Generalizations of Relation Algebras from the perspective of (semi)lattices with operators

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Relation algebras

Definition (Tarski 1941)

Relation algebras are algebras $(A, \wedge, \vee, ', \bot, \top, \cdot, ^{\smile}, 1)$ such that

- $(A, \wedge, \vee, ', \bot, \top)$ is a Boolean algebra
- $(A, \cdot, 1)$ is a monoid and
- for all $x, y, z \in A$, $\quad (x \vee y)z = xz \vee yz \qquad (x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$

  $x^{\smile\smile} = x \qquad (xy)^{\smile} = y^{\smile}x^{\smile} \qquad x^{\smile}(xy)' \le y'$

The axioms were intended to capture the equational theory of algebras of binary relations: For a set U

$Re(U) = (\mathcal{P}(U^2), \cap, \cup, ', \emptyset, U^2, \circ, ^{\smile}, id_U)$ is the *full relation algebra* on U

$\circ$ is composition, $\quad R^{\smile} = \{(u, v) : (v, u) \in R\}, \quad id_U = \{(u, u) : u \in U\}$

E.g. $(u, v) \in x^{\smile}(xy)' \;\Rightarrow\; \exists w (w, u) \in x, (w, v) \notin x \circ y \;\Rightarrow\; (u, v) \notin y$

Properties of relation algebras

The variety **RRA** of *representable relation algebras* is generated by the class of all full relation algebras

Monk [1964] proved that **RRA** is a *nonfinitely axiomatizable* subvariety of the variety **RA** of all relation algebras

Hirsch and Hodkinson [1997] proved that it is *undecidable* whether a finite relation algebra is in **RRA**

Relation algebras are *Boolean algebras with operators* ($\circ, ^{\smile}$ distr. over $\vee$)

Relation algebras can model relational semantics of computer programs

But both the varieties **RA** and **RRA** have *undecidable* equational theories

Can this be fixed by weakening the axioms, *keeping associativity*?

Conjugates and residuals

The five identities are equivalent to

$$xy \le z' \iff x^{\smile} z \le y' \iff zy^{\smile} \le x'$$

Proof.

From $x^{\smile}(xy)' \le y'$ we get $xy \le z' \ \Rightarrow\ x^{\smile}z \le x^{\smile}(xy)' \le y'$ and

from $x^{\smile}(xy)' \le y'$ we get $y \le (x^{\smile}z)' \ \Rightarrow\ xy \le x^{\smile\smile}(x^{\smile}z)' \le z'$

Conversely from the $\iff$ we get $xy \le (xy)'' \ \Rightarrow x^{\smile}(xy)' \le y'$ □

So defining *conjugates* $x \triangleright z = x^{\smile}z$ and $z \triangleleft y = zy^{\smile}$ we have

$$xy \le z' \iff x \triangleright z \le y' \iff z \triangleleft y \le x'$$

or replacing z by z' and defining *residuals* $x\backslash z = (x \triangleright z')'$ and $z/y = (z' \triangleleft y)'$ we get the equivalent *residuation property*

$$xy \le z \iff y \le x\backslash z \iff x \le z/y$$

Residuated Boolean monoids

Definition (Birkhoff 1948, Jónsson 1991)

Residuated Boolean monoids are algebras $(A, \wedge, \vee, ', \bot, \top, \cdot, \triangleright, \triangleleft, 1)$ s. t.

- $(A, \wedge, \vee, ', \bot, \top)$ is a Boolean algebra
- $(A, \cdot, 1)$ is a monoid and
- for all $x, y, z \in A$, $\quad xy \le z' \iff x \triangleright z \le y' \iff z \triangleleft y \le x'$

Examples: For any monoid $\mathbf{M} = (M, *, e)$ the powerset monoid $\mathcal{P}(\mathbf{M}) = (\mathcal{P}(M), \cap, \cup, ', \emptyset, M, \cdot, \triangleright, \triangleleft, \{e\})$ is a residuated Boolean monoid

where $XY = \{x * y : x \in X, y \in Y\}$,
$X \triangleright Y = \{z : x * z = y \text{ for some } x \in X, y \in Y\}$,
$X \triangleleft Y = \{z : z * y = x \text{ for some } x \in X, y \in Y\}$

