. In *Description Logic Handbook*, pages 219–261. Cambridge University Press, 2003.

10. F. Baader, S. Ghilardi, and C. Lutz. LTL over description logic axioms. In *Proc. of the 11th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR)*, pages 684–694. AAAI Press, 2008.

11. D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. *DL-Lite*: Tractable description logics for ontologies. In *Proc. of the 10th Nat. Conf. on Artificial Intelligence and the 17th Innovative Applications of Artificial Intelligence Conf.*, pages 602–607, AAAI Press/The MIT Press, 2005.

12. D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. Data complexity of query answering in description logics. In *Proc. of the 10th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR)*, pages 260–270. AAAI Press, 2006.

13. D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. Tractable reasoning and efficient query answering in description logics: The *DL-Lite* family. *J. of Automated Reasoning*, 39(3):385–429, 2007.

14. D. Gabbay, I. Hodkinson and M. Reynolds. *Temporal Logic: Mathematical Foundations and Computational Aspects, Volume 1*. Oxford University Press, 1994.

15. D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*. Elsevier, 2003.

16. G. Hall and R. Gupta. Modeling transition. In *Proc. of the 7th Int. Conf. on Data Engineering (ICDE)*, pages 540–549. IEEE Computer Society, 1991.

17. C. Lutz, F. Wolter, and M. Zakharyaschev. Temporal description logics: A survey. In *Proc. of the 14th Int. Symposium on Temporal Representation and Reasoning (TIME)*, pages 3–14. IEEE Computer Society, 2008.

18. A. Poggi, D. Lembo, D. Calvanese, G. De Giacomo, M. Lenzerini, and R. Rosati. Linking data to ontologies. *J. on Data Semantics*, X:133–173, 2008.

19. K. Schild. Combining terminological logics with tense logic. In *Proc. of the 6th Portuguese Conf. on Artificial Intelligence (EPIA)*, volume 727 of LNCS, pages 105–120. Springer, 1993.

20. S. Spaccapietra, C. Parent, and E. Zimanyi. *Conceptual Modeling for Traditional and Spatio-Temporal Applications—The MADS Approach*. Springer, 2006.

A   Proof of Theorem 7

Theorem 7. *The satisfiability problem for $T_\mathcal{U}DL\text{-}Lite_{core}$ KBs is* PSPACE-*hard.*

Proof. We show that the unsatisfiability problem for $T_\mathcal{U}DL\text{-}Lite_{core}$ KBs is PSPACE-hard. The proof is by reduction of the halting problem of deterministic Turing machines that require polynomial space $s(n)$ given an input of length n. Without loss of generality, assume $s(n)$ is computable in time polynomial of n and that the Turing machine M halts on all inputs (either in accepting or rejecting state) and never runs outside the first $s(n)$ tape cells (we also assume it never goes to the left of the input).

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \#, q_a, q_r \rangle$, where Q is a finite set of states, Γ is a set of tape symbols, $\# \in \Gamma$ is the blank symbol, $\Sigma \subseteq \Gamma$ is a set of input symbols, $\delta \colon (Q \setminus \{q_r, q_a\}) \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is a transition function, $q_0, q_r, q_a \in Q$ are the initial, accepting and rejecting states, respectively.

Let $\boldsymbol{a} = a_1 \ldots a_n$ be an input for M. We define a KB $\tau(M, \boldsymbol{a})$ that is unsatisfiable iff M accepts $\boldsymbol{a}$. We use the following concept names, for $1 \leq i \leq s(n)$,

- $H_i q$, $q \in Q$, with the intended meaning that $H_i q$ is true iff the head points to the cell i and the current control state is q;
- $S_i a$, $a \in \Gamma$, with the intended meaning that $S_i a$ is true iff the tape cell i contains symbol a;
- D_i with the intended meaning that D_j is false, for all $j \neq i$, if in the previous configuration the head pointed to the cell i
- $P_i q$, $q \in Q$, with the intended meaning that $P_i q$ is true if in the previous configuration the head pointed to the cell i and the state was q; $\overline{P_i q}$ has the opposite meaning.

