Canonical extensions and the intermediate structure

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Let L be a lattice and C a complete lattice with L isomorphic to a sublattice of C. Then C is a completion of L and

- C is a dense completion if any element of C can be expressed as both a meet of joins and join of meets of elements of L,
- C is a compact completion if for any filter F, and ideal I, of L if $\bigwedge F \leqslant \bigvee I$ then $F \cap I \neq \emptyset$.

If C is both a dense and compact completion of L it is called a canonical extension of L.

Theorem (Gehrke & Harding, 2001)

Every bounded lattice L has a canonical extension, and this is unique up to an isomorphism which fixes L.

History of canonical extensions:

1951 Jónsson & Tarski: canonical extensions for Boolean algebras with operators

1994 Gehrke & Jónsson: bounded distributive lattices with operators

2000 Gehrke & Jónsson: bounded distributive lattices with monotone operations

2001 Gehrke & Harding: bounded lattice expansions

2004 Gehrke & Jónsson: distributive lattices with arbitrary operations

2005 Dunn, Gehrke, Palmigiano: partially ordered sets

2009 Moshier & Jipsen: topological duality theorem for bounded lattices

Construction of the canonical extension

Using the filters, $\mathcal{F}(L)$, and ideals, $\mathcal{I}(L)$, of L, form $\mathcal{F}(L) \cup \mathcal{I}(L)$. This is the *intermediate structure*, ordered by:

- $F_1 \leqslant^* F_2 \iff F_2 \subseteq F_1$
- $I_1 \leqslant^* I_2 \iff I_1 \subseteq I_2$
- $F \leqslant^* I \iff F \cap I \neq \emptyset$
- $I \leqslant^* F \iff x \in I, y \in F \Longrightarrow x \leqslant y$

$\big(IM(L), \leqslant^*\big)$ is the intermediate structure.

The canonical extension, L^δ, is the MacNeille completion of the intermediate structure.

That is, $L^\delta = \overline{IM(L)}$.

Filter and ideal elements of L^δ

$p = \bigwedge F$, where $F \in \mathcal{F}(L)$, is a *filter* element

$u = \bigvee I$, where $I \in \mathcal{I}(L)$, is an *ideal* element

$F(L^\delta)$: filter elements of L^δ

$I(L^\delta)$: ideal elements of L^δ

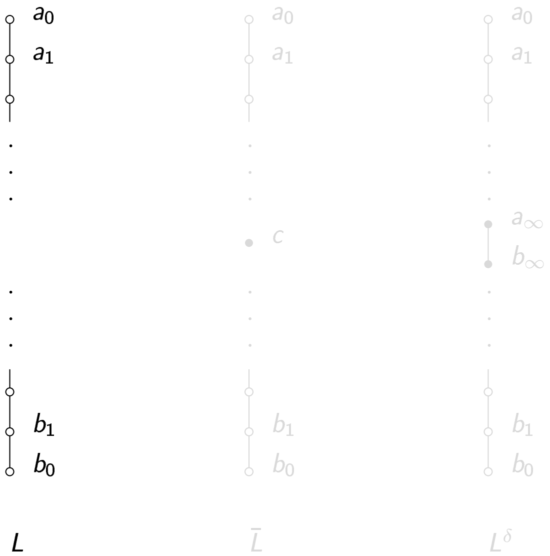
$F(L^\delta)$ is order isomorphic to $(\mathcal{F}(L), \supseteq)$, and $I(L^\delta)$ is order isomorphic to $(\mathcal{I}(L), \subseteq)$.

$$\alpha : \mathcal{F}(L) \longrightarrow C,\ F \longmapsto \bigwedge e[F]$$

$$\beta : \mathcal{I}(L) \longrightarrow C,\ I \longmapsto \bigvee e[I]$$

This gives $\left(\mathcal{F}(L) \cup \mathcal{I}(L), \leqslant^*\right)$ order isomorphic to $\left(F(L^\delta) \cup I(L^\delta), \leqslant\right)$.