If $\mathbf{G} = (G, *, ^{-1})$ is a group, $\mathcal{P}(\mathbf{G})$ is a relation algebra, $X^{\smile} = \{x^{-1} : x \in X\}$

RM = the variety of residuated Boolean monoids

RA = the variety of relation algebras

Theorem (Jónsson and Tsinakis 1993)

RA *is termequivalent to the subvariety of* **RM** *defined by* $(x \triangleright y)z = x \triangleright (yz)$

The termequivalence is given by $x \triangleright y = x^{\smile} y$, $x \triangleleft y = xy^{\smile}$ *and* $x^{\smile} = x \triangleright 1$

Aim to lift this result to residuated lattices and FL-algebras

RA and **RM** have undecidable equational theories

Want to find a larger variety "close to" RA that has a decidable equational theory, but ...

Kurucz, Nemeti, Sain and Simon [1993] proved that the variety of all Boolean algebras with an associative operator, as well as a "large number" of expanded subvarieties have undecidable equational theories

Positive Relation Algebras

Basically the theory of relation algebras without complementation

Subalgebras of complementation-free reducts of relation algebras

Subreducts of a variety are always a quasivariety (closed under S, P, P_U)

Is **pRA** a variety? (i.e. closed under H?)

Is **pRA** finitely based? (i.e. has fin. many equational or q-equat. axioms)

Does **pRA** have a decidable equational theory or universal theory?

[Andreka 1990] Representable pRAs have a decidable theory

Residuals are not definable in **pRA**

Lattice reducts are distributive; $xy = x \wedge y$ for $x, y \leq 1$

Sequential algebras

Definition (Hoare and Von Karger 1994)

A *sequential algebra* is a residuated Boolean monoid that is

- *balanced*: $x \triangleright 1 = 1 \triangleleft x$ and
- *euclidean*: $x(y \triangleright z) \leq (xy) \triangleright z$

Ex: Any relation algebra **A** relativized with a reflexive transitive element

For $t \in A$ with $1 \leq t = t^2$ define $\mathbf{A}|_t = (\downarrow t, \wedge, \vee, {}'^t, \bot, t, \cdot, \triangleright, \triangleleft, 1)$

where $x'^t = x' \wedge t$, $\quad x \triangleright y = (x^\smile y) \wedge t \quad$ and $\quad x \triangleleft y = (xy^\smile) \wedge t$

Problem: Does every sequential algebra arise in this way? True for **RSeA**

[KNSS 1993] The equational theory of sequential algebras is undecidable

[J. and Maddux 1997] Representable sequential algebras are not finitely axiomatizable

Unisorted allegories

Definition (Freyd and Scedrov 1990, Gutierrez 1998)

Unisorted allegories are algebras of the form $(A, \wedge, \cdot, ^\smallsmile, 1)$ such that

- $(A, \wedge)$ is a semilattice
- $(A, \cdot, 1)$ is a monoid
- $x^{\smallsmile\smallsmile} = x, \quad (xy)^\smallsmile = y^\smallsmile x^\smallsmile, \quad (x \wedge y)^\smallsmile = y^\smallsmile \wedge x^\smallsmile,$

  $x(y \wedge z) \wedge xy = x(y \wedge z)$ and $(x \wedge (zy^\smallsmile))y \wedge z = xy \wedge z$

They are generalizations of relation algebras without $\vee, ', \bot, \top$

Is the equational theory of allegories decidable?

Consider the graphical calculi of Andreka and Bredekhin 1995, Curtis and Lowe 1995, de Freitas and Viana 2010

(Anti)domain-range monoids

Definition (J. and Struth 2009)

A *domain-range monoid* is an algebra $(A, \cdot, 1, d, r)$ such that $(A, \cdot, 1)$ is a monoid and

(D1)	$d(x)x = x$	(R1)	$xr(x) = x$
(D2)	$d(xy) = d(xd(y))$	(R2)	$r(xy) = r(r(x)y)$
(D3)	$d(d(x)y) = d(x)d(y)$	(R3)	$r(xr(y)) = r(x)r(y)$
(D4)	$d(x)d(y) = d(y)d(x)$	(R4)	$r(x)r(y) = r(y)r(x)$
(D5)	$d(r(x)) = r(x)$	(R5)	$r(d(x)) = d(x)$

