On homomorphisms of extended-order algebras

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Workshop on Lattices, Relations and Kleene Algebras

University College London, London, UK
September 21 - 23, 2010

Outline

Implicative algebras

1974: H. Rasiowa considers implicative algebras as a possible tool for a uniform algebraic treatment of various logics.

Definition 1

An implicative algebra is an abstract algebra $(A, \Rightarrow, V)$, where V is a nullary operation and $\Rightarrow$ is a binary operation such that for every $a, b, c \in A$, the following conditions hold:

1. $a \Rightarrow a = V$;
2. if $a \Rightarrow b = V$ and $b \Rightarrow c = V$, then $a \Rightarrow c = V$;
3. if $a \Rightarrow b = V$ and $b \Rightarrow a = V$, then $a = b$;
4. $a \Rightarrow V = V$.

d-algebras

1999: J. Neggers and H. S. Kim introduce the notion of d-algebra as a generalization of BCK-algebras.

Definition 2

A d-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying for every $x, y \in X$ the following axioms:

1. $x * x = 0$;
2. $0 * x = 0$;
3. if $x * y = 0$ and $y * x = 0$, then $x = y$.

A d-algebra $(X, *, 0)$ is called d-transitive provided that for every $x, y, z \in X$, $x * y = 0$ and $y * z = 0$ imply $x * z = 0$.

Weak extended-order algebras

2008: C. Guido and P. Toto provide the concept of weak extended-order algebra to serve as a common framework for the majority of algebraic structures used in many-valued mathematics.

Definition 3

A weak extended-order algebra (w-eo algebra) is a triple $(L, \rightarrow, \top)$, where L is a non-empty set, $L \times L \xrightarrow{\rightarrow} L$ is a binary operation on L, and $\top$ is a distinguished element of L such that for every $a, b, c \in L$ the following conditions are satisfied:

1. $a \rightarrow \top = \top$ (upper bound);
2. $a \rightarrow a = \top$ (reflexivity);
3. if $a \rightarrow b = \top$ and $b \rightarrow a = \top$, then $a = b$ (antisymmetry);
4. if $a \rightarrow b = \top$ and $b \rightarrow c = \top$, then $a \rightarrow c = \top$ (transitivity).

W-eo algebras and partially ordered sets

Lemma 4

- *Given a w-eo algebra* $(L, \rightarrow, \top)$, *the binary relation* $\leqslant$ *on* L *with*

$$a \leqslant b \text{ iff } a \rightarrow b = \top$$

provides an upper-bounded partially ordered set $(L, \leqslant, \top)$.
- *Given an upper-bounded partially ordered set* $(L, \leqslant, \top)$, *every binary operation* $\rightarrow$ *on* L, *extending the relation* $\leqslant$ ($a \rightarrow b = \top$ *iff* $a \leqslant b$), *provides a w-eo algebra* $(L, \rightarrow, \top)$.

! Lemma 4 backs the use of the term extended-order algebra.

Quantales . . .

Elements of the theory of quantales

- A quantale Q is a $\bigvee$-semilattice equipped with an associative binary operation $\otimes$ (multiplication) distributing across $\bigvee$ from both sides: $a \otimes (\bigvee S) = \bigvee_{s \in S}(a \otimes s)$, $(\bigvee S) \otimes a = \bigvee_{s \in S}(s \otimes a)$.
- The multiplication operation $\otimes$ gives rise to two residuations: $a \rightarrow_r b = \bigvee\{c \in Q \mid a \otimes c \leqslant b\}$, $a \rightarrow_l b = \bigvee\{c \in Q \mid c \otimes a \leqslant b\}$.
- A special case of the residuations provides two $\otimes$-pseudocomplementations: $a^{\perp} = a \rightarrow_r \perp$, ${}^{\perp}a = a \rightarrow_l \perp$.

! The basic operation $\otimes$ gives rise to a variety of derived ones.

