Some results about wRRA

Ian Hodkinson

Thanks to the organisers for inviting me!
Let’s hope they don’t regret it.

Relation algebras and representations

Relation algebras have been discussed today already.

Relation algebras were originally introduced (by Tarski & students Chin and Thompson, 1940s) as an *abstract class.*

The corresponding *concrete class* they had in mind was RRA.

This is the class of *representable* relation algebras: those isomorphic to algebras of genuine binary relations.

I guess it was hoped that RA $=$ RRA.

A few facts about RRA

RRA is rather wild.

- RRA $\subseteq$ RA
- RRA $\neq$ RA (Lyndon, 1950)
- RRA is a variety (Tarski, $\sim$1955)
- RRA is *canonical* — closed under canonical extensions (Monk, in McKenzie's thesis, 1966)
- RRA is not finitely axiomatisable (Monk 1964)
- RRA is not closed under Monk completions (IH 1997)
- more...

Weakly representable relation algebras

Jónsson (1959) introduced an alternative concrete class: wRRA, the *weakly representable relation algebras*.

A *weak representation* of a relation algebra $\mathcal{A}$ is an embedding of $\mathcal{A}$ into an algebra of binary relations that respects all the algebra operations *except perhaps* $+, -$. Note: $\cdot$ is respected.

Possible motivation: *every* relation algebra has a representation respecting all operations *except perhaps* $\cdot, -$ *but including* $+$ (Jónsson–Tarski 1951).

Jónsson (1959) also looked at *subreducts* of RRA to signature without $+, -$.

$$\text{wRRA} = \{\mathcal{A} \in \text{RA} : \mathcal{A} \text{ has a weak representation}\}.$$

Could wRRA be better behaved than RRA...?

A few facts about wRRA

Not all that much is known about wRRA.

But so far, it seems even worse than RRA.

- RRA $\subseteq$ wRRA $\subseteq$ RA by definition
- wRRA is a variety (Pécsi, 2009)
- RRA not finitely axiomatisable over wRRA (Andréka 1994)
- wRRA not finitely axiomatisable (Haiman 1991/IH–Mikulás 2000)

  So RRA $\subsetneq$ wRRA $\subsetneq$ RA.
- wRRA, RRA recursively inseparable on finite algebras (Hirsch–IH 2002)
- $\mathrm{RA}_n \not\subseteq$ wRRA $\not\subseteq \mathrm{RA}_n$ all $n \geq 5$ (Hirsch–IH–Maddux 2009)

Two more items for the list

- wRRA is not canonical (answers question of Goldblatt)
- wRRA is not closed under taking Monk completions.

Joint work with Szabolcs Mikulás, 2010 (thanks also to Rob Goldblatt, Robin Hirsch, Roger Maddux).

I'll only discuss canonicity (Monk completions use nearly the same proof).

Why prove that wRRA is not canonical?

How prove that wRRA is not canonical?

We (probably) need to find a relation algebra $\mathcal{A}$ such that

- $\mathcal{A} \in \mathsf{wRRA}$,
- the canonical extension $\mathcal{A}^\sigma$ of $\mathcal{A}$ is not in wRRA.

So we need to know...

1. what canonical extensions are,
2. some property of relation algebras (or maybe just some special relation algebras), that is
    - equivalent to weak representability of such relation algebras,
    - provably NOT preserved by canonical extensions.

Canonical extensions of relation algebras (revision)

$\mathcal{A} = (A, +, \cdot, -, 0, 1, 1', \smile, ;)$: a relation algebra.

$\mathcal{A}$ is a completely additive BAO. We can use standard duality (JT51).

$\mathcal{A}_+$: set of all *ultrafilters* of (the boolean reduct of) $\mathcal{A}$.

Make this an *atom structure* [terminology abuse] called the *canonical frame of $\mathcal{A}$,* with relations dual to the relation algebra operators.
Eg $R_;(\mu, \nu, \theta)$ iff complex product $\mu\,;\nu \subseteq \theta$.

Take *full complex algebra* $(\mathcal{A}_+)^+$ over this atom structure. (Domain is $\wp(\mathcal{A}_+)$, operations induced from $\mathcal{A}_+$-relations by complete additivity.)
This is the canonical extension $\mathcal{A}^\sigma$.

Notes: $\mathcal{A}$ embeds in $\mathcal{A}^\sigma$ via $a \mapsto \{\mu \in \mathcal{A}_+ : a \in \mu\}$.

For a *finite* relation algebra atom structure $\mathcal{S}$, we have $(\mathcal{S}^+)_+ \cong \mathcal{S}$.

Inverse systems of atom structures (over $(\omega, \leq)$)

Reminder: an *inverse system* of relation algebra atom structures is an object

$$\mathfrak{I} = (\mathcal{S}_n, f_n^m : n \leq m < \omega),$$

where the $\mathcal{S}_n$ are relation algebra atom structures, and each $f_n^m : \mathcal{S}_m \to \mathcal{S}_n$ is a surjective bounded morphism (p-morphism) of atom structures (defined as in modal logic).

The *inverse limit* of $\mathfrak{I}$ is the substructure $\lim_{\leftarrow} \mathfrak{I} \subseteq \prod_{n<\omega} \mathcal{S}_n$ consisting of all sequences x such that $f_n^m(x(m)) = x(n)$ for all $n \leq m < \omega$.

Inverse systems and canonicity

We wanted a property of (some) relation algebras that is

1. provably NOT preserved by canonical extensions,
2. equivalent to weak representability.

Restrict attention to properties $\mathcal{P}$ of *complete atomic relation algebras,* so of the form $\mathcal{S}^+$ up to isomorphism.

Essentially, $\mathcal{P}$ is a property of the atom structure, $\mathcal{S}$.

Assuming (2), *it's enough if $\mathcal{P}$ is provably NOT preserved by inverse limits of inverse systems of finite relation algebra atom structures.*

Why is this enough?

Let $\mathcal{P}$ be a property of (some) relation algebra atom structures such that $\mathcal{P}(\mathcal{S}) \iff \mathcal{S}^+ \in \text{wRRA}$.

Let $\mathfrak{I} = (\mathcal{S}_n, f_n^m : n \leq m < \omega)$ be an inverse system of finite relation algebra atom structures and surjective bounded morphisms.

Suppose all the $\mathcal{S}_n$ have property $\mathcal{P}$, but $\lim_{\leftarrow} \mathfrak{I}$ doesn't.

Let $\mathfrak{D} = (\mathcal{S}_n^+, (f_n^m)^+ : n \leq m < \omega)$. Here, $(f_n^m)^+(X) = (f_n^m)^{-1}[X]$.

$\mathfrak{D}$ is a direct system of (finite) relation algebras and embeddings.
Let $\mathcal{A} = \lim_{\rightarrow} \mathfrak{D}$. (like union of chain)

All $\mathcal{S}_n$ have $\mathcal{P}$, so $\mathcal{S}_n^+ \in \text{wRRA}$. So $\mathcal{A} \in \text{wRRA}$ (variety, $\lim_{\rightarrow}$-closed).

As the $\mathcal{S}_n$ are finite, $(\mathcal{S}_n^+)_+ \cong \mathcal{S}_n$.

Goldblatt 1976: $\mathcal{A}_+ \cong \lim_{\leftarrow} \mathfrak{I}$. Hence $\mathcal{A}^\sigma \cong (\lim_{\leftarrow} \mathfrak{I})^+$.

But $\lim_{\leftarrow} \mathfrak{I}$ doesn't have $\mathcal{P}$. So $\mathcal{A}^\sigma \notin \text{wRRA}$ as required.

So what $\mathcal{P}$ do we use?

We use (undirected loop-free) *graphs.*

Chromatic number of a graph Γ: smallest k s.t. Γ is union of k independent (edge-free) sets.

GIven a graph Γ, we build a relation algebra atom structure $\alpha(\Gamma)$.

α : $\langle$graphs,graph bounded morphisms$\rangle \to \langle$relation algebra atom structures, bounded morphisms$\rangle$ is a covariant functor preserving surjective maps and commuting with inverse limits.

Define $\mathcal{P}(\alpha(\Gamma))$ iff Γ *has chromatic number* > 2.

So enough to show

P1. 'chromatic number > 2' is not preserved by inverse limits of inverse systems of finite graphs and graph bounded morphisms,

P2. Γ has chromatic number > 2 iff $\alpha(\Gamma)^+ \in$ wRRA.

P1. 'Chromatic number > 2' not preserved by inverse limits

For $n \geq 1$ let $\mathcal{C}_n$ be a *cycle of length* 3^n*.* It has chromatic number 3.

For $n \leq m$, let $f_n^m : \mathcal{C}_m \to \mathcal{C}_n$ 'wrap' $\mathcal{C}_m$ onto $\mathcal{C}_n$.

Then $\mathfrak{I} = (\mathcal{C}_n, f_n^m : 1 \leq n \leq m < \omega)$ is an inverse system of finite graphs of chromatic number > 2, and surjective graph bounded morphisms.

But (exercise) $\lim_{\leftarrow} \mathfrak{I}$ has no cycles so has chromatic number ≤ 2.

We are done.

P2. Γ has chromatic number > 2 iff $\alpha(\Gamma)^+ \in$ wRRA

Tougher. We need to define α now. Complicated...

Fix a graph $\Gamma = (V, E)$ (so $E \subseteq V \times V$ is the edge relation).
The atoms of the relation algebra atom structure $\alpha(\Gamma)$ are:

$$1', \ \mathsf{g}_i, \ \mathsf{w}_i, \ \breve{\mathsf{w}}_i \ (i \in \{0,1,2,3\}), \ \mathsf{v}, \ \mathsf{y}_x, \ \mathsf{r}_x \ (x \in \Gamma).$$

The w_i are white, the g_i green, the y_x yellow, and the r_x red.

$1'$ is the sole identity atom. The converse of w_i is $\breve{\mathsf{w}}_i$, and vice versa (each $i < 4$). All other atoms are self-converse.

Composition can be specified by listing the *forbidden triples* (a, b, c) of atoms — those such that $c \not\leq a \,;\, b$ in $\alpha(\Gamma)^+$.