A *domain monoid* $(A, \cdot, 1, d)$ is a monoid that satisfies (D1)-(D4)

An *antidomain monoid* $(A, \cdot, 1, a)$ is a monoid that satisfies

$a(x)x = a(1)$	$xa(1) = a(1)$	$a(x)a(y) = a(y)a(x)$
$a(a(x))x = x$	$a(x) = a(xy)a(xa(y))$	$a(xy)x = a(xy)xa(y)$

Defining $d(x) = a(a(x))$ in an antidomain monoid gives a domain monoid

(Anti)domain-range semirings

Definition (Desharnais, Möller and Struth, 2003)

A *domain-range semiring* is an algebra $(A, \cdot, 1, +, 0, d, r)$ such that

- $(A, \cdot, 1, d, r)$ is a domain-range monoid
- $(A, +, 0)$ is a semilattice with bottom
- $\cdot, d, r$ distribute over $+$
- $x0 = 0x = 0 \qquad d(0) = 0 \qquad r(0) = 0 \qquad$ d(x)+1=1

An *antidomain semiring* $(A, \cdot, 1, +, 0, a)$ is an antidomain monoid such that $(A, +, 0)$ is a semilattice with bottom, $\cdot$ distributes over $+$, $a(x + y) = a(x)a(y)$, $x0 = 0x = 0$, $a(1) = 0$ and $a(x) + 1 = 1$

RAs have antidomain-range semiring reducts with $d(x) = xx^{\smile} \wedge 1$ etc

[J., Struth] The equational theory of domain-range semirings is decidable

[Hirsch Mikulas 2010] The class of representable (anti)domain(-range) monoids is not finitely axiomatizable

Residuated lattices and FL-algebras

Definition (Ward and Dilworth 1939, Ono 1990)

A *Residuated lattices* is of the form $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ where

- $(A, \wedge, \vee)$ is a lattice
- $(A, \cdot, 1)$ is a monoid
- the *residuation property* holds, i. e., for all $x, y, z \in A$

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x \backslash z$$

A *Full Lambek* (or *FL-*)*algebra* $(A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ is a residuated lattice expanded with a constant 0 (no properties assumed about it)

Examples: BAs, Heyting algebras, MV-algebras, BL-algebras, intuitionistic linear logic algebras, ... are FL-algebras

Generalized BAs, Brouwerian algebras, Wajsberg hoops, basic hoops, l-groups, GMV-algebras, GBL-algebras, ... are residuated lattices

Relational semantics for lattices with operators

Atom structures for BAOs = Kripke frames = $(W, R_i(i \in I))$

For lattices with operators = Galois frames = $(W, W', N, R_i, \epsilon_i(i \in I))$

E.g. for residuated lattices: $\mathbf{W} = (W, W', N, \circ, \backslash\!\backslash, /\!/, E)$ such that

- N is a binary relation from W to W', called the *Galois relation*,
- $X^{\triangleright} = \{y \in W' : XNy\} \quad Y^{\triangleleft} = \{x \in W : xNY\} \quad \gamma_N(X) = X^{\triangleright\triangleleft}$
- $\circ \subseteq W^3, \quad \backslash\!\backslash \subseteq W \times W' \times W', \quad /\!/ \subseteq W' \times W \times W'$
- $x \circ y = \{z : (x, y, z) \in \circ\}$ and similarly for $\backslash\!\backslash, /\!/$
- $(u \circ v)\ N\ w$ iff $v\ N\ (u \backslash\!\backslash w)$ iff $u\ N\ (w /\!/ v)$ all $u, v \in W, w \in W'$
- $E \subseteq W$ such that $(x \circ E)^{\triangleright} = \{x\}^{\triangleright} = (E \circ x)^{\triangleright}$, for all $x \in W$
- $[(x \circ y) \circ z]^{\triangleright} = [x \circ (y \circ z)]^{\triangleright}$ for all $x, y, z \in W$

Then $\mathbf{W}^+ = (\gamma_N[\mathcal{P}(W)], \cap, \vee, \circ, \backslash\!\backslash, /\!/, E)$ is a residuated lattice

Conversely, from a residuated lattice we get a Galois frame by taking

$W =$ filters, $W' =$ ideals, $N = \{(F, I) : F \cap I \neq \emptyset\}$

$(F, G, H) \in \circ$ iff $F \cdot G \subseteq H$, $(F, I, J) \in \backslash\!\!\backslash$ iff $F \backslash I \subseteq J$, $E = {\downarrow}1$