Consider the TBox $\mathcal{T}$ containing the following concept inclusions, for all $a, a' \in \Gamma$, $q \in Q \setminus \{q_r, q_a\}$, $q' \in Q$ and $1 \leq i \leq s(n)$,

$$H_i q \sqsubseteq \bot \, \mathcal{U} \, P_i q, \tag{6}$$

$$P_i q \sqsubseteq \neg \overline{P_i q}, \tag{7}$$

$$P_i q \sqsubseteq \neg D_j \qquad \text{for all } 1 \leq j \leq s(n), \quad j \neq i, \tag{8}$$

$$S_i a \sqsubseteq S_i a \, \mathcal{U} \, D_i, \tag{9}$$

$$S_i a \sqsubseteq S_i a' \, \mathcal{U} \, \overline{P_i q}, \qquad \text{if } \delta(q, a) = (q', a', R) \text{ or } \delta(q, a) = (q', a', L), \tag{10}$$

$$S_i a \sqsubseteq H_{i+1} q' \, \mathcal{U} \, \overline{P_i q}, \qquad \text{if } \delta(q, a) = (q', a', R) \text{ and } i < s(n), \tag{11}$$

$$S_i a \sqsubseteq H_{i-1} q' \, \mathcal{U} \, \overline{P_i q}, \qquad \text{if } \delta(q, a) = (q', a', L) \text{ and } i > 1, \tag{12}$$

and the following ABox $\mathcal{A}$:

$$H_1 q_0(d), \qquad S_i a_i(d), \ 1 \leq i \leq n, \qquad S_i \#(d), \ n < i \leq s(n), \qquad \Box R(d).$$

Note that although $\bot$ is not available in $T_\mathcal{U}DL\text{-}Lite_{core}$, it can be easily introduced as a concept name B with the following concept inclusions: $B \sqsubseteq A$ and $B \sqsubseteq \neg A$, for some fresh concept name A. Let

$$\tau(M, \boldsymbol{a}) = (\mathcal{T} \cup \{R \sqsubseteq \neg H_i q_a \mid 1 \leq i \leq s(n)\}, \mathcal{A}).$$

We are going to show that $\tau(M, \boldsymbol{a})$ is unsatisfiable iff M accepts $\boldsymbol{a}$. We represent configurations of M as tuples of the form $\mathfrak{c} = \langle b_1 \ldots b_{s(n)}, i, q \rangle$, where $b_1 \ldots b_{s(n)}$ is the contents of the first $s(n)$ tape cells ($b_j \in \Gamma$, for $1 \leq j \leq s(n)$), i is the head position, $1 \leq i \leq s(n)$, and q is the control state, $q \in Q$. We say that a point $d \in \Delta^{\mathcal{I}}$ *encodes a configuration* $\mathfrak{c} = \langle b_1 \ldots b_{s(n)}, i, q \rangle$ at moment k in $\mathcal{I}$ if $d \in (H_i q)^{\mathcal{I}(k)}$ and $d \in (S_j b_j)^{\mathcal{I}(k)}$, for all $1 \leq j \leq s(n)$.

Claim. Let $\mathfrak{c}_0, \ldots, \mathfrak{c}_m$ be a sequence of configurations representing a partial computation of M on $\boldsymbol{a}$. Then for any model $\mathcal{I}$ of $(\mathcal{T}, \mathcal{A})$, $d^{\mathcal{I}}$ encodes $\mathfrak{c}_k$ at moment k in $\mathcal{I}$, for all $0 \leq k \leq m$.

Proof. The proof is by induction on k. For $k = 0$ the claim follows from $\mathcal{I} \models \mathcal{A}$. For the induction step, let $d^{\mathcal{I}}$ encode $\mathfrak{c}_k = \langle b_1 \ldots b_i \ldots b_{s(n)}, i, q \rangle$ at moment k in $\mathcal{I}$ and let $\mathfrak{c}_{k+1} = \langle b_1 \ldots b_i' \ldots b_{s(n)}, i', q' \rangle$. Consider first $\delta(q, b_i) = (q', b_i', L)$, in which case $i > 1$ and $i' = i - 1$. Then $d^{\mathcal{I}} \in H_i q^{\mathcal{I}(k)}$ for $q \notin \{q_r, q_a\}$ and, by (6), $d^{\mathcal{I}} \in P_i q^{\mathcal{I}(k+1)}$. By (7), $d^{\mathcal{I}} \notin \overline{P_i q}^{\mathcal{I}(k+1)}$, and, as $d^{\mathcal{I}} \in S_i b_i^{\mathcal{I}(k)}$, we obtain, by (12), $d^{\mathcal{I}} \in H_{i'} q'^{\mathcal{I}(k+1)}$ and, by (10), $d^{\mathcal{I}} \in S_i b_i'^{\mathcal{I}(k+1)}$. Let b_j for $j \neq i$, $1 \leq j \leq s(n)$. By (8), $d^{\mathcal{I}} \notin D_j^{\mathcal{I}(k+1)}$, and so, since $d^{\mathcal{I}} \in S_j b_j^{\mathcal{I}(k)}$, we get, by (9), $d^{\mathcal{I}} \in S_j b_j^{\mathcal{I}(k+1)}$. Hence, $d^{\mathcal{I}}$ encodes $\mathfrak{c}_{k+1}$ at moment $k + 1$ in $\mathcal{I}$. The case of $\delta(q, b_i) = (q', b_i', R)$ is similar. ❑