Example



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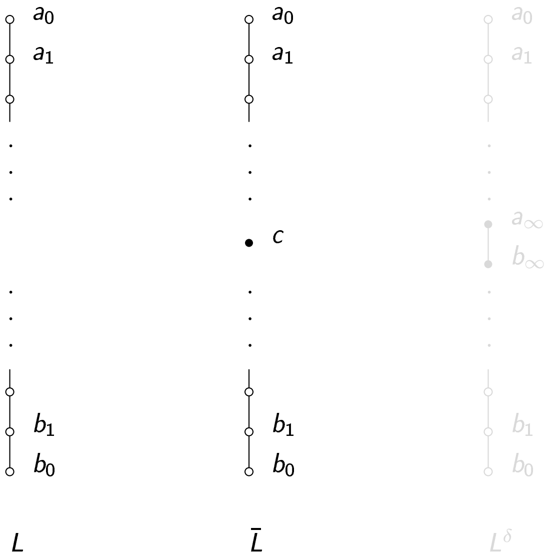

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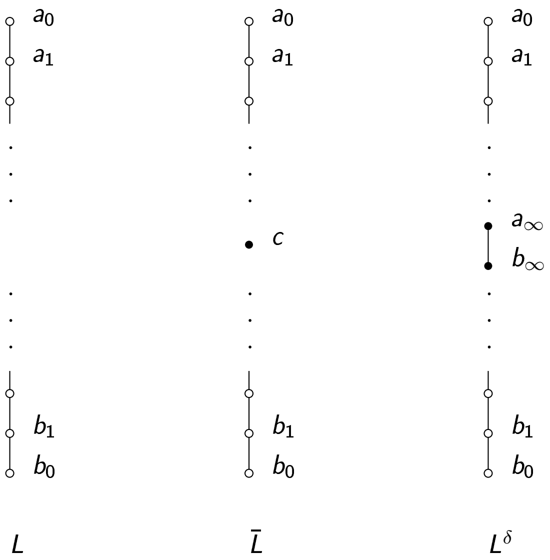
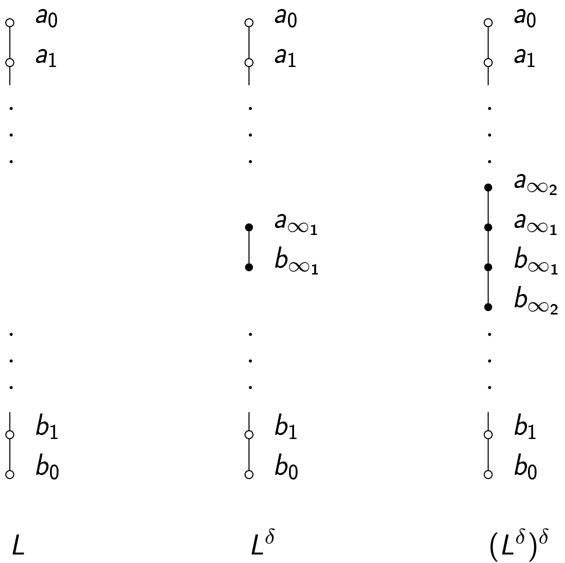


Canonical extensions destroy existing infinite meets and joins:



The intermediate structure

- Intermediate structure first described by Ghilardi and Meloni (1997).
- Used by Dunn, Gehrke and Palmigiano (2005) to describe canonical extensions of posets.
- Gehrke and Priestley (2008) use the intermediate structure to explain why the canonical extension is functorial.

This is done using the following concept.

Definition (Erné, 1991)

An order-preserving map $f : P \longrightarrow Q$ is *cut-stable* if for all $q_1, q_2 \in Q$,

$$q_1 \nleq q_2 \Longrightarrow$$

$$\exists\, p_1, p_2 \in P \text{ s.t. } p_1 \nleq p_2 \text{ and } f^{-1}({\uparrow}q_1) \subseteq {\uparrow}p_1 \text{ and } f^{-1}({\downarrow}q_2) \subseteq {\downarrow}p_2.$$

Theorem (Erné, 1991)

The category of complete lattices with complete lattice homomorphisms is a full reflective subcategory of the category of posets with cut-stable maps.

Define the extension of a map $f : L \longrightarrow M$ to $IM(f) : IM(L) \longrightarrow IM(M)$:

$$IM(f)(x) = \begin{cases} \bigvee\{f(a) \ : \ \downarrow a \subseteq x, a \in L\} \text{ if } x \in \mathcal{I}(L) \\ \bigwedge\{f(a) \ : \ \uparrow a \subseteq x, a \in L\} \text{ if } x \in \mathcal{F}(L) \end{cases}$$

Theorem (Gehrke & Priestley, 2008)

For $f : L \longrightarrow M$ a lattice homomorphism, the extension $IM(f) : IM(L) \longrightarrow IM(M)$ is a cut-stable map.