$\mathbf{W}^+$ gives the *canonical extension* of the residuated lattice

For semilattices only need filters (or ideals)

Galois frames can be built from a *Gentzen system* **G** (sequent calculus)

$W = T(Var)^* =$ sequences of terms over $\wedge, \vee, \cdot, \backslash, /, 1, Var$

$W' = T(Var) \times W^2$

$N = \{(w, (t, u, v)) : \mathbf{G} \vdash uwv \leq t\}$, $\circ =$ concatenation, $E = \{()\}$

$zwN(t, u, v)$ iff $\mathbf{G} \vdash uzwv \leq t$ iff $wN(t, uz, v)$, so $z \backslash\!\!\backslash (t, u, v) = \{(t, uz, v)\}$

Consequences of this construction

[J. and Tsinakis 2002] Algebraic proof of *eq. decidability* for RL, FL

[Blok and van Alten 2003] *Finite embeddability property* for integral RL

[Belardinelli, J. and Ono 2004] *Finite model property* for FL_{ew}

[Wille 2005] Algebraic proof of *equational decidability* of cyclic InFL

[J. and Galatos 2010] Algebraic proofs of cut-elimination, FMP and eq. decidability for RL, FL and "structural subvarieties", InFL, distributive FL

The construction can be adapted to *many subvarieties* of residuated lattices and other lattice ordered algebras, gives FEP in integral case

Whenever the Gentzen system gives a decision procedure then $\mathbf{W}^+$ contains the *Var*-generated *free algebra* of the variety

[J. and Moshier] Adding topology to Galois frames gives a *duality* for LOs

Returning to FL-algebras and relation algebras

- Complementation free reducts of residuated Boolean monoids
- *Symmetric relation algebras* are a subvariety of **RA** defined by $x^{\smile} = x$

If we let $0 = 1'$, $x\backslash y = (xy')'$ and $x/y = (x'y)'$ then symmetric RAs are FL-algebras

In this case $x' = x\backslash 0 = 0/x$

But for relation algebras in general $x\backslash 0 = (x^{\smile}1'')' = x^{\smile\prime}$ so complementation is not recovered by this term

In an FL-algebra there are two *linear negations*

$$\sim x = x\backslash 0 \qquad -x = 0/x$$

but they need not coincide

Definition of FL'-algebras

To interpret relation algebras into FL-algebras we expand FL-algebras with a unary operation:

Definition

An *FL′-algebra* is an expansion of an FL-algebra with a unary operation $'$ that satisfies $x'' = x$. Also define the following terms:

- *converses* $x^{\smile} = ({\sim}x)'$ and $x^{\sqcup} = (-x)'$,
- *conjugates* $x \triangleright y = (x\backslash y')'$ and $y \triangleleft x = (y'/x)'$

and consider the identities

(In) ${\sim}{-}x = x = -{\sim}x$ (involutive law)

(Cy) ${\sim}x = -x$ (cyclic law)

(Dm) $(x \wedge y)' = x' \vee y'$ (De Morgan, equivalent to $(x \vee y)' = x' \wedge y'$)

Properties of FL$'$-algebras

Proposition

In an FL$'$-algebra the following properties hold:

1. $(xy) \triangleright z = y \triangleright (x \triangleright z)$ *and* $z \triangleleft (yx) = (z \triangleleft x) \triangleleft y$
2. $(xy)^{\smile} = y \triangleright x^{\smile}$ *and* $(xy)^{\sqcup} = y^{\sqcup} \triangleleft x$
3. $1 \triangleright x = x$ *and* $x \triangleleft 1 = x$
4. $\sim x = -x$ *iff* $x^{\smile} = x^{\sqcup}$ *(cyclic/balanced)*

If (Dm): $(x \wedge y)' = x' \vee y'$ *is assumed then we also have*

- $xy \le z' \Leftrightarrow x \triangleright z \le y' \Leftrightarrow z \triangleleft y \le x'$ *(conjugation)*
- $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$ *and* $(x \vee y)^{\sqcup} = x^{\sqcup} \vee y^{\sqcup}$
- $(x \vee y) \triangleright z = (x \triangleright z) \vee (y \triangleright z)$ *and* $z \triangleleft (x \vee y) = (z \triangleleft x) \vee (z \triangleleft y)$
- $(x \vee y) \triangleleft z = (x \triangleleft z) \vee (y \triangleleft z)$ *and* $z \triangleright (x \vee y) = (z \triangleright x) \vee (z \triangleright y)$

