Complete Representations for Distributive Lattices

Rob Egrot, with thanks to Robin Hirsch

January 13, 2011

Representations of lattices 1

L is a bounded, distributive lattice.

- A *representation* of L is an embedding $h : L \to \mathscr{P}(X)$ for some set X, where $\mathscr{P}(X)$ is considered as a ring of sets.
- When such an h exists we say L is *representable*.

Representations 2: alternative view

Let $h : L \to \mathscr{P}(X)$ be a representation. Then for each $x \in X$:

- Define $h^{-1}[x] = \{a \in L : x \in h(a)\}$.
- $h^{-1}[x]$ is a prime filter of L.
- h is an embedding so if $a \neq b \in L$ there is some $x \in X$ with $x \in h(a) \triangle h(b)$.
- For this x either $a \in h^{-1}[x]$ and $b \notin h^{-1}[x]$ or vice versa.
- We say the set $\{h^{-1}[x] : x \in X\}$ is *distinguishing* over L.

Representations 3

Conversely, suppose set P of prime filters is distinguishing over L:

- Easy to show the map $h :\longrightarrow \mathscr{P}(P)$, $h : a \mapsto \{p \in P : a \in p\}$ is an embedding.
- Therefore L is representable.

Representations 4

The previous two slides combine to:

Theorem
A distributive lattice is representable if and only if it has a distinguishing set of prime filters.

In view of the prime ideal theorem we have:

Theorem
Every distributive lattice is representable.

These results were first proved by Birkhoff in [1]

Preservation of arbitrary meets and joins

- $f : L_1 \to L_2$ is *meet-complete* when $f(\bigwedge S) = \bigwedge f[S]$ whenever $\bigwedge S$ is defined in L_1.
- *join-complete* defined similarly. When f is both meet and join complete we say it is *complete*.
- When L has a meet-complete representation we say it is meet-completely representable etc.

Duality for complete representations

Theorem

L has a meet-complete representation iff L^δ has a join-complete representation.

Proof.

If $h : L \to \mathscr{P}(P)$ is a representation, where P is some distinguishing set of prime filters of L, then the map $\bar{h} : L^\delta \to \mathscr{P}(P)$, $a \mapsto -h(a)$ is also a representation. If h is meet-complete then by De Morgan
$\bar{h}(\bigvee_\delta) = -h(\bigwedge S) = -\bigcap h[S] = -\bigcap -\bar{h}[S] = \bigcup \bar{h}[S]$. □

Preservation of arbitrary meets and joins 2

Theorem

Let L be a bounded, distributive lattice. Then:

1. *L has a meet-complete representation iff L has a distinguishing set of complete, prime filters,*
2. *L has a join-complete representation iff L has a distinguishing set of completely-prime filters,*
3. *L has a complete representation iff L has a distinguishing set of complete, completely-prime filters,*

This result was known at least as far back as 1948 [2].

Preservation of arbitrary meets and joins 3

Proof.

We prove 1), the rest follows from duality: If h is a representation of L we can assume wlog that $h : L \to \mathscr{P}(K)$ for some distinguishing set K of prime filters.
It is always the case that $h(\bigwedge S) \subseteq \bigcap h[S]$.
Now, $p \in \bigcap h[S] \iff \forall s \in S(p \in h(s)) \iff \forall s \in S(s \in p)$, so $\bigcap h[S] \subseteq h(\bigwedge S)$ if and only if

$$\forall p \in K \big(\forall s \in S(s \in p) \to \bigwedge S \in p\big)$$

and the demand that this hold for every $S \subseteq L$ such that $\bigwedge S$ exists in L is precisely the demand that every prime filter in K is complete. □

Boolean algebras

Lemma

If L is complemented then its prime filters are precisely its ultrafilters, moreover the following are equivalent:

1. *P is a principal ultrafilter of L,*
2. *P is a complete ultrafilter of L,*
3. *P is a completely-prime ultrafilter of L.*

Proof.

Easy to see ultrafilters of a BA are precisely its prime filters. Clearly 1) $\implies$ 2). Let U be an ultrafilter. If U is complete it must contain a non-zero lower bound and thus is principle so 2) $\implies$ 1). De Morgan gives $-\bigvee S = \bigwedge -S$ so if U is complete then $S \cap U = \emptyset \implies \bigwedge -S \in U \implies -\bigvee S \in U \implies \bigvee S \notin U$, so 2) $\implies$ 3). Similarly, if U is completely-prime then $\bigwedge S \notin U \implies -\bigwedge S \in U \implies \bigvee -S \in U \implies -s \in U$ for some $s \in S \implies S \not\subseteq U$, so 3) $\implies$ 2). □

Boolean algebras 2

Corollary

For a Boolean algebra B the following are equivalent:

1. *B is atomic,*
2. *B is completely representable,*
3. *B is meet-completely representable,*
4. *B is join-completely representable.*

(that a BA is completely representable iff it is atomic was first proved in Hirsch and Hodkinson [3]).