Forbidden triples

F1. $(1', a, b)$ whenever $a \neq b$ (we have to put this in),

F2. $(\mathsf{g}_i, \mathsf{g}_j, \mathsf{g}_k)$ for each pairwise distinct $i, j, k < 4$,

F3. $(\mathsf{y}_x, \mathsf{y}_y, \mathsf{r}_z)$ for each pairwise distinct $x, y, z \in \Gamma$,

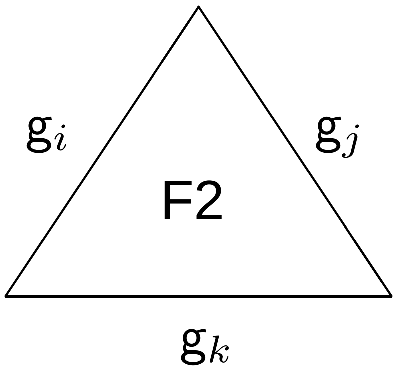
F4. $(\mathsf{g}_i, \mathsf{y}_x, b)$ for each $x \in \Gamma$, $i < 4$, and

$$\begin{aligned} b \quad \in \quad & \{\mathsf{g}_j : j < 4,\ j \neq i\} \cup \{\mathsf{w}_k : k < 4,\ k = i \bmod 2\} \\ & \cup \{\breve{\mathsf{w}}_i, \mathsf{v}\} \cup \{\mathsf{y}_y : y \in \Gamma \setminus \{x\}\} \cup \{\mathsf{r}_z : z \in \Gamma, \neg E(z, x)\}, \end{aligned}$$

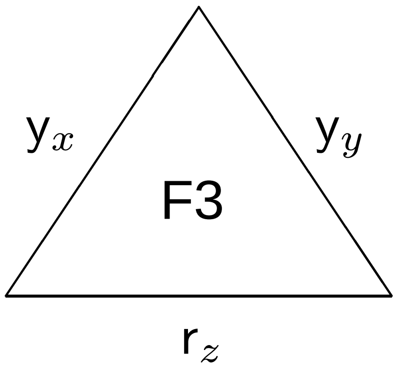
F5. $(\breve{\mathsf{w}}_i, \mathsf{w}_j, \mathsf{v})$ for each distinct $i, j < 4$.

All Peircean transforms of cycles forbidden by a rule are also forbidden by the same rule: i.e., if (a, b, c) is forbidden by a rule then so are $(\breve{b}, \breve{a}, \breve{c})$, $(\breve{a}, c, b)$, $(\breve{c}, a, \breve{b})$, $(c, \breve{b}, a)$, and $(b, \breve{c}, \breve{a})$.

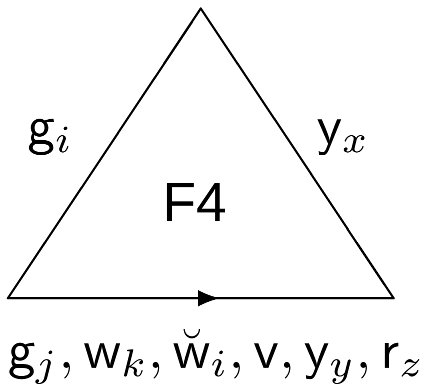
The main forbidden triples in pictures



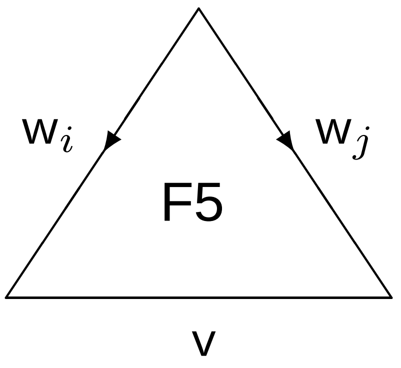
$i, j, k < 4$ distinct



$x, y, z \in \Gamma$ distinct



$j \neq i, k = i \bmod 2, x \neq y, \neg E(x, z)$



$i, j < 4$ distinct

$\alpha(\Gamma)^+$ not weakly representable for Γ of chromatic number ≤ 2

Write $\alpha(\Gamma)^+$ as $\mathcal{A}(\Gamma)$.

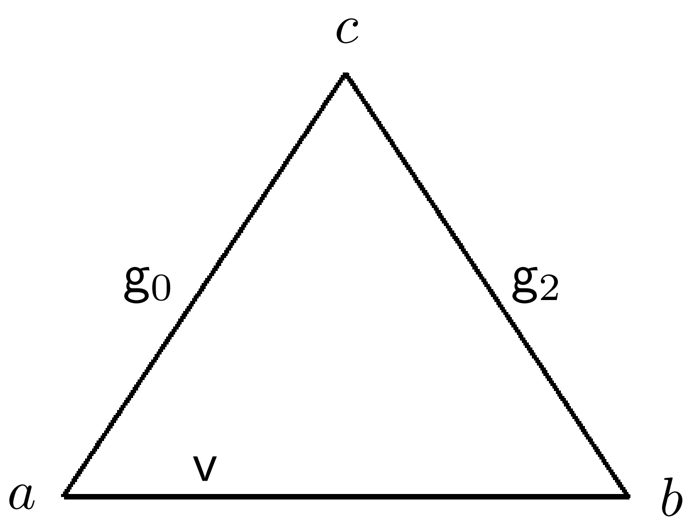
For $P \subseteq \Gamma$ write $\mathsf{y}_P = \sum_{x \in P} \mathsf{y}_x \in \mathcal{A}(\Gamma)$.
Define r_P similarly.

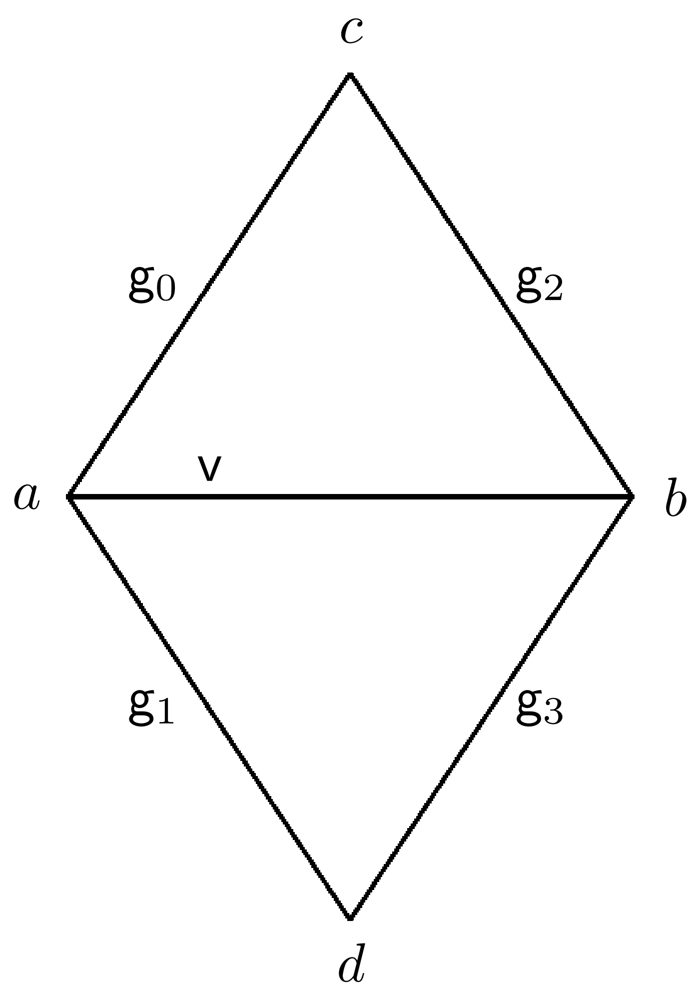
Proposition 1 *Suppose that Γ is a graph with at least two nodes, and chromatic number ≤ 2. Then $\mathcal{A}(\Gamma) \notin$ wRRA.*

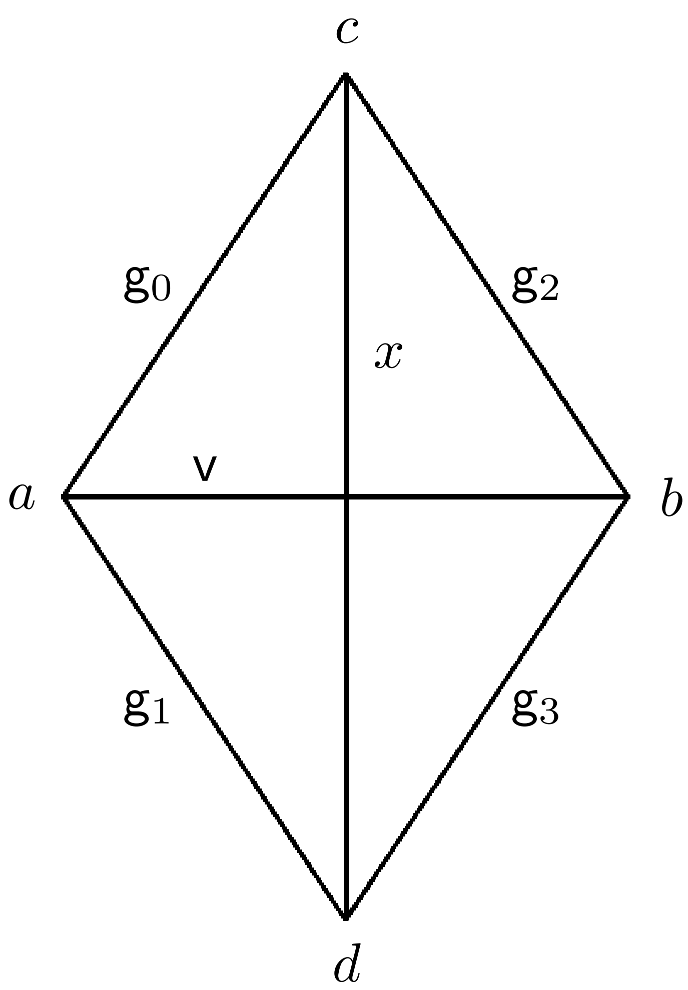
Proof. Assume for contradiction that $h : \mathcal{A}(\Gamma) \to \mathcal{B}$ is a weak representation of $\mathcal{A}(\Gamma)$, with $\mathcal{B}$ an algebra of binary relations over some set M.

Now watch. . .