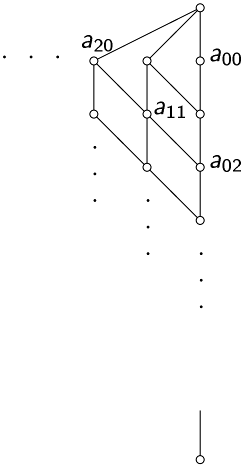
Algebraicity of canonical extensions

Gehrke and Jónsson (1994) showed that the canonical extension of a distributive lattice is always doubly algebraic. That is, both L^δ and $(L^\delta)^\partial$ are algebraic.

The following example is originally due to Harding (1998). It is a non-distributive lattice whose canonical extension is not algebraic. The result below helps us to construct its canonical extension.

Lemma (Gehrke & Vosmaer)

If a lattice L satisfies ACC, then $L^\delta = F(L^\delta)$. Dually, if L satisfies DCC, then $L^\delta = I(L^\delta)$.



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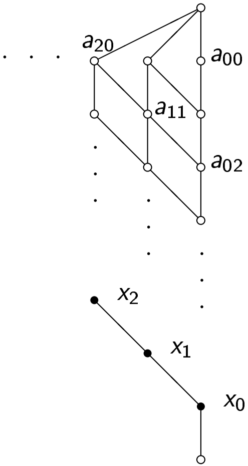

$L^{\delta} = F(L^{\delta})$

Define the following: $F \perp I$ if $F \not\leq^* I$ and $I \not\leq^* F$.

$F_\perp I = \{x \in F \ : \ \exists y \in I, x \perp y\}$

$I_\perp F = \{y \in I \ : \ \exists x \in F, y \perp x\}$.

Theorem

If $F \perp I$ and either $F_\perp I$ or $I_\perp F$ is finite, then both $F \vee I$ and $F \wedge I$ exist.

Lemma

Consider the embeddings $e_1 : (\mathcal{F}(L), \supseteq) \longrightarrow (IM(L), \leqslant^*)$, $F \longmapsto F$ *and* $e_2 : (\mathcal{I}(L), \subseteq) \longrightarrow (IM(L), \leqslant^*)$, $I \longmapsto I$. *Then* e_1 *preserves arbitrary meets and finite joins, and* e_2 *preserves arbitrary joins and finite meets.*

Question: Is the canonical extension of a lattice always the same as the intermediate structure?

Answer: No!

Lemma

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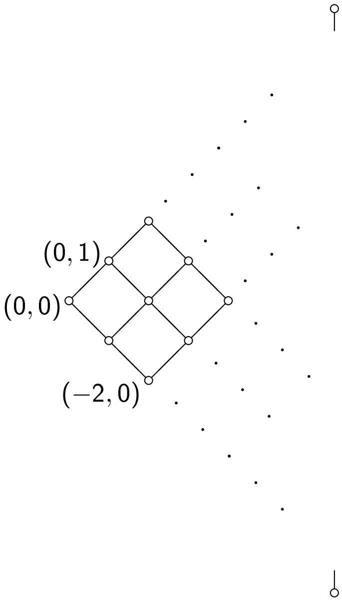
A distributive lattice whose canonical extension is not the intermediate structure:



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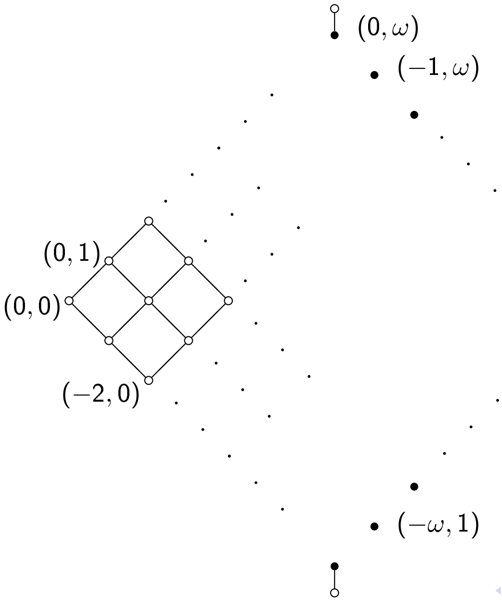