. . . and w-eo algebras

Elements of the theory of w-eo algebras

- A w-eo algebra $(L, \rightarrow, \top)$ is called complete (w-ceo algebra) provided that the set L with the partial order obtained from $\rightarrow$ is a complete lattice.
- A w-ceo algebra $(L, \rightarrow, \top)$ is called right-distributive (w-crdeo algebra) provided that $a \rightarrow \bigwedge S = \bigwedge_{s \in S}(a \rightarrow s)$ for every $a \in L$ and every $S \subseteq L$.
- Given a w-crdeo algebra $(L, \rightarrow, \top)$, the operation $\rightarrow$ provides a binary operation $\otimes$ on L with $a \otimes b = \bigwedge\{c \in L \mid b \leqslant a \rightarrow c\}$.
- Every w-ceo algebra $(L, \rightarrow, \top)$ comes equipped with a unary operation $(-)^{\perp}$ defined by $a^{\perp} = a \rightarrow \perp$.

! The basic operation $\rightarrow$ gives rise to a variety of derived ones, whose properties can be investigated through those of $\rightarrow$.

Contribution of this talk

2008 - 2010: C. Guido, M. E. Della Stella and P. Toto investigate properties of the operation $\rightarrow$ of a given w-eo algebra $(L, \rightarrow, \top)$, paying much attention to its derived operations.

The main idea

Base all the algebraic structures of many-valued mathematics on a single binary operation $\rightarrow$ obtained as an extension of partial order.

! During their studies, C. Guido *et al.* never consider the topic of homomorphisms of w-eo algebras.

The purpose of the talk

Provide a categorical approach to w-eo algebras, thereby studying properties of homomorphisms of the structures in question.

The category of partially ordered sets

Definition 5

Pos is the category, whose

objects are partially ordered sets (posets) $(X, \leqslant)$, and whose

morphisms are order-preserving (monotone) maps $(X,\leqslant) \xrightarrow{f} (Y,\leqslant)$.

Definition 6

$\mathbf{Pos}^{\top}$ is the non-full subcategory of **Pos**, whose

objects are upper-bounded posets $(X, \leqslant, \top)$, and whose

morphisms are monotone maps preserving the top element.

W-eo algebras as generalized posets

Definition 7

$\mathbf{WEOAlg}^{\top}$ is the category, whose

objects are w-eo algebras $(A, \to, \top)$, and whose

morphisms $(A, \to, \top) \xrightarrow{\varphi} (B, \to, \top)$ are maps $A \xrightarrow{\varphi} B$ such that

1. for every $a_1, a_2 \in A$, if $a_1 \to a_2 = \top$, then $\varphi(a_1) \to \varphi(a_2) = \top$;
2. $\varphi(\top) = \top$.

! The category $\mathbf{WEOAlg}^{\top}$ provides a direct generalization of the category $\mathbf{Pos}^{\top}$.

Categorical equivalence

Theorem 8

- *There exists a functor* $\mathbf{WEOAlg}^{\top} \xrightarrow{\|-\|} \mathbf{Pos}^{\top}$ *which is defined by* $\|(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)\| = (A, \leqslant, \top) \xrightarrow{\varphi} (B, \leqslant, \top)$, *where* $c_1 \leqslant c_2$ *iff* $c_1 \rightarrow c_2 = \top$.
- *There exists a functor* $\mathbf{Pos}^{\top} \xrightarrow{F} \mathbf{WEOAlg}^{\top}$ *which is defined by* $F((X, \leqslant, \top) \xrightarrow{f} (Y, \leqslant, \top)) = (X, \rightarrow, \top) \xrightarrow{f} (Y, \rightarrow, \top)$, *where*

$$z_1 \rightarrow z_2 = \begin{cases} \top, & z_1 \leqslant z_2 \\ z_2, & \textit{otherwise.} \end{cases}$$

- *The functors* $\| - \|$ *and* F *provide an equivalence between the categories* $\mathbf{WEOAlg}^{\top}$ *and* $\mathbf{Pos}^{\top}$ *such that* $\| - \| \circ F = 1_{\mathbf{Pos}^{\top}}$.

A more sophisticated approach

! Given a w-eo algebra $(A, \rightarrow, \top)$ and $a, b \in A$, $a \rightarrow b = \top$ is occasionally denoted by $a \leqslant b$.