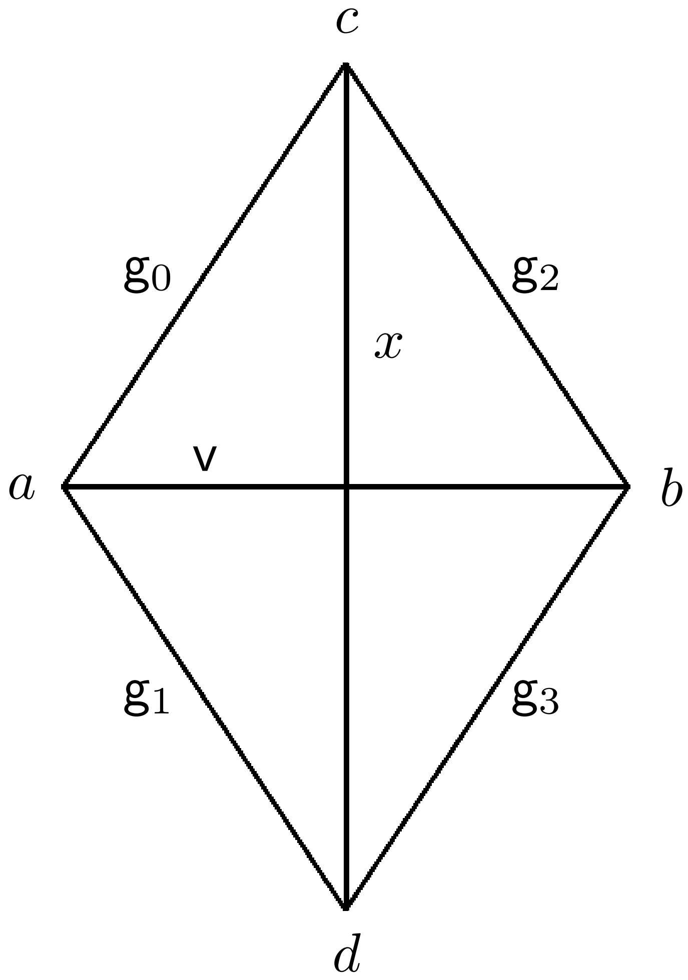

a
v
b


c
g0
g2
a
v
b

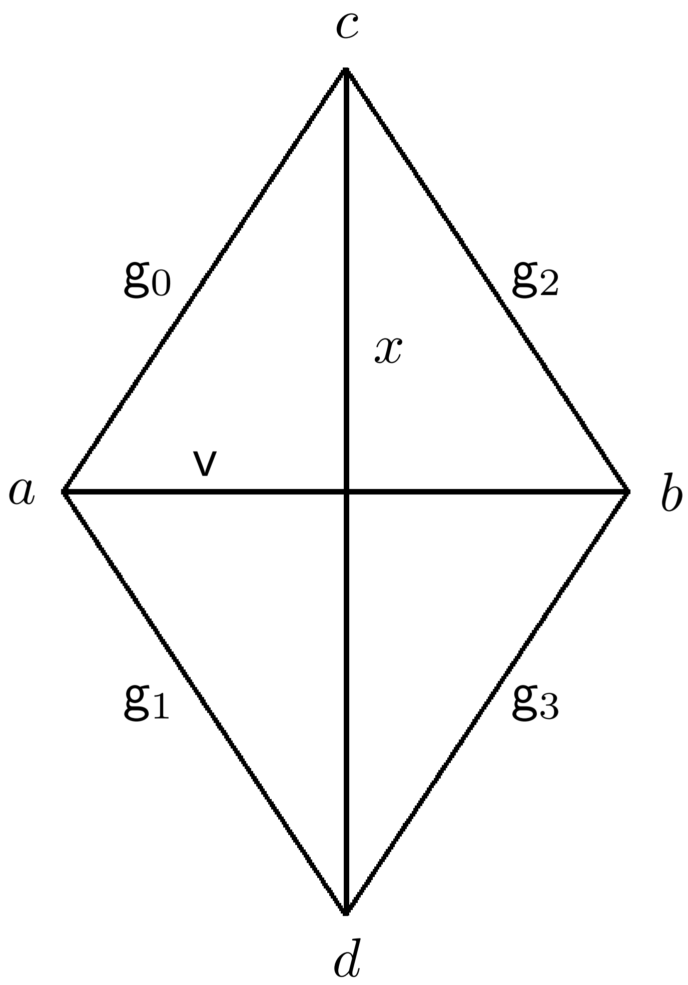

c
g0
g2
a
v
b
g1
g3
d



$$x = g_0\,;g_1 \cdot g_2\,;g_3$$



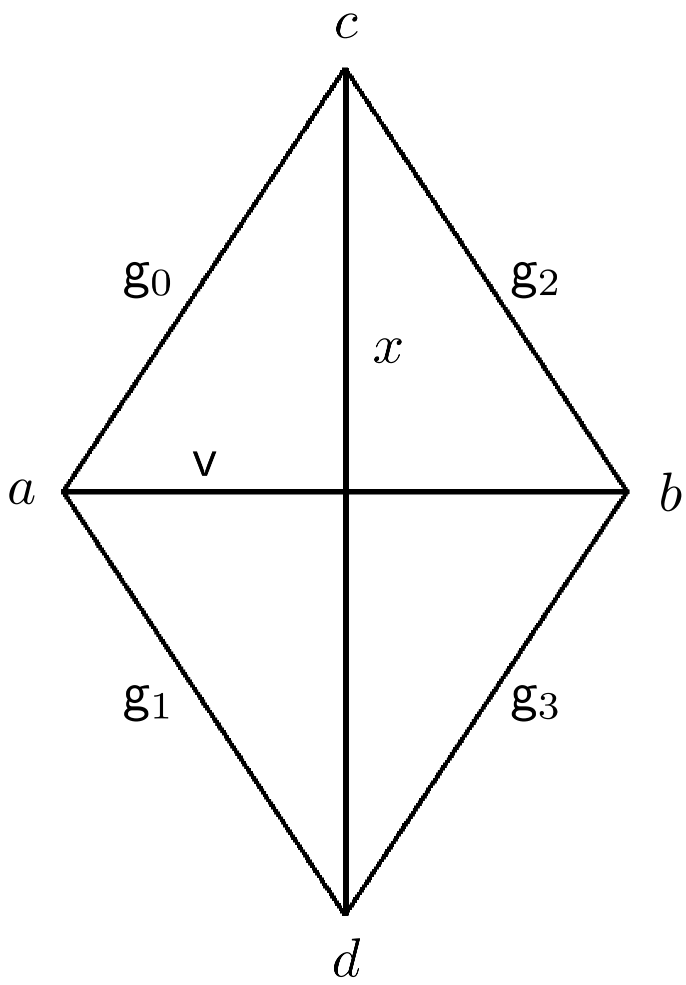
$$x = \mathsf{g}_0 \,;\, \mathsf{g}_1 \cdot \mathsf{g}_2 \,;\, \mathsf{g}_3$$
$$x = 1 - 1' - \sum_i \mathsf{g}_i - \mathsf{y}_\Gamma$$



$x = \mathsf{g}_0 \,;\mathsf{g}_1 \cdot \mathsf{g}_2 \,; \mathsf{g}_3$

$x = 1 - 1' - \sum_i \mathsf{g}_i - \mathsf{y}_\Gamma$

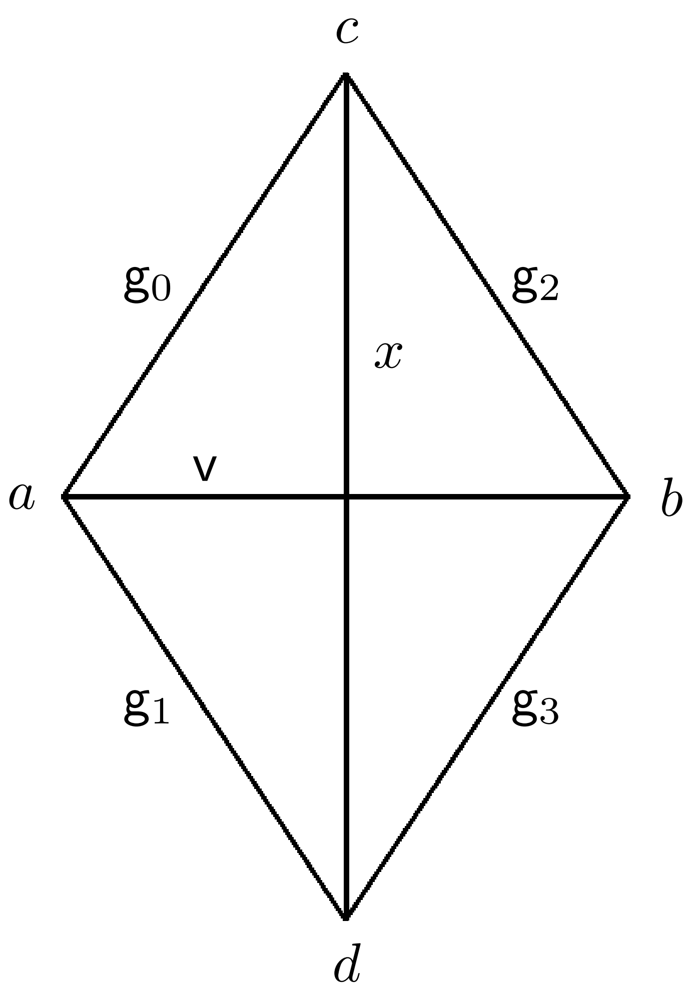
Pick a 2-colouring $\Gamma = P \mathbin{\dot\cup} Q$
(with $P, Q \neq \emptyset$).



$$x = \mathsf{g}_0 \,;\, \mathsf{g}_1 \cdot \mathsf{g}_2 \,;\, \mathsf{g}_3$$
$$x = 1 - 1' - \textstyle\sum_i \mathsf{g}_i - \mathsf{y}_\Gamma$$

Pick a 2-colouring $\Gamma = P \mathbin{\dot\cup} Q$
(with $P, Q \neq \emptyset$).

Then $\mathsf{y}_P \,;\, \mathsf{y}_Q = 1 - 1' - \sum_i \mathsf{g}_i$.