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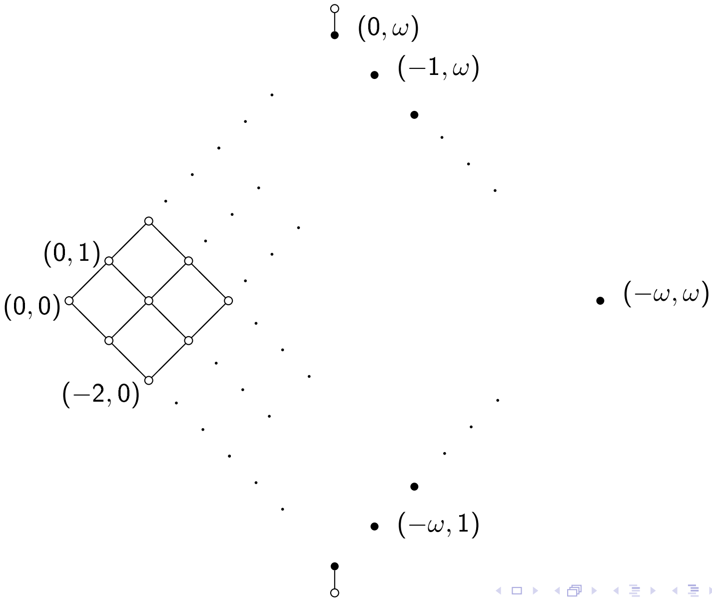


Distributivity of canonical extensions

The canonical extension of a distributive lattice is always distributive. By contrast, the MacNeille completion does not always preserve distributivity.

Counter-example due to Funayama (1944) makes use of the concept of neutral elements.

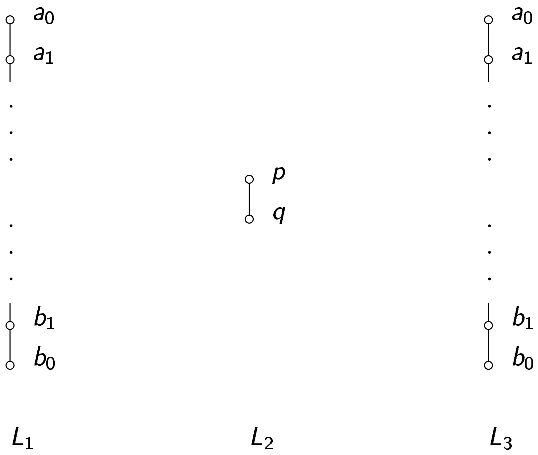
Definition (Ore, 1935)

An element a of a lattice L is *neutral* if, for any $x, y \in L$, the lattice generated by $\{a, x, y\}$ is distributive.

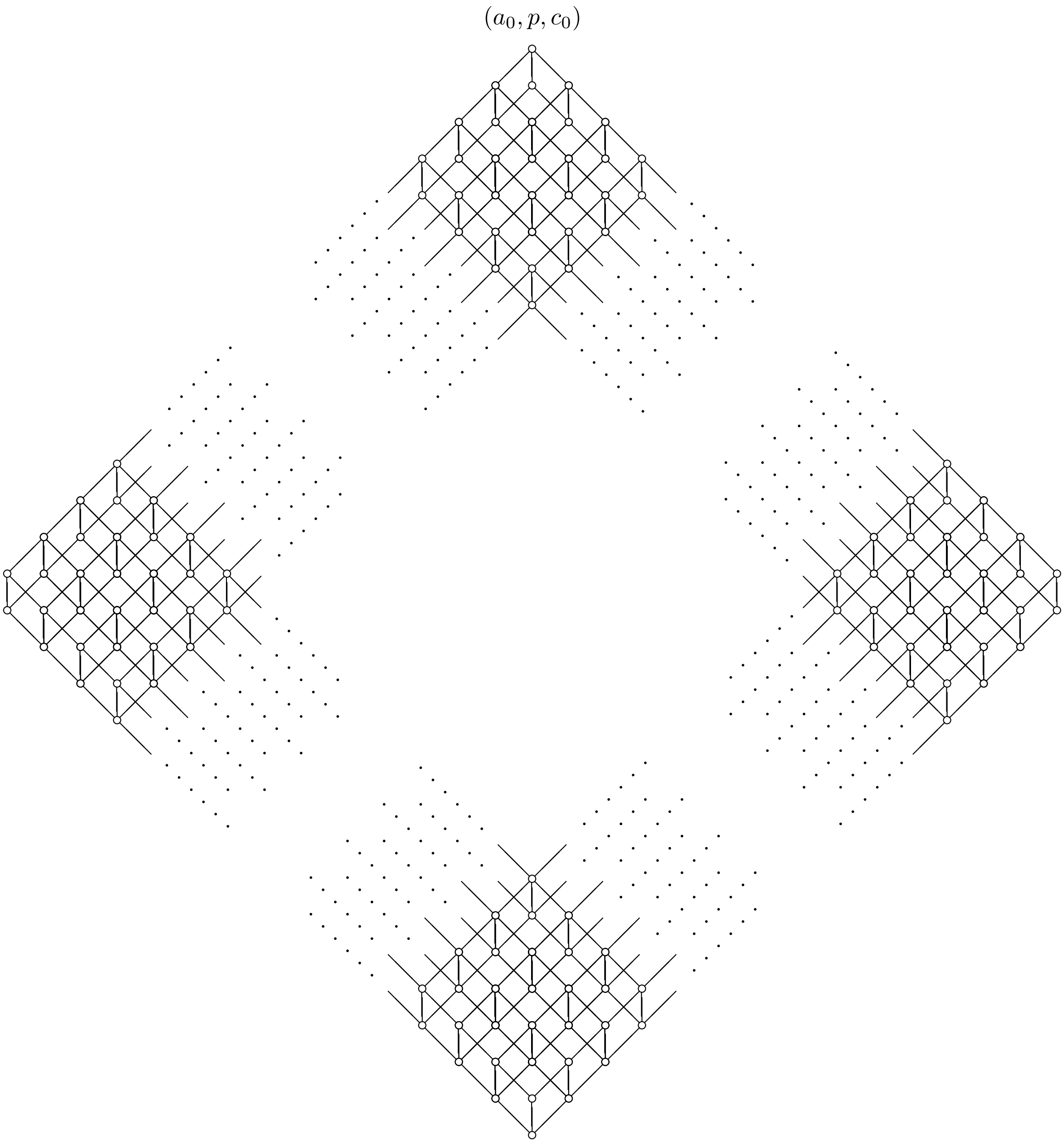
The neutral elements of a lattice L form a distributive sublattice of L.