It follows that if M accepts $\boldsymbol{a}$ then $\tau(M, \boldsymbol{a})$ is unsatisfiable. Indeed, if M accepts $\boldsymbol{a}$ then the computation is a sequence of configurations $\mathfrak{c}_0, \ldots, \mathfrak{c}_k$ such that $\mathfrak{c}_k = \langle b_1 \ldots b_i \ldots b_{s(n)}, i, q_a \rangle$. Suppose $\tau(M, \boldsymbol{a})$ is satisfiable, i.e., there is a model $\mathcal{I}$ of $\tau(M, \boldsymbol{a})$, which is also a model of $(\mathcal{T}, \mathcal{A})$. By the above claim, we have $d^{\mathcal{I}} \in H_i q_a^{\mathcal{I}(k)}$, which implies $d^{\mathcal{I}} \notin R^{\mathcal{I}(k)}$ contrary to our assumption.

Conversely, if M rejects $\boldsymbol{a}$ then $\tau(M, \boldsymbol{a})$ is satisfiable. Let $\mathfrak{c}_0, \ldots, \mathfrak{c}_m$ be a sequence of configurations representing the rejecting computation of M on $\boldsymbol{a}$, $\mathfrak{c}_k = \langle b_{1,k}, \ldots, b_{s(n),k}, i_k, q_k \rangle$, for $0 \leq k \leq m$. We define an interpretation $\mathcal{I}$ with domain $\Delta^{\mathcal{I}} = \{w\}$. Let $d^{\mathcal{I}} = w$. For every $0 \leq k \leq m$, $1 \leq j \leq s(n)$, $a \in \Gamma$, $q \in Q$, we set

- $R^{\mathcal{I}(k)} = \Delta^{\mathcal{I}}$,
- $H_j q^{\mathcal{I}(k)} = \Delta^{\mathcal{I}}$ if $j = i_k$ and $q = q_k$, $\emptyset$ otherwise,
- $S_j a^{\mathcal{I}(k)} = \Delta^{\mathcal{I}}$ if $a = b_{j,k}$ and $\emptyset$ otherwise,
- $P_j q^{\mathcal{I}(0)} = \emptyset$ and $P_j q^{\mathcal{I}(k+1)} = H_j q^{\mathcal{I}(k)}$ if $k + 1 \leq m$, $\quad \overline{P_j q}^{\mathcal{I}(k)} = \Delta^{\mathcal{I}} \setminus P_j q^{\mathcal{I}(k)}$,
- $D_j^{\mathcal{I}(k)} = \bigcup_{q' \in Q} P_j q'^{\mathcal{I}(k)}$,

For every $k > m$, $1 \leq j \leq s(n)$, $a \in \Gamma$, $q \in Q$, we set

- $R^{\mathcal{I}(k)} = \Delta^{\mathcal{I}}$,
- $H_j q^{\mathcal{I}(k)} = \Delta^{\mathcal{I}}$ if $q = q_r$ and $\emptyset$ otherwise,
- $S_j a^{\mathcal{I}(k)} = \emptyset$,
- $P_j q^{\mathcal{I}(k)} = \emptyset$ and $\overline{P_j q}^{\mathcal{I}(k)} = \Delta^{\mathcal{I}}$,
- $D_j^{\mathcal{I}(k)} = \Delta^{\mathcal{I}}$.