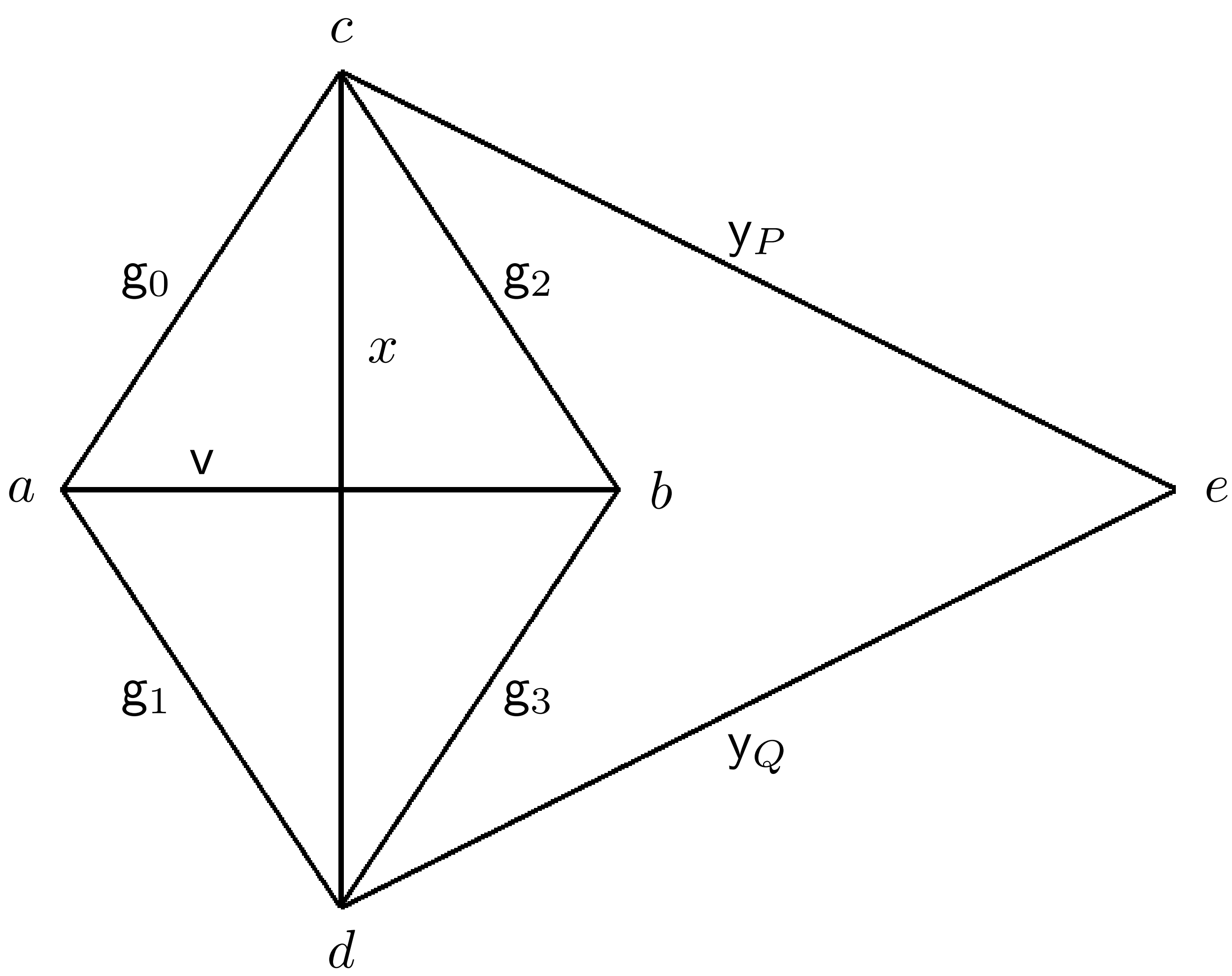


$$x = \mathsf{g}_0 \mathbin{;} \mathsf{g}_1 \cdot \mathsf{g}_2 \mathbin{;} \mathsf{g}_3$$
$$x = 1 - 1' - \textstyle\sum_i \mathsf{g}_i - \mathsf{y}_\Gamma$$

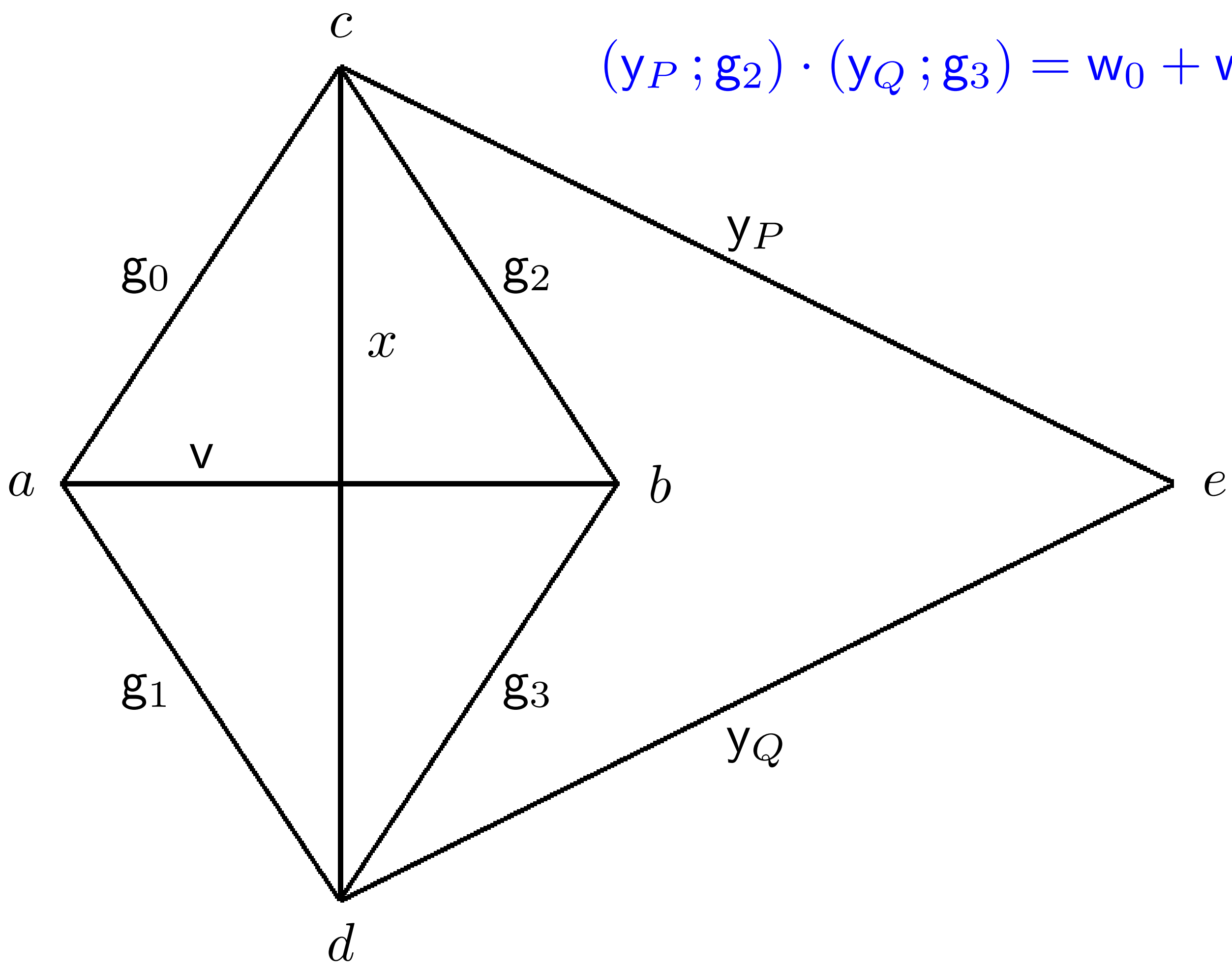
Pick a 2-colouring $\Gamma = P \,\dot{\cup}\, Q$ (with $P, Q \neq \emptyset$).

Then $\mathsf{y}_P \mathbin{;} \mathsf{y}_Q = 1 - 1' - \sum_i \mathsf{g}_i$.

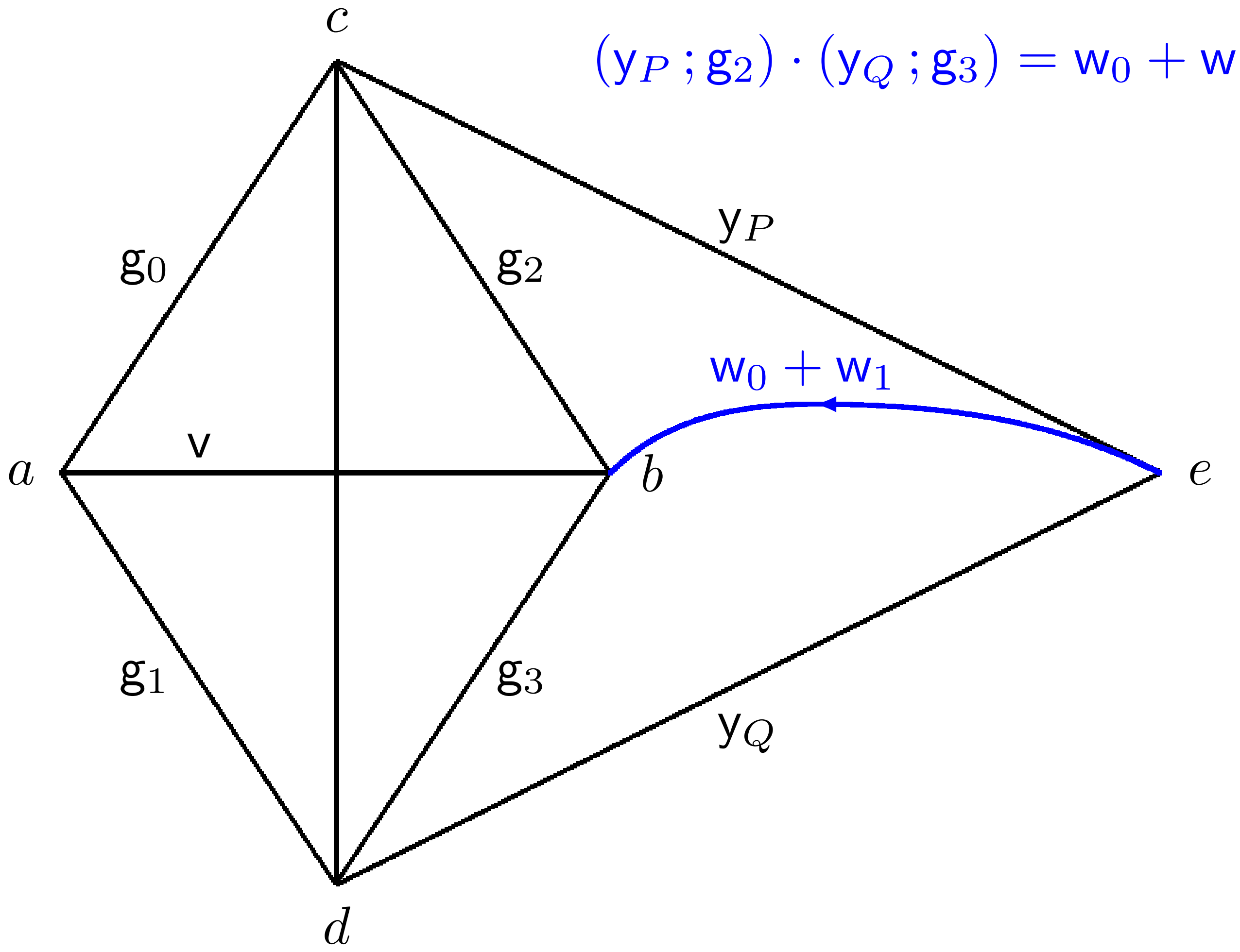
So $x \leq \mathsf{y}_P \mathbin{;} \mathsf{y}_Q$.