Definition 9

$\mathbf{WEOAlg}^{\leqslant}$ is the non-full subcategory of $\mathbf{WEOAlg}^{\top}$ having the same objects, and whose

morphisms $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ are maps $A \xrightarrow{\varphi} B$ such that

1. $\varphi(a_1 \rightarrow a_2) \leqslant \varphi(a_1) \rightarrow \varphi(a_2)$ for every $a_1, a_2 \in A$;
2. $\varphi(\top) = \top$.

Definition 10

$\mathbf{WEOAlg}^{\leqslant\rightarrow}$ is the full subcategory of $\mathbf{WEOAlg}^{\leqslant}$, whose

objects are w-eo algebras $(A, \rightarrow, \top)$, which satisfy the condition

- $a \rightarrow (b \rightarrow a) = \top$ for every $a, b \in A$.

Obtained adjunction

Theorem 11

- *There exists the restriction* $\mathbf{WEOAlg}^{\leqslant\to} \xrightarrow{\|-\|^{\leqslant\to}} \mathbf{Pos}^{\top}$ *of the functor* $\mathbf{WEOAlg}^{\top} \xrightarrow{\|-\|} \mathbf{Pos}^{\top}$.
- *There exists the restriction* $\mathbf{Pos}^{\top} \xrightarrow{F^{\leqslant\to}} \mathbf{WEOAlg}^{\leqslant\to}$ *of the functor* $\mathbf{Pos}^{\top} \xrightarrow{F} \mathbf{WEOAlg}^{\top}$.
- $F^{\leqslant\to}$ *is a left-adjoint-right-inverse to* $\| - \|^{\leqslant\to}$.

! The category $\mathbf{Pos}^{\top}$ embeds into the category $\mathbf{WEOAlg}^{\leqslant\to}$.

More bounds in play

Definition 12

BPos is the non-full subcategory of $\mathbf{Pos}^{\top}$, whose

objects are bounded posets $(X, \leqslant, \bot, \top)$, and whose

morphisms are monotone maps preserving the bounds.

Definition 13

$\mathbf{WEOAlg}^{\leqslant\bot}$ is the non-full subcategory of $\mathbf{WEOAlg}^{\leqslant}$, whose

objects are w-eo algebras $(A, \to, \top)$ having some $\bot \in A$ such that

- $\bot \to a = \top$ for every $a \in A$,

and whose

morphisms are $\bot$-preserving $\mathbf{WEOAlg}^{\leqslant}$-morphisms.

Application to w-eo algebras

Theorem 14

- *There exists the restriction* $\mathbf{WEOAlg}^{\leqslant\bot} \xrightarrow{\|-\|^{\leqslant\bot}} \mathbf{BPos}$ *of the functor* $\mathbf{WEOAlg}^{\top} \xrightarrow{\|-\|} \mathbf{Pos}^{\top}$.
- *There exists a functor* $\mathbf{BPos} \xrightarrow{G} \mathbf{WEOAlg}^{\leqslant\bot}$ *which is given by* $G((X,\leqslant,\bot,\top) \xrightarrow{f} (Y,\leqslant,\bot,\top)) = (X,\to,\top) \xrightarrow{f} (Y,\to,\top)$, *where*

$$z_1 \to z_2 = \begin{cases} \top, & z_1 \leqslant z_2 \\ \bot, & \textit{otherwise}. \end{cases}$$

- G *is a left-adjoint-right-inverse to* $\| - \|^{\leqslant\bot}$.

! The category **BPos** embeds into the category $\mathbf{WEOAlg}^{\leqslant\bot}$.

Making a preordered set partially ordered

Definition 15

Prost is the category, whose

objects are preordered sets $(X, \leqslant)$ (the relation $\leqslant$ is reflexive and transitive), and whose

morphisms are monotone maps.

! **Pos** is the full subcategory of **Prost**, with the embedding E.

Theorem 16

The embedding $\mathbf{Pos} \overset{E}{\hookrightarrow} \mathbf{Prost}$ *has a left adjoint.*

Proof.

For a preordered set $(X, \leqslant)$, define an equivalence relation $x_1 \sim x_2$ iff $x_1 \leqslant x_2$ and $x_2 \leqslant x_1$, and consider the quotient set $(X/\sim, \leqslant_\sim)$.