It can be easily verified that $\mathcal{I} \models \tau(M, \boldsymbol{a})$. ❑

B Decision procedure for $T^0_\mathcal{U}DL\text{-}Lite^\mathcal{N}_{bool}$

First we guess and store in binary numbers K, K_0, L' and P such that

$$0 < K < K_0 < L' \leq N_\mathcal{A} + T^{|role^\pm(\mathcal{K})|} \cdot (|role^\pm(\mathcal{K})| + 1) + T + 2 + 2^N,$$
$$0 < P \leq N \cdot 2^N,$$

where $N \leq (|role^\pm(\mathcal{K})| + |ob(\mathcal{A})|) \cdot |cl(\mathcal{K})|$. Then we guess a set $\Omega \subseteq role^\pm(\mathcal{K})$ and numbers $\{i_R \mid R \in \Omega\}$ such that $i_R < L'$, for each $R \in \Omega$. Let $\Delta = \Omega \cup ob(\mathcal{A})$.

Let $\{r_x(i) \mid x \in \Delta\}$ be a state of an ultimately periodic quasimodel (cf. Theorem 2). Consider the following conditions:

1. $B \in r_a(i)$, for all $\bigcirc^i B(a) \in \mathcal{A}$; $B \in r_a(i)$, for all $\square B(a) \in \mathcal{A}$;
2. $r_x(i)$ is realisable, for all $x \in \Omega \cup ob(\mathcal{A})$;
3. for all $R \in role^\pm(\mathcal{K})$, if $i_R = i$ then $\exists R \in r_R(i)$.

It should be clear that these conditions can be verified in polynomial space. In particular, condition 2 can be verified by calling a satisfiability checking algorithm for the concept $\sqcap_{C \in r_x(i)} \overline{C}$ w.r.t. the TBox $\overline{\mathcal{T}}$, which can be done in NP [1, 2]. We assume that at every step of the algorithm presented below these conditions are checked after a quasimodel state is guessed, and if any of them fails the algorithm terminates with the negative answer.

Our algorithm works as follows. We guess an initial state $\{r_x(0) \mid x \in \Delta\}$. Then we guess another state $\{r_x(1) \mid x \in \Delta\}$ and check the *suitability* condition for each pair of types $(r_x(0), r_x(1))$: for all $C \, \mathcal{U} \, D \in ev(\mathcal{K})$, $C \, \mathcal{U} \, D \in r_x(0)$ iff either $C \, \mathcal{U} \, D, C \in r_x(1)$ or $D \in r_x(1)$ (if it fails the algorithm answers 'no'). Then, for all $R \in role^\pm(\mathcal{K})$, we check: if $\exists R = r_x(1)$, for some x, then $i_{R^-} \geq 1$ (if not, the algorithm terminates with the negative answer). After that we erase the previous state $\{r_x(0) \mid x \in \Delta\}$.

We continue in such a fashion until we reach step K. From now on the check on roles is different: for all $R \in role^\pm(\mathcal{K})$, if $\exists R \in r_x(K)$ then $i_{R^-} \geq K$. After checking the suitability condition we erase the state $K-1$ of the quasimodel and guess $\{r_x(K+1) \mid x \in \Delta\}$. Then we verify: for all $R \in role^\pm(\mathcal{K})$, if $\exists R \in r_x(K+1)$ then $i_{R^-} \geq K$. We never erase $\{r_x(K) \mid x \in \Delta\}$ from the memory.

We proceed in this way until we reach step L'. (At step K_0 we need to verify whether $\{r_x(K_0) \mid x \in \Delta\}$ is equal to $\{r_x(K) \mid x \in \Delta\}$). We perform all the same checks as at the previous steps. Now, for every x, we store the set $\mathsf{ev}_x = \{D \in cl(\mathcal{K}) \mid C\mathcal{U}D \in r_x(L')\}$. After that we guess $\{r_x(L'+1) \mid x \in \Delta\}$ and perform the same checks as before. If $D \in r_x(L'+1)$ and $D \in \mathsf{ev}_x$ then we update $\mathsf{ev}_x = \mathsf{ev}_x \setminus \{D\}$. Next we guess $\{r_x(L'+2) \mid x \in \Delta\}$ and perform the same checks and the same update of ev_x as before. Then erase $\{r_x(L'+1) \mid x \in \Delta\}$. We proceed like that until step P. Now, the algorithm answers 'yes' if $\mathsf{ev}_x = \emptyset$ and $r_x(L') = r_x(P)$ for all $x \in \Delta$.