RL'-algebras

FL-algebras are a subvariety of FL$'$-algebras if we define $x' = x$

Residuated lattices (**RL**) are a subvariety of **FL** if we define $0 = 1$

RL$'$ is the subvariety of **FL$'$** defined by $1' = 0$

Lemma

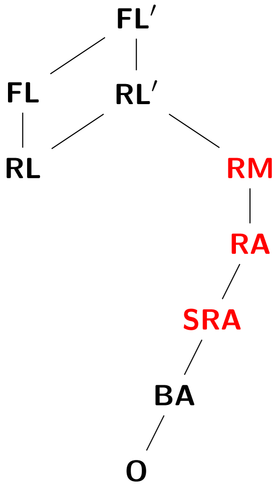
In an RL$'$-algebra the following properties hold:

- $x \triangleright 1 = x^{\smile}$ *and* $1 \triangleleft x = x^{\sqcup}$
- $1^{\smile} = 1^{\sqcup} = 1$

Proof.

$x \triangleright 1 = (x \backslash 0)' = x^{\smile}$. Likewise, $1 \triangleleft x = x^{\sqcup}$
By previous Prop. $1 \triangleright x = x$, hence $1^{\smile} = 1 \triangleright 1 = 1$ □

Some subvarieties of **FL**′



How $'$ interacts with the linear negations

Recall the definitions $x^{\smile} = ({\sim}x)'$ and $x^{\sqcup} = (-x)'$

Proposition

In a DmFL$'$-algebra $\mathbf{A}$ (1a)-5(b) *are equivalent:*

(1a) $({\sim}x)' = -(x')$	(1b) $(-x)' = {\sim}(x')$	*De Morgan involution*
(2a) $x^{\smile\prime} = x^{\prime\sqcup}$	(2b) $x^{\sqcup\prime} = x^{\prime\smile}$	*De Morgan converses*
(3a) ${\sim}x = x^{\prime\sqcup}$	(3b) $-x = x^{\prime\smile}$	*De Morgan converses*
(4a) $x^{\sqcup\sqcup} \leq x \leq x^{\smile\smile}$	(4b) $x^{\smile\smile} \leq x \leq x^{\sqcup\sqcup}$	*converses involutive*
(5a) ${\sim}x^{\smile} \leq x' \leq -x^{\sqcup}$	(5b) $-x^{\sqcup} \leq x' \leq {\sim}x^{\smile}$	

Moreover, each of these properties implies

(In) ${\sim}{-x} = x = -{\sim}x$ *(linear) involutive.*

Proof.

To see that (1a)$\Leftrightarrow$(1b), replace x by x' in (1a) to get $-x = ({\sim}(x'))'$ and apply $'$ to both sides. Since $x'' = x$, this calculation is reversible. □

Proof continued.

The equivalence of (1a), (2a), (3a), (1b), (2b) and (3b) follows directly from the definition of the converses

(1a)⇒(4a): By definition of $x^{\smile}$ we have $x^{\smile\smile} = [\sim((\sim x)')]' = -((\sim x)'') = -\sim x \geq x$, where he second equality follows from (1a). By (Dm) we deduce $x^{\smile\smile\prime} \leq x'$, hence $x'^{\sqcup\sqcup} \leq x'$ by (2a). Replacing x by x' we get $x^{\sqcup\sqcup} \leq x$.

(4a)⇒(1a): $x^{\sqcup\prime} \leq x^{\sqcup\prime\smile\smile} = x^{\sqcup\prime\smile\prime\prime\smile} = (\sim -x)'^{\smile} \leq x'^{\smile}$, where the last inequality follows from $x \leq \sim -x$ and the fact that $'$ is order reversing and $^{\sqcup}$ is order preserving. For the reverse inclusion we use the assumption $x^{\sqcup\sqcup} \leq x$, which gives $x'^{\smile} \leq x^{\sqcup\sqcup\prime\smile} = (-(x^{\sqcup}))^{\smile\prime\prime} = (\sim -(x^{\sqcup}))' \leq x^{\sqcup\prime}$.