Hence the MacNeille completion, $\overline{L}$ of a distributive lattice L is distributive if and only if every element of $\overline{L}$ is neutral.

Consider the following lattices



Let $L = L_1 \times L_2 \times L_3$.



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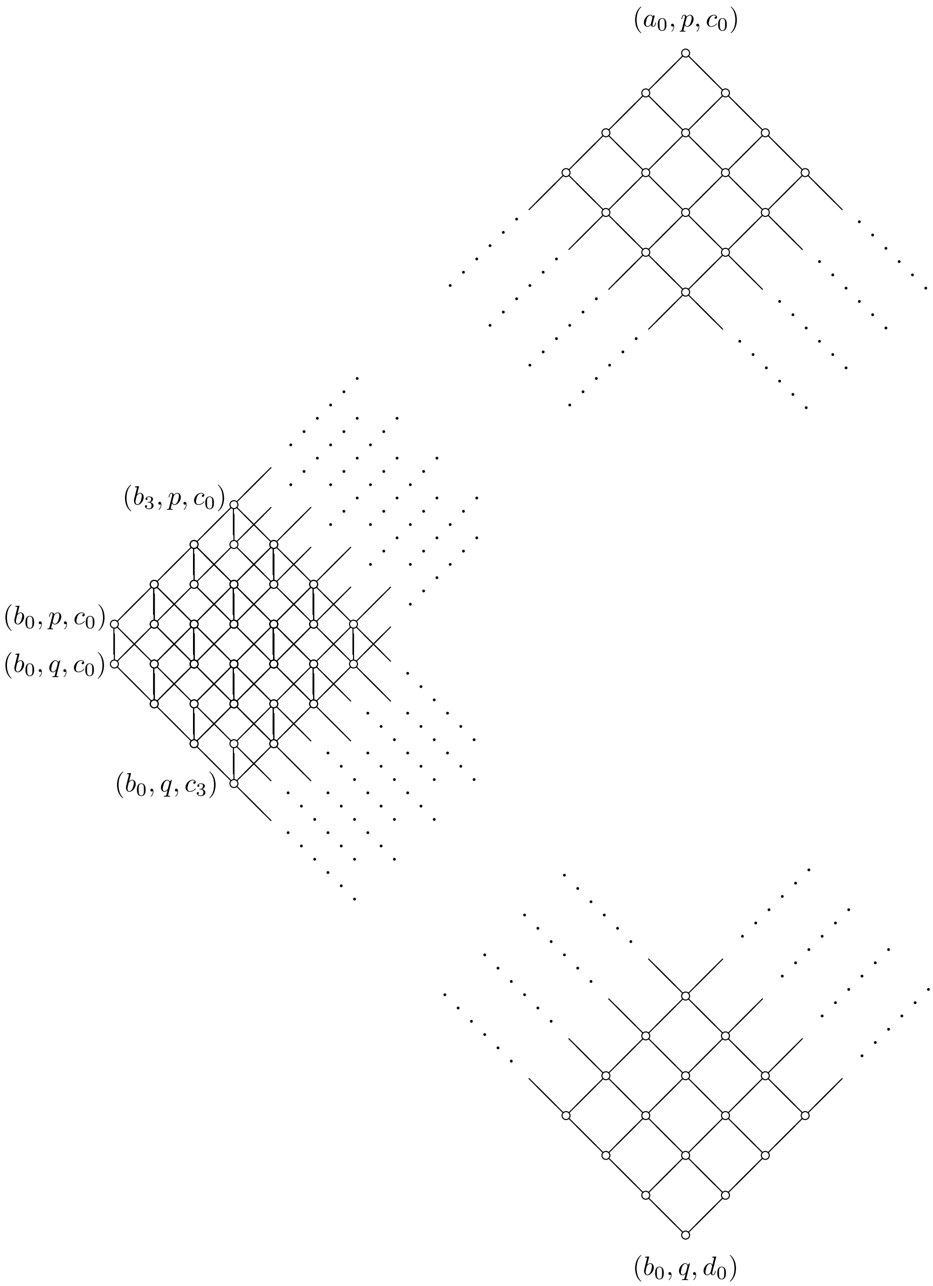
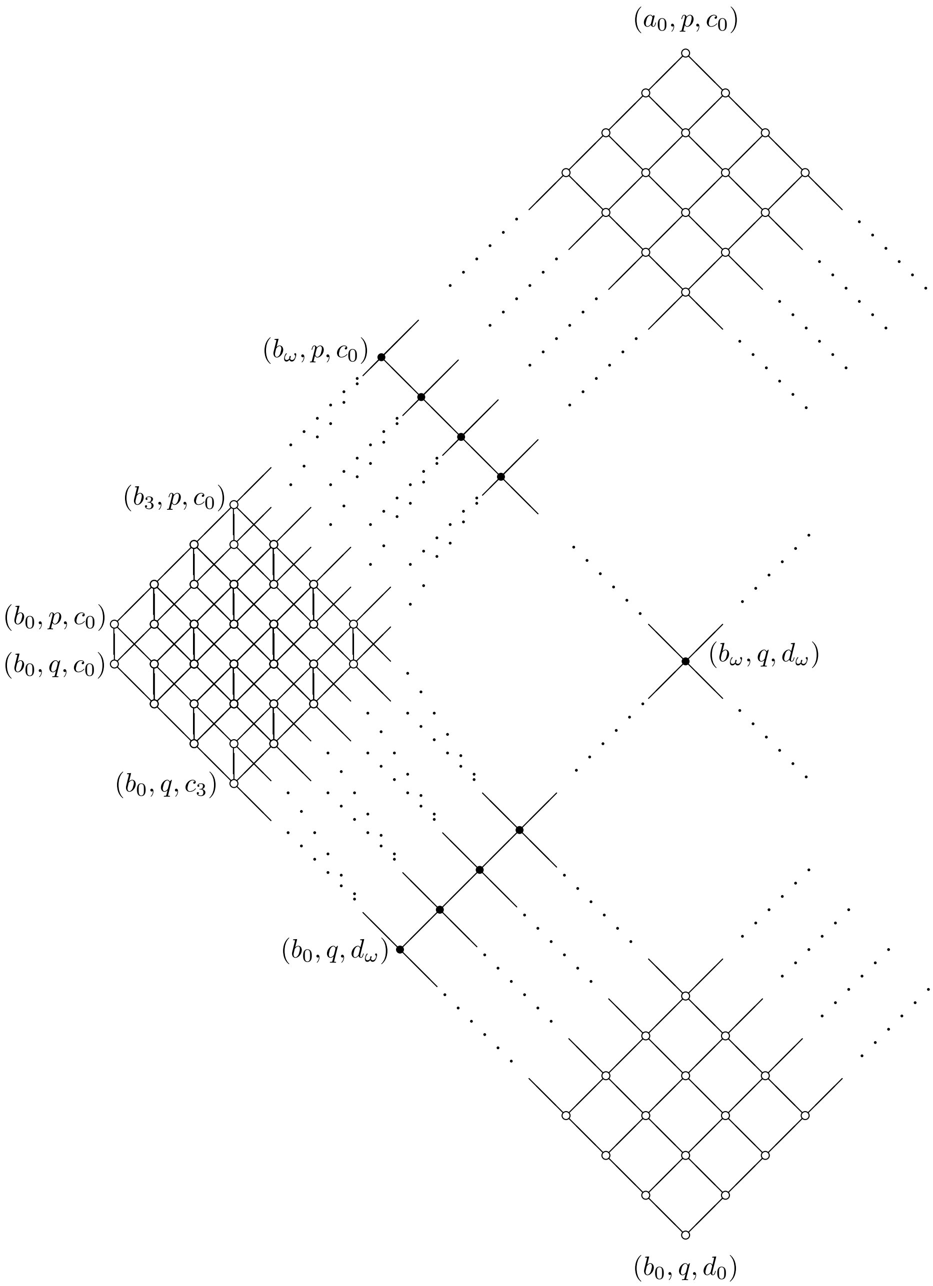
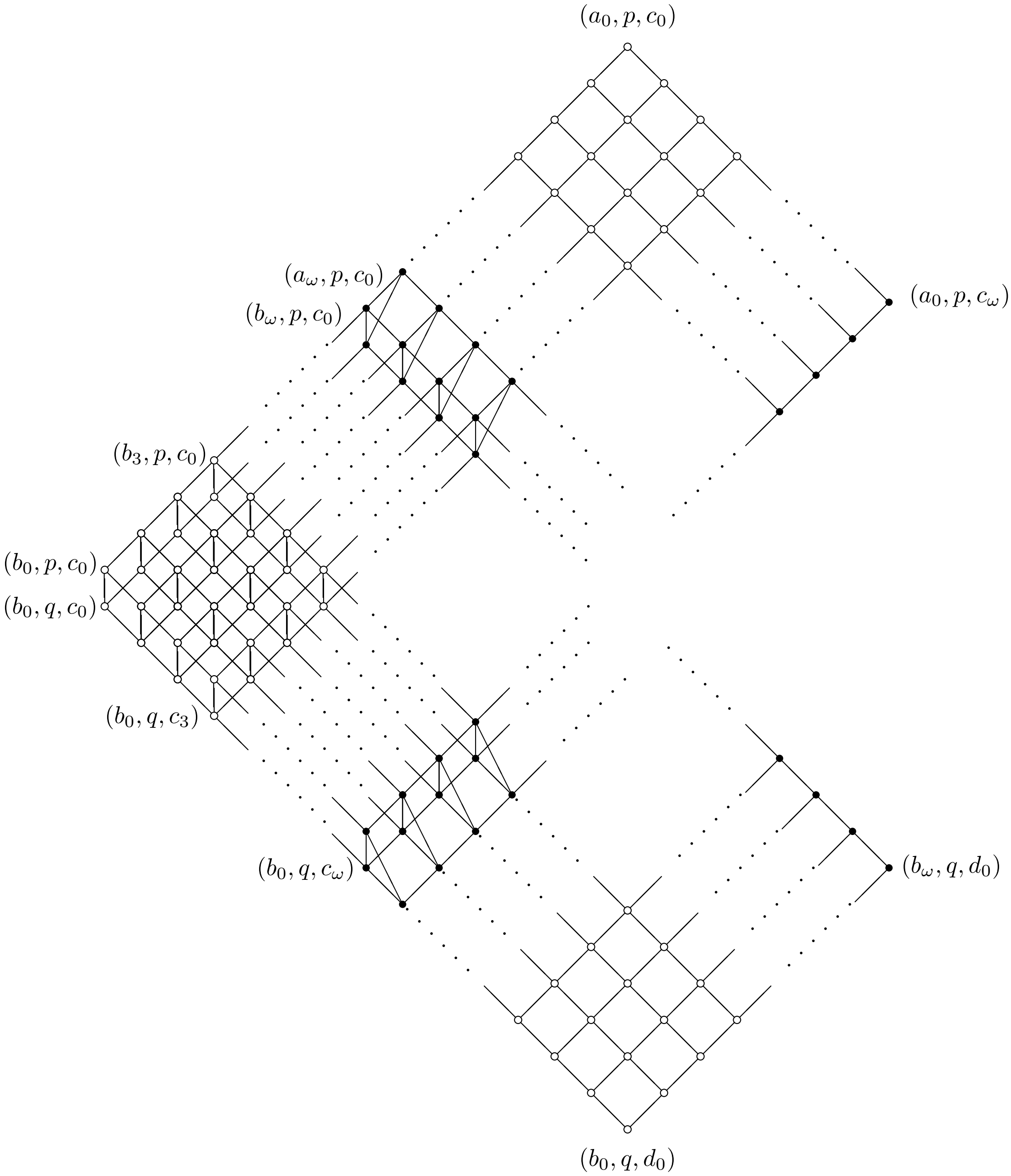