c
y_P
g_0
g_2
x
v
a
b
e
g_1
g_3
y_Q
d

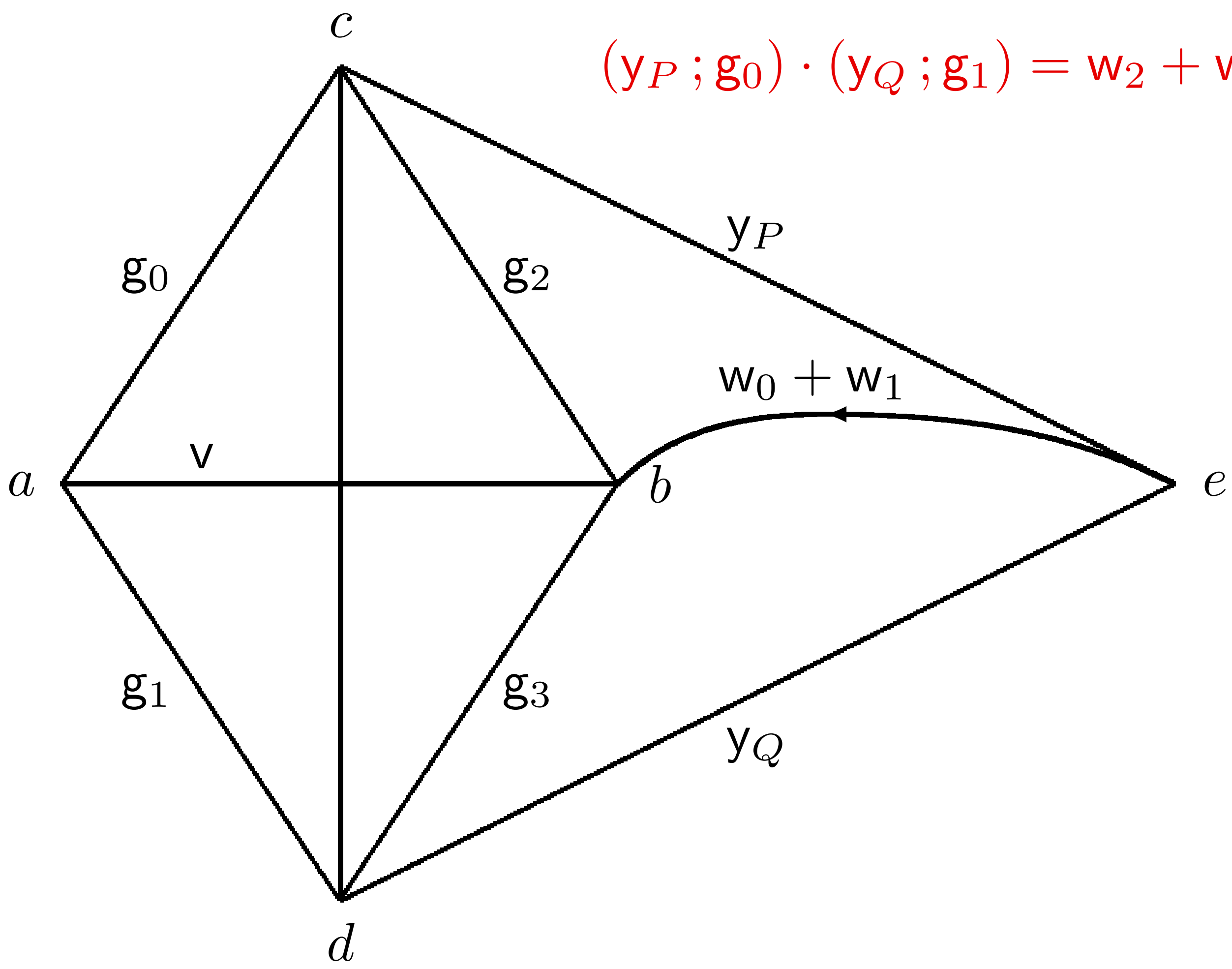
$$(\mathsf{y}_P\,;\mathsf{g}_2)\cdot(\mathsf{y}_Q\,;\mathsf{g}_3)=\mathsf{w}_0+\mathsf{w}_1$$

