Axiomatizability of Algebras of Binary Relations

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Algebras of binary relations

Let Λ be a signature and $\mathfrak{A} = (A, \Lambda)$ be an algebra. We say that $\mathfrak{A}$ is an *algebra of binary relations* if $A \subseteq \mathcal{P}(U \times U)$ for some set U and each operation in Λ is interpreted as a "natural" operation on relations.

For instance, $+$ is union, $\cdot$ is intersection, $-$ is complement, ; is interpreted as *composition* of relations

$$x \,;\, y = \{(u, v) \in U \times U : \exists w((u, w) \in x \text{ and } (w, v) \in y)\}$$

$\smile$ is interpreted as *converse* of relations

$$x^{\smile} = \{(u, v) \in U \times U : (v, u) \in x\}$$

$1'$ is the *identity* constant

$$1' = \{(u, v) \in U \times U : u = v\}$$

0 is the empty set.

Other possible operations include reflexive-transitive closure *, the residuals $\backslash$ and $/$ of composition, domain d and range r, etc.

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RRA and RKA

We denote the class of algebras of binary relations of the signature Λ by $\mathsf{R}(\Lambda)$. The quasivariety and the variety generated by $\mathsf{R}(\Lambda)$ are denoted by $\mathsf{Q}(\tau)$ and $\mathsf{V}(\Lambda)$.
The class of *representable relation algebras* is

$$\mathsf{RRA} = \mathsf{Q}(+, \cdot, -, 0, ;, \smile, 1') = \mathsf{V}(+, \cdot, -, 0, ;, \smile, 1')$$

The class of *relational Kleene algebras* is

$$\mathsf{RKA} = \mathsf{R}(+, 0, ;, ^*, 1')$$

The question

For which Λ is the (quasi)equational theory of $\mathsf{R}(\Lambda)$ finitely axiomatizable?

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Motivations

- The (quasi)equational theory of RRA is not finitely axiomatizable (Monk). For which fragment of RRA is there a finite axiomatization?
- Dynamic semantics: Lambek calculus (van Benthem), Situation theory: channel algebras (Barwise, Seligman)
- Completeness of (fragments of) substructural logics: relevance logic (Dunn, Kowalski, Maddux), linear logic (Dunn)
- Program semantics: domain algebras (Desharnais, Jipsen, Struth, etc., Kleene algebras (Conway, Kozen, etc.)

Variations on finite axiomatizability

Is the *quasivariety* Q(Λ) generated by R(Λ) finitely axiomatizable?

Quasiequational theory — representability of all algebras of Mod(Qeq) — strong completeness (semantical consequence)

Is the *variety* V(Λ) generated by R(Λ) finitely axiomatizable?

Equational theory — representability of the *free* algebra of Mod(Eq) — weak completeness (validities)

Positive RRA fragments

Λ is a positive RRA-subsignature containing composition ; and at least one of the lattice operations join $+$ or meet $\cdot$. (Including 0 does not change the results.)

	$Q(\Lambda)$	$V(\Lambda)$
$\Lambda = \{\cdot, ;\}$	Yes	Yes
$\Lambda = \{\cdot, ;, 1'\}$	No	Yes
$\Lambda \supseteq \{\cdot, ;, \smile\}$	No	No
$\Lambda \supseteq \{+, ;\}$	No	
$\Lambda \not\supseteq \{\cdot, ;, \smile\}$		Yes

Table: Finite axiomatizability of positive RRA fragments, Andréka and Mikulás *AU* to appear

Term graphs

For $\Lambda \subseteq \{\cdot, ;, ^\smile, 1', 0)$, we define *term graphs*

$$G(\sigma) = (V(\sigma), E(\sigma), \iota(\sigma), o(\sigma))$$

as special 2-pointed, labelled graphs by induction on the complexity of Λ-terms. Let $G(0)$ be the empty graph,

$$G(1') = (\{\iota\}, \{(\iota, 1', \iota)\}, \iota, \iota)$$

and for variable x,

$$G(x) = (\{\iota, o\}, \{(\iota, 1', \iota), (\iota, x, o), (o, 1', o)\}, \iota, o)$$

For terms σ and τ, we set

$$\begin{aligned} G(\sigma \cdot \tau) &= G(\sigma) \cdot G(\tau) \quad \text{(almost) disjoint union} \\ G(\sigma \,;\, \tau) &= G(\sigma) \,;\, G(\tau) \quad \text{concatenation} \end{aligned}$$

and $G(\sigma^\smile)$ is $G(\sigma)$ with ι and o interchanged.