Weak extended-preorder algebras

Definition 17

WEPOAlg is the category, whose

objects weak extended-preorder algebras (w-epo algebras) are triples $(A, \rightarrow, \top)$, where L is a non-empty set, $\rightarrow$ is a binary operation on L, and $\top$ is an element of L such that for every $a, b, c \in L$, the following conditions are satisfied:

1. $a \rightarrow \top = \top$;
2. $a \rightarrow a = \top$;
3. if $a \rightarrow b = \top$ and $b \rightarrow c = \top$, then $a \rightarrow c = \top$;

and whose

morphisms $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ are maps $A \xrightarrow{\varphi} B$ such that

- $\varphi(a_1 \rightarrow a_2) = \varphi(a_1) \rightarrow \varphi(a_2)$ for every $a_1, a_2 \in A$.

! Every w-epo algebra homomorphism is $\top$-preserving.

Important subcategories

Definition 18

WEOAlg is the full subcategory of **WEPOAlg** of w-eo algebras.

Definition 19

WEPOAlg* is the full subcategory of **WEPOAlg**, whose objects (w-epo* algebras) are all w-epo algebras $(A, \rightarrow, \top)$, which satisfy for every $a, b, c, d \in A$ the following conditions:

1. if $a \rightarrow b = \top$, $b \rightarrow a = \top$ and $c \rightarrow d = \top$, $d \rightarrow c = \top$, then $(a \rightarrow c) \rightarrow (b \rightarrow d) = \top$ and $(b \rightarrow d) \rightarrow (a \rightarrow c) = \top$;
2. if $\top \rightarrow (a \rightarrow b) = \top$, $\top \rightarrow (b \rightarrow c) = \top$, then $\top \rightarrow (a \rightarrow c) = \top$;
3. if $\top \rightarrow (a \rightarrow b) = \top$, $\top \rightarrow (b \rightarrow a) = \top$, then $a \rightarrow b = \top$ and $b \rightarrow a = \top$.

! **WEOAlg** is the full subcategory of **WEPOAlg***, E standing for the embedding functor.

Making a w-eo algebra out of w-epo algebra

Theorem 20

The embedding $\textbf{WEOAlg} \xhookrightarrow{E} \textbf{WEPOAlg}^*$ *has a left adjoint.*

Proof.

- Given a w-epo* algebra $(A, \rightarrow, \top)$, define a congruence $\sim$ on A by $a \sim b$ iff $a \rightarrow b = \top$ and $b \rightarrow a = \top$.
- Define $(A/\sim) \times (A/\sim) \xrightarrow{\rightsquigarrow} (A/\sim)$ by $[a] \rightsquigarrow [b] = [a \rightarrow b]$, where $[a] = \{c \in A \mid a \sim c\}$ is the congruence class of a, and obtain a w-eo algebra $(A/\sim, \rightsquigarrow, [\top])$.
- Easy computations show that the quotient map $A \xrightarrow{p} (A/\sim)$, $p(a) = [a]$ is the required E-universal arrow for $(A, \rightarrow, \top)$.

Completion of posets

Definition 21

CSLat$(\bigvee)$ is the (non-full) subcategory of **Pos** (the embedding functor denoted by E), whose

objects are $\bigvee$-semilattices (posets having arbitrary $\bigvee$), and whose

morphisms are $\bigvee$-preserving maps.

Theorem 22

The embedding **CSLat**$(\bigvee) \overset{E}{\hookrightarrow}$ **Pos** *has a left adjoint.*

Proof.

Given a poset $(X, \leqslant)$, let $\mathcal{P}_{\downarrow}(X)$ be the collection of all lower sets S of X ($s \in S$ and $x \leqslant s$ imply $x \in S$). The map $X \xrightarrow{\downarrow(-)} \mathcal{P}_{\downarrow}(X)$, $\downarrow x = \{y \in X \mid y \leqslant x\}$ provides an E-universal arrow for $(X, \leqslant)$.

Left-distributive eo algebras

! Recall that every w-eo algebra $(A, \rightarrow, \top)$ comes equipped with a partial order induced by the operation $\rightarrow$.

