Modal correspondence theory on quantales

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joint work with Johannes Martí and Pedro Resende

21 September 2010

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- Observation: Sahlqvist correspondence-type arguments underly each such instance;
- Aim: Develop a three-sided Sahlqvist-style correspondence theory, involving
  - a modal logic language,
  - its associated first-order frame correspondence language,
  - the language of stably supported quantales.

Unital involutive quantales

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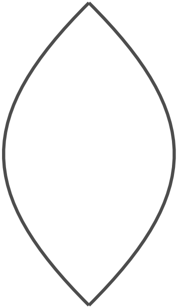

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quantale

Quantales: complete $\bigvee$-semilattices

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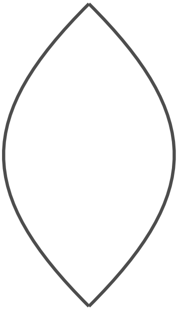

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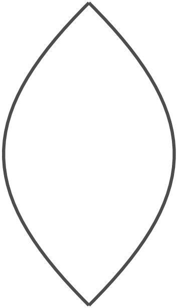

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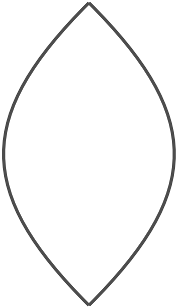

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Every quantale is a complete (non distributive) lattice

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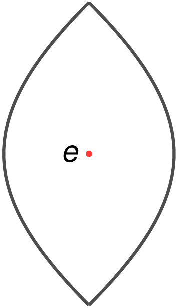

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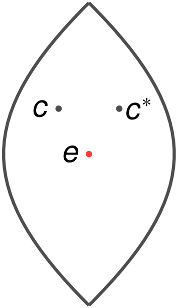

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involution: $c^{**} = c$

$$(c \cdot q)^* = q^* \cdot c^*$$

$$(\bigvee S)^* = \bigvee_{s \in S} s^*$$

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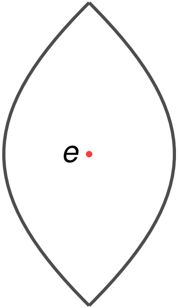

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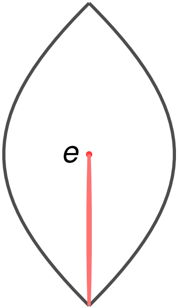

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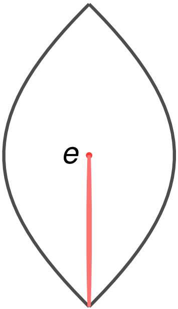

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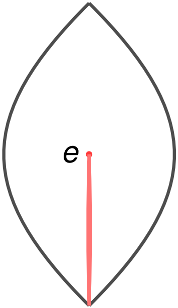

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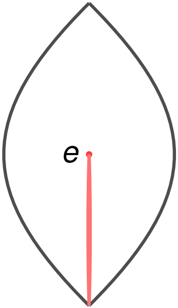

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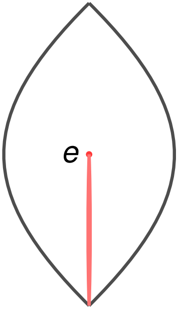

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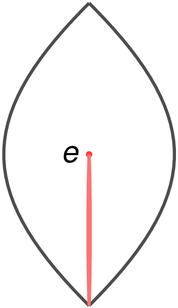

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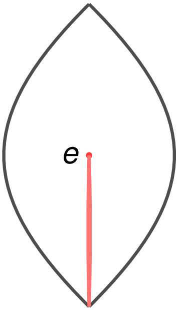

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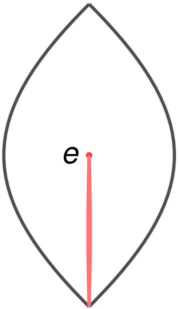

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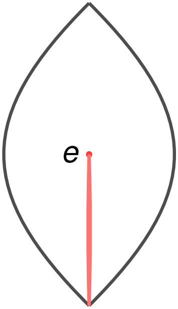

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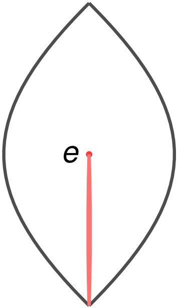

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A property rather than extra structure

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- Bounded morphisms, bisimulations can be extended to pointed ssq's.

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Extension map: For every model N, $[\![\cdot]\!]_N : Fm \to Q_e$

$$
\begin{array}{rcl}
[\![p]\!] & = & V(P) \\
[\![\varphi \vee \psi]\!] & = & [\![\varphi]\!] \vee [\![\psi]\!] \\
[\![\varphi \wedge \psi]\!] & = & [\![\varphi]\!][\![\psi]\!] \\
[\![\varphi \to \psi]\!] & = & [\![\varphi]\!] \to [\![\psi]\!] \\
[\![\Diamond\varphi]\!] & = & \varsigma(a[\![\varphi]\!]) \\
[\![\Box\varphi]\!] & = & \bigvee\{d \in Q_e \mid \varsigma(a^* d) \leq [\![\varphi]\!]\}
\end{array}
$$

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- for all $x, y \in B_K$, (α is the selected element):

$$\begin{array}{rcl} [x \vee y] & = & [x] \vee [y] \\ [\neg x] \cdot [x] & = & 0 \\ [\neg x] \vee [x] & = & e \\ [\Diamond x] & = & [\varsigma(\alpha[x])]. \end{array}$$

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Lindenbaum quantales for T, K4, S4, and S5: pointed ssqs $\mathcal{Q}_T$, $\mathcal{Q}_{K4}$, $\mathcal{Q}_{S4}$, $\mathcal{Q}_{S5}$ presented by generators and relations, with the additional relations:

$$\mathcal{Q}_T : \; e \leq \alpha \qquad \mathcal{Q}_{K4} : \alpha\alpha \leq \alpha$$
$$\mathcal{Q}_{S4} : \; e \leq \alpha \geq \alpha\alpha \qquad \mathcal{Q}_{S5} : e \leq \alpha = \alpha^* \geq \alpha\alpha.$$

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[Marcelino, Resende 2007]

For a bimodal frame $(L, \Diamond, \blacklozenge)$, its associated Lindenbaum pointed ssq $T_K(L)$ and its quotients $T_T(L)$, $T_{K4}(L)$... can be constructed, in such a way that:

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- Same for S4 and S5...

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- Every inequality in the language of ssqs containing only finite joins is equivalent to a Kracht formula, hence morally corresponds to a "Sahlqvist" formula.