The equivalence of (4a) and (5a) is a simple consequence of the definition of the converses and (Dm).

(1a)⇒(In): We always have $x \leq \sim -x$. Hence by (Dm), $(\sim -x)' \leq x'$, so by (1a) and its equivalent (1b) $-\sim(x') \leq x'$, for all x. Consequently $-\sim x \leq x$, for all x. Since the reverse inequality always holds, this establishes half of (In); the other half follows by symmetry. □

How $'$ interacts with multiplication

The prefix (Di), for *De Morgan involution*, is used for an algebras that satisfies (1a) or any of its other 9 equivalent forms.

A 4-element counterexample shows that (In) is not equivalent to (Di), even in the commutative case.

Define the term $x + y = {\sim}(-y \cdot -x) \quad (= -({\sim}y \cdot {\sim}x)$ if (In) is assumed)

Proposition

In every InFL$'$-algebra the following are equivalent and they imply $0 = 1'$

(1) $(xy)^{\smile} = y^{\smile}x^{\smile}$ (2) $(xy)^{\sqcup} = y^{\sqcup}x^{\sqcup}$

(3) $x \triangleright y = x^{\smile}y$ (4) $y \triangleleft x = yx^{\sqcup}$

(5) $(xy)' = x' + y'$

The prefix (Dp) for *De Morgan product* is used for (5)

Quasi relation algebras

A *quasi relation algebra* (*qRA*) is an FL$'$-algebra that satisfies (Dm): $(x \wedge y)' = x' \vee y'$, (Di): $(\sim x)' = -(x')$ and (Dp): $(xy)' = x' + y'$

Proposition

$\mathbf{RA} = \mathbf{qRA} +$ *Boolean, i.e. it suffices to add*
distributivity*:* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *and*
complementation*:* $x \wedge x' = \bot \;\; (= 1 \wedge 1')$ *and* $x \vee x' = \top \;\; (= 1 \vee 1')$

Proof.

$x \wedge x' = \bot$ implies $\sim(x \wedge x') = \top$, hence $\sim x \vee \sim(x') = \top$.

By distributivity and (Di) $(\sim x)' \leq \sim(x') = (-x)'$, so $x^{\smile} \leq x^{\sqcup}$.

The reverse is similar, and the remaining axioms of RA follow from their qRA versions. □

qRAs from lattice ordered groups

Let $G = \mathrm{Aut}(C)$ be the ℓ-group of all order-automorphisms of a chain C, and assume that C has a dual automorphism ${}^\partial : C \to C$

G is a involutive FL-algebra with $\sim x = -x = x^{-1}$, $x + y = xy$, and $0 = 1$

For $g \in G$, define $g'(x) = g(x^\partial)^\partial$. Then $g'' = g, \quad 1' = 1$

$$y = g^{-1\prime}(x) \quad \Leftrightarrow \quad y = g^{-1}(x^\partial)^\partial \quad \Leftrightarrow \quad y^\partial = g^{-1}(x^\partial)$$
$$g(y^\partial)^\partial = x \quad \Leftrightarrow \quad g'(y) = x \quad \Leftrightarrow \quad y = g'^{-1}(x)$$

$(g \vee h)'(x) = (g(x^\partial) \vee h(x^\partial))^\partial = g(x^\partial)^\partial \wedge h(x^\partial)^\partial = (g' \wedge h')(x)$ and

$(gh)'(x) = (g(h(x^\partial)))^\partial = g(h(x^\partial)^{\partial\partial})^\partial = (g'h')(x) = (g' + h')(x)$.

Hence G expanded with $'$ is a quasi relation algebra.

Constructing qRAs from InFL-algebras

For InFL-algebra $(A, \wedge, \vee, \cdot, \sim, -, 1, 0)$ define $\mathbf{A}^\partial = (A, \vee, \wedge, +, -, \sim, 0, 1)$

$\mathbf{A}^\partial$ is also an InFL-algebra called the *dual* of $\mathbf{A}$

Define $F : \mathbf{InFL} \to \mathbf{InFL}'$ by $F(\mathbf{A}) = \mathbf{A} \times \mathbf{A}^\partial$ expanded with $(a, b)' = (b, a)$

For a homomorphism $h : \mathbf{A} \to \mathbf{B}$ define $F(h) : F(\mathbf{A}) \to F(\mathbf{B})$ by $F(h)(a, b) = (h(a), h(b))$.