Definition 23

$\mathbf{LDEOAlg}^{\leqslant}$ is the full subcategory of $\mathbf{WEOAlg}^{\leqslant}$, whose objects are left-distributive eo algebras (ldeo algebras), i.e., w-eo algebras $(A, \rightarrow, \top)$, which satisfy for every $a, b, c \in A$ and every $S \subseteq A$ the following conditions:

1. $(\bigvee S) \rightarrow a = \bigwedge_{s \in S}(s \rightarrow a)$, if the respective $\bigvee$ and $\bigwedge$ exist;
2. if $b \rightarrow c = \top$, then $(a \rightarrow b) \rightarrow (a \rightarrow c) = \top$.

Complete left-distributive eo algebras

Definition 24

$\mathbf{LDEOAlg}^{\leqslant}(\bigvee)$ is the (non-full) subcategory of $\mathbf{LDEOAlg}^{\leqslant}$ (with the embedding denoted by E), whose

objects are ldeo algebras, which are also $\bigvee$-semilattices, and whose

morphisms are $\bigvee$-preserving ldeo algebra homomorphisms.

! The category $\mathbf{LDEOAlg}^{\leqslant}(\bigvee)$ provides a substitution for the category $\mathbf{CSLat}(\bigvee)$.

Completion of w-eo algebras

Theorem 25

The functor $\mathbf{LDEOAlg}^{\leqslant}(\bigvee) \overset{E}{\hookrightarrow} \mathbf{LDEOAlg}^{\leqslant}$ *has a left adjoint.*

Proof.

- Define a completion of an ldeo algebra $(A, \to, \top)$ as follows:
  1. $\mathcal{P}_{\downarrow}(A) = \{\downarrow S \mid S \subseteq A\}$, where $\downarrow S = \{a \in A \mid a \to s = \top$ for some $s \in S\}$;
  2. for $T_1, T_2 \in \mathcal{P}_{\downarrow}(A)$ let $T_1 \rightsquigarrow T_2 = \bigcap_{t_1 \in T_1} \bigcup_{t_2 \in T_2} \downarrow(t_1 \to t_2)$;
  3. given a family $(T_i)_{i \in I} \subseteq \mathcal{P}_{\downarrow}(A)$ let $\bigvee_{i \in I} T_i = \bigcup_{i \in I} T_i$.
- Easy computations show that the map $A \xrightarrow{\downarrow(-)} \mathcal{P}_{\downarrow}(A)$ provides an E-universal arrow for $(A, \to, \top)$.

No chance for improvement

- The map $A \xrightarrow{\downarrow(-)} \mathcal{P}_{\downarrow}(A)$, obtained in Theorem 25, has the property $\downarrow(a \to b) = \downarrow a \rightsquigarrow \downarrow b$, motivating the change from $\mathbf{LDEOAlg}^{\leqslant}$ to $\mathbf{LDEOAlg}$. The next lemma dismisses the modification.

Lemma 26

The adjunction of Theorem 25 does not allow the restriction to the category $\mathbf{LDEOAlg}$.

Proof.

Consider the ldeo algebra $(\mathbf{2} = \{\bot, \top\}, \leqslant, \top)$. If the restriction is possible, there exists a **WEOAlg**-morphism $\mathcal{P}_{\downarrow}(\mathbf{2}) \xrightarrow{\varphi} \mathbf{2}$ defined by $\varphi(T) = \bigvee T$. On the other hand, $T_1 = \{\bot\}$ and $T_2 = \varnothing$ provide $\varphi(T_1 \rightsquigarrow T_2) = \bot < \top = \varphi(T_1) \to \varphi(T_2)$.

Comparison with the result of C. Guido *et al.*

Definition 27

An eo algebra is a w-eo algebra $(A, \to, \top)$, which satisfies for every $a, b, c \in A$ the following conditions:

1. if $a \to b = \top$, then $(c \to a) \to (c \to b) = \top$;
2. if $a \to b = \top$, then $(b \to c) \to (a \to c) = \top$.