Examples

A distributive lattice both meet-completely representable and join-completely representable but not completely representable:

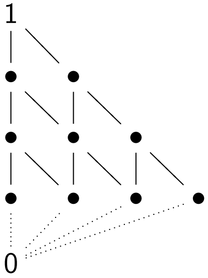
Let $L = [0, 1] \subseteq \mathbb{R}$. Then by taking $\{\{y : y \geq x\} : x \in L\}$ we obtain a distinguishing set of complete, prime filters, and by taking $\{\{y : y > x\} : x \in L\}$ we obtain a distinguishing set of completely-prime filters.

However, if F is a complete filter of L then $\bigwedge F \in F$ (by completeness properties of L and F) and, since $\bigwedge F = \bigvee\{x \in L : x < \bigwedge F\}$, F cannot be completely prime.

Examples 2

A distributive lattice neither meet-completely nor join-completely representable:
We can take any BA that fails to be atomic.

A distributive lattice join-completely representable but not meet-completely representable:



Class definitions

- **DL**: the class of bounded, distributive lattices.
- **CRL**: the class of completely representable bounded, distributive lattices.
- **mCRL**: the class of meet-completely representable bounded, distributive lattices.
- **jCRL**: the class of join-completely representable bounded, distributive lattices.
- **biCRL**: the class of bounded, distributive lattices both meet-completely and join-completely representable.

The situation

The previous examples show the following:

$$\mathbf{CRL} \subset \mathbf{biCRL} = \mathbf{mCRL} \cap \mathbf{jCRL} \subset \mathbf{mCRL} \neq \mathbf{jCRL} \subset \mathbf{DL}$$

The question

As ‘being atomic’ is a first order property of Boolean algebras we have (re)proved* that the class of completely representable BAs is elementary.

Question

What about **CRL**, **mCRL**, **jCRL** *and* **biCRL***?*

CRL is not elementary

Theorem

CRL *is not closed under elementary equivalence.*

Proof.

The lattice $L = [0,1]$ from a previous example is not in **CRL**, however the lattice $L' = [0,1] \cap \mathbb{Q}$ *is* in **CRL** as for every irrational r the set $\{a \in L' : a > r\}$ is a complete, completely-prime filter. L and L' are elementarily equivalent as $\mathbb{R}$ and $\mathbb{Q}$ are. $\square$

CRL etc. are pseudoelementary

We shall see that **mCRL** is precisely the algebra sorted first order reduct of the class of models of a (finite) theory in two-sorted FOL, and thus is pseudo-elementary. The proof can be adapted easily for the other classes under consideration.

CRL etc. are pseudoelementary 2

First some basic definitions:

- $\mathscr{L} = \{+, \cdot, 0, 1\}$ is the language of bounded lattices in FOL.
- $\mathscr{L}^+ = \mathscr{L} \cup \{\in (\mathbf{A}, \mathbf{S})\}$ is a two sorted language with sorts **A** and **S** and additional two sorted binary predicate $\in$.

Here the **A** sort will specify lattice elements and the **S** sort sets of these elements.

CRL etc. are pseudoelementary 3

Define additional predicates in $\mathscr{L}^+$ as follows:

- $P(\mathbf{S})$ holds whenever s is a 'prime filter' with regards $\in$ and the **A** sorted lattice operations.
- $I(\mathbf{A}, \mathbf{S})$ holds when a is the infimum of s with regards $\in$ and the **A** sorted lattice operations.
- $C(\mathbf{S})$ holds iff $\forall t \forall a \Big(\big((t \subseteq s) \wedge (I(a, s) \big) \rightarrow (a \in s) \Big)$, so C says roughly that s is complete with respect to the **S** sort.

CRL etc. are pseudoelementary 4

Let T be the $\mathscr{L}$ theory of bounded, distributive lattices. Let T^+ be the natural translation of T into $\mathscr{L}^+$ with the following additional axioms:

1. $\forall ab\Big(a \neq b \rightarrow \exists s\Big(\big(P(s) \wedge C(s)\big) \wedge \big(((a \in s) \wedge (b \notin s)) \vee ((b \in s) \wedge (a \notin s))\big)\Big)\Big)$
2. $\forall a \exists s\Big((b > a) \leftrightarrow (b \in s)\Big)$
3. $\forall st \exists u \forall a\Big(\big((a \in s) \wedge (a \in t)\big) \leftrightarrow (a \in u)\Big)$

CRL etc. are pseudoelementary 5

Lemma

*The class $\{M^{\mathbf{A}} \restriction_{\mathscr{L}} : M \models T^+\}$ of **A** sort $\mathscr{L}$-reducts of models of T^+ is precisely **mCRL**.*

Proof.