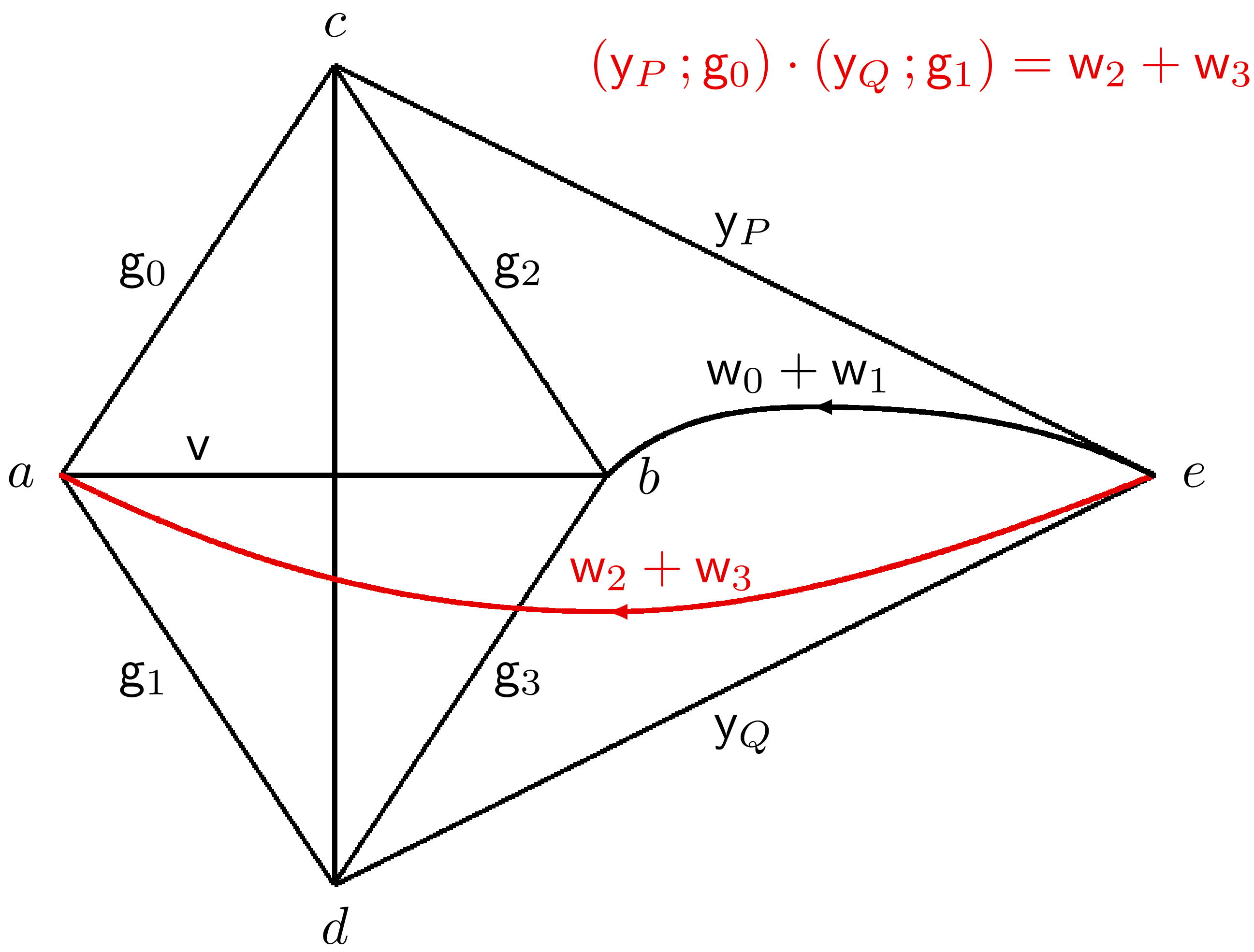


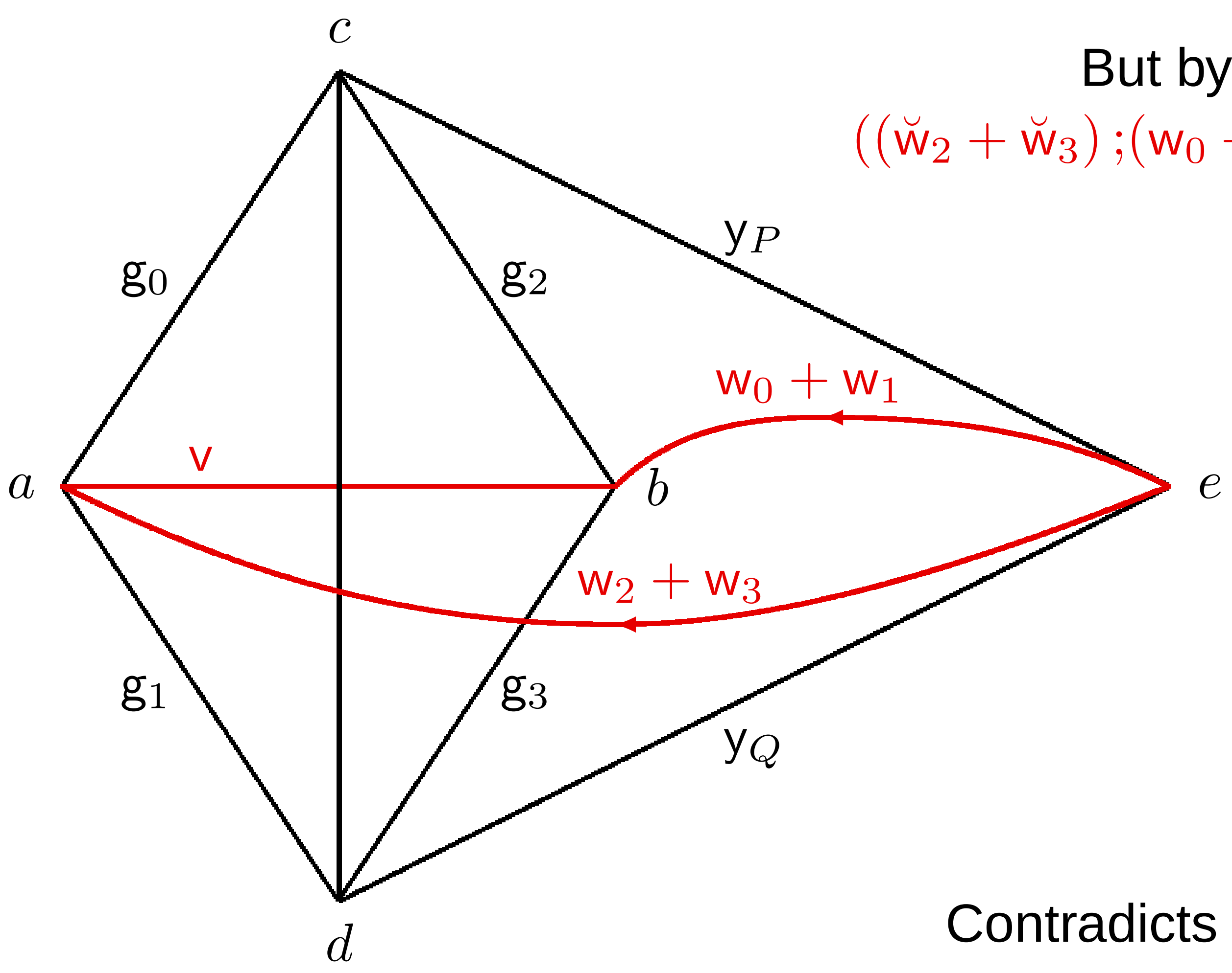
$$(\mathsf{y}_P\,;\mathsf{g}_2)\cdot(\mathsf{y}_Q\,;\mathsf{g}_3)=\mathsf{w}_0+\mathsf{w}_1$$



$$(\mathsf{y}_P\,;\mathsf{g}_0)\cdot(\mathsf{y}_Q\,;\mathsf{g}_1) = \mathsf{w}_2 + \mathsf{w}_3$$




$(y_P\,;g_0)\cdot(y_Q\,;g_1) = w_2 + w_3$
c
g_0
g_2
y_P
$w_0 + w_1$
a
v
b
e
$w_2 + w_3$
g_1
g_3
y_Q
d



But by F5,

$$((\breve{\mathsf{w}}_2 + \breve{\mathsf{w}}_3)\,;(\mathsf{w}_0 + \mathsf{w}_1)) \cdot \mathsf{v} = 0$$

Contradicts $h(0) = \emptyset$.

QED

Weak representability of $\mathcal{A}(\Gamma)\ldots$

Proposition 2 *Let Γ be a connected graph with chromatic number > 2. Then $\mathcal{A}(\Gamma) \in$ wRRA.*

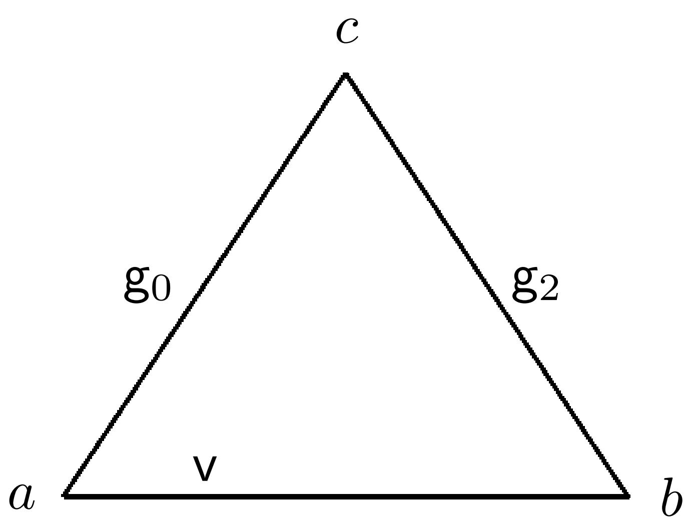
Proof.

Too long to go into, sorry.
Proved by games.
The ‘connected’ requirement is harmless as 3^n-cycles are connected. ■

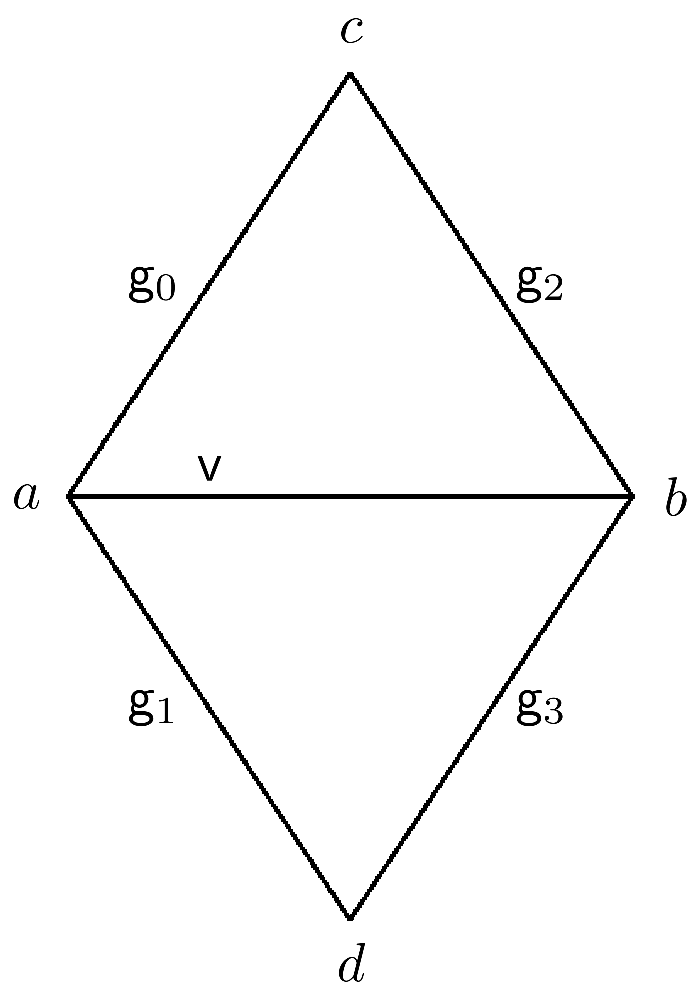
However I can say what goes wrong with the previous proof when Γ is connected & has chromatic number > 2.

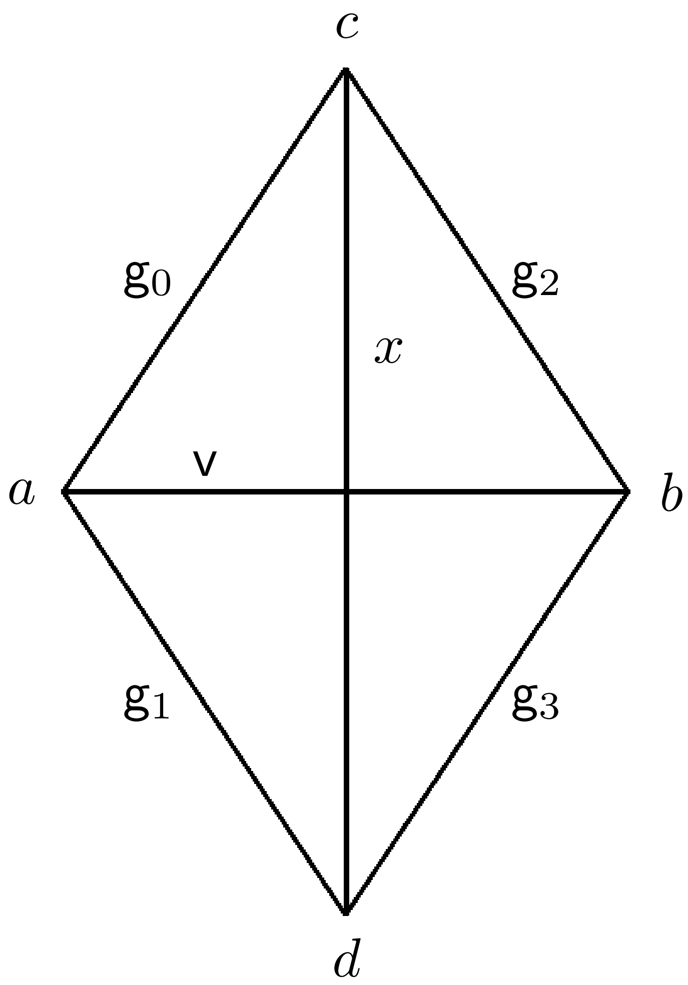
And this is the heart of the proof of proposition 2.