- C. Guido *et al.* constructed the MacNeille completion of an eo algebra $(A, \to, \top)$ such that the new operation $\rightsquigarrow$ provides an extension of the original one.
- The construction of Theorem 25 provides a larger (in terms of cardinality) completion of eo algebras, the additional condition of distributivity used to extend the result to homomorphisms.
- ! The object part of the new framework simplifies the respective procedure of C. Guido *et al.*

Free partially ordered sets over sets

- There exists (the obvious) forgetful functor $\mathbf{Pos} \xrightarrow{|-|} \mathbf{Set}$.

Theorem 28

The functor $\mathbf{Pos} \xrightarrow{|-|} \mathbf{Set}$ *has a left adjoint.*

Proof.

Given a set X, the map $X \xrightarrow{1_X} |(X, =)|$ provides a $|-|$-universal arrow for X.

Application to w-eo algebras

Theorem 29

The forgetful functor $\mathbf{WEOAlg}^{\leqslant\to} \xrightarrow{|-|} \mathbf{Set}$ *has a left adjoint.*

Proof.

- Given a set X, define $F(X) = X \uplus \{\top\}$ and let

$$x \to y = \begin{cases} \top, & x = y \\ y, & \text{otherwise.} \end{cases}$$

- $(F(X), \to, \top)$ is in $\mathbf{WEOAlg}^{\leqslant\to}$, and the map $X \xrightarrow{\eta} F(X)$ with $\eta(x) = x$ is a $|-|$-universal arrow for X.

Adding more restrictions

Definition 30

$\mathbf{WEOAlg}^{\leqslant\rightarrow *}$ is the full subcategory of $\mathbf{WEOAlg}^{\leqslant\rightarrow}$, whose objects $(A, \rightarrow, \top)$ satisfy for every $a, b, c \in A$ the next condition:

- if $a \rightarrow b = \top$ and $a \rightarrow c \neq \top$, then $a \rightarrow (b \rightarrow c) \neq \top$.

Theorem 31

There exists the restriction of the adjunction of Theorem 29 to the category $\mathbf{WEOAlg}^{\leqslant\rightarrow *}$.

Corollary 32

The monomorphism in both $\mathbf{WEOAlg}^{\leqslant\rightarrow}$ *and* $\mathbf{WEOAlg}^{\leqslant\rightarrow *}$ *are precisely the injective maps.*

Coseparators in the category of posets

Definition 33

An object C of a category $\mathbf{C}$ is called coseparator provided that for every distinct morphisms $B \xrightarrow{f} A$, $B \xrightarrow{g} A$, there exists a morphism $A \xrightarrow{h} C$ such that $B \xrightarrow{f} A \xrightarrow{h} C \neq B \xrightarrow{g} A \xrightarrow{h} C$.

Lemma 34

Coseparators in **Pos** *are precisely the non-discrete (the order is not given by equality) posets.*

Coseparators in the category of w-eo algebras

Theorem 35

The coseparators in $\mathbf{WEOAlg}^{\leqslant\rightarrow *}$ are precisely the objects having at least two elements.

Proof.

- Given distinct $B \xrightarrow{\varphi} A$, $B \xrightarrow{\psi} A$, choose $b \in B$ with $\varphi(b) \neq \psi(b)$.
- Take some $(C, \rightarrow, \top)$ in $\mathbf{WEOAlg}^{\leqslant\rightarrow *}$ with $c \in C$, $c \neq \top$.
- Define $A \xrightarrow{\phi} C$ by

$$\phi(a) = \begin{cases} \top, & \phi(b) \rightarrow a = \top \\ c, & \text{otherwise.} \end{cases}$$

- ϕ is in $\mathbf{WEOAlg}^{\leqslant\rightarrow *}$ and $\phi \circ \varphi \neq \phi \circ \psi$.

! $(\mathbf{2}, \leqslant, \top)$ is a coseparator in $\mathbf{WEOAlg}^{\leqslant\rightarrow *}$.

Epimorphisms in the category of posets

Definition 36

A morphism $A \xrightarrow{f} B$ of a category $\mathbf{C}$ is said to be an epimorphism provided that for all pairs $B \xrightarrow{h} C$, $B \xrightarrow{k} C$ of morphisms such that $h \circ f = k \circ f$, it follows that $h = k$.