Theorem (generalization of Brzozowski 2001)

F is a functor from InFL to qRA.
If G is the reduct functor from qRA to InFL then for any quasi relation algebra $\mathbf{C}$, the map $\sigma_{\mathbf{C}} : \mathbf{C} \to FG(\mathbf{C})$ given by $\sigma_{\mathbf{C}}(a) = (a, a')$ is an embedding.

Proving that $F(\mathbf{A})$ is a qRA

Proof.

Let $\mathbf{A}$ be an InFL-algebra. Since $\mathbf{A}^\partial$ is also an InFl-algebra, it will follow that $F(\mathbf{A})$ is a qRA as soon as we observe that (Dm), (Dp) and (Di) hold.

(Dm): $((a, b) \wedge (c, d))' = (a \wedge c, b \vee d)' = (b \vee d, a \wedge c) = (b, a) \vee (d, c) = (a, b)' \vee (c, d)'$.

(Dp): $((a, b) \cdot (c, d))' = (ac, b + d)' = (b + d, ac) = ({\sim}(-d \cdot -b), {\sim}(-c + -a)) = {\sim}((-d, -c) \cdot (-b, -a)) = {\sim}(-(d, c) \cdot -(b, a)) = (b, a) + (d, c) = (a, b)' + (c, d)'$.

(Di): ${\sim}(a, b)' = {\sim}(b, a) = ({\sim}b, -a) = (-a, {\sim}b)' = (-(a, b))'$ and similarly $-(a, b)' = ({\sim}(a, b))'$. □

Corollary

The equational theory of **qRA** *is a conservative extension of that of* **InFL**; *i.e., every equation over the language of* **InFL** *that holds in* **qRA**, *already holds in* **InFL**.

Lifting the Jónsson-Tsinakis result to qRAs

Theorem

qRAs *are term-equivalent to the subvariety of* **DiDmRL**$'$ *defined by* $(x \triangleright y)z = x \triangleright (yz)$

The term-equivalence is given by $x \triangleright y = x^{\smile} y$, $x \triangleleft y = xy^{\sqcup}$ *and* $x^{\smile} = x \triangleright 1$, $x^{\sqcup} = 1 \triangleleft x$

Proof.

By (Dp) $x \triangleright y = x^{\smile} y$, hence $(x \triangleright y)z = x^{\smile} yz = x \triangleright (yz)$
Conversely, if $(x \triangleright y)z = x \triangleright (yz)$ holds then $x^{\smile} z = (x \triangleright 1)z = x \triangleright z$, hence (Dp) holds.

□

qRAs have a decidable equational theory

We make use of the following result:

Theorem (J. and Galatos)

The variety **InFL** *is generated by its finite members, hence has a decidable equational theory*

For an **InFL**-term t, we define the *dual* term t^∂ inductively by

$$\begin{array}{rcl@{\qquad}rcl} x^\partial &=& x & (s \wedge t)^\partial &=& s^\partial \vee t^\partial \\ 0^\partial &=& 1 & (s \vee t)^\partial &=& s^\partial \wedge t^\partial \\ 1^\partial &=& 0 & (s \cdot t)^\partial &=& s^\partial + t^\partial \\ ({\sim}s)^\partial &=& -s^\partial & (s + t)^\partial &=& s^\partial \cdot t^\partial \\ (-s)^\partial &=& {\sim}s^\partial & & & \end{array}$$

We also define $(s = t)^\partial$ to be $s^\partial = t^\partial$.

Lemma

An equation ε is valid in **InFL** *iff ε^∂ is also valid in* **InFL**.

We fix a partition of the denumerable set of variables into two denumerable sets X and $X^\bullet$, and fix a bijection $x \mapsto x^\bullet$ from the first set to the second (hence $x^{\bullet\bullet}$ denotes x).