Validity and derivability

Andréka and Bredikhin *AU* 1995

$\mathsf{R}(\Lambda) \models \sigma \leq \tau$ iff there is a homomorphism $G(\tau) \to G(\sigma)$.

For $\Lambda \not\supseteq \{\cdot, ;, ^\smile\}$, there is a finite Eq_Λ such that

Andréka and Mikulás *AU* to appear

$\mathsf{Eq}_\Lambda \vdash \sigma \leq \tau$ iff there is a homomorphism $G(\tau) \to G(\sigma)$.

Hence $\mathfrak{Fr}(\mathsf{V}(\Lambda)) = \mathfrak{Fr}(\mathsf{Mod}(\mathsf{Eq}_\Lambda))$.

The free algebra $\mathfrak{Fr}(\mathsf{V}(\Lambda))$

Let

$$TG(\Lambda) = (V, E) = \biguplus_{\sigma} (V(\sigma), E(\sigma))$$

disjoint union of (non-pointed reducts of) all Λ-term graphs. Define

$$R_x = \{(u, v) : (u, x, v) \in E\}$$

and let $\mathfrak{TG}(\Lambda)$ be the Λ-algebra generated by R_x.

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$\mathfrak{TG}(\Lambda)$ is the free algebra $\mathfrak{Fr}(\mathsf{V}(\Lambda))$ of $\mathsf{V}(\Lambda)$.

Using additivity of the operations this can be extended to $+ \in \Lambda$: close $\mathfrak{TG}(\Lambda)$ under union (and define $G(\sigma + \tau)$ as union of graphs).
This helps to find out what are the validities in the variety (e.g., $1' \leq x + y$ iff $1' \leq x$ or $1' \leq y$).

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Residuals

Recall the interpretation of the residuals of composition in representable algebras

$$x \setminus y = \{(u,v) \in U \times U : \forall w((w,u) \in x \text{ implies } (w,v) \in y)\}$$
$$x \,/\, y = \{(u,v) \in U \times U : \forall w((u,w) \in y \text{ implies } (v,w) \in x)\}$$

Main properties:

$$y \leq x \setminus z \iff x \,;\, y \leq z \iff x \leq z \,/\, y$$

But also:

$$x \leq y \Rightarrow z \leq z \,;\, x \setminus y \text{ etc.}$$

Lower semilattice-ordered residuated semigroups, (Andréka and Mikulás *JoLLI* 1994)

$\mathsf{Q}(\cdot, ;, \setminus, /) = \mathsf{V}(\cdot, ;, \setminus, /)$ is finitely axiomatizable.

Distributive lattice-ordered residuated semigroups

Hirsch and Mikulás *RSL* to appear

For $\Lambda \supseteq \{+, \cdot, \backslash\}$, the (quasi)equational theory of $\mathsf{R}(+, \cdot, \backslash)$ is not finitely axiomatizable.

The same holds if we assume *commutativity*: for every $x, y \in A$ and u, v, w,

$$(u, w) \in x \text{ and } (w, v) \in y \text{ imply } (u, w') \in y \text{ and } (w', v) \in x \text{ for some } w'$$

and/or *density*: for every $x \in A$ and u, v,

$$(u, v) \in x \text{ implies } (u, w) \in x \text{ and } (w, v) \in x \text{ for some } w$$

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NFA of $V(+, \cdot, \backslash)$ with composition

For every $n \in \omega$, $\mathfrak{A}_n$ is a finite, integral, symmetric, commutative and dense relation algebra.
$\mathfrak{A}_n$ has (among others) the following atoms: greens g_i, yellows y_j and reds r_j for $i \in n+1$ and $j \in n$.
Composition is defined so that

$$\begin{aligned} g_i ; g_j \cdot g_k = y_i ; y_j \cdot y_k &= 0 \text{ unless } i = j = k \\ g_i ; g_j \cdot y_k &= 0 \text{ unless } |i - j| = 1 \\ g_i ; g_j \cdot r_k &= 0 \text{ unless } k = |i - j| \leq 5 \text{ or } 5 < |i - j| \equiv_5 k \\ r_i ; r_j \cdot r_k &= 0 \text{ unless } i = j = k \text{ or } i + j = k \text{ etc.} \end{aligned}$$

$\mathfrak{A}_n$ is not representable.