Lemma 37

Epimorphisms in **Pos** *are precisely the morphisms with surjective underlying maps.*

Epimorphisms in the category of w-eo algebras

Theorem 38

Epimorphisms in $\mathbf{WEOAlg}^{\leqslant\rightarrow}$ *are the surjective morphisms.*

Proof (the necessity).

- Take a non-surjective $A \xrightarrow{\varphi} B$ in $\mathbf{WEOAlg}^{\leqslant\rightarrow}$.
- Choose some $b_0 \in B \backslash \varphi^{\rightarrow}(A)$, define $B_* = B \biguplus \{*\}$ and let

$$b_1 \rightarrow_* b_2 = \begin{cases} b_1 \rightarrow_B b_2, & b_1 \neq * \neq b_2 \\ \top, & b_1 = b_2 = * \text{ or } (b_1 = *,\ b_2 = b_0) \\ *, & b_1 = b_0,\ b_2 = * \\ b_0 \rightarrow_B b_2, & b_1 = *,\ b_2 \in B \backslash \{b_0, *\} \\ b_1 \rightarrow_B b_0, & b_1 \in B \backslash \{b_0, *\},\ b_2 = *. \end{cases}$$

- Let $B \xrightarrow{\psi_1} B_*$, $\psi_1(b) = b$ and $B \xrightarrow{\psi_2} B_*$, $\psi_2(b_0) = *$; otherwise, $\psi_2(b) = b$. It follows that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$ and $\psi_1 \neq \psi_2$.

Further restriction is not possible

Lemma 39

The $\mathbf{WEOAlg}^{\leqslant\to}$*-object* $(B_*, \to_*, \top)$ *constructed in Theorem 38 does not belong to the category* $\mathbf{WEOAlg}^{\leqslant\to *}$*.*

Proof.

Define $b = b_0$, $b_1 = \top$ and $b_2 = *$. Then $b \to_* b_1 = \top$, $b \to_* b_2 \neq \top$, but $b \to_* (b_1 \to_* b_2) = \top$.

Initial morphisms in the category of posets

Definition 40

Let $(\mathbf{A}, |-|)$ be a concrete category over $\mathbf{X}$. An $\mathbf{A}$-morphism $A \xrightarrow{f} B$ is called initial provided that for every $\mathbf{A}$-object C, an $\mathbf{X}$-morphism $|C| \xrightarrow{g} |A|$ is an $\mathbf{A}$-morphism whenever $|C| \xrightarrow{f \circ g} |B|$ is an $\mathbf{A}$-morphism.

Theorem 41

In the category **Pos***, a morphism* $(X, \leqslant) \xrightarrow{f} (Y, \leqslant)$ *is initial iff the equivalence* $x_1 \leqslant x_2 \Leftrightarrow f(x_1) \leqslant f(x_2)$ *holds.*

Corollary 42

Initial morphisms in **Pos** *have injective underlying maps.*

Initial morphisms in the category of w-eo algebras

Theorem 43

A $\mathbf{WEOAlg}^{\leqslant \rightarrow}$*-morphism* $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ *is initial iff for every* $a_1, a_2 \in A$*, the following condition holds:*

- $a_1 \rightarrow a_2 = \bigvee\{a \in A \mid \varphi(a_2) \leqslant \varphi(a) \leqslant \varphi(a_1) \rightarrow \varphi(a_2)\}$.

Corollary 44

Initial $\mathbf{WEOAlg}^{\leqslant \rightarrow}$*-morphisms have injective underlying maps.*

Proof.

Every initial $\mathbf{WEOAlg}^{\leqslant \rightarrow}$-morphism $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ has the property $a_1 \rightarrow a_2 = \top$ iff $\varphi(a_1) \rightarrow \varphi(a_2) = \top$ for every $a_1, a_2 \in A$.

Products of w-eo algebras

Theorem 45

The category **WEOAlg** *has products of objects.*

Proof.

Given some family $((A_i, \rightarrow_i, \top_i))_{i \in I}$ of w-eo algebras, the cartesian product $\prod_{i \in I} A_i$ of the underlying sets, equipped with the pointwise structure, provides the required product in the category **WEOAlg**.