For a **qRA**-term t, we define the term t° inductively by

$$\begin{array}{cc}
x^\circ = x & (s'')^\circ = s \\
0^\circ = 0, \quad 1^\circ = 1, & ((s \wedge t)')^\circ = s'^\circ \vee t'^\circ, \\
(0')^\circ = 1, \quad (1')^\circ = 0, & ((s \vee t)')^\circ = s'^\circ \wedge t'^\circ, \\
(s \diamond t)^\circ = s^\circ \diamond t^\circ, \text{ for all } \diamond \in \{\wedge, \vee, \cdot, +\}, & ((s \cdot t)')^\circ = s'^\circ + t'^\circ, \\
(\sim s)^\circ = \sim s^\circ, \quad (-s)^\circ = -s^\circ, & ((s + t)')^\circ = s'^\circ \cdot t'^\circ, \\
((\sim s)')^\circ = -(s'^\circ), \quad ((-s)')^\circ = \sim(s'^\circ), & (x')^\circ = x^\bullet
\end{array}$$

Lemma

For every **qRA***-term* t, $t^{\circ\partial}(x_1, \ldots, x_n) = t'^{\circ}(x_1^\bullet, \ldots, x_n^\bullet)$.

For a substitution σ, we define a substitution σ° by $\sigma^\circ(x) = (\sigma(x))^\circ$, if $x \in X$, and $\sigma^\circ(x) = (\sigma(x)')^\circ$, if $x \in X^\bullet$.

Lemma

For every **qRA***-term* t *and* **qRA***-substitution* σ, $(\sigma(t))^\circ = \sigma^\circ(t^\circ)$.

Theorem

An equation ε *over* X *holds in* **qRA** *iff the equation* ε° *holds in* **InFL**.

Corollary

The equational theory of **qRA** *is decidable.*

qRA has the finite model property

Theorem

The variety **qRA** *is generated by its finite members. Actually, the finite members of the form* $F(\mathbf{A})$*, for* $\mathbf{A} \in \mathbf{InFL}$*, generate the variety.*

Proof.

Let $\varepsilon = (s = t)$ be an equation in the language of **qRA**, over the variables $x_1, \ldots, x_n$, that fails in the variety.

Then the equation $s^\circ = t^\circ$ (over the variables $x_1, \ldots, x_n, x_1^\bullet, \ldots, x_n^\bullet$) fails in **InFL**

Since the variety **InFL** is generated by its finite members, there is a finite $\mathbf{A} \in \mathbf{InFL}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, such that $(s^\circ)^{\mathbf{A}}(\bar{a}, \bar{b}) \neq (t^\circ)^{\mathbf{A}}(\bar{a}, \bar{b})$.

We can assume that negations in $s = t$ have been pushed down to the variables

Then s and s° are almost identical, except for occurrences of variables x' and $x^\bullet$. □

qRA has the finite model property

Proof continued.

Therefore, $s(x_1, \ldots, x_n) = s^\circ(x_1, \ldots, x_n, x_1', \ldots, x_n')$, and the same for t. We have

$$
\begin{aligned}
&s^{F(\mathbf{A})}((a_1, b_1), \ldots, (a_n, b_n)) = \\
&(s^\circ)^{F(\mathbf{A})}((a_1, b_1), \ldots, (a_n, b_n), (a_1, b_1)', \ldots, (a_n, b_n)') = \\
&(s^\circ)^{F(\mathbf{A})}((a_1, b_1), \ldots, (a_n, b_n), (b_1, a_1), \ldots, (b_n, a_n)) = \\
&((s^\circ)^{\mathbf{A}}(\bar{a}, \bar{b}), (s^\circ)^{\mathbf{A}^\partial}(\bar{b}, \bar{a})) \neq \\
&((t^\circ)^{\mathbf{A}}(\bar{a}, \bar{b}), (t^\circ)^{\mathbf{A}^\partial}(\bar{b}, \bar{a})) = \\
&t^{F(\mathbf{A})}((a_1, b_1), \ldots, (a_n, b_n)).
\end{aligned}
$$

In other words, the equation $s = t$ fails in $F(\mathbf{A})$, which is a finite qRA. □

Conclusion

By expanding FL-algebras with a unary De Morgan operation one can interpret relation algebras with FL'-algebras

This leads to the variety of quasi relation algebras that has many properties in common with RA

In addition **qRA** has a decidable equational theory and the FMP

Problem: Do distributive qRAs have a decidable equational theory?

Problem: Are positive relation algebras finitely based or eq. decidable?

Problem: Is every sequential algebra a relativization of a relation algebra?

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