a
v
b

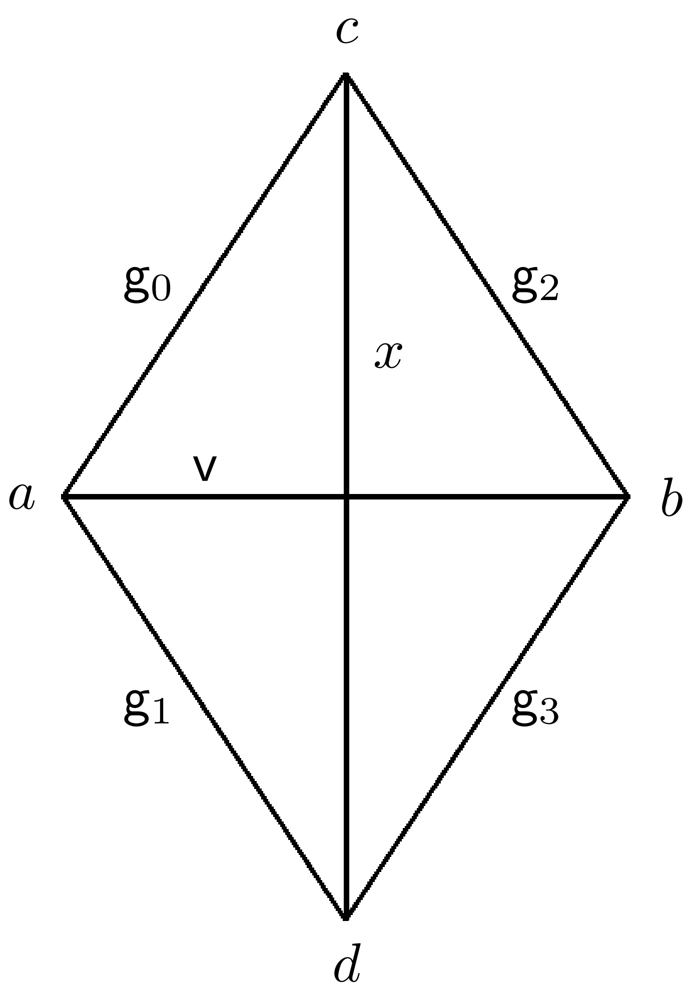


c
g_0
g_2
a
v
b


c
g0
g2
a
v
b
g1
g3
d

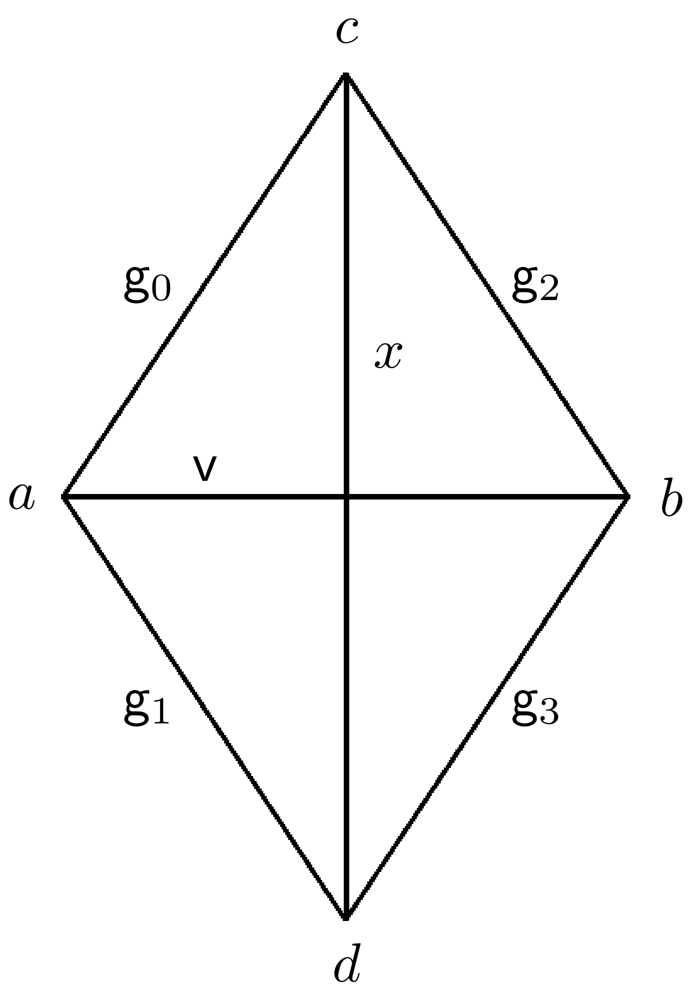


$$x = \mathsf{g}_0 \,;\, \mathsf{g}_1 \cdot \mathsf{g}_2 \,;\, \mathsf{g}_3$$
$$x = 1 - 1' - \sum_i \mathsf{g}_i - \mathsf{y}_\Gamma \geq \mathsf{r}_\Gamma$$



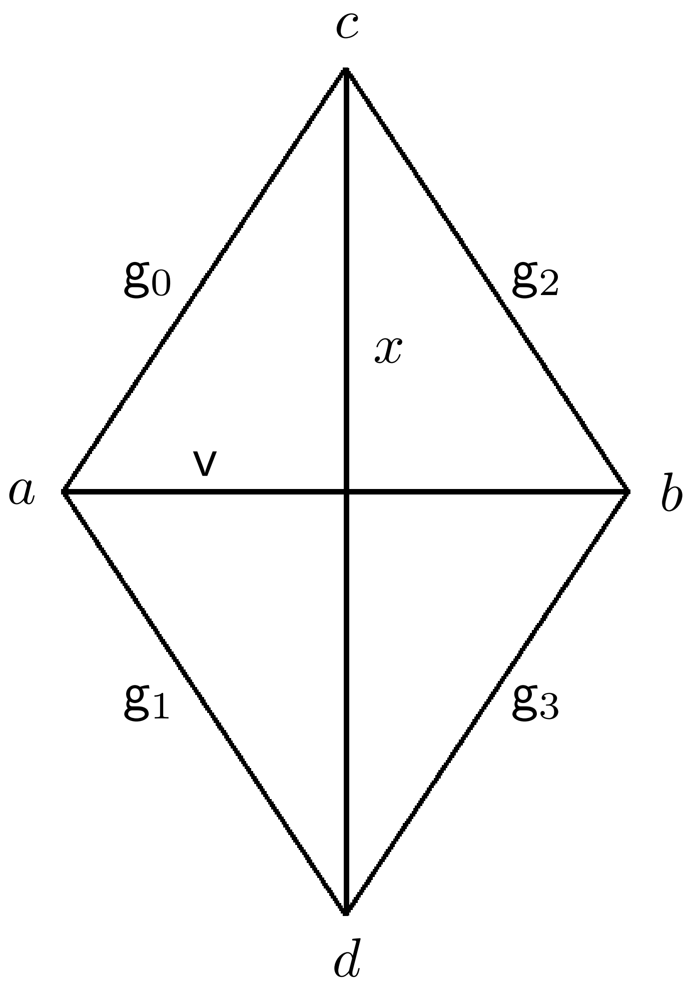
$$x = \mathsf{g}_0 \,;\, \mathsf{g}_1 \cdot \mathsf{g}_2 \,;\, \mathsf{g}_3$$
$$x = 1 - 1' - \textstyle\sum_i \mathsf{g}_i - \mathsf{y}_\Gamma \geq \mathsf{r}_\Gamma$$

If $x \leq \mathsf{y}_P \,;\, \mathsf{y}_Q$ and $P \cap Q = \emptyset$,
then by F3, $P \cup Q = \Gamma$.



$$x = g_0 \,;\, g_1 \cdot g_2 \,;\, g_3$$
$$x = 1 - 1' - \sum_i g_i - y_\Gamma \geq r_\Gamma$$

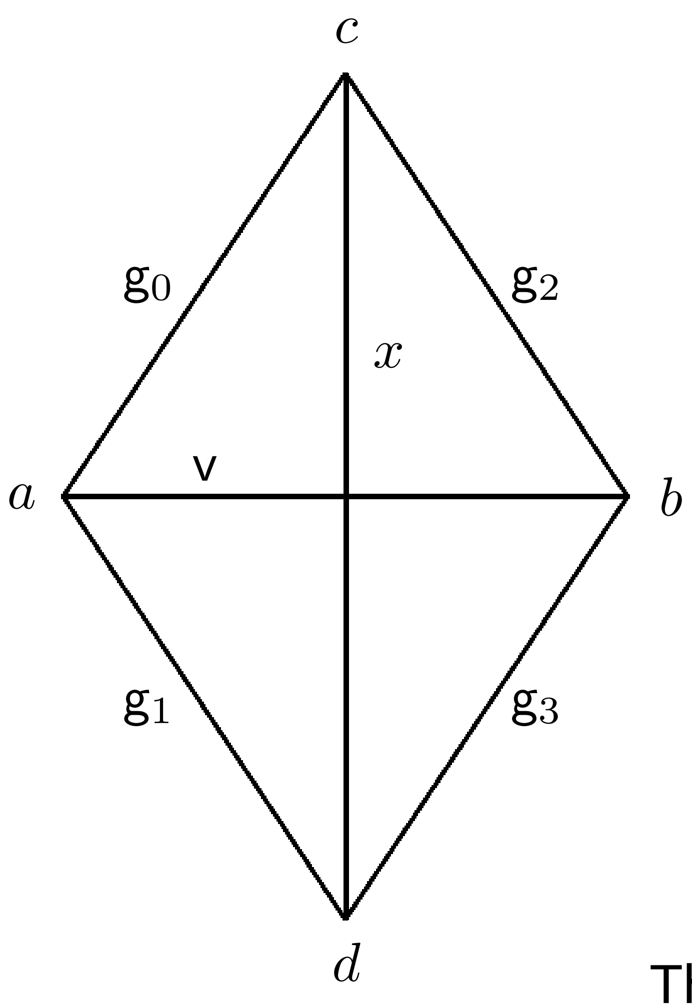
If $x \leq y_P \,;\, y_Q$ and $P \cap Q = \emptyset$,
then by F3, $P \cup Q = \Gamma$.
So P, Q not both independent.