! The construction applies to, e.g., the categories $\mathbf{WEOAlg}^{\top}$, $\mathbf{WEOAlg}^{\leqslant}$, $\mathbf{WEOAlg}^{\leqslant\rightarrow}$ and $\mathbf{WEOAlg}^{\leqslant\rightarrow *}$ as well.

Coproducts of w-eo algebras

Theorem 46

The category $\mathbf{WEOAlg}^{\leqslant \rightarrow}$ *has coproducts of objects.*

Proof.

- Take a family $((A_i, \rightarrow_i, \top_i))_{i \in I}$ of $\mathbf{WEOAlg}^{\leqslant \rightarrow}$-objects.
- Let $\bigoplus_{i \in I} A_i = (\biguplus_{i \in I}(A_i \setminus \{\top_i\})) \uplus \{\top\}$ and $\coprod_{i \in I}(A_i, \rightarrow_i, \top_i) = (\bigoplus_{i \in I} A_i, \rightarrow, \top)$, where

$$a \rightarrow b = \begin{cases} a \rightarrow_i b, & a, b \in A_i \text{ for some } i \in I \\ b, & a \in A_i, b \in A_j \text{ and } i \neq j. \end{cases}$$

- For $j \in I$ let $(A_j, \rightarrow_j, \top_j) \xrightarrow{\mu_j} \coprod_{i \in I}(A_i, \rightarrow_i, \top_i)$, $\mu_j(a) = a$.
- $((\mu_i)_I, \coprod_{i \in I}(A_i, \rightarrow_i, \top_i))$ provides the required coproduct in the category $\mathbf{WEOAlg}^{\leqslant \rightarrow}$.

w-eo algebras versus d-algebras

Definition 47

Given a w-eo algebra $(A, \to, \top)$, its dual (denoted by $(A, \to, \top)^d$) is the triple $(A, \rightsquigarrow, \bot)$, where $\bot = \top$ and $a \rightsquigarrow b = b \to a$.

Lemma 48

Every dual w-eo algebra $(A, \to, \top)^d$ has the following properties:

1. $\bot \rightsquigarrow a = \bot$;
2. $a \rightsquigarrow a = \bot$;
3. *if* $a \rightsquigarrow b = \bot$ *and* $b \rightsquigarrow a = \bot$, *then* $a = b$;
4. *if* $a \rightsquigarrow b = \bot$ *and* $b \rightsquigarrow c = \bot$, *then* $a \rightsquigarrow c = \bot$.

! The category $\mathbf{WEOAlg}^d$ of dual w-eo algebras arises, which is isomorphic to the category of d-transitive d-algebras provided by J. Neggers and H. S. Kim.

Different categories of w-eo algebras

- The talk introduced several approaches to homomorphisms of w-eo algebras based on different categories of the structures.
- The two main categories (with w-eo algebras as objects) are:
  1. **WEOAlg**, whose morphisms $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ are maps $A \xrightarrow{\varphi} B$ with $\varphi(a_1 \rightarrow a_2) = \varphi(a_1) \rightarrow \varphi(a_2)$ for every $a_1, a_2 \in A$. The additional property $\varphi(\top) = \top$ comes as a consequence.
  2. **WEOAlg**$^{\leqslant}$, whose morphisms $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ are maps $A \xrightarrow{\varphi} B$ with $\varphi(a_1 \rightarrow a_2) \leqslant \varphi(a_1) \rightarrow \varphi(a_2)$ for every $a_1, a_2 \in A$, and $\varphi(\top) = \top$.
- Approach 1 backs the algebraic viewpoint on w-eo algebras.
- Approach 2 considers w-eo algebras as an extension of posets.

Open problems

- The talk considered several subcategories of the category $\mathbf{WEOAlg}^{\leqslant}$, to provide a convenient framework to match different properties of the category **Pos**.
- The abundance of available subcategories motivates the following problems.

Problem 49

What is the best subcategory of $\mathbf{WEOAlg}^{\leqslant}$ *to get a "convenient" analogue of the category* **Pos***?*

Problem 50

Does there exist a better starting point than the above-mentioned category $\mathbf{WEOAlg}^{\leqslant}$*?*

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Thank you for your attention!