Soundness of the presented algorithm follows from Theorem 1 and completeness from Theorems 1 and 2. It is easy to see that the algorithm runs in polynomial space.

C Decision procedure for $T^0_\diamond DL\text{-}Lite^{\mathcal{N}}_{bool}$

Our algorithm for checking satisfiability of $T^0_\diamond DL\text{-}Lite^{\mathcal{N}}_{bool}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ essentially guesses the 'prefix' of length $L + 1$ and the period of length P of an ultimately periodical quasimodel $\mathfrak{Q}' = \langle W', K, L \rangle$ for $\mathcal{K}$ as in Theorem 5, and then checks whether conditions **(runs)**, **(stuttr)**, **(obj)** in Section 4 hold and whether the types in positions $L + 1$ and $L + P + 1$ of the prefix coincide for every run.

More precisely, first we guess and store some numbers L, K and P such that $L \leq N_{\mathcal{K}} + |ev(\mathcal{K})| \cdot (|role^{\pm}(\mathcal{K})| + 2) + 3$, $K \leq L$ and $P \leq |ev(\mathcal{K})|$. Then we guess a set $\Omega \subseteq role^{\pm}(\mathcal{K})$ and numbers $\{i_R \mid R \in \Omega\}$ such that $i_R < L$, for each $R \in \Omega$. For every $R \in \Omega$, we then guess a sequence r_R of length $L + P + 2$ of types for $\mathcal{K}$ and, for every $a \in ob(\mathcal{A})$, a sequence r_a of length $L + P + 2$ of types for $\mathcal{K}$.

Let $W_0 = \{(r_R, \{i_R\}) \mid R \in \Omega\} \cup \{(r_a, \emptyset) \mid a \in ob(\mathcal{A})\}$. The set W_0 can be regarded as a finite representation of the witnesses W' from $\mathfrak{Q}'$. Now we check that the following conditions hold:

1. $r(K)$ and the $r(i)$, for $L \leq i \leq L + P + 1$, are stutter-invariant for each $(r, \Xi) \in W_0$;
2. if $\bigcirc^n B(a) \in \mathcal{A}$ then $B \in r_a(n)$; if $\Box B(a) \in \mathcal{A}$ then $B \in r_a(i)$, for all $0 < i \leq L + P + 1$;
3. for all $i \leq L + P + 1$ and $R \in role^{\pm}(\mathcal{K})$, if $\exists R^- \in r(i)$, for some $(r, \Xi) \in W_0$, then $(r_R, \{i_R\}) \in W_0$, $\exists R \in r_R(i_R)$ and either $i \leq i_R < K$ or $K \leq i_R < L$;
4. $r(L + 1) = r(L + P + 1)$, for all $(r, \Xi) \in W_0$;
5. $r(i)$ is realisable, for all $(r, \Xi) \in W_0$ and $i \leq L + P + 1$;
6. for all $(r, \Xi) \in W_0$, $i \leq L + P + 1$ and $\Diamond C \in r(i)$,
   - if $i \leq L$ then there is j, $i < j \leq L + P + 1$, such that $C \in r(j)$;
   - if $L < i \leq L + P + 1$ then there is j, $L < j \leq L + P + 1$, such that $C \in r(j)$;
7. for all $(r, \Xi) \in W_0$, $\Diamond C \in ev(\mathcal{K})$ and all $i \leq L + P + 1$, if $C \in r(i)$ then $\Diamond C \in r(j)$, for all $j < i$.

The algorithm returns 'yes' iff all the conditions above are satisfied.

The presented algorithm is sound: indeed, if conditions 1–7 are satisfied we can construct an ultimately periodical quasimodel for $\mathcal{K}$ which, by Theorem 4, means that $\mathcal{K}$ is satisfiable. The algorithm is also complete: if $\mathcal{K}$ is satisfiable then, by Theorems 4 and 5, there exists an ultimately periodical quasimodel $\mathfrak{Q} = \langle W', K, L \rangle$ with period P and K, L, P bounded by polynomial functions in $|\mathcal{K}|$ as above; then W_0 consisting of the prefixes of length $L + P + 2$ of runs in W' satisfies conditions 1–7 and thus the algorithm returns 'yes.' Finally, it is easy to see that L, K, P and W_0 can be constructed and conditions 1–7 checked by a non-deterministic algorithm that runs in time polynomial in $|\mathcal{K}|$.