Clearly if L is in **mCRL** its elements satisfy T and L, $\mathscr{P}(L)$ and set theoretic $\in$ satisfy T^+.
Conversely, axiom 1 ensures such a model has a distinguishing set of prime filters each satisfying the form of completeness specified by our C predicate. Axioms 2 and 3 ensure there are enough sets governed by T^+ for C to give actual completeness. □

Corollary

mCRL *is pseudoelementary.*

Some well known facts about classes

The following are true of any class $\mathscr{C}$:

Fact

*$\mathscr{C}$ is elementary if and only if it is closed under isomorphism, ultraproducts and ultraroots.**

Fact

$\mathscr{C}$ is pseudoelementary $\implies$ $\mathscr{C}$ is closed under ultraproducts.

*this is a corollary of the main result of Shelah [4].

Ultraroots

Since **CRL** is pseudoelementary and closed under isomorphism, but is not elementary, it cannot be closed under ultraroots. **mCRL**, **jCRL** and **biCRL** will be elementary if and only if they are closed under ultraroots.

Question

Which, if any, of **mCRL**, **jCRL** *and* **biCRL** *are closed under ultraroots?*

Note that **mCRL** is elementary iff **jCRL** is elementary (by duality), and therefore **mCRL** is elementary $\implies$ **biCRL** is elementary (as $\mathbf{biCRL} = \mathbf{mCRL} \cap \mathbf{jCRL}$).

Ultraroots 2

Some notation:

- For a lattice L, an ordinal I and a non-principle ultrafilter U over $\mathscr{P}(I)$ we denote the ultrapower of L over U by $\prod_U L$.
- For $a \in L$ define $\bar{a} \in \prod_I L$ by $\bar{a}(i) = a$ for all $i \in I$.
- For $S \subseteq L$ define $S^* = \{[x] \in \prod_U L : \{i \in I : x(i) \in S\} \in U\}$

Is there an L with $\prod_U L$ in **mCRL** but L not in **mCRL**?

I don't know, but *if* such an L does exist it must have certain properties:

Proposition

If $\prod_U L$ has a meet-complete representation then L is $\vee(\bigwedge)$-distributive.

Proof.

It is straightforward to show that if $S \subseteq L$ and $\bigwedge S$ exists in L then $\bigwedge(S^*)$ exists in $\prod_U L$ and equals $[\overline{\bigwedge S}]$. Moreover, in light of this if there is some $A \cup \{b\} \subseteq L$ with $b \vee \bigwedge A \neq \bigwedge(b \vee A)$ then $\bigwedge A^* \vee [\bar{b}] = [\overline{\bigwedge A}] \vee [\bar{b}] = [\overline{(\bigwedge A) \vee b}] \neq [\overline{(\bigwedge(A \vee b)}] = \bigwedge(A^* \vee [\bar{b}])$, so if L is not $\vee(\bigwedge)$-distributive then neither is $\prod_U L$. Since when $\prod_U L$ is in **mCRL** it inherits $\vee(\bigwedge)$-distributivity from its representation we have the result. □

Note that the converse to this is false as, for example, every BA is $\vee(\bigwedge)$-distributive but not nec. atomic.

Proposition

If $\prod_U L$ has a meet-complete representation but L does not then there is a pair $x < y$ such that for every pair $a < b \in [x, y]$ there is some c with $a < c < b$.

Proof.

Since $\prod_U L$ is in **mCRL**, for each pair $a, b \in L$ there is a cpf γ distinguishing $[\bar{a}]$ and $[\bar{b}]$. It's easy to show that if $a < b$ and $(a, b) = \emptyset$ the set $\gamma_* = \{c \in L : [\bar{c}] \in \gamma\}$ is a cpf of L with $b \in \gamma_*$ and $a \notin \gamma_*$. Since to distinguish arbitrary a and b it is sufficient to distinguish e.g. a and $a \vee b$, if for every pair $a < b$ we have $(a, b) = \emptyset$ the meet-complete representability of $\prod_U L$ passes back to L via the cpfs γ_*. □

Note that by duality the same result holds for join-complete representations.

Work in progress

- Find and examine the elementary closures of **CRL**, **mCRL**, **jCRL** and **biCRL**. Since they are pseudoelementary there is a procedure for calculating these.
- Continue investigation of closure of **mCRL**, **jCRL** and **biCRL** under ultraroots.

References

G. Birkhoff.
On the combination of subalgebras.
Proc. Camb. Philos. Soc., 29:441–464, 1933.

G. Birkhoff and O. Frink Jr.
Representations of lattices by sets.
Transactions of the American Mathematical Society, 64:299–316, 1948.

R. Hirsch and I. Hodkinson.
Complete representations in algebraic logic.
Journal of Symbolic Logic, 62:816–847, 1997.

S. Shelah.
Every two elementarily equivalent models have isomorphic ultrapowers.
Israel Journal of Mathematics, 10:224–233, 1971.