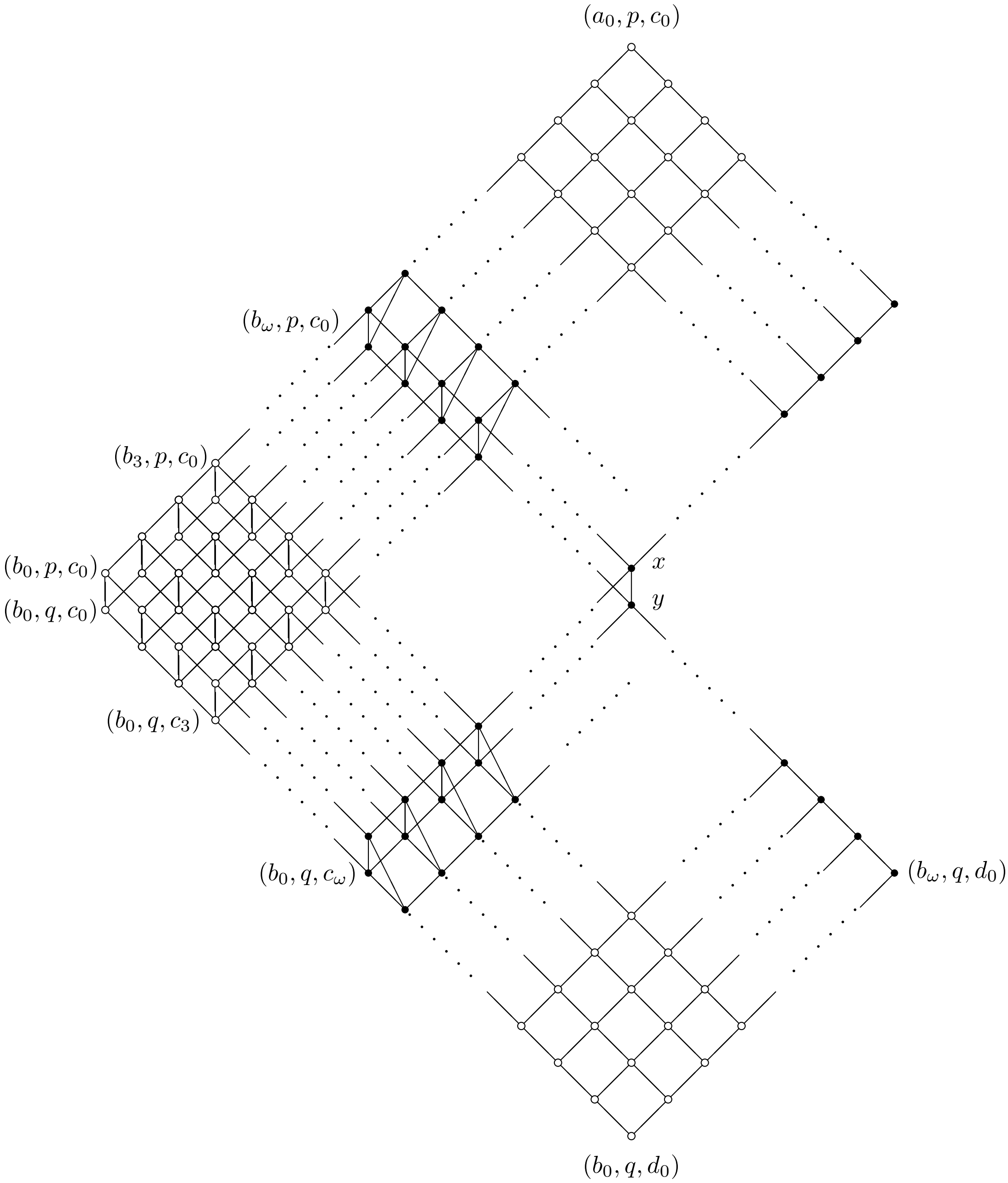
Distributive sublattice of L. The MacNeille completion of this is *not* distributive.



This is the MacNeille completion of the sublattice of L. This completion is *not* distributive.



The intermediate structure.



The canonical extension.

Further work:

Use the intermediate structure to find when L^δ will be:

- algebraic;
- continuous;
- meet-continuous.