By an indirect argument: $g_i \leq y_i \,;\, g_{i+1}$, whence there are u_i, v such that $(u_i, v) \in g_i$ and $(u_i, u_{i+1}) \in y_i$. Then $(u_i, u_{i+2}) \in r_2$ (there are no yellow and green triangles). Similarly, $(u_i, u_{i+j}) \in r_j$ for $j \leq 5$.
Consider, say, the triangle u_0, u_5, u_7 where $(u_0, u_5) \in r_5$ and $(u_5, u_7) \in r_2$.
We have $(u_0, u_7) \in r_i$ for some i such that $i \equiv_5 7$ and $i = 5 + 2$ or $5 = i + 2$ or $2 = 5 + i$. Thus $i = 7$. Similarly, $(u_0, u_n) \in r_n$, a contradiction.

Nontrivial ultraproducts of $\mathfrak{A}_n$ are representable.

The contradiction disappears in the infinity (10 pages).
The same argument can be told without using composition (a more indirect argument) and using an equation ("residuals are implications", see Pratt).

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Interpreting relevance logic

$\mathfrak{A}$ a commutative and dense family of binary relations closed under $\cdot, +, \backslash$, v a valuation such that $\wedge$, $\vee$ and $\rightarrow$ are interpreted as $\cdot$, $+$ and $\backslash$, respectively.

Sound semantics for $\mathbf{R}_+$:

$$\mathfrak{A} \models \varphi \iff \mathsf{Id} \subseteq v(\varphi)$$

Incompleteness of $\mathbf{R}_+$

The relevance logic $\mathbf{R}_+$ is not complete w.r.t. binary relations even if we expand it with finitely many axioms and standard derivation rules.

By the previous theorem and noting that $\sigma \leq \tau$ is valid iff $\mathsf{Id} \subseteq \sigma \backslash \tau$.

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Open problems for the residuals

Probably not, see nfa of $Q(\cdot, ;, 1')$ (Hirsch and Mikulás *AU* 2007):

??? Lower semilattice-ordered residuated monoids ???

Is the equational theory of $R(\cdot, ;, \backslash, /, 1')$ finitely axiomatizable?

Would be nice, cf. nfa of $Q(+, ;)$ (Andréka *AU* 1991):

??? Upper semilattice-ordered residuated semigroups/monoids ???

Are the equational theories of $R(+, ;, \backslash, /)$ and $R(+, ;, \backslash, /, 1')$ finitely axiomatizable?

When everything else fails

Finite quasiaxiomatization

Is there a finitely axiomatizable quasivariety K such that $\mathbf{V}(K) = V(\Lambda)$?

Equational theory using quasiequations — weak completeness with additional rules (preserve validities, are not valid in individual algebras). NOT irreflexivity rule!

Kleene algebras, e.g., Kozen *IC* 1994

There is a finitely axiomatizable quasivariety generating the variety $V(+, 0, ;, ^*, 1')$

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Kleene challenges

Can the graph-method be used for the following?

??? Kleene lattices ???

Find a finitely axiomatizable quasivariety that generates the variety $V(+, \cdot, 0, ;, ^*, 1')$.

The free algebra $\mathfrak{Fr}(V(+, \cdot, 0, ;, ^*, 1'))$ of $V(+, \cdot, 0, ;, ^*, 1')$ can be described as $\mathfrak{TG}(+, \cdot, 0, ;, ^*, 1')$. That is,

$$G(\sigma^*) = \bigcup_n G(\sigma^n)$$

Find a finite set Qeq of quasiequations such that $\mathfrak{Fr}(\mathrm{Mod}(\mathrm{Qeq})) = \mathfrak{Fr}(V(+, \cdot, 0, ;, ^*, 1'))$.
Must be even harder:

??? Action algebras and action lattices ???

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