$x = g_0 \,;\, g_1 \cdot g_2 \,;\, g_3$
$x = 1 - 1' - \sum_i g_i - y_\Gamma \geq r_\Gamma$

If $x \leq y_P \,;\, y_Q$ and $P \cap Q = \emptyset$,
then by F3, $P \cup Q = \Gamma$.
So P, Q not both independent.

So $\exists x \in P, y \in Q, z \in \Gamma$
with $E(z, x) \wedge E(z, y)$.

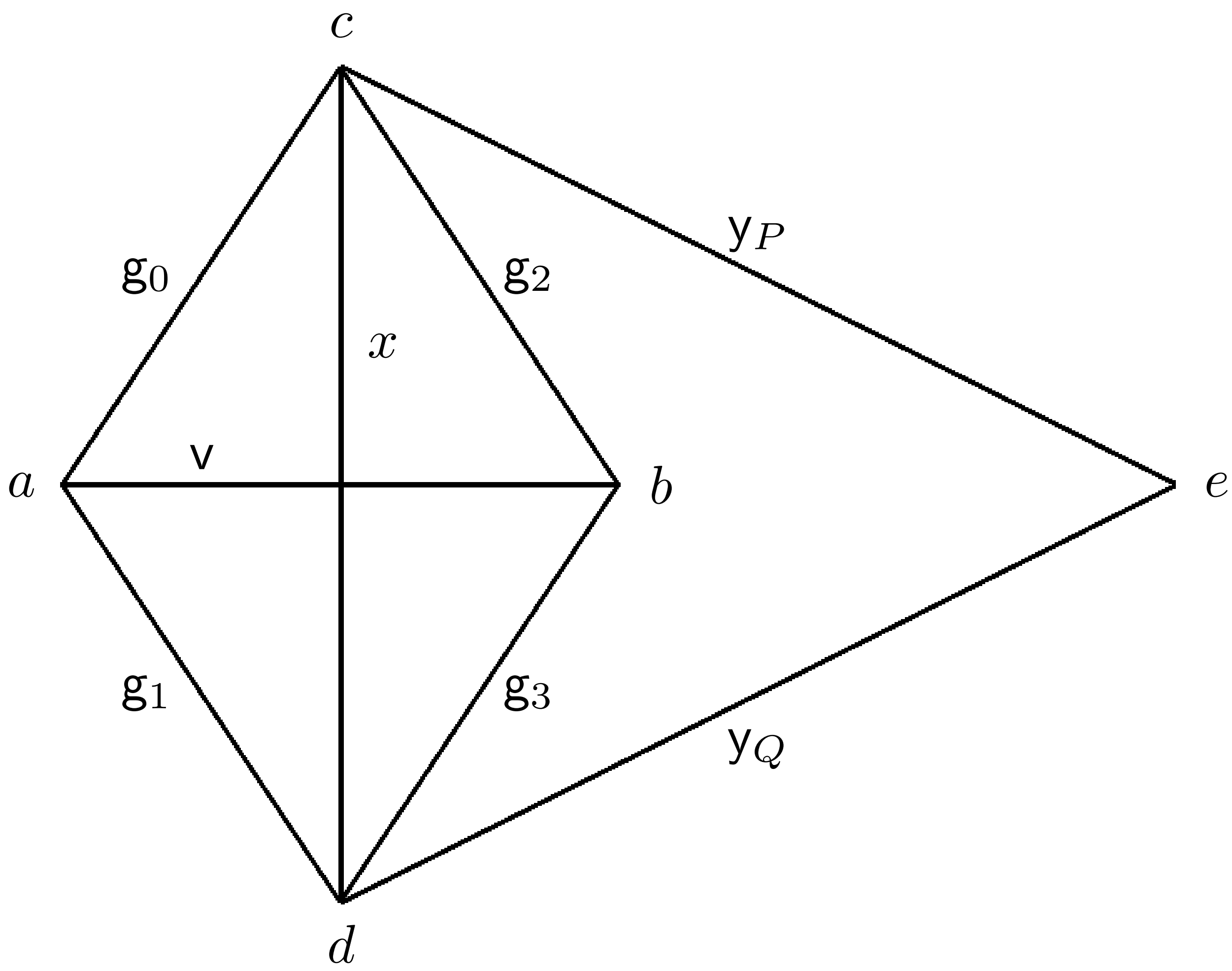


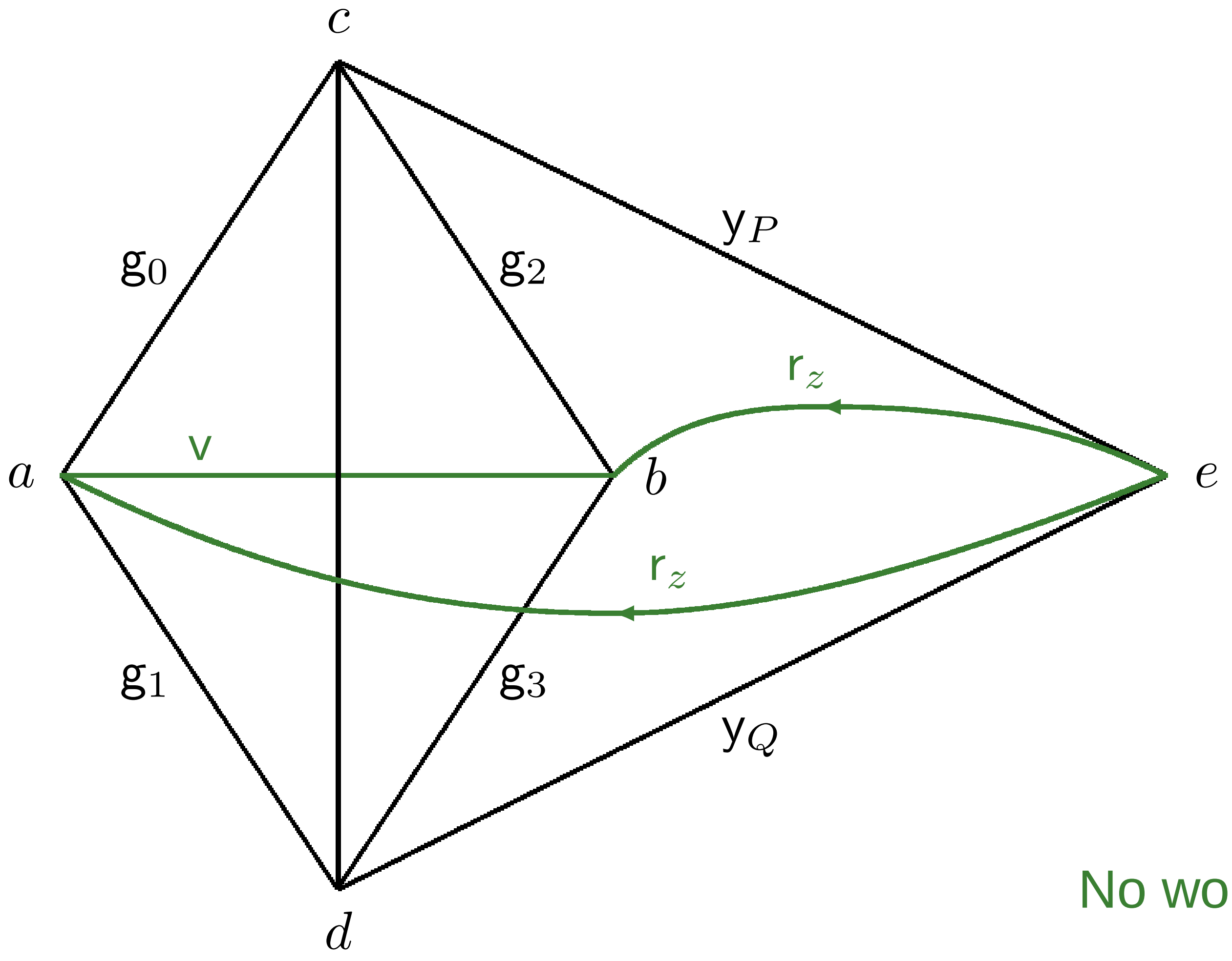
$x = \mathsf{g}_0 \,;\, \mathsf{g}_1 \cdot \mathsf{g}_2 \,;\, \mathsf{g}_3$
$x = 1 - 1' - \sum_i \mathsf{g}_i - \mathsf{y}_\Gamma \geq \mathsf{r}_\Gamma$

If $x \leq \mathsf{y}_P \,;\, \mathsf{y}_Q$ and $P \cap Q = \emptyset$,
then by F3, $P \cup Q = \Gamma$.
So P, Q not both independent.

So $\exists x \in P, y \in Q, z \in \Gamma$
with $E(z, x) \wedge E(z, y)$.

Then $\mathsf{r}_z \leq (\mathsf{y}_P \,;\, \mathsf{g}_i) \cdot (\mathsf{y}_Q \,;\, \mathsf{g}_i)$ all $i < 4$.


c
y_P
g_0
g_2
x
v
a
b
e
g_1
g_3
y_Q
d



No worries

Summary

$\mathcal{C}_n$: cycle of length 3^n.

We made relation algebras $\mathcal{A}(\mathcal{C}_1) \subseteq \mathcal{A}(\mathcal{C}_2) \subseteq \cdots \in$ wRRA.

The union $\mathcal{A}$ of the chain is in wRRA as well (variety).

But by Goldblatt, $\mathcal{A}^\sigma \cong \mathcal{A}(\Gamma)$ for an acyclic Γ (inverse limit of the $\mathcal{C}_n$).

So Γ has chromatic number ≤ 2. Therefore, $\mathcal{A}^\sigma \notin$ wRRA.

And there we are.

Open problems

Is there any class $\mathcal{K}$ of atom structures (frames) whatsoever such that

- wRRA $= \mathbf{HSPCm}\mathcal{K}$?
- wRRA $= \mathbf{SCm}\mathcal{K}$?

You might also investigate *complete weak representations:* preserving all existing meets.

Enjoy your lunch

and thank you for your patience.

Ian Hodkinson and Szabolcs Mikulás

On canonicity and completions of weakly representable relation algebras

`http://www.doc.ic.ac.uk/~imh/papers/wrra